Calculus: practice questions

Your actual exam is 2 hours and will consist of 11 questions, all compulsory and comparable to those appearing here. It is hoped that these specimen questions will be a useful aid to you in your revision, but you need to revise the notes and exercise sheets as well.

1. Assuming that $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ where $i = \sqrt{-1}$, show that

$$\cos^5 x = \tfrac{1}{16}\cos 5x + \tfrac{5}{16}\cos 3x + \tfrac{5}{8}\cos x$$

Using the above result and Osborn's rule, state a formula for $\cosh^5 x$ in terms of $\cosh 5x$, $\cosh 3x$ and $\cosh x$.

2. Find an identity for $\cos^3 x$ in terms of $\cos 3x$ and $\cos x$, and hence find $\displaystyle\int_0^{\pi/2} \cos^3 x \, dx$.

3. Calculate the Maclaurin expansion of $\cos x$ up to and including the x^4 term. Use the series and the result of Question 1 to find the Maclaurin series of $\cos^5 x$ up to and including the x^4 term. All coefficients must be expressed as fractions in their lowest terms.

4. Find the binomial expansion of $(1 + t^2)^{-1/2}$ up to and including the term in t^6. Using the fact that

$$\sinh^{-1} x = \int_0^x \frac{1}{\sqrt{1+t^2}} \, dt$$

show that the Maclaurin expansion of $\sinh^{-1} x$ is

$$\sinh^{-1} x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \cdots$$

Given that $\sin^{-1} x = -i \sinh^{-1}(ix)$ where $i = \sqrt{-1}$, deduce that

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots.$$

5. Find the Maclaurin series for $\sin x$ up to and including the x^5 term and deduce the Maclaurin series for $\sin(x^2)$ up to and including the term in x^{10}. Hence find an approximate value for

$$\int_0^1 \sin(x^2) \, dx$$

giving your answer to four decimal places.

6. Give the definitions of the functions $\cosh x$ and $\sinh x$ in terms of exponentials and prove from these definitions that (i) $\sinh x$ is an odd function; (ii) $\cosh 2x = 2\sinh^2 x + 1$, (iii) $\cosh x + \sinh x = e^x$ and (iv) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$. Also find $\displaystyle\int \sinh^2 x \, dx$.

7. Show that
$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$
Hence show that if $\sinh y = \tan x$ then
$$y = \ln(\sec x + \tan x).$$

8. Show that $\dfrac{d}{dx}\left(\sinh^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{x^2+a^2}}$ and evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+20}}$.

9. If $\sinh x = -2$ find the values of $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\coth x$. Show also that $\cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(1+\cosh x)}$.

10. Find the following limits:

$$\text{(i)}\ \lim_{x\to 2}\frac{x^3-8}{x-2} \qquad \text{(ii)}\ \lim_{x\to 0}\frac{\sin(\sin x)}{\sin x} \qquad \text{(iii)}\ \lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{1}{2}x}{x^2}$$

11. Find the following limits:

$$\text{(i)}\ \lim_{x\to 1}\frac{x^{1/3}-1}{\sqrt{x}-1} \qquad \text{(ii)}\ \lim_{x\to 0}\frac{x^2}{\cos x-\cosh x} \qquad \text{(iii)}\ \lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}$$

12. Find the following limits, using a substitution for part (iii):

$$\text{(i)}\ \lim_{x\to 1}\frac{e^{x-1}-1}{\ln x} \qquad \text{(ii)}\ \lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)} \qquad \text{(iii)}\ \lim_{x\to\infty} x\sinh(3/x)$$

13. Find the sum to infinity of the geometric series $\displaystyle\sum_{n=1}^{\infty} x^{n+1}$. By differentiating your answer twice and then multiplying by x show that, for $-1 < x < 1$,
$$\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$$
and hence show that $\displaystyle\sum_{n=1}^{\infty}\frac{n(n+1)}{3^n} = \frac{9}{4}$.

14. Find the sum to infinity of $\displaystyle\sum_{n=0}^{\infty} t^n$ assuming $|t| < 1$. By integrating your result with respect to t between limits 0 and x, deduce the Maclaurin series for $\ln(1-x)$:
$$\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$
Hence show that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{e^2-1}{e^2}\right)^{n+1} = 2.$

15. Evaluate the integrals

$$\text{(i)} \int_0^{\pi/4} \frac{\sin x}{\cos^3 x}\,dx \qquad \text{(ii)} \int_0^{\pi/2} x \sin 4x\,dx \qquad \text{(iii)} \int_2^4 \frac{x-2}{x(x+2)}\,dx$$

16. Evaluate the integrals

$$\text{(i)} \int_0^1 \frac{x^2}{(1+x^3)^2}\,dx \qquad \text{(ii)} \int_1^4 \sqrt{x}\ln x\,dx \qquad \text{(iii)} \int_0^{\sqrt{3}} \frac{3-x}{(x+2)(x^2+1)}\,dx$$

17. Show that, if n is a positive integer, then

$$\int_0^{\pi/4} \tan^n x\,dx = \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2} x\,dx.$$

[You will find the identity $\tan^2 x = \sec^2 x - 1$ useful].
Hence evaluate $\int_0^{\pi/4} \tan^5 x\,dx$.

18. Suppose that $\lambda > 0$ and n is an integer with $n \geq 1$. Show that

$$\int_0^\infty x^n e^{-\lambda x}\,dx = (n/\lambda)\int_0^\infty x^{n-1}e^{-\lambda x}\,dx.$$

[You may assume that $x^n e^{-\lambda x} \to 0$ as $x \to \infty$]. Hence show that

$$\int_0^\infty x^4 e^{-\lambda x}\,dx = \frac{24}{\lambda^5}$$

19. Solve the following differential equations, expressing the solution in the form $y = f(x)$:

(i) $\dfrac{dy}{dx} = -2y^2 x$

(ii) $\dfrac{dy}{dx} = (y^2+1)x^2$ subject to $y(0) = 1$

(iii) $\dfrac{dy}{dx} + 3y = x$

20. Solve the following differential equations, expressing the solution in the form $y = f(x)$:

(i) $\dfrac{dy}{dx} = yx^3$

(ii) $e^{2x}\dfrac{dy}{dx} = y^2$ subject to $y(0) = 1$

(iii) $x\dfrac{dy}{dx} - 2y = x^4 e^{-x}$

21. Solve the differential equations:

(i) $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 10y = 0$

(ii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = x^2 - 1$

22. Solve the differential equations:

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$ subject to $y(0) = -2$ and $y'(0) = 1$

(ii) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = \cos 2x$

23. By introducing the new variable $u = y/x$, solve the differential equation

$$x\frac{dy}{dx} = xe^{y/x} + y$$

24. Transform the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^3y^2$$

into a new differential equation involving only x, v and dv/dx, where $v = 1/y$. Hence solve the original differential equation.

25. Find $\partial z/\partial x$ and $\partial z/\partial y$ when

(a) $z = x^2 \ln y$, (b) $z = x^2e^{-3y}$, (c) $z = \dfrac{x}{x^2+y^2}$, (d) $z = \sin(x^2+5y)$, (e) $z = e^{xy} \ln y$

26. Show that $w = \ln(x^2 + y^2 + z^2)$ satisfies

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{2}{x^2 + y^2 + z^2}.$$

Answers/hints:

1. Since there are no products or implied products of sine terms, the hyperbolic equivalent is the same identity with cos replaced by cosh.

2. $\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$, $\displaystyle\int_0^{\pi/2} \cos^3 x\,dx = \frac{2}{3}$.

3. $\cos^5 x = 1 - \frac{5}{2}x^2 + \frac{65}{24}x^4 + \cdots$.

4. $(1+t^2)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \cdots$.

5. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, $\sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots$, $\displaystyle\int_0^1 \sin(x^2)\,dx \approx 0.3103$.

6. $\displaystyle\int \sinh^2 x\,dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x + c$.

8. $\sinh^{-1}\left(\frac{x-2}{4}\right) + c$

9. $\cosh x = \sqrt{5}$, $\tanh x = -2/\sqrt{5}$, $\operatorname{sech} x = 1/\sqrt{5}$, $\operatorname{cosech} x = -1/2$, $\coth x = -\sqrt{5}/2$. For the last part find an identity that relates $\cosh x$ with $\cosh 2x$ and replace x by $x/2$.

10. (i) 12, (ii) 1, (iii) $-\frac{1}{8}$.

11. (i) $\frac{2}{3}$, (ii) -1, (iii) $\frac{1}{2}$.

12. (i) 1, (ii) $\frac{1}{2}$, (iii) substitute $x = 1/y$, answer 3.

13. At A-Level the formula for summing an infinite geometric series is usually taught in the form

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \qquad \text{for } |r| < 1.$$

15. (i) $\frac{1}{2}$, (ii) $-\frac{\pi}{8}$, (iii) $2\ln 3 - 3\ln 2$.

16. (i) $\frac{1}{6}$, (ii) $\frac{32}{3}\ln 2 - \frac{28}{9}$, (iii) $\ln(\sqrt{3}+2) - 2\ln 2 + \pi/3$.

17. For the first part, write the integrand as $\tan^{n-2} x \tan^2 x$.

$\displaystyle\int_0^{\pi/4} \tan^5 x\,dx = \ln(\sqrt{2}) - \frac{1}{4} = \frac{1}{2}\ln 2 - \frac{1}{4}$.

19. (i) $y = \frac{1}{x^2 - c}$, (ii) $y = \tan\left(\frac{x^3}{3} + \frac{\pi}{4}\right)$, (iii) $y = \frac{x}{3} - \frac{1}{9} + c\,e^{-3x}$.

20. (i) $y = Ae^{x^4/4}$, (ii) $y = \frac{2}{1+e^{-2x}}$, (iii) $y = -x^2(x+1)e^{-x} + c\,x^2$.

21. (i) $y = A\,e^{-5x} + B\,e^{-2x}$, (ii) $y = e^{-x}(A\cos 2x + B\sin 2x) + \frac{1}{5}x^2 - \frac{4}{25}x - \frac{27}{125}$.

22. (i) $y = e^{-x}(-2\cos 2x - \frac{1}{2}\sin 2x)$, (ii) $y = Ae^{-2x} + Be^{x} - \frac{3}{20}\cos 2x + \frac{1}{20}\sin 2x$.

23. $y = -x\ln(-\ln x + c)$.

24. $y = \dfrac{1}{cx - x^4/3}$

25. (a) $2x\ln y$, x^2/y, (b) $2x\,e^{-3y}$, $-3x^2\,e^{-3y}$, (c) $(y^2 - x^2)/(x^2 + y^2)^2$, $-2xy/(x^2 + y^2)^2$, (d) $2x\cos(x^2 + 5y)$, $5\cos(x^2 + 5y)$, (e) $y\,e^{xy}\ln y$, $e^{xy}/y + x\,e^{xy}\ln y$.