Answer all questions. All working must be shown.

The marks for each question are shown in brackets; you should note that some questions carry more marks than others.
1. Solve the following differential equations, expressing the solution in the form 
\( y = f(x) \):

(i) \( \frac{dy}{dx} = \frac{e^{-y}}{x} \) \[3\]

(ii) \( \frac{dy}{dx} = \frac{\sin x}{y^2} \) subject to \( y(0) = 4 \) \[5\]

(iii) \( (x + 1)\frac{dy}{dx} + 2y = 1 \) \[5\]

2. A thief has stolen a can of lemonade from a fridge. When he is arrested for this
defence at 1.30pm, the temperature of the lemonade is 11°C. Half an hour later the
temperature of the lemonade has risen to 14°C. The air temperature is 18°C and the
temperature in the fridge is 2°C. Calculate, to the nearest minute, when the lemonade
was stolen (giving the answer as a time of day).
[Assume that the temperature \( T \) of the lemonade satisfies \( \frac{dT}{dt} = k(T - 18) \) for some
number \( k \)]. \[8\]

3. Solve the differential equations:

(i) \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 26y = 0 \). \[4\]

(ii) \( \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 5 - 6x \) subject to \( y(0) = 3 \) and \( y'(0) = 11 \). \[9\]

4. Calculate the partial derivatives \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) of each of the following

(i) \( z = 3y^2 + xy + 3y^2x^4 \) \[4\]

(ii) \( z = \frac{2x^2}{x + y} \) \[4\]

5. An open-top rectangular cardboard box is to have a volume of 3 m\(^3\). If \( A \) is the
total area of cardboard and \( x \) and \( y \) are the length and breadth of the base show that
\[
A = xy + \frac{6}{x} + \frac{6}{y}
\]

Find the dimensions of the box which minimise the amount of cardboard required,
giving each measurement to three decimal places. [NB: you must confirm that you
have a minimum]. \[13\]

6. For each of the following systems of simultaneous equations determine whether
the system has a unique solution, an infinite number of solutions or no solution and
solve where possible. \[14\]

(i) \[
\begin{align*}
2x + y - 3z &= -5 \\
x - y + 2z &= 12
\end{align*}
\]

(ii) \[
\begin{align*}
x + y + 5z &= -2 \\
4x + 7y + 8z &= 2 \\
3x + 5y + 7z &= 1
\end{align*}
\]

[NB: careless errors will be heavily penalised.]
7.

(i) Find the rank of the matrix

\[
\begin{pmatrix}
5 & -2 & 5 & 6 & 1 \\
-2 & 0 & 1 & -1 & 3 \\
-1 & -2 & 8 & 3 & 10
\end{pmatrix}
\]

(ii) Show that

\[
\begin{vmatrix}
 x + 1 & -1 & 2 \\
 2 & x - 1 & 1 \\
 1 & -1 & x + 2
\end{vmatrix} = x^3 + 2x^2
\]

8. A triangular lamina is bounded by the x-axis, the line \( x = 1 \) and the line \( y = 2x \). Sketch this lamina. The lamina rotates about the x-axis. Find its moment of inertia in terms of its total mass \( M \). [Hint: divide the lamina up into thin vertical rods, and use the fact that the moment of inertia of a rod of length \( y \) rotating about one end is \( \frac{1}{3}my^2 \) where \( m \) is the rod’s mass.]

9. Evaluate

(i) \( \int_0^1 \int_{x^2}^x (1 - xy) \, dy \, dx \)

(ii) \( \int \int_D (x^2 + y^2) \, dA \) where \( D \) is the triangle with vertices (0, 0), (2, 0) and (1, 1).

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