Spring 2010: Answers to ENG1002 Mathematics 1b exam

1.
(i) $y = A\,\exp(-2x^3/3)$, (ii) $y = \tan(e^x + c)$, (iii) $y = (x-2)e^{-x^2}$.

2. $\dfrac{1}{T-2}\,dT = -k\,dt$ so T can be put in the form $T = 2 + Ae^{-kt}$. Taking $t = 0$ to be at midnight, we have $T = 20$ when $t = 0$ so $20 = 2 + A$ giving $A = 18$. Hence $T = 2 + 18e^{-kt}$. Also $T = 14$ when $t = 30$ so $12 = 18e^{-30k}$ giving $k = 0.0135155$. Hence $T = 2 + 18e^{-0.0135155t}$. When $t = 60$ (i.e. at 1am) the temperature is therefore $T = 2 + 18e^{-0.0135155\times 60} = 10^oC$.

3. (i) $y = \frac{9}{4}e^{2x} - \frac{1}{4}e^{-2x}$, (ii) $y = e^{3x}(A\cos 2x + B\sin 2x) + \frac{1}{25}\sin 2x + \frac{4}{75}\cos 2x$

4. (i) $\dfrac{\partial z}{\partial x} = 1 + 6xy - 4y^3$, $\dfrac{\partial z}{\partial y} = 3x^2 - 12xy^2 - 1/(y+1)$.

(ii) $\dfrac{\partial z}{\partial x} = -\dfrac{1}{(xy-1)^2}$, $\dfrac{\partial z}{\partial y} = -\dfrac{x^2}{(xy-1)^2}$.

5. First part elementary trigonometry.
Second part follows immediately from $\delta H \approx \dfrac{\partial H}{\partial x}\,\delta x + \dfrac{\partial H}{\partial \theta}\,\delta\theta$.
$H = 11.92\,\text{m}$ and $\delta H = 0.155\,\text{m}$.

6. The heating cost C is given by $C = \underbrace{xy}_{\text{roof}} + 2yz + 2(2xz) = xy + 2yz + 4xz$. But $xyz = 8000$ since the volume is 8000, so $z = 8000/xy$ and therefore the cost is given by

$$C(x,y) = xy + 2y\frac{8000}{xy} + 4x\frac{8000}{xy} = xy + \frac{16000}{x} + \frac{32000}{y}$$

The partial derivatives of this are

$$\frac{\partial C}{\partial x} = y - \frac{16000}{x^2}, \qquad \frac{\partial C}{\partial y} = x - \frac{32000}{y^2}, \qquad \frac{\partial^2 C}{\partial x^2} = \frac{32000}{x^3}, \qquad \frac{\partial^2 C}{\partial y^2} = \frac{64000}{y^3},$$

$$\frac{\partial^2 C}{\partial x \partial y} = 1.$$

We have $C_x = C_y = 0$ when $y = 16000/x^2$ and $x = 32000/y^2$, so

$$y = \frac{16000}{(32000/y^2)^2} = \frac{y^4}{64000}$$

Since $y \neq 0$, this means that $y = 40$. Then $x = 20$ and $z = 10$.
To confirm it's a minimum, note that, with these values, $C_{xx} = 4$, $C_{yy} = 1$ and $C_{xy} = 1$. So $C_{xx}C_{yy} - (C_{xy})^2 = 3 > 0$. Since also $C_{xx} > 0$ and $C_{yy} > 0$ this confirms

it's a minimum.

7. In matrix form:

$$\begin{pmatrix} 1 & 1 & 2 \\ -4 & -1 & 3 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \\ -1 \end{pmatrix}$$

Row echelon form of the augmented matrix $A|\mathbf{b}$ is

$$\begin{pmatrix} 1 & 1 & 2 & | & -7 \\ 0 & -1 & -5 & | & 13 \\ 0 & 0 & -4 & | & 12 \end{pmatrix}$$

or equivalent. So the system of simultaneous equations can be written

$$x + y + 2z = -7, \qquad -y - 5z = 13, \qquad -4z = 12$$

so that $x = -3$, $y = 2$ and $z = -3$.

8.

$$A^2 = \begin{pmatrix} 8 & -4 & 2 \\ 13 & 26 & -11 \\ 24 & 4 & 2 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 26 & 36 & -18 \\ 36 & -100 & 54 \\ 90 & 36 & -10 \end{pmatrix}$$

It is easily checked that $A^3 - 18A + 28I = 0$.

9. (i) Row echelon form of the matrix is

$$\begin{pmatrix} 1 & 0 & 6 & 1 \\ 0 & 2 & 23 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or equivalent. Hence the rank is 2.
(ii) Result can either be obtained by direct expansion, or by expansion of the following slightly simpler determinant obtained from the original using properties of determinants:

$$b^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a & a^2 + b \end{vmatrix}$$

10. (i)

$$\text{integral} = \int_0^2 \left[\frac{y^3 \sin 2x}{2} \right]_{x=0}^{x=\pi/4} dy = \tfrac{1}{2} \int_0^2 y^3 \, dy = 2$$

(ii)

$$\text{integral} = \int_0^1 \left[x^2 \ln y \right]_{y=1}^{y=e^x} dx = \int_0^1 x^3 \, dx = \frac{1}{4}.$$