Mathematics 1b: Sheet 1 solutions

1. (a) $2x \ln y$, x^2/y

(b) $2xe^{-3y}$, $-3x^2e^{-3y}$

(c) $\dfrac{y^2 - x^2}{(x^2+y^2)^2}$, $\dfrac{-2xy}{(x^2+y^2)^2}$

(d) $2x\cos(x^2+5y)$, $5\cos(x^2+5y)$

(e) $ye^{xy}\ln y$, $\frac{e^{xy}}{y} + xe^{xy}\ln y$

2. (a) $yz - 6x^2$, $xz + 2y$, $xy - 1$

(b) $\dfrac{2xy^3}{z}$, $\dfrac{3x^2y^2}{z}$, $-\dfrac{x^2y^3}{z^2}$

3.

$$\begin{aligned} z_x &= 4xy + y^3 + 12x^3y^3 \\ z_y &= 2x^2 + 3xy^2 + 9x^4y^2 \\ z_{xx} &= 4y + 36x^2y^3 \\ z_{yy} &= 6xy + 18x^4y \\ z_{xy} &= 4x + 3y^2 + 36x^3y^2 \\ z_{yx} &= 4x + 3y^2 + 36x^3y^2 \end{aligned}$$

4. $\dfrac{\partial T}{\partial t} = -4ke^{-4kt}(\cos 2x + \cos 2y)$, $\dfrac{\partial^2 T}{\partial x^2} = -4e^{-4kt}\cos 2x$, $\dfrac{\partial^2 T}{\partial y^2} = -4e^{-4kt}\cos 2y$.

5. $z = \dfrac{x}{x^2+y^2} = \dfrac{r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \dfrac{\cos\theta}{r}$. The partial derivatives with respect to x and y are $z_x = \dfrac{y^2-x^2}{(x^2+y^2)^2}$, $z_y = \dfrac{-2xy}{(x^2+y^2)^2}$ while $z_r = -\dfrac{\cos\theta}{r^2}$, $z_\theta = -\dfrac{\sin\theta}{r}$ so

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \frac{1}{r^4}$$

6. We have

$$\delta Q \approx \frac{\partial Q}{\partial p}\,\delta p + \frac{\partial Q}{\partial r}\,\delta r + \frac{\partial Q}{\partial \eta}\,\delta\eta + \frac{\partial Q}{\partial l}\,\delta l$$

Evaluating the partial derivatives and dividing by Q leads to the stated result. For the last part, one possibility is to identify the worst case scenario (ie the maximum possible value for $\dfrac{\delta Q}{Q}$) given the constraints that $\left|\dfrac{\delta p}{p}\right| \le 0.02$, $\left|\dfrac{\delta r}{r}\right| \le 0.01$, $\left|\dfrac{\delta\eta}{\eta}\right| \le 0.03$, $\left|\dfrac{\delta l}{l}\right| \le 0.02$. Since $\dfrac{\delta\eta}{\eta}$ and $\dfrac{\delta l}{l}$ have negative coefficients, $\dfrac{\delta Q}{Q}$ is maximised when $\dfrac{\delta p}{p} = 0.02$, $\dfrac{\delta r}{r} = 0.01$, $\dfrac{\delta\eta}{\eta} = -0.03$, $\dfrac{\delta l}{l} = -0.02$, giving $\dfrac{\delta Q}{Q} = 0.11$ or 11%.

Another possibility is to apply the triangle inequality (which the students may not know) to the formula for $\frac{\delta Q}{Q}$. This gives

$$\left|\frac{\delta Q}{Q}\right| \leq \left|\frac{\delta p}{p}\right| + 4\left|\frac{\delta r}{r}\right| + \left|\frac{\delta \eta}{\eta}\right| + \left|\frac{\delta l}{l}\right|$$

which gives the same answer, without worrying about the minus signs.