Faculty of Engineering and Physical Sciences

Mathematics 1b: tutorial sheet 2

Exercise 1:

If $z = e^{xy}$ and $x = 2u + v$ and $y = \frac{u}{v}$

This is obtained using the chain rule for partial differentials. Alternatively you may replace (this is only to check whether you get the correct result here) x and y by their values in u and v in the expression of z and obtain the partial differentials in u and v this way. You should obtain the same result as when using the chain rule.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u} = ye^{xy} \times 2 + xe^{xy} \times \frac{1}{v} = e^{xy}(2y + \frac{x}{v}) = e^{(2u+v)\times\frac{u}{v}}(2 \times \frac{u}{v} + \frac{(2u+v)}{v}) =$$

$$\frac{\partial z}{\partial u} = e^{(2u+v)\times\frac{u}{v}}(\frac{2u + 2u}{v} + \frac{v}{v}) = e^{(2u+v)\times\frac{u}{v}}(4 \times \frac{u}{v} + 1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v} = ye^{xy} \times 1 + xe^{xy} \times (\frac{-u}{v^2}) = e^{xy}(y - \frac{xu}{v^2}) = e^{(2u+v)\times\frac{u}{v}}(\frac{u}{v} - \frac{u(2u+v)}{v^2}) =$$

$$\frac{\partial z}{\partial v} = e^{(2u+v)\times\frac{u}{v}}(\frac{uv}{v^2} - \frac{u(2u+v)}{v^2}) = e^{(2u+v)\times\frac{u}{v}}(\frac{uv - 2u^2 - uv}{v^2}) = \frac{-2u^2}{v^2}e^{(2u+v)\times\frac{u}{v}}$$

Exercise 2:

$z = f(x - y, y - x)$ show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

You have to substitute u for x-y (u = x-y) and v for y-x (v = y-x). This allows for expressing the equations with only one variable u or v.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \times 1 + \frac{\partial z}{\partial v} \times (-1) = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \times (-1) + \frac{\partial z}{\partial v} \times 1 = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

Therefore $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0$

Exercise 3: Find the stationary points of the function:
$f(x,y) = y^2 + xy + 3y + 2x + 3$

To find stationary points, you have to obtain the partial differentials of the function f in x and y. Those differentials are equal to 0 for those particular points.

$$\frac{\partial f}{\partial x} = y + 2 = 0 \Rightarrow y = -2$$
$$\frac{\partial f}{\partial y} = 2y + x + 3 = 0 \Rightarrow -4 + x + 3 = 0 \Rightarrow x = 1$$

According to the above (1,-2) is a stationary point. You then have to determine its nature.

$$\Delta = (\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2}) - (\frac{\partial^2 f}{\partial x \partial y})^2$$

Δ<0 saddle point
Δ>0 minimum or maximum

$$\frac{\partial^2 f}{\partial x^2} = 0$$
$$\frac{\partial^2 f}{\partial y^2} = 2$$
$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

$$\Delta = 0 \times 2 - 1^2 = -1$$
$$\Delta < 0$$

(1,-2) is a saddle point.

Exercise 4: Find the stationary points of the function:
$f(x,y) = xy^2 - 2y^3 - x^2 + 2xy$

$$\frac{\partial f}{\partial x} = y^2 - 2x + 2y = 0 \Rightarrow 2x = y^2 + 2y$$
$$\frac{\partial f}{\partial y} = 2xy - 6y^2 + 2x = 0 = 2x(y+1) - 6y^2 \Rightarrow (y^2 + 2y)(y+1) - 6y^2 = 0$$
$$\frac{\partial f}{\partial y} = y(y+2)(y+1) - 6y^2 = y(y^2 + 3y + 2 - 6y) = y(y^2 - 3y + 2) = y(y-1)(y-2)$$

Therefore y = 0 or y = 1 or y = 2.
If y = 0 x = 0; y = 1 x = 3/2; y = 2 x = 4.

You then have to repeat the same operation as for exercise 3 above.

$$\frac{\partial^2 f}{\partial x^2} = -2$$

$$\frac{\partial^2 f}{\partial y^2} = 2x - 12y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y + 2$$

$(0,0)$

$\Delta = 0 - 2^2$

$\Delta < 0$

Saddle point

$(\frac{3}{2},1)$

$\Delta = (-2)(3-12) - (2+2)^2 = 2$

$\Delta > 0$

Minimum or maximum, you have to look at the value of the second order derivative in x and y.

$$\frac{\partial^2 f}{\partial x^2} = -2$$

$$\frac{\partial^2 f}{\partial x^2} < 0$$

$$\frac{\partial^2 f}{\partial y^2} = -9$$

$$\frac{\partial^2 f}{\partial y^2} < 0$$

Both are negative (0,0) is a maximum.

$(4,2)$

$\Delta = (-2)(2\times 4 - 12\times 2) - (2\times 2 + 2)^2 = -20$

$\Delta < 0$

Saddle point

Exercise 5: Find the stationary points of the function:

$$f(x, y) = x^2 + y^2 + \frac{2}{xy}$$

$$\frac{\partial f}{\partial x} = 2x + \frac{2}{y} \times (-\frac{1}{x^2}) = 2(x - \frac{1}{x^2 y}) = 0 \Rightarrow x = \frac{1}{x^2 y} \Rightarrow x^3 y = 1 \Rightarrow y = \frac{1}{x^3}$$

$$\frac{\partial f}{\partial y} = 2y + \frac{2}{x} \times (-\frac{1}{y^2}) = 2(y - \frac{1}{xy^2}) = 0 \Rightarrow y = \frac{1}{xy^2} \Rightarrow xy^3 = 1 \Rightarrow x \times \frac{1}{(x^3)^3} \Rightarrow x^8 = 1$$

This implies that x = 1 or -1.
If x = 1, y = 1 and if x = -1, y = -1

$$\frac{\partial^2 f}{\partial x^2} = 2 - \frac{2}{y} \times \frac{(-2)}{x^3} = 2 + \frac{4}{x^3 y}$$

$$\frac{\partial^2 f}{\partial y^2} = 2 - \frac{2}{x} \times \frac{(-2)}{y^3} = 2 + \frac{4}{xy^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2 \times (\frac{-1}{y^2}) \times (\frac{-1}{x^2}) = \frac{2}{x^2 y^2}$$

$(1,1)$

$\Delta = (6)(6) - 2^2 = 36 - 4 = 32$

$\Delta > 0$

Minimum or maximum

$$\frac{\partial^2 f}{\partial x^2} = 2 + 4 = 6$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + 4 = 6$$

Both second order partial derivatives are positive, so (1,1) is a minimum.

$(-1,-1)$

$\Delta = (6)(6) - 2^2 = 32$

$\Delta > 0$

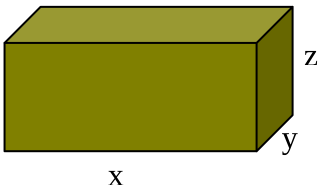
$$\frac{\partial^2 f}{\partial x^2} = 2 + 4 = 6$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + 4 = 6$$

Similarly (-1,-1) is a minimum

Exercise 6:

The first thing you have to do is to draw a schematic of your crate. The second is to read the text properly. Several of you went of to get the smallest area whereas the function for which you have to calculate a minimum is the cost: *"the most economical … crate"*; in other words the crate costing you the smallest amount of money.



The volume is equal to xyz and is 96 m^3. To calculate the cost, one has to separate the areas you obtain for the sides and the bottom because their cost is different. The sides areas are 2(xz + yz), their cost, in pence will be 20(xz + yz). Similarly, for the base (remember it is an open top, just think of it like a skip), the cost will be 30(xy). The total cost is therefore equal to 20(xz + yz)+30xy. This defines your function "cost" for which we need to find a minimum value. Before applying any method as for the previous exercises to find this minimum, you have to transform the equation "cost" into an equation of two variables x and y as follows:

$$C = 30xy + 20xz + 20yz$$

$$xyz = 96 \Rightarrow z = \frac{96}{xy}$$

$$C = 30xy + 20 \times \frac{96x}{xy} + 20 \times \frac{96y}{xy} = 30xy + \frac{1920}{y} + \frac{1920}{x}$$

$$\frac{\partial C}{\partial x} = 0 = 30y - \frac{1920}{x^2} \Rightarrow y = \frac{64}{x^2}$$

$$\frac{\partial C}{\partial y} = 0 = 30x - \frac{1920}{y^2} \Rightarrow x = \frac{64}{y^2}$$

Substitute y as a function of x from the first equation into the second:

$$x = \frac{64}{(\frac{64}{x^2})^2} = \frac{64}{64 \times 64} x^4 = \frac{x^4}{64}$$

With such an equation x may be equal to zero, which does not make much sense for a crate or to the cubic root of 64, i.e. 4 meters. From x you may deduce y as 64 divided by 16 or 4 meters and finally z as 96 divided by 16 or 6 meters.

The actual cost will be:

$$C = 30xy + 20xz + 20yz = 30 \times 4 \times 4 + 20(4 \times 6 + 4 \times 6) = 1440$$

1440 in pence or £14.40 in pounds.

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