Mathematics 1b: Sheet 5 solutions

1.

(a)

$$y^4\,dy = \sinh 4x\,dx$$

so that

$$\frac{y^5}{5} = \frac{\cosh 4x}{4} + c$$

Hence

$$y = \left(\frac{5}{4}\cosh 4x + d\right)^{1/5}$$

where d is a constant.

(b)

$$\frac{1}{y^2}\,dy = (x+1)\,dx$$

Integrating

$$-\frac{1}{y} = \frac{x^2}{2} + x + c$$

so that

$$y = \frac{1}{-\frac{1}{2}x^2 - x - c} \qquad \left(\text{or} \quad y = \frac{1}{-\frac{1}{2}x^2 - x + c}\right)$$

(c)

$$\frac{1}{y}\,dy = \frac{4}{1+x}\,dx$$

Integrating,

$$\ln y = 4\ln(1+x) + c$$

so that

$$y = A(1+x)^4$$

where A is another constant. Since $y = 1$ when $x = 1$ it follows that $A = \frac{1}{16}$ and hence

$$y = \frac{(1+x)^4}{16}$$

(d)

$$\frac{1}{y^2+1}\,dy = e^{2x}\,dx$$

so that

$$\tan^{-1} y = \tfrac{1}{2}e^{2x} + c$$

and therefore $y = \tan(\frac{1}{2}e^{2x} + c)$. Since $y(0) = 1$ we have $c + \frac{1}{2} = \tan^{-1}(1) = \pi/4$ and therefore

$$y = \tan(\tfrac{1}{2}e^{2x} + \pi/4 - \tfrac{1}{2})$$

(e)

$$\frac{1}{y(y-1)}\,dy = dx$$

Using partial fractions,

$$\left(\frac{1}{y-1} - \frac{1}{y}\right) dy = dx.$$

Integrating,

$$\ln(y-1) - \ln y = x + c$$

so that

$$y = \frac{1}{1 - Ae^x}$$

Since $y(0) = 2$ it follows that $A = \frac{1}{2}$ and therefore

$$y = \frac{2}{2 - e^x}$$

2. Separating the variables,

$$\frac{dy}{(N-y)(M-y)} = k\,dt$$

or, using partial fractions,

$$\frac{1}{M-N}\left(\frac{1}{N-y} - \frac{1}{M-y}\right) dy = k\,dt.$$

Integrating,

$$\frac{1}{M-N}\Big(-\ln(N-y) + \ln(M-y)\Big) = kt + c$$

where c is a constant, so that

$$\ln\left(\frac{M-y}{N-y}\right) = (M-N)kt + d$$

where d is another constant. Taking exponentials, rearranging and solving for y gives

$$y = \frac{NAe^{(M-N)kt} - M}{Ae^{(M-N)kt} - 1}$$

where A is another constant ($A = e^d$). Since $y(0) = 0$, $A = M/N$ and so

$$y = \frac{M(e^{(M-N)kt} - 1)}{\frac{M}{N}e^{(M-N)kt} - 1}$$

If $M > N$, $y(t) \to N$ as $t \to \infty$. If $M < N$, then $y(t) \to M$.

3. (i) $M^4 - T^4 = (M-T)(M+T)(M^2+T^2)$. The partial fractions for $\dfrac{1}{M^4 - T^4}$ will have the form

$$\frac{A}{M-T} + \frac{B}{M+T} + \frac{CT+D}{M^2+T^2}$$

and the usual techniques show that $A = B = \dfrac{1}{4M^3}$, $C = 0$ and $D = \dfrac{1}{2M^2}$. Therefore

$$\frac{1}{M^4 - T^4} = \frac{1}{4M^3}\left(\frac{1}{M-T} + \frac{1}{M+T} + \frac{2M}{M^2+T^2}\right).$$

(ii) Separating the variables in the differential equation and then using the above partial fraction representation gives

$$\left(\frac{1}{M-T} + \frac{1}{M+T} + \frac{2M}{M^2+T^2}\right) dT = 4kM^3\, dt.$$

Integrating,

$$-\ln(M-T) + \ln(M+T) + 2M\,\frac{1}{M}\tan^{-1}\frac{T}{M} = 4kM^3 t + c$$

which leads to the stated result.