Hyperbolic Functions

Notice: this material must not be used as a substitute for attending the lectures

0.1 Hyperbolic functions sinh and cosh

The hyperbolic functions sinh (pronounced "shine") and cosh are defined by the formulae

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \tag{1}$$

The function $\cosh x$ is an **even** function, and $\sinh x$ is **odd**.
On modern calculators hyperbolic functions are usually accessed using a button marked **hyp**. So the sinh function would be accessed by typically using a sequence of keystrokes of the form **hyp** → **sin**.
The function $y = \cosh x$ (or more precisely $y = a \cosh x/a$ for a suitable value of a) gives the exact shape of a heavy chain or rope hanging between two fixed points.
The cosh and sinh functions arise commonly in wave and heat theory. Many identities for them look similar to identities for the ordinary trigonometric functions cos and sin, but sometimes with a change of sign.
Important identity: $\boxed{\cosh^2 x - \sinh^2 x = 1}$
This looks like the well known trigonometric identity $\cos^2 x + \sin^2 x = 1$, but note that there is a change of sign.
The above identity can easily be derived from the basic definitions (1) as follows:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{(e^{2x} - 2 + e^{-2x})}{4} \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 1\end{aligned}$$

Another identity:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \tag{2}$$

This, too, can be derived from the basic definitions (1):

$$\begin{aligned}\text{RHS of (2)} &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)}}{4} + \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)}}{4} \\ &= \frac{e^{x+y}}{2} - \frac{e^{-(x+y)}}{2} \\ &= \sinh(x + y)\end{aligned}$$

0.2 The hyperbolic tangent function: tanh

The hyperbolic function $\tanh x$ is defined by

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$\tanh x$ is an **odd function**.
As in ordinary trigonometry, if we know the sinh or cosh of a number we can work out the other hyperbolic functions of that number, as the following example demonstrates.

0.3 Example

If $\sinh x = \frac{8}{15}$ calculate $\cosh x$ and $\tanh x$.
Solution. We know that $\cosh^2 x - \sinh^2 x = 1$. Lets use this identity to find $\cosh x$. We have

$$\cosh^2 x = 1 + \sinh^2 x = 1 + \left(\frac{8}{15}\right)^2 = \frac{289}{15^2}$$

From the basic definition of cosh in (1) we see that the cosh of anything is always positive. Hence $\cosh x = \frac{17}{15}$.
Finally, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{8/15}{17/15} = \frac{8}{17}$.

0.4 Example

Solve for x the equation $5\cosh x + 3\sinh x = 4$.
Solution. From the basic definitions of sinh and cosh, the equation can be written as

$$5\left(\frac{e^x + e^{-x}}{2}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right) = 4$$

which simplifies to $4e^x + e^{-x} = 4$. Multiplying by e^x gives $4e^{2x} + 1 = 4e^x$, which can be written in the form of a quadratic equation

$$4(e^x)^2 - 4e^x + 1 = 0$$

From the formula for solving a quadratic equation, we find that

$$e^x = \frac{1}{8}\left(4 \pm \sqrt{0}\right)$$

i.e. $e^x = \frac{1}{2}$. Hence $x = \ln\frac{1}{2} = \ln 2^{-1} = -\ln 2$.

0.5 An identity for tanh

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \qquad (3)$$

This looks a bit like the corresponding identity for the ordinary trigonometric function tan, namely:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

but note there is a difference in sign in the denominator.
Let us derive formula (3). We already have

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

and we can also show that

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Hence

$$\begin{aligned}
\tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} \\
&= \frac{\frac{\sinh x}{\cosh x} + \frac{\sinh y}{\cosh y}}{1 + \frac{\sinh x}{\cosh x}\frac{\sinh y}{\cosh y}} \\
&= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}
\end{aligned}$$

As commented on previously, identities for hyperbolic functions often look like those for the ordinary trigonometric functions sin, cos, tan, but there is often a change of sign. There is a general rule for deriving an identity for hyperbolic functions from the corresponding identity for ordinary trigonometric functions. It is called **Osborn's rule.**

0.6 Osborn's rule

To get a formula for hyperbolic functions from the corresponding identity for ordinary trigonometric functions, replace every ordinary trigonometric function by the corresponding hyperbolic function, **and** change the sign of every product or implied product of sine terms.
Implied product means things like $\sin^2 x$ which can be written as $\sin x \sin x$ and is therefore a product of sines.
Example. We know that $\cos(x+y) = \cos x \cos y - \sin x \sin y$. Replace cos by cosh and sin by sinh. Also $\sin x \sin y$ is a product of sines so change the sign from $-$ to $+$. In this way we derive the identity

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Example. We know that $\cos^2 x + \sin^2 x = 1$.
Now $\sin^2 x = \sin x \sin x$ so is a product of sine terms. We write the above identity as

$$\cos x \cos x + \sin x \sin x = 1$$

Now change cos to cosh, sin to sinh, and change the sign of the $\sin x \sin x$ term which is a product of sines. We derive the identity

$$\cosh^2 x - \sinh^2 x = 1$$

Example. We know that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

which can be written as

$$\frac{\sin(x+y)}{\cos(x+y)} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}}$$

In the above expression there appears the product $\sin x \sin y$ which is a product of sines so its sign should be changed. Also, replace sin by sinh and cos by cosh to derive the following:

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

0.7 Derivatives of hyperbolic functions

It can be straightforwardly shown from the basic definitions (1) that

$$\begin{aligned}
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x
\end{aligned}$$

The third of these can be derived from the quotient rule for derivatives:

$$\begin{aligned}
\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x} \\
&= \operatorname{sech}^2 x
\end{aligned}$$

0.8 Other hyperbolic functions and their derivatives

The function $\coth x$ is defined by

$$\coth x = \frac{\cosh x}{\sinh x}$$

and is analogous to the ordinary trigonometric function $\cot x$, which equals $\cos x / \sin x$. The derivative of $\coth x$ can be found using the quotient rule as follows:

$$\begin{aligned}
\frac{d}{dx}(\coth x) &= \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) \\
&= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} \\
&= \frac{-1}{\sinh^2 x} \\
&= -\operatorname{cosech}^2 x
\end{aligned}$$

where

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

The function $\operatorname{sech} x$ is defined by

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

and its derivative is given by

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

The function $\operatorname{cosech} x$ is defined by

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

and its derivative is given by

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

0.9 Example

Find $\frac{d}{dt}(\tanh \sqrt{1+t^2})$.
Solution. To do this we shall have to use the chain rule. Let $u = \sqrt{1+t^2}$. Then

$$\begin{aligned}
\frac{d}{dt}(\tanh \sqrt{1+t^2}) &= \frac{d}{dt}(\tanh u) \\
&= \left(\frac{d}{du}(\tanh u)\right)\frac{du}{dt} \\
&= (\operatorname{sech}^2 u)\, \tfrac{1}{2}(1+t^2)^{-1/2}\,(2t) \\
&= \frac{t \operatorname{sech}^2 \sqrt{1+t^2}}{\sqrt{1+t^2}}
\end{aligned}$$

0.10 Integrals of hyperbolic functions

$$\begin{aligned}
\int \sinh x \, dx &= \cosh x + c \\
\int \cosh x \, dx &= \sinh x + c \\
\int \operatorname{sech}^2 x \, dx &= \tanh x + c
\end{aligned}$$

0.11 Example

Find $\int \coth 5x\, dx$.

Solution.

$$\begin{aligned}\int \coth 5x\, dx &= \int \frac{\cosh 5x}{\sinh 5x}\, dx \qquad \text{let } u = \sinh 5x \quad \text{so} \quad du = 5\cosh 5x\, dx \\ &= \int \frac{1/5}{u}\, du \\ &= \frac{1}{5}\int \frac{du}{u} \\ &= \frac{1}{5}\ln u + c \\ &= \frac{1}{5}\ln(\sinh 5x) + c\end{aligned}$$

0.12 Example

Find $\int \sinh^2 x\, dx$.

Solution. Need to re-write $\sinh^2 x$ as something that can be more easily integrated. Start with the identity

$$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$$

Put $B = A$ in this to get $\cosh 2A = \cosh^2 A + \sinh^2 A$. Recall also that $\cosh^2 A - \sinh^2 A = 1$. Putting these things together gives

$$\cosh 2A = 1 + 2\sinh^2 A$$

so that

$$\sinh^2 A = \frac{1}{2}(\cosh 2A - 1)$$

Hence

$$\begin{aligned}\int \sinh^2 x\, dx &= \frac{1}{2}\int (\cosh 2x - 1)\, dx \\ &= \frac{1}{2}\left(\frac{\sinh 2x}{2} - x\right) + c\end{aligned}$$

0.13 Inverse hyperbolic functions

If $y = \sinh x$ then we say $x = \sinh^{-1} y$, the inverse hyperbolic sin of y.
NB: $\sinh^{-1} y$ is just notation for the inverse sinh of y. It does **not** mean the same as $(\sinh y)^{-1}$.
The function $y = \cosh x$ is not *one to one*. This is because each y value has two corresponding x values. However, if we restrict to $x \geq 0$ then the function $\cosh x$ does have an inverse, written $\cosh^{-1} x$.

On modern calculators inverse hyperbolic functions are usually accessed using a **shift** and a **hyp** button. A typical sequence for $\sinh^{-1}$ might be something like **shift** → **hyp** → **sin**.

0.14 Derivatives of inverse hyperbolic functions

Let $y = \cosh^{-1} x$ and suppose we want to find dy/dx.
Since $y = \cosh^{-1} x$, it follows that $\cosh y = x$. Implicit differentiation of this equation gives

$$\sinh y \, \frac{dy}{dx} = 1$$

so that $\frac{dy}{dx} = \frac{1}{\sinh y}$. But recall that $\cosh^2 y - \sinh^2 y = 1$. Using this fact gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sinh y} \\ &= \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}} \end{aligned}$$

We have shown that

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$$

Similarly,

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$$

and

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}$$

0.15 Example

Evaluate $\displaystyle\int_0^1 \frac{2}{\sqrt{3 + 4x^2}}\, dx$.
Solution. The appearance of the integrand suggests that we should aim to use the result

$$\int \frac{1}{\sqrt{x^2 + 1}}\, dx = \sinh^{-1} x + c$$

but we shall first need to make a substitution. Note that
$3 + 4x^2 = 3\left(1 + \frac{4}{3}x^2\right)$; this suggests we should use the substitution $x = \frac{\sqrt{3}}{2}y$, so that $dx = \frac{\sqrt{3}}{2}\, dy$. Making this substitution, we get

$$\begin{aligned} \int_0^1 \frac{2}{\sqrt{3 + 4x^2}}\, dx &= \int_0^{2/\sqrt{3}} \frac{2\left(\frac{\sqrt{3}}{2}\right) dy}{\sqrt{3 + 4\left(\frac{3}{4}y^2\right)}} \\ &= \sqrt{3} \int_0^{2/\sqrt{3}} \frac{dy}{\sqrt{3 + 3y^2}} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{3}\int_0^{2/\sqrt{3}} \frac{dy}{\sqrt{3}\sqrt{1+y^2}} \\
&= \left[\sinh^{-1} y\right]_0^{2/\sqrt{3}} \\
&= \sinh^{-1}\frac{2}{\sqrt{3}} - \sinh^{-1} 0 \\
&= 0.9866
\end{aligned}
$$

0.16 Example

Find $\displaystyle\int \frac{dx}{\sqrt{x^2-2}}$.

Solution. Substitute $x = \sqrt{2}\,y$. Then $dx = \sqrt{2}\,dy$. We have

$$
\begin{aligned}
\int \frac{dx}{\sqrt{x^2-2}} &= \int \frac{\sqrt{2}\,dy}{\sqrt{2y^2-2}} = \int \frac{\sqrt{2}\,dy}{\sqrt{2}\sqrt{y^2-1}} \\
&= \int \frac{dy}{\sqrt{y^2-1}} = \cosh^{-1} y + c \\
&= \cosh^{-1}\frac{x}{\sqrt{2}} + c
\end{aligned}
$$

0.17 Example

Find $\displaystyle\int \sqrt{a^2+x^2}\,dx$.

Solution. Recall $\cosh^2\theta - \sinh^2\theta = 1$, so that $\cosh^2\theta = 1 + \sinh^2\theta$. This suggests we should substitute $x = a\sinh\theta$ in the integral. Carrying out the substitution gives

$$
\begin{aligned}
\int \sqrt{a^2+x^2}\,dx &= \int \sqrt{a^2+a^2\sinh^2\theta}\,a\cosh\theta\,d\theta \\
&= \int a\sqrt{1+\sinh^2\theta}\,a\cosh\theta\,d\theta \\
&= \int a^2\cosh^2\theta\,d\theta
\end{aligned}
$$

We may apply Osborn's rule to the ordinary trigonometric identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, to deduce that

$$
\cosh^2\theta = \frac{1}{2} + \frac{1}{2}\cosh 2\theta
$$

Therefore

$$
\begin{aligned}
\int \sqrt{a^2+x^2}\,dx &= \frac{1}{2}a^2\int (1+\cosh 2\theta)\,d\theta \\
&= \frac{1}{2}a^2\left(\theta + \frac{\sinh 2\theta}{2}\right) + c \\
&= \frac{1}{2}a^2(\theta + \sinh\theta\cosh\theta) + c
\end{aligned}
$$

$$= \frac{1}{2}a^2\left(\sinh^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}\right) + c$$

0.18 Example

Find $\int \frac{dx}{\sqrt{x^2-6x+5}}$.
Solution.

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2-6x+5}} &= \int \frac{dx}{\sqrt{(x-3)^2-4}} \qquad \text{Substitute } x-3=2t \\
&= \int \frac{2\,dt}{\sqrt{4t^2-4}} = \int \frac{dt}{\sqrt{t^2-1}} \\
&= \cosh^{-1} t + c \\
&= \cosh^{-1}\left(\frac{x-3}{2}\right) + c
\end{aligned}$$