<u>Calculus: further exercises</u>

This sheet must not be used in isolation. It does **not** *aim to cover everything in the exam - only to give you an idea of what to expect. You need to thoroughly revise the material covered in lectures and exercise sheets as well. The examination questions do* **not** *remind you of any relevant formulae nor do they contain any hints.*
This sheet does not have the actual format of the examination paper. In the actual exam you will answer 4 questions from 6, and each question is broken up into sub-questions with no further choice.
Please further note that **Q1-6** below are not technically part of this syllabus any more. However I leave them here because the techniques involved to answer them come up in other topics that **are** in the current syllabus.

1. Calculate $(1,3,-2) \times (-1,5,7)$ and $(1,3,-2) \cdot (-1,5,7)$

2. Find a unit vector that is perpendicular to both $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{i} + \mathbf{k}$.

3. Find the equation of the plane that contains the point $(-1,2,7)$ and is parallel to the vectors $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 5\mathbf{k}$.

4. Find the distance from the point $(2,-1,3)$ to the plane $2x + 4y - z + 1 = 0$.

5. Find the cartesian equation of the plane through the points $(1,2,1)$, $(3,1,-2)$ and $(2,1,4)$.

6. If two planes are parallel, with normal vector $\mathbf{n}$, show that the distance between them is

$$\frac{|\mathbf{n} \cdot (\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{n}|}$$

where $\mathbf{a}_1$ is the position vector of a point on one of the planes, and $\mathbf{a}_2$ the position vector of a point on the other plane.

7. Let $w = f(u,v)$ where $u = x/y$ and $v = z/y$. Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0.$$

8. Find the directional derivative of the function $f(x,y) = \tan^{-1}(y/x)$ at $(-2,2)$ in the direction $-\mathbf{i} - \mathbf{j}$.

9. Find the stationary points of the function $z = 3xy - x^3 - y^3 + 2$ and determine their nature.

10. Evaluate $\displaystyle\iint_D \frac{1}{x^2+y^2}\,dA$ where D is the region bounded by the lines $y = x$, $y = 2x$, $x = 1$ and $x = 2$.

11. Sketch the region of integration giving rise to the double integral

$$\int_0^1 \int_y^{2-y} \frac{x+y}{x^2}\, e^{x+y}\, dx\, dy.$$

Show that the transformation $u = x + y$, $v = y/x$ transforms the integral into

$$\int_0^1 \int_0^2 e^u \, du\, dv$$

and hence evaluate it.

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12. Show that

$$\int_a^b \int_a^b (y-x)(f(y)-f(x))\,dx\,dy = (b-a)\left(2\int_a^b xf(x)\,dx - (b+a)\int_a^b f(x)\,dx\right).$$

Deduce that if f is increasing on $[a,b]$ and $\int_a^b f(x)\,dx > 0$ then

$$\frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx} > \frac{a+b}{2}.$$

13. In terms of spherical polar coordinates describe the region V, where V is (a) the region inside the sphere $x^2+y^2+z^2=25$, (b) the region inside the sphere $x^2+y^2+z^2=25$ and above the plane $z=3$.

14. Use cylindrical polar coordinates to find the volume of the solid bounded by the paraboloid $z=x^2+y^2$ and the plane $z=9$.

15. Let k be some given constant. Find the maximum value of the function $f(x,y,z)=(xyz)^{1/3}$ subject to the constraint that the nonnegative numbers x, y and z satisfy $x+y+z=k$. Use your result to show that

$$(xyz)^{1/3} \le \frac{x+y+z}{3}.$$

16. Let C be a closed curve in the (x,y) plane and D be the region enclosed by C. If u and v are functions of x and y show that

$$\int_C (u\nabla v)\cdot d\mathbf{r} = \iint_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) dA.$$

17. Show that if f is harmonic (i.e. it satisfies Laplace's equation $\nabla^2 f = 0$) then

$$\iint_{\partial V} f\,\nabla f\cdot\mathbf{n}\,dS = \iiint_V |\nabla f|^2\,dV.$$

18. Show that

$$\iint_{\partial V} (f\,\nabla g - g\,\nabla f)\cdot\mathbf{n}\,dS = \iiint_V (f\nabla^2 g - g\nabla^2 f)\,dV.$$

19. Show that

$$\iint_{\partial V} (\nabla f\times\nabla g)\cdot\mathbf{n}\,dS = 0.$$

20. Evaluate $\iint_{\partial V} \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$ and V is the part of the sphere $x^2+y^2+z^2\le 4$ which lies in the first octant.

Answers/hints:

1. $(31,-5,8)$, 0
2. $\frac{1}{3\sqrt{3}}(1,-5,-1)$ or $\frac{1}{3\sqrt{3}}(-1,5,1)$
3. $15x+11y+3z=28$
4. $2/\sqrt{21}$
5. $6x+9y+z=25$
6. Draw a diagram. Let A_1 be the point with position vector $\mathbf{a}_1$, and P the point on the other plane closest to A_1. Let A_2 be the point with position vector $\mathbf{a}_2$. Consider the round trip

$O \to A_1 \to P \to A_2 \to O$. Proceed similarly to the proof of the formula for the minimum distance between skew lines.

8. $\frac{1}{2\sqrt{2}}$

9. $(0,0)$ saddle, $(1,1)$ maximum

10. Do the integration with respect to y first. Answer: $(\tan^{-1}2 - \pi/4)\ln 2$, which can be simplified to $(\tan^{-1}\frac{1}{3})\ln 2$.

11. Region of integration is the triangle with vertices $(0,0)$, $(2,0)$ and $(1,1)$. Answer: $e^2 - 1$.

12. Start by expanding out the integrand.

13. (a) $0 \le r \le 5$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. (b) $3\sec\theta \le r \le 5$, $0 \le \theta \le \cos^{-1}\frac{3}{5}$, $0 \le \phi \le 2\pi$.

14. $81\pi/2$

15. $k/3$

16. Use Green's theorem.

17-18. Apply Gauss's theorem with $\mathbf{F}$ chosen suitably.

19. Apply Gauss's theorem and then use some of the formulae for div and curl presented in lectures.

20. Apply Gauss's theorem. You then need to find $\iiint_V 3x\,dV$. Do this by transforming to spherical polars. Answer: 3π.