Enhancement of the Benjamin-Feir instability with dissipation

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Abstract

It is shown that there is an overlooked mechanism whereby some kinds of dissipation can enhance the Benjamin-Feir instability of water waves. This observation is new, and although it is counterintuitive, it is due to the fact that the Benjamin-Feir instability involves the collision of modes with opposite energy sign (relative to the carrier wave), and it is the *negative energy perturbations* which are enhanced.

The discovery of the Benjamin-Feir (BF) instability of travelling waves was a milestone in the history of water waves. Before 1960 the idea that a Stokes wave could be unstable does not appear to be given much thought. The possibility that the Stokes wave could be unstable was pointed out in the early 1960s [1–4], but it was the seminal work of Benjamin and Feir [5, 6] which combined experimental evidence with a weakly nonlinear theory that convinced the scientific community.

Indeed, Benjamin & Feir started their experiments in 1963 *assuming that the Stokes wave was stable.* After several frustrating years watching their waves disintegrate – in spite of equipment and laboratory changes and improvements – they finally came to the conclusion that they were witnessing a new kind of instability. The appearance of "sidebands" in the experiments suggested the form that the perturbations should take. A history of these experiments and the outcome is reported in [7].

The theory of the BF instability is based on inviscid fluid mechanics, and the assumption that the system is conservative. Therefore it is natural to study the implication of perturbations on the system. The implications of a range of perturbations on the BF instability have been studied in the literature: for example the effect of wind [8, 9] and the effect of viscosity [8, 10–13]. Some perturbations have been shown to stabilize and others destabilize the BF instability.

However, there is a fundamental overlooked mechanism in all this work. Mathematically, the BF instability can be characterized as a collision of two pairs of purely imaginary eigenvalues of opposite energy sign as shown in Figure 1. In [14], this observation is implicit but the demonstration and implications have not been given heretofore. This characterization of the BF instability also appears in the nonlinear Schrödinger (NLS) model for modulation of dispersive travelling waves [15, 16]. The eigenvalue with smaller positive imaginary part in the figure – just before collision – has negative energy, whereas the eigenvalue with larger imaginary part has positive energy. This energy is relative to the energy of the carrier wave E^{Stokes}: $E_- < E^{\text{Stokes}} < E_+$ where $E_\pm$ are the energies of the modes associated with the
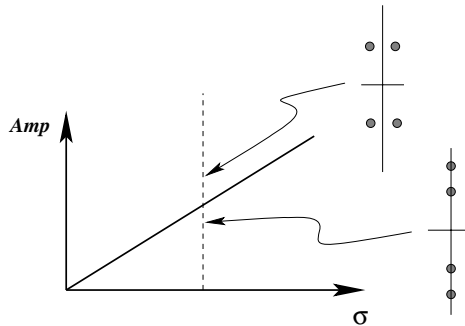
FIG. 1: Schematic of the eigenvalue movement associated with the BF instability, as a function of the amplitude of the basic carrier wave (Amp) and the sideband wavenumber σ. For a fixed σ (vertical dashed line), there is a threshold amplitude. Below the threshold, the eigenvalues are purely imaginary. At the threshold a collision occurs and above the threshold, the eigenvalues are complex.

respective purely imaginary eigenvalues in Figure 1. Hence "negative energy" means that $E_- - E^{\text{Stokes}} < 0$.

Once these facts are established we can appeal to the result that dissipation can *destabilize* negative energy modes [18–20]. There are many examples in fluid mechanics where negative energy modes – which are stable in the inviscid limit – are destabilized by the addition of dissipation [21]: Kelvin-Helmholtz instability [18, 19, 22], interaction of a fluid with a flexible boundary [23, 24], stability of a fluid-loaded elastic plate [25], Euler modes perturbed by the Navier-Stokes equations [26].

The book of Fabrikant & Stepanyants [22] reports on experimental results for interfacial waves near the Kelvin-Helmholtz instability illustrating the enhancement of the instability of negative energy waves due to dissipation. See Figure 3.5 on page 83, and the discussion on pp. 82-83 in [22].

In this brief communication, we sketch the basic result for water waves and then use a nonlinear Schrödinger model perturbed by dissipation for illustration. The NLS equation has shortcomings (e.g. symmetry which enables the phase to be factored out, lack of validity for all time [27]) but it provides a simple example of the

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phenomenon.

Davey [10] gives a general argument for the form of a dissipation-perturbed NLS model, and Blennerhassett [8] starts with the full Navier-Stokes equations for a free-surface flow with viscous free-surface boundary conditions and derives a similar perturbed NLS equation. For the dissipatively perturbed Stokes wave in deep water, these NLS models take the form

$$iA_t + (\alpha - ia)A_{xx} + ibA + (\gamma + ic)|A|^2A = 0, \tag{1}$$

where A is the envelope of the wave carrier, and the modulations are restricted to one space dimension x. When $a = b = c = 0$, equation (1) reduces to the NLS equation for the modulations of Stokes waves in deep water; hence α and γ are positive real numbers. This NLS model has a BF instability and one can show explicitly that it involves a collision of eigenvalues of the form shown in Figure 1. We show below that when $a > 0$, there is always dissipation induced instability (before the BF instability), no matter how small a is. The parameter a is the perturbation of the rate of change of the group velocity dc_g/dk due to dissipation.

First, consider the linear stability problem for gravity waves in deep water. As the wave amplitude increases we show that there is a threshold value at which two eigenvalues of the linear stability problem collide, and these two modes have negative and positive energy.

With $\theta = x - ct$, the speed c and amplitude η of the basic gravity wave of wavelength $2\pi/k$, to leading order, are $c = c_0(1 + k^2\varepsilon^2 + \cdots)$, $c_0^2 = gk^{-1}$,

$$\eta(\theta) = \varepsilon\eta_1(\theta) + \varepsilon^2\eta_2(\theta) + \mathcal{O}(\varepsilon^3),$$

where ε is a measure of the amplitude,

$$\eta_1(\theta) = \sqrt{2}\cos(k\theta - \theta_0), \quad \eta_2(\theta) = k\,\cos(2k\theta - 2\theta_0),$$

with θ_0 an arbitrary phase shift. Using standard results on integral properties of Stokes waves, the total energy *relative to the moving frame* is

$$E^{\text{Stokes}} = T + V - cI = V - T, \quad \text{using} \quad 2T = cI,$$

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where T and V are the kinetic and potential energies respectively, and I is the momentum [28]. Substitution of the Stokes expansion shows that $E^{\text{Stokes}} = 0 + \varepsilon^3 E_3 + \mathcal{O}(\varepsilon^4)$. Although the actual value of E_3 is not important for the argument below, it is noteworthy that it is negative, and, using Table 2 of [28], one can confirm that E^{Stokes} is negative at finite amplitude as well.

To formulate the linear stability for gravity waves take

$$\eta(\theta, x, t) \mapsto \widehat{\eta}(\theta, \varepsilon) + \eta(\theta, x, t) \,,$$

where $\widehat{\eta}(\theta, \varepsilon)$ is the basic carrier wave. Take a similar expression for the velocity potential $\phi(\theta, x, y, t)$, where y denotes the vertical space dimension. Next one substitutes this form into the water wave equations, linearizes about the carrier wave and takes $\eta(\theta, x, t)$ of the form

$$\eta(\theta, x, t) = \text{Re}\left(\Sigma(\theta, \sigma)\mathrm{e}^{\mathrm{i}\sigma x + \lambda t}\right) \,,$$

where σ is real (the modulation wavenumber), and $\Sigma(\theta, \sigma)$ is periodic of the same period as the Stokes wave. The result is an eigenvalue problem for the eigenfunction Σ and eigenvalue λ.

The BF instability corresponds to a solution of this eigenvalue problem with $0 < \sigma \ll 1$ and $\text{Re}(\lambda) > 0$. When σ is fixed – but nonzero and small – and the amplitude of the Stokes wave is increased, there is a threshhold amplitude where the BF instability occurs, and it corresponds to a collision of two eigenvalues as shown in Figure 1. To leading order the eigenvalues collide at $\lambda = \pm \mathrm{i} c_g \sigma$, where $c_g = \frac{1}{2}\sqrt{g/k}$ is the group velocity.

To show that the colliding modes have opposite energy sign, we need a definition of the energy of the perturbation. This definition requires some consideration because the perturbation is quasiperiodic in space: $2\pi/k-$periodic in θ and $2\pi/\sigma-$periodic in x. The total energy relative to the moving frame is

$$E^{\text{total}} = \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} \frac{k}{2\pi} \int_0^{2\pi/k} \widehat{E} \, \mathrm{d}\theta \, \mathrm{d}x \,,$$

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where $\widehat{E} = \widehat{T} - \widehat{V} - c\widehat{I}$, $\widehat{V} = \frac{1}{2}g\eta^2$,

$$\widehat{T} = \int_{-\infty}^{\eta} \frac{1}{2}(\phi_\theta^2 + 2\phi_\theta\phi_x + \phi_x^2 + \phi_y^2)\mathrm{d}y\,, \quad \text{and} \quad \widehat{I} = \int_{-\infty}^{\eta} (\phi_\theta + \phi_x)\mathrm{d}y\,.$$

Evaluating the perturbation energy for the two modes that collide leads to $E^{\text{total}} = E^{\text{Stokes}} + \varepsilon^2 E_2^\pm + \cdots$, with

$$E_2^\pm = 2(k \pm \sigma)\left(1 - \sqrt{1 \mp \frac{\sigma}{k}}\right)|C_\pm|^2\,.$$

Here $C_\pm$ are scale factors associated with the eigenfunctions. Clearly $\text{sign}(E_2^+ E_2^-) < 0$ for $0 < \sigma \ll 1$.

Having shown that the colliding modes have opposite energy sign, we consider a simple example which illustrates the mechanism for destabilization of negative energy modes by damping. A prototype for a conservative system, where the linearization has a collision of eigenvalues of opposite energy sign, which is perturbed by Rayleigh damping is

$$\mathbf{q}_{tt} + 2b\mathbf{J}\mathbf{q}_t + (\chi - \tau^2)\mathbf{q} + 2\delta\mathbf{q}_t = \mathbf{0}\,, \quad \mathbf{q} \in \mathbb{R}^2\,, \quad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \quad (2)$$

where $\tau > 0$ is the "gyroscopic coefficient", χ a real parameter with $|\chi| \ll \tau^2$, and $\delta \geq 0$.

The energy of the system (2) is strictly decreasing when $\delta > 0$ and $\|\mathbf{q}_t\| > 0$. Let $\mathbf{q}(t) = \widehat{\mathbf{q}}e^{\lambda t}$; then substitution into (2) leads to the roots

$$\lambda = \mathrm{i}\tau - \delta \pm \mathrm{i}\sqrt{\chi + 2\mathrm{i}\tau\delta - \delta^2} \quad \text{and} \quad \lambda = -\mathrm{i}\tau - \delta \pm \mathrm{i}\sqrt{\chi - 2\mathrm{i}\tau\delta - \delta^2}\,. \quad (3)$$

When $\delta = 0$ there are four roots $\lambda = \pm\mathrm{i}(\tau \pm \sqrt{\chi})$. The eigenvalue movement shown in Figure 1 is realized as χ decreases from a positive value to a negative value, the collision occurring at $\chi = 0$. Suppose now χ is small and positive (just before the collision) and look at the effect of dissipation on the two modes $\lambda_0 = \mathrm{i}\tau \pm \mathrm{i}\sqrt{\chi}$. Substitution of the eigenfunctions for these two eigenvalues into the energy shows that the mode associated with $\mathrm{i}\tau - \mathrm{i}\sqrt{\chi}$ has negative energy while the mode associated with $\mathrm{i}\tau + \mathrm{i}\sqrt{\chi}$ has positive energy.
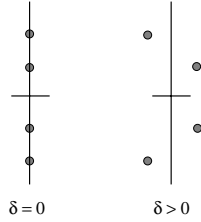
$$\delta = 0 \qquad \delta > 0$$

FIG. 2: Schematic of the effect of dissipation on the eigenvalues associated with (2).

With δ small, expand the first pair of roots in (3) in a Taylor series

$$\lambda(\delta) = \mathrm{i}\tau \pm \mathrm{i}\sqrt{\chi} \mp \frac{\delta}{\sqrt{\chi}}(\tau \pm \sqrt{\chi}) + \mathcal{O}(\delta^2)\,.$$

With $0 < \delta \ll 1$ the eigenvalues are perturbed as shown to the right in Figure 2. The negative energy mode, $\lambda_0 = \mathrm{i}(\tau - \sqrt{\chi})$, has positive real part when dissipatively perturbed, and the positive energy mode, $\lambda_0 = \mathrm{i}(\tau + \sqrt{\chi})$, has negative real part under perturbation. Consequently, when small dissipation is added to the otherwise *stable* system (that is, $0 < \chi \ll \tau^2$), the mode with negative energy will destabilize. After the collision (when $\chi < 0$) the growth rate of the instability is enhanced.

It should be noted that other mathematically consistent forms of damping can be used. For example the uniform damping

$$\mathbf{q}_t = \frac{\partial H}{\partial \mathbf{p}} - \delta \mathbf{q}\,, \quad \mathbf{p}_t = -\frac{\partial H}{\partial \mathbf{q}} - \delta \mathbf{p}\,, \tag{4}$$

makes mathematical sense. But it leads to uniform contraction of the phase space, and does not destabilize negative energy modes.

In order to study the effect of dissipation on water waves, one could start with the Navier-Stokes equations and perturb about the Stokes wave solution (see [8] for instance for the case of wind forcing). Another approach is to add viscous perturbations to the potential flow in various forms [29, 30]. From the modified equations one can derive a dissipative NLS equation. There are two issues to highlight: negative energy modes can be destabilized and so the BF instability can be enhanced by dissipation, and secondly, the form of the damping is important. It is known

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that negative energy modes of the Euler equations can be destabilized by the form of damping found in the Navier–Stokes equations [26].

Following [8, 10], a general perturbed NLS equation for various types of physical situations can be written in the form (1). The parameters a, b and c are taken to be non-negative. When they are positive, they represent dissipative perturbations, since the norm of the solution is strictly decreasing in time when $a^2 + b^2 + c^2 > 0$.

When $a = b = c = 0$, the resulting NLS equation is a Hamiltonian partial differential equation; with $A = u_1 + iu_2$ and $\mathbf{u} = (u_1, u_2)$,

$$\mathbf{J}\mathbf{u}_t = \nabla H(\mathbf{u}) + a\mathbf{J}\mathbf{u}_{xx} - b\mathbf{J}\mathbf{u} - c\|\mathbf{u}\|^2\mathbf{J}\mathbf{u}, \tag{5}$$

where $\mathbf{J}$ was defined in (2) and

$$H(\mathbf{u}) = \int_{\mathbb{R}} \left[\tfrac{1}{2}\alpha\|\mathbf{u}_x\|^2 - \tfrac{1}{4}\gamma\|\mathbf{u}\|^4 \right] \, \mathrm{d}x. \tag{6}$$

Let $\theta(x, t) = kx - \omega t + \theta_0$, and consider the basic travelling wave solution to (5) when dissipation is neglected,

$$\widehat{\mathbf{u}}(x, t) = \mathbf{R}_{\theta(x,t)}\mathbf{u}_0, \quad \mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{7}$$

Then $\mathbf{u}_0, \omega, k$ satisfy $-\omega + \alpha k^2 = \gamma\|\mathbf{u}_0\|^2$.

It is assumed that the Stokes travelling wave exists for a sufficiently long time before any dissipation can affect it: dissipation is taken to be a second order effect.

Next we check the energetics of the BF stability problem in NLS. Linearize the partial differential equation (5) with $a = b = c = 0$ about the basic travelling wave (7). Letting $\mathbf{u}(x, t) = \mathbf{R}_{\theta(x,t)}(\mathbf{u}_0 + \mathbf{v}(x, t))$, substituting into the conservative version of (5), linearizing about $\mathbf{u}_0$, and simplifying leads to

$$\mathbf{J}\mathbf{v}_t + 2\alpha k\mathbf{J}\mathbf{v}_x + \alpha\mathbf{v}_{xx} + 2\gamma\langle\mathbf{u}_0, \mathbf{v}\rangle\mathbf{u}_0 = \mathbf{0}, \tag{8}$$

where $\langle\cdot, \cdot\rangle$ is the standard scalar product on $\mathbb{R}^2$.

The class of solutions of interest are solutions which are periodic in x with wavenumber σ. The parameter σ represents the sideband. The BF instability will

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be associated with the limit $|\sigma| \ll 1$. Therefore let

$$\mathbf{v}(x,t) = \tfrac{1}{2}\mathbf{v}_0(t) + \sum_{n=1}^{\infty}(\mathbf{v}_n(t)\cos n\sigma x + \mathbf{w}_n(t)\sin n\sigma x)\,.$$

Neglecting the $\sigma-$independent modes (superharmonic instability), the $\sigma-$dependent modes decouple into 4$-$dimensional subspaces for each n, and satisfy

$$\mathbf{J}\dot{\mathbf{v}}_n + 2\alpha kn\sigma\mathbf{J}\mathbf{w}_n - \alpha(n\sigma)^2\mathbf{v}_n + 2\gamma\mathbf{u}_0\mathbf{u}_0^T\mathbf{v}_n = \mathbf{0}$$
$$\mathbf{J}\dot{\mathbf{w}}_n - 2\alpha kn\sigma\mathbf{J}\mathbf{v}_n - \alpha(n\sigma)^2\mathbf{w}_n + 2\gamma\mathbf{u}_0\mathbf{u}_0^T\mathbf{w}_n = \mathbf{0}\,. \tag{9}$$

When the amplitude $\|\mathbf{u}_0\| = 0$, it is easy to show that all eigenvalues of the above system (i.e. taking solutions of the form $\mathrm{e}^{\lambda t}$ and computing λ) are purely imaginary. Considering all other parameters fixed, and increasing $\|\mathbf{u}_0\|$, we find that there is a critical amplitude where the $n = 1$ mode becomes unstable first through a collision of eigenvalues of opposite signature.

To analyze this instability, take $n = 1$ and study the reduced four dimensional system

$$\mathbf{J}\dot{\mathbf{v}}_1 + 2\alpha k\sigma\mathbf{J}\mathbf{w}_1 - \alpha\sigma^2\mathbf{v}_1 + 2\gamma\mathbf{u}_0\mathbf{u}_0^T\mathbf{v}_1 = \mathbf{0}$$
$$\mathbf{J}\dot{\mathbf{w}}_1 - 2\alpha k\sigma\mathbf{J}\mathbf{v}_1 - \alpha\sigma^2\mathbf{w}_1 + 2\gamma\mathbf{u}_0\mathbf{u}_0^T\mathbf{w}_1 = \mathbf{0}\,. \tag{10}$$

To determine the spectrum, let $(\mathbf{v}_1, \mathbf{w}_1) = (\mathbf{q}, \mathbf{p})\mathrm{e}^{\lambda t}$. Then (λ, σ) are determined by roots of

$$\Delta(\lambda, \sigma) = \lambda^4 + 2(p^2 + 4k^2\alpha^2\sigma^2)\lambda^2 + (p^2 - 4k^2\alpha^2\sigma^2)^2\,,$$

where $p^2 = \alpha^2\sigma^4 - 2\alpha\gamma\|\mathbf{u}_0\|^2\sigma^2$. Suppose $p^2 > 0$, then all four roots are purely imaginary (see Figure 1) and given by

$$\lambda = \mathrm{i}2\alpha k\sigma \pm \mathrm{i}p \quad \text{and} \quad \lambda = -\mathrm{i}2\alpha k\sigma \pm \mathrm{i}p\,.$$

These modes are purely imaginary as long as $p^2 > 0$; equivalently $2\gamma\alpha\|\mathbf{u}_0\|^2 < \alpha^2\sigma^2$. Since $\alpha\gamma > 0$, the instability threshold is achieved when the amplitude reaches

$$\|\mathbf{u}_0\| = \frac{|\alpha\sigma|}{\sqrt{2\alpha\gamma}}\,. \tag{11}$$

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At this threshold, a collision of eigenvalues occurs at the points $\lambda = \pm 2\mathrm{i}k\alpha\sigma$; see Figure 1 for a schematic of this collision.

It will be assumed henceforth that $k \neq 0$. Then instability is through a collision of eigenvalues of opposite energy sign, which reproduces the instability mechanism for the full water-wave problem.

Purely imaginary eigenvalues of a Hamiltonian system have a signature associated with them, and this signature is related to the sign of the energy [17, 18, 31]. Collision of eigenvalues of opposite signature is a necessary condition for the collision resulting in instability.

It is straightforward to compute the signature of the modes in the NLS model. Suppose that the amplitude $\|\mathbf{u}_0\|$ of the basic state is smaller than the critical value (11) for instability. Then there are two pairs of purely imaginary eigenvalues, and they each have a signature. Let us concentrate on the eigenvalues on the positive imaginary axis

$$\lambda = \mathrm{i}\Omega_\pm \quad \text{with} \quad \Omega_\pm = c_g\sigma \pm p\,, \quad c_g = 2\alpha k\,. \tag{12}$$

Then

$$\mathsf{Sign}(\Omega_\pm) = \mathrm{i}\langle \overline{\mathbf{q}}, \mathbf{J}\mathbf{q}\rangle + \mathrm{i}\langle \overline{\mathbf{p}}, \mathbf{J}\mathbf{p}\rangle\,,$$

where the inner product is real in order to make the conjugation explicit. One can also show that this signature has the same sign as the energy perturbation restricted to this mode. A straightforward calculation shows that $\mathsf{Sign}(\Omega_\pm) = \pm 4$ when $\|\mathbf{u}_0\| = 0$. Since p^2 decreases as the amplitude increases, the two modes will have opposite signature for all $\|\mathbf{u}_0\|$ between $\|\mathbf{u}_0\| = 0$ and the point of collision.

Now consider the effect of the damping terms. Consider the reduced system (10) for the BF instability with the $abc-$damping terms included:

$$\begin{aligned} \mathbf{J}\dot{\mathbf{v}}_1 + 2\alpha k\sigma \mathbf{J}\mathbf{w}_1 - \alpha\sigma^2\mathbf{v}_1 + 2\gamma\mathbf{u}_0\mathbf{u}_0^T\mathbf{v}_1 + \mathcal{D}_1 = \mathbf{0} \\ \mathbf{J}\dot{\mathbf{w}}_1 - 2\alpha k\sigma \mathbf{J}\mathbf{v}_1 - \alpha\sigma^2\mathbf{w}_1 + 2\gamma\mathbf{u}_0\mathbf{u}_0^T\mathbf{w}_1 + \mathcal{D}_2 = \mathbf{0}\,, \end{aligned} \tag{13}$$

with

$$\begin{aligned} \mathcal{D}_1 &= 2k a\sigma\mathbf{w}_1 + a\sigma^2\mathbf{J}\mathbf{v}_1 + b\mathbf{J}\mathbf{v}_1 + 2c\langle\mathbf{u}_0, \mathbf{v}_1\rangle\mathbf{J}\mathbf{u}_0 \\ \mathcal{D}_2 &= -2k a\sigma\mathbf{v}_1 + a\sigma^2\mathbf{J}\mathbf{w}_1 + b\mathbf{J}\mathbf{w}_1 + 2c\langle\mathbf{u}_0, \mathbf{w}_1\rangle\mathbf{J}\mathbf{u}_0\,. \end{aligned} \tag{14}$$

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Now, let $(\mathbf{v}_1(t), \mathbf{w}_1(t)) = (\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{w}}_1)e^{\lambda t}$. Then the eigenvalue problem for the stability exponent reduces to studying the roots of a determinant showing (with the help of MAPLE) that the two roots in the upper half plane are given by

$$\lambda_{\pm} = 2ik\sigma\alpha - (b + a\sigma^2 + c\|\mathbf{u}_0\|^2) \pm i\sqrt{S}\,, \tag{15}$$

with

$$S = 4ia\sigma^3 k\alpha - c^2\|\mathbf{u}_0\|^4 - 4k^2\sigma^2 a^2 - 4ia\sigma k\gamma\|\mathbf{u}_0\|^2 - 2\alpha\gamma\sigma^2\|\mathbf{u}_0\|^2 + \alpha^2\sigma^4\,.$$

When $a = b = c = 0$, these stability exponents reduce to

$$\lambda_{\pm} = 2ik\sigma\alpha \pm i\sqrt{\alpha^2\sigma^4 - 2\alpha\gamma\sigma^2\|\mathbf{u}_0\|^2}\,.$$

Now suppose these two eigenvalues are purely imaginary: the amplitude $\|\mathbf{u}_0\|$ is below the critical value (11). To determine the leading order effect of dissipation, expand (15) in a Taylor series with respect to a, b and c and take the real part

$$\text{Re}(\lambda_{\pm}) = -(a\sigma^2 + b + c\|\mathbf{u}_0\|^2) \mp \frac{2ak\sigma(\alpha\sigma^2 - \gamma\|\mathbf{u}_0\|^2)}{\sqrt{\alpha^2\sigma^4 - 2\alpha\gamma\sigma^2\|\mathbf{u}_0\|^2}} + \cdots. \tag{16}$$

For any $a > 0$ there is an open region of parameter space where these two real parts have opposite sign since their product to leading order is

$$\text{Re}(\lambda_-)\text{Re}(\lambda_+) = (a\sigma^2 + b + c\|\mathbf{u}_0\|^2)^2 - \frac{4a^2k^2\sigma^2(\alpha\sigma^2 - \gamma\|\mathbf{u}_0\|^2)^2}{\alpha^2\sigma^4 - 2\alpha\gamma\sigma^2\|\mathbf{u}_0\|^2} + \cdots.$$

For any a, b, c with $a \neq 0$ there is an open set of values of $\|\mathbf{u}_0\|$ where this expression is strictly negative, showing that $\text{Re}(\lambda_-)$ and $\text{Re}(\lambda_+)$ perturb in opposite directions. In this parameter regime the dissipation perturbs the negative energy mode as shown schematically in Figure 2.

It is clear that when only the $b-$term is present all eigenvalues shift to the left. Therefore the $b-$term does not produce any enhancement of the instability, in agreement with [12]. This damping is analogous to the *uniform damping* in (4). It is the $a-$term which leads to enhancement. However the NLS is a simplified model for water waves.

In summary, the fundamental observation is that BF instability is associated with a collision of eigenvalues of positive and negative energy, and there are physically realizable forms of damping which enhance this instability. It remains to be seen how this effect can be revealed in laboratory experiments, in numerical experiments based on the full water-wave equations, and in the open ocean.

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