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Geometric lift of paths of Hamiltonian equilibria and homoclinic bifurcation

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A saddle-center transition of eigenvalues in the linearization about Hamiltonian equilibria, and the attendant planar homoclinic bifurcation, is one of the simplest and most well known bifurcations in dynamical systems theory. It is therefore surprising that anything new can be said about this bifurcation. In this tutorial, the classical view of this bifurcation is reviewed and then it is shown that lifting the planar system to four dimensions gives a new view. The principal practical outcome is a new formula for the nonlinear coefficient in the normal form which generates the homoclinic orbit. The new formula is based on the intrinsic curvature of the lifted path of equilibria.

Keywords: homoclinic, saddle-center, Hamiltonian.

1. Introduction

Consider a one-parameter family of Hamiltonian systems on $\mathbb{R}^2$

$$-\dot{p} = \frac{\partial H}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p}\,, \tag{1.1}$$

where $H(q,p,\alpha)$ is a given smooth function. Suppose there exists a family of equilbria

$$\frac{\partial H}{\partial q}(q,p,\alpha) = \frac{\partial H}{\partial p}(q,p,\alpha) = 0\,. \tag{1.2}$$

It is natural to consider this family of equilibria as a parameterized curve

$$(q(\alpha), p(\alpha))\,, \quad \alpha \in \mathscr{A}\,, \tag{1.3}$$

for some interval $\mathscr{A}$. (In general, the solution set of (1.2) consists of multiple disconnected branches.) This viewpoint is shown schematically in Figure 1(a). An additional parameter can be introduced so that the planar curve is lifted to a parameterized curve in three dimensions

$$(q(c), p(c), \alpha(c))\,, \quad c \in \mathscr{C}\,, \tag{1.4}$$

for some interval $\mathscr{C}$. This viewpoint is shown schematically in Figure 1(b).

At this point there is no advantage of one representation over the other. Moreover there is no obvious choice of parameter c. It will be argued that c should be defined by

$$c = \frac{\partial H}{\partial \alpha}\,. \tag{1.5}$$

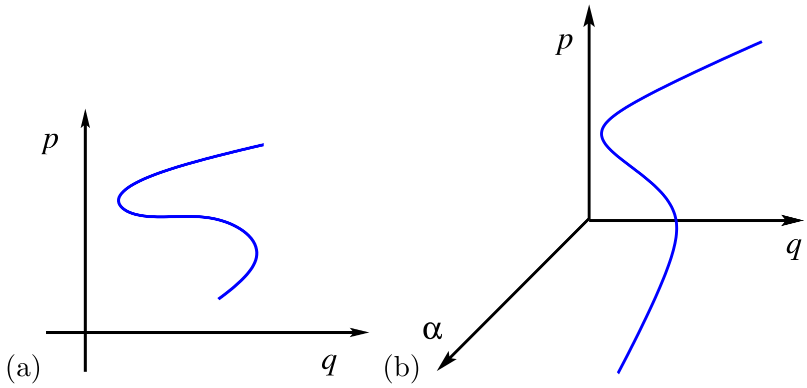

Fig. 1. Curve of equilibria: (a) parameterized by α and (b) parameterized by c.

The representation (1.5) can be interpreted as a *Legendre transform* of parameter space, but at this point it can be viewed as just a definition of the parameter c. When combined with (1.2) it gives three equations for the three functions $(q(c), p(c), \alpha(c))$. It is necessary for the Legendre transform (1.5) to be non-degenerate in order for there to be a one-to-one correspondence between the representations (1.3) and (1.4). This latter point is addressed in §3, particularly §3.1.

Now, suppose that the path of equilibria has a saddle-center transition of eigenvalues: for some value of α, denoted by α_0, the Hessian is degenerate

$$\det(\mathbf{L}) = 0\,, \quad \text{but} \quad \tau := \mathrm{Trace}(\mathbf{L}) \neq 0\,, \tag{1.6}$$

where

$$\mathbf{L} := \begin{bmatrix} \frac{\partial^2 H}{\partial q^2} & \frac{\partial^2 H}{\partial q \partial p} \\ \frac{\partial^2 H}{\partial p \partial q} & \frac{\partial^2 H}{\partial p^2} \end{bmatrix}\Bigg|_{\alpha=\alpha_0} . \tag{1.7}$$

The linearization of (1.1) about the equilibrium point at $\alpha = \alpha_0$ is

$$\mathbf{J}\dot{U} = \mathbf{L}U\,, \quad \mathbf{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \tag{1.8}$$

Here and throughout the paper, a dot is used to indicate a derivative with respect to time. Let $U(t) = \mathrm{e}^{\lambda t}\widehat{U}$, then the eigenvalue problem for $(\lambda, \widehat{U})$ is

$$\left[\mathbf{J}^{-1}\mathbf{L} - \lambda \mathbf{I}\right]\widehat{U} = 0\,.$$

With the conditions (1.6), zero is an eigenvalue of algebraic multiplicity two and geometric multiplicity one, at $\alpha = \alpha_0$, of this eigenvalue problem. The movement of the eigenvalues, for α near α_0, is shown schematically in Figure 2.

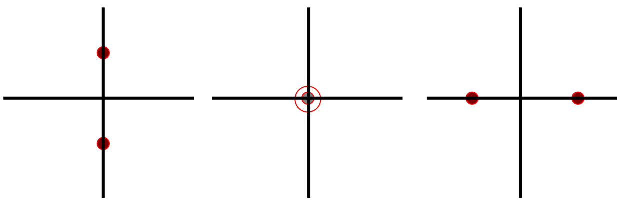

Fig. 2. Movement of the eigenvalues of $\mathbf{J}^{-1}\mathbf{L}$ near a saddle-center transition, plotted in the complex $\lambda-$plane.

Because the system is planar, a saddle-center transition of eigenvalues generates a robust homoclinic orbit in the nonlinear problem. The generation of the homoclinic orbit is shown by transforming the nonlinear system (1.1) into nonlinear normal form [Arnold *et al.*, 1993; Broer *et al.*, 1995; Meyer & Hall, 1992]. The nonlinear normal form to leading order is

$$-\dot{p} = \mu - \tfrac{1}{2}\kappa q^2 \quad \text{and} \quad \dot{q} = s\,p\,, \tag{1.9}$$

where $s = \pm 1$ is a symplectic invariant associated with the linearization, μ is an unfolding parameter proportional to $\alpha - \alpha_0$, and κ is a parameter depending on the nonlinearity of the original system (the classical formula for κ is given in §2). The homoclinic orbit, shown schematically in Figure 3, is formed

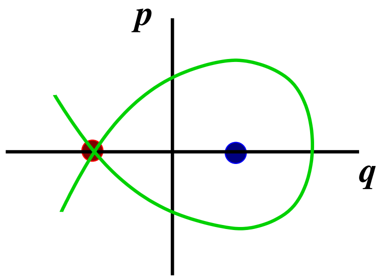

Fig. 3. The homoclinic orbit appearing in the nonlinear unfolding of the saddle-center transition.

around a pair of equilibria, one of which is hyperbolic (unstable) and one elliptic (stable), that coalesce as $\alpha \to \alpha_0$.

The main result of this tutorial is to prove that the first derivative $\alpha'(c_0)$ and second derivative $\alpha''(c_0)$ determine the nonlinear normal form at $c = c_0$. When the branch of equilibria is parameterized by c in (1.5) a saddle-center transition of eigenvalues occurs in the linearization if and only if

$$\alpha'(c_0) = 0\,. \tag{1.10}$$

This observation is not so surprising and could be inferred using other methods. The surprising result is that

$$\kappa = a^3\alpha''(c_0)\,, \tag{1.11}$$

where a is a positive scale factor. Instead of (1.9), the nonlinear normal form is

$$-\dot{p} = \mu - \tfrac{1}{2}a^3\alpha''(c_0)q^2 \quad \text{and} \quad \dot{q} = s\,p\,. \tag{1.12}$$

The parameter a can be scaled away by taking $\tau = at$, $u = aq$ and $\mu = a\widetilde{\mu}$. Then the scaled nonlinear normal form (1.12) is

$$-\frac{dp}{d\tau} = \widetilde{\mu} - \tfrac{1}{2}\alpha''(c_0)u^2 \quad \text{and} \quad \frac{du}{d\tau} = sp\,. \tag{1.13}$$

Noting that μ is proportional to $\alpha - \alpha_0$, the right hand side of the first equation is a second-order Taylor expansion of $\alpha(c)$ with $c - c_0$ represented by u, and $\alpha'(c_0) = 0$. This observation suggests that the function $u(t)$ may be interpretable as a modulation on the c parameter space.

The parameter s can also be given a geometric interpretation. It will be proved that

$$s = \text{sign}\left(H_{pp}\big|_{c=c_0} \right)\,. \tag{1.14}$$

In other words all the principal coefficients of the nonlinear normal form are determined by the geometry of the underlying path of equilibria. κ is the curvature of the projection of the path of equilibria onto the (c, α) plane. It is the intrinsic curvature at $c = c_0$ of the plane curve $\alpha(c)$. (It is intrinsic because $\alpha'(c_0) = 0$.)

The strategy of the tutorial is as follows. First the classical normal form is reviewed in §2, with particular attention to a formula for κ in (1.9). In §3 and §4 the first implications of the definition (1.5) are proved. In §3 the role of the first derivative (1.10) is proved and in §4 the formula (1.11) is proved.

In §5 a natural argument for the definition (1.5) is presented. By lifting the system to $\mathbb{R}^4$, the family of equilibria is converted to a family of relative equilibria and c turns out to be the speed of the relative equilibria! In §6 some of the implications of the lift are presented. In the Concluding Remarks section, the extension to multi-parameter families of equilibria is sketched as well as other generalizations of the geometric viewpoint.

1.1. *History*

The study of *parameter-dependent* potential functions

$$V(\mathbf{q},\alpha)\,,\quad \mathbf{q}\in\mathbb{R}^n\,,\quad \alpha\in\mathbb{R}\,, \tag{1.15}$$

has a rich history going back to [Poincaré, 1885]. Our main interest is the history of the use of the geometric lift (1.5). This lift appears to have first been introduced by [Katz, 1978, 1979] in the context an astrophysical application. He calls the parameter c in (1.5) a *conjugate parameter* and notes that $\alpha'(c)=0$ indicates a change of critical point type for functionals of the type (1.15). [Thompson, 1979] concurrently applied this idea to the analysis of beam buckling and elastic stability. [Maddocks, 1987] extends the theory to infinite dimensions, specifically studying parameter-dependent potential functions (1.15) with $\mathbf{q}$ an element of a Hilbert space, and derives an explicit formula for the critical eigenvalue in the second variation of the functional, thereby giving an explicit relation between curvature in (c,α) space and critical point type. This work was further extended in the astrophysical literature by [Kaburaki, 1994] and [Sorkin, 1981, 1982]. [Kaburaki, 1994] was the first to point out that (1.5) is in fact a Legendre transformation of parameter space, and he showed that $c'(\alpha)=0$ does not indicate a change of critical point type. [Sorkin, 1982] extends the theory to the case where the parameter space is multi-dimensional.

All the above work is for potential functions. Here we are interested in equilibria of Hamiltonian systems. The above theory can be embedded in a Hamiltonian context by taking

$$H(\mathbf{q},\mathbf{p},\alpha)=\tfrac{1}{2}\mathbf{p}\cdot\mathbf{p}+V(\mathbf{q},\alpha)\,. \tag{1.16}$$

Then critical points of V in (1.15) correspond to equilibria of $H(\mathbf{q},\mathbf{p},\alpha)$ since all equilibria of H have $\mathbf{p}=0$. The advantage of studying equilibria of (1.15) using (1.16) or studying Hamiltonian equilibria in general is that the geometric lift (1.5) can be related to the dynamics of the Hamiltonian system.

2. Linear and nonlinear normal form

In this section a summary of the classical normal form theory for deriving (1.9) is given. In particular the main results needed are the linear normal form and a formula for the coefficient κ in the nonlinear normal form. The theory for deriving linear and nonlinear normal forms can be found in [Meyer & Hall, 1992].

Start with the planar Hamiltonian system (1.1) assuming the existence of a path of equilibria (1.2)-(1.3). Assume further that the conditions for a saddle-center transition (1.6) are satisfied. Then there exists eigenvectors $\widehat{\xi}_1$ and $\widehat{\xi}_2$ satisfing

$$\mathbf{L}\widehat{\xi}_1=0\quad\text{and}\quad\mathbf{L}\widehat{\xi}_2=\mathbf{J}\widehat{\xi}_1\,.$$

Transform the linear system (1.8) to canonical form. In order for the transformation matrix to be symplectic, scale the eigenvectors to

$$\xi_1=a\widehat{\xi}_1\quad\text{and}\quad\xi_2=a\widehat{\xi}_2\,,\quad\text{where}\quad a=\frac{1}{\sqrt{|\langle\mathbf{J}\widehat{\xi}_1,\widehat{\xi}_2\rangle|}}\,,$$

where here and throughout $\langle\cdot,\cdot\rangle$ is a standard inner product. Then the appropriate transformation matrix is

$$\Xi=\left[\,\xi_1\,|\,s\,\xi_2\,\right].$$

The parameter s is the *symplectic invariant* of the linearization, defined in this case,

$$s:=\mathrm{sign}\big(\langle\mathbf{J}\widehat{\xi}_1,\widehat{\xi}_2\rangle\big)=\langle\mathbf{J}\xi_1,\xi_2\rangle\,. \tag{2.1}$$

Now Ξ is symplectic and so under the transformation $U = \Xi\widehat{U}$ the form of (1.8) is preserved and $\mathbf{L}$ is transformed to

$$\Xi^T \mathbf{L} \Xi = \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} ,$$

transforming (1.8) to

$$\dot{p} = 0 \quad \text{and} \quad \dot{q} = s\,p\,.$$

This construction is a special case of *Williamson normal form* [Williamson, 1936] for Hamiltonian systems linearized about equilibria.

Now the aim is to transform the nonlinear part of (1.1) to normal form. First expand the right-hand side of (1.1) in a Taylor series

$$\mathbf{J}\dot{\mathbf{u}} = \mathbf{L}\mathbf{u} + \tfrac{1}{2} D^3 H(\mathbf{u},\mathbf{u}) + \cdots\,, \qquad \mathbf{u} = \begin{pmatrix} q \\ p \end{pmatrix} . \tag{2.2}$$

Now introduce a quadratic nonlinear transformation taking this equation to the form (1.9). The details of this construction are given in Chapter 7 §C of [Meyer & Hall, 1992]. Here our interest is in deriving a formula for κ.

To this end express $\mathbf{u}$ up to quadratic order in (q,p),

$$\mathbf{u} = q\boldsymbol{\xi}_1 + s\,p\boldsymbol{\xi}_2 + q^2\boldsymbol{\eta}_1 + qp\boldsymbol{\eta}_2 + p^2\boldsymbol{\eta}_3 + \cdots\,.$$

Substitute into (2.2), replace $\dot{q}$ and $\dot{p}$ using (1.9) and equate terms proportional to like powers of q and p. The linear terms cancel, and the quadratic term proportional to q^2 is

$$\tfrac{1}{2} s\,\kappa \mathbf{J}\boldsymbol{\xi}_2 = \mathbf{L}\boldsymbol{\eta}_1 + \tfrac{1}{2} D^3 H(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)\,.$$

The equations proportional to qp and p^2 are also easily written down, but only the above equation is needed to obtain a formula for κ. The solvability condition for

$$\mathbf{L}\boldsymbol{\eta}_1 = \tfrac{1}{2} s\,\kappa \mathbf{J}\boldsymbol{\xi}_2 - \tfrac{1}{2} D^3 H(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)\,,$$

is

$$\langle \boldsymbol{\xi}_1, \tfrac{1}{2} s\,\kappa \mathbf{J}\boldsymbol{\xi}_2 - \tfrac{1}{2} D^3 H(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)\rangle = 0\,,$$

or

$$\tfrac{1}{2} s\,\kappa \langle \boldsymbol{\xi}_1, \mathbf{J}\boldsymbol{\xi}_2\rangle - \tfrac{1}{2}\langle \boldsymbol{\xi}_1, D^3 H(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)\rangle = 0\,,$$

But $s = \langle \mathbf{J}\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\rangle$ and so

$$\kappa = -\langle \boldsymbol{\xi}_1, D^3 H(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)\rangle\,. \tag{2.3}$$

Another form of this expression will be needed for the comparison with $\alpha''(c_0)$. Let $\xi_1 = (\xi_{11}, \xi_{12})$ and suppose H is viewed as a function of q and p. Then the formula (2.3) is equivalent to

$$\kappa = -H_{qqq}\xi_{11}^3 - 3H_{qqp}\xi_{11}^2\xi_{12} - 3H_{qpp}\xi_{11}\xi_{12}^2 - H_{ppp}\xi_{12}^3\,. \tag{2.4}$$

Equivalently

$$\frac{\kappa}{a^3} = -H_{qqq}\widehat{\xi}_{11}^3 - 3H_{qqp}\widehat{\xi}_{11}^2\widehat{\xi}_{12} - 3H_{qpp}\widehat{\xi}_{11}\widehat{\xi}_{12}^2 - H_{ppp}\widehat{\xi}_{12}^3\,. \tag{2.5}$$

3. Significance of $\alpha'(c_0) = 0$

Consider the path of equilibria parameterized by c in (1.4) and suppose that

$$H_{pp} \neq 0\,, \tag{3.1}$$

along the path of equilbria. Differentiating (1.2) and (1.5) with respect to c gives

$$\begin{bmatrix} H_{qq} & H_{qp} & H_{q\alpha} \\ H_{pq} & H_{pp} & H_{p\alpha} \\ H_{\alpha q} & H_{\alpha p} & H_{\alpha\alpha} \end{bmatrix} \begin{pmatrix} q'(c) \\ p'(c) \\ \alpha'(c) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.2}$$

Suppose the matrix on the left-hand side is invertible

$$\det(\mathbf{M}(c)) \neq 0 \quad \text{with} \quad \mathbf{M}(c) := \begin{bmatrix} H_{qq} & H_{qp} & H_{q\alpha} \\ H_{pq} & H_{pp} & H_{p\alpha} \\ H_{\alpha q} & H_{\alpha p} & H_{\alpha\alpha} \end{bmatrix}. \tag{3.3}$$

Sufficient conditions for (3.3) to be satisfied will be given below.

With the hypothesis (3.3), Cramer's rule can be used to solve for the derivatives

$$\begin{aligned} q'(c) &= +\frac{1}{\det(\mathbf{M})}\det\begin{bmatrix} H_{qp} & H_{q\alpha} \\ H_{pp} & H_{p\alpha} \end{bmatrix} \\ p'(c) &= -\frac{1}{\det(\mathbf{M})}\det\begin{bmatrix} H_{qq} & H_{q\alpha} \\ H_{pq} & H_{p\alpha} \end{bmatrix} \\ \alpha'(c) &= +\frac{1}{\det(\mathbf{M})}\det\begin{bmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{bmatrix}. \end{aligned} \tag{3.4}$$

It is clear from these expressions that at $c = c_0$,

$$\alpha'(c_0) = 0 \quad \Leftrightarrow \quad \det\begin{bmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{bmatrix}\Bigg|_{c=c_0} = 0\,, \tag{3.5}$$

if $\det(\mathbf{M}(c_0)) \neq 0$.

3.1. *Invertibility of* $\mathbf{M}(c_0)$

To check the invertibility of $\mathbf{M}(c_0)$, write $\mathbf{M}(c_0)$ as a bordered matrix

$$\mathbf{M}(c_0) = \begin{bmatrix} \mathbf{L} & \mathbf{v} \\ \mathbf{v}^T & r \end{bmatrix}, \quad \mathbf{v} = \begin{pmatrix} H^o_{q\alpha} \\ H^o_{p\alpha} \end{pmatrix}, \quad r = H^0_{\alpha\alpha}\,,$$

where the superscript o indicates evaluation at $c = c_0$. Now use the standard formula for the determinant of a bordered matrix

$$\det\begin{bmatrix} \mathbf{L} & \mathbf{v} \\ \mathbf{v}^T & r \end{bmatrix} = r\det(\mathbf{L}) - \langle \mathbf{v}, \mathbf{L}^{\#}\mathbf{v}\rangle\,, \tag{3.6}$$

where $\mathbf{L}^{\#}$ is the adjugate. With $\det(\mathbf{L}) = 0$, and the assumption $H^o_{pp} \neq 0$, which follows from (3.1) and implies $\tau \neq 0$, the first term vanishes and

$$\mathbf{L}^{\#} = \tau\frac{\widehat{\xi}_1\widehat{\xi}_1^T}{\langle \widehat{\xi}_1, \widehat{\xi}_1\rangle}, \quad \tau = \mathrm{Trace}(\mathbf{L})\,. \tag{3.7}$$

Both the formulas (3.6) and (3.7) are proved in the book [Magnus & Neudecker, 1988].

Hence a sufficient condition for $\det(\mathbf{M}(c_0)) \neq 0$ is $\langle \mathbf{v}, \widehat{\xi}_1\rangle \neq 0$. But it follows from (3.2) that when $\alpha'(c_0) = 0$,

$$\mathrm{Ker}(\mathbf{L}) = \mathrm{span}\left\{ \begin{pmatrix} q'(c_0) \\ p'(c_0) \end{pmatrix} \right\} \quad \Rightarrow \quad \widehat{\xi}_1 = \begin{pmatrix} q'(c_0) \\ p'(c_0) \end{pmatrix}. \tag{3.8}$$

Hence

$$\langle \mathbf{v}, \widehat{\xi}_1 \rangle = H_{q\alpha} q'(c_0) + H_{p\alpha} p'(c_0) = 1\,,$$

where the second equality follows from the third equation in (3.2). In summary, with the assumption $H^o_{pp} \neq 0$, the definition (1.5) is sufficient to conclude invertibility of $\mathbf{M}(c_0)$.

3.2. *Geometric definition of the generalized eigenvector*

A geometric definition of $\widehat{\xi}_2$ can also be given. Solving $\mathbf{L}\widehat{\xi}_2 = \mathbf{J}\widehat{\xi}_1$ with $\widehat{\xi}_1$ in (3.8) gives

$$\widehat{\xi}_2 = \frac{q'(c_0)}{\tau H^o_{pp}} \begin{pmatrix} H^o_{qp} \\ H^o_{pp} \end{pmatrix} . \tag{3.9}$$

That $q'(c_0)$ is nonzero is assured by the hypothesis $H^o_{pp} \neq 0$.

Using (3.8) and (3.9) we can now prove (1.14)

$$\begin{aligned} \langle \mathbf{J}\widehat{\xi}_1, \widehat{\xi}_2 \rangle &= \frac{q'(c_0)}{\tau H^o_{pp}} \Big(-p'(c_0) H^o_{qp} + q'(c_0) H^o_{pp} \Big) \\ &= \frac{q'(c_0)}{\tau H^o_{pp}} \Big(q'(c_0) H^o_{qq} + q'(c_0) H^o_{pp} \Big) \\ &= \frac{q'(c_0)}{\tau H^o_{pp}} \Big(\tau q'(c_0) \Big) \\ &= \frac{q'(c_0)^2}{H^o_{pp}}\,, \end{aligned}$$

and so $s = \mathrm{sign}(\langle \mathbf{J}\widehat{\xi}_1, \widehat{\xi}_2 \rangle) = \mathrm{sign}(H^o_{pp})$.

The expression for $\widehat{\xi}_2$ in (3.9) is not unique. An arbitrary multiple of $\widehat{\xi}_1$ can be added:

$$\widehat{\xi}_2 \mapsto \widehat{\xi}_2 + \gamma \widehat{\xi}_1\,. \tag{3.10}$$

Now choose γ so that $\langle \widehat{\xi}_2, \mathbf{v} \rangle = 0$. With this normalization, the modified symplectic matrix $\widehat{\Xi} = [\widehat{\xi}_1 \,|\, s\widehat{\xi}_2]$, with $\widehat{\xi}_1$ defined in (3.8) and $\widehat{\xi}_2$ defined in (3.9) and (3.10), has the property

$$\widehat{\Xi}^T \begin{pmatrix} H^o_{q\alpha} \\ H^o_{p\alpha} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

assuring that the sign of μ in (1.9) has the sign of $(\alpha - \alpha_0)$.

4. $\alpha''(c_0)$ and the normal form

In this section it is proved that when $\alpha'(c_0) = 0$, then

$$\alpha''(c_0) = -H_{qqq} q_c^3 - H_{qqp} p_c q_c^2 - H_{qpp} q_c p_c^2 - H_{ppp} p_c^3\,. \tag{4.1}$$

By combining this expression with (2.5) and (3.8), the following formula for κ is deduced

$$\kappa = a^3 \alpha''(c_0)\,. \tag{4.2}$$

The parameter κ is the coefficient of the nonlinear term in the leading-order nonlinear normal form near a saddle-center transition (1.9).

To prove (4.1), start with (3.2) written out

$$\begin{aligned} H_{qq} q(c) + H_{qp} p(c) + H_{q\alpha} \alpha'(c) &= 0 \\ H_{pq} q(c) + H_{pp} p(c) + H_{p\alpha} \alpha'(c) &= 0 \\ H_{\alpha q} q(c) + H_{\alpha p} p(c) + H_{\alpha\alpha} \alpha'(c) &= 1\,. \end{aligned}$$

Differentiate again, starting with the first equation,

$$\begin{aligned} H_{qqq}q'(c)q'(c) + H_{pqq}q'(c)p'(c) + H_{\alpha qq}q'(c)\alpha'(c) \\ +H_{qqp}p'(c)q'(c) + H_{pqp}p'(c)p'(c) + H_{\alpha qp}p'(c)\alpha'(c) \\ +H_{qq\alpha}\alpha'(c)q'(c) + H_{pq\alpha}\alpha'(c)p'(c) + H_{\alpha q\alpha}\alpha'(c)\alpha'(c) = 0\,, \end{aligned}$$

with similar expressions for the second derivative of the second and third equations. Set $\alpha'(c) = 0$ to simplify, and rearrange,

$$-\begin{bmatrix} H_{qq} & H_{qp} & H_{q\alpha} \\ H_{pq} & H_{pp} & H_{p\alpha} \\ H_{\alpha q} & H_{\alpha p} & H_{\alpha\alpha} \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \ddot{p} \\ \ddot{\alpha} \end{pmatrix} = \begin{pmatrix} H_{qqq}q'(c)q'(c) + 2H_{pqq}q'(c)p'(c) + H_{pqp}p'(c)p'(c) \\ H_{qpq}q'(c)q'(c) + 2H_{ppq}q'(c)p'(c) + H_{ppp}p'(c)p'(c) \\ H_{q\alpha q}q'(c)q'(c) + 2H_{p\alpha q}q'(c)p'(c) + H_{p\alpha p}p'(c)p'(c) \end{pmatrix}.$$

Express the right-hand side as (f_1, f_2, f_3). Then, setting $c = c_0$ and using Cramer's rule to solve for $\alpha''(c_0)$,

$$\begin{aligned} \alpha''(c_0) &= -\tfrac{1}{\det(\mathbf{M}(c_0))} \det \begin{pmatrix} H_{qq} & H_{qp} & f_1 \\ H_{pq} & H_{pp} & f_2 \\ H_{\alpha q} & H_{\alpha p} & f_3 \end{pmatrix} \\ &= -\tfrac{1}{\det(\mathbf{M}(c_0))} \left(f_1 \det \begin{pmatrix} H_{pq} & H_{pp} \\ H_{\alpha q} & H_{\alpha p} \end{pmatrix} - f_2 \det \begin{pmatrix} H_{qq} & H_{qp} \\ H_{\alpha q} & H_{\alpha p} \end{pmatrix} \right), \end{aligned}$$

using $\alpha'(c_0) = 0$. Now simplify using the expressions for q' and p' in (3.4), and noting that the kernel of $\mathbf{L}$ can be expressed in terms of $q'(c_0)$ and $p'(c_0)$ using (3.8),

$$\alpha''(c_0) = -f_1 q'(c_0) - f_2 p'(c_0)\,.$$

Substituting for f_1 and f_2 then proves the formulas (4.1) and (4.2).

5. Lifting the planar system to $\mathbb{R}^4$

A new argument is presented for why the definition (1.5) is natural. Since α is a constant, the system (1.1) is equivalent to

$$\begin{aligned} -\dot{\alpha} &= 0 \\ -\dot{p} &= H_q \\ \dot{q} &= H_p\,. \end{aligned}$$

However this system is not symplectic. Symplecticity requires dual pairs, so add a fourth equation to balance H_α,

$$\begin{aligned} -\dot{\alpha} &= 0 \\ \dot{\phi} &= H_\alpha \\ -\dot{p} &= H_q \\ \dot{q} &= H_p\,. \end{aligned} \tag{5.1}$$

This system is completely equivalent to (1.1), but it has a shadow two-dimensional system with coordinates (ϕ, α). Moreover it has a symmetry: $\phi \mapsto \phi + \phi_0$ for any $\phi_0 \in \mathbb{R}$. The original planar system (1.1) has been lifted to a four-dimensional system with a one-parameter symmetry. Symmetric systems have relative equilibria: solutions that travel at constant speed along the group. For this system relative equilibria are of the form

$$(\phi(t), \alpha(t), q(t), p(t)) = (ct + \phi_0, \alpha_0, q_0, p_0)\,.$$

Substitution into (5.1) gives $\dot{\alpha}_0 = 0$ and

$$\begin{aligned} c &= H_\alpha \\ 0 &= H_q \\ 0 &= H_p\,, \end{aligned} \tag{5.2}$$

which is precisely (1.2) combined with (1.5).

In summary, the introduction of the parameter c in (1.5) can be interpreted as generated by a lift of the planar system (1.1) to a four-dimensional system, replacing the one-parameter family of equilibria by a one-parameter family of relative equilibria.

The generalization of the leading order normal form (1.9) is

$$\begin{aligned} -\dot{\alpha} &= 0 \\ \dot{\phi} &= q \\ -\dot{p} &= \mu - \tfrac{1}{2}\kappa q^2 \\ \dot{q} &= s\,p\,. \end{aligned} \tag{5.3}$$

A proof of this normal form for the general case of saddle-center transition of relative equilibria is given in [Bridges, 2008].

6. Stability of the branching equilibria

A complete classification of the stability of the branches of equilibria that coalesce in the saddle-center transition can be given using the leading-order normal form in (5.3). The curve of equilibria near the transition is a relative equilibrium of the normal form (5.3)

$$(\phi(t), \alpha(t), q(t), p(t)) = (ct + \phi_0, \alpha_0, q_0, 0)\,.$$

Substitution gives $q_0 = c$ and

$$\mu = \tfrac{1}{2}\kappa c^2\,,$$

where μ can be interpreted as proportional to $\alpha - \alpha_0$. (The use of c here should be interpreted as $c - c_0$ for the original system.)

There are two cases $\kappa > 0$ and $\kappa < 0$ and they are shown in upper and lower diagram pairs, respectively, in Figure 4. On the other hand there are two possible elliptic (stable) configurations in each case. The

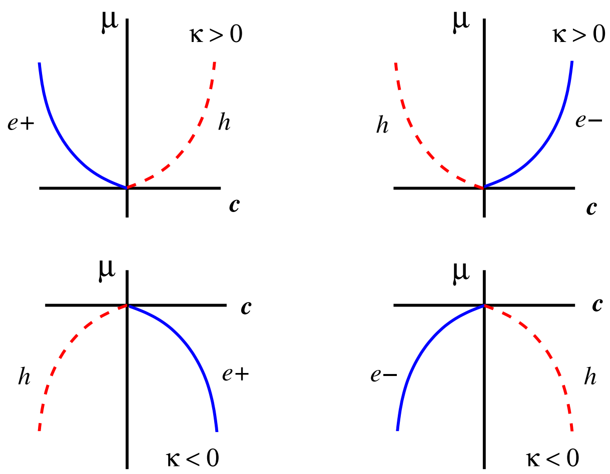

Fig. 4. Four cases in the saddle-center transition. h denotes hyperbolic (unstable) branch of equilibria, and $e\pm$ denoted elliptic (stable) branch with Krein signature $s = \pm 1$.

distinction is due to the *Krein signature.*

The Krein signature, $\mathcal{S}$, associated with a purely imaginary eigenvalue, with eigenvector ζ, in the linearization about an equilibrium is defined by (see [Kapitula, 2010])

$$\Omega(\overline{\zeta}, \zeta) := \langle \mathbf{J}\overline{\zeta}, \zeta \rangle = 2\mathrm{i}\mathcal{S}\,. \tag{6.1}$$

By scaling ζ, the sign $\mathcal{S} = \pm 1$.

Now compute the Krein signature along the stable branches of equilibria. The linearization of (5.3) about the basic state is

$$\mathbf{J}\dot{W} = \mathrm{D}^2 H(Z_0) W \,,$$

with $Z_0 = (\phi_0, \alpha_0, c, 0)$, and

$$\mathrm{D}^2 H(Z_0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -\kappa c & 0 \\ 0 & 0 & 0 & s \end{bmatrix} .$$

Take $W(t) = \mathrm{e}^{\lambda t}\widehat{W}$, then the characteristic equation for λ is

$$\Delta(\lambda) = \lambda^2(\lambda^2 - s\kappa c) = 0 \,.$$

Hence the equilibrium is hyperbolic (unstable) if $s\kappa c > 0$ and elliptic (stable) if $s\kappa c < 0$. Now suppose we are in the elliptic region, $-s\kappa c > 0$, and define

$$\sigma = \sqrt{-s\kappa c}\,,$$

and look at the elliptic eigenvalue $\lambda = \mathrm{i}\sigma$. The complex eigenvector ζ is

$$\zeta = \mathbb{C} \begin{pmatrix} 1 \\ \mathrm{i}\sigma \\ 0 \\ -s\sigma^2 \end{pmatrix} ,$$

where $\mathbb{C}$ is an arbitrary complex scale factor. Compute the Krein signature (6.1)

$$2\mathrm{i}\mathcal{S} = \langle \mathbf{J}\overline{\zeta}, \zeta \rangle = |\mathbb{C}|^2 2\mathrm{i}s\sigma^3 \,.$$

Since $\sigma > 0$, take $\mathbb{C} = \sigma^{-3/2}$. Then the Krein sign $\mathcal{S}$ equals s. The resulting Krein signs of each branch are then labelled in Figure 4 as $e\pm$.

6.1. *A hidden geometric phase*

An additional curiosity that shows up in the lifted system is a *geometric phase*. Geometric phases appear naturally in symmetric systems (see e.g. [Marsden *et al.*, 1990]) and a symmetry has been created here due to the lift. The phase variable ϕ can be shifted by an arbitrary constant.

To see the geometric phase in this case, look at the exact solution of the leading-order normal form (5.3)

$$\phi = \nu t - 3\frac{\nu}{\beta}\tanh(\beta t) \,.$$

with $q = \dot{\phi}$ and $p = s\dot{q}$, and

$$\nu = \pm\sqrt{\frac{2\mu}{\kappa}} \quad \text{and} \quad \beta = \tfrac{1}{2}\sqrt{s\nu\kappa}\,.$$

The sign of ν is chosen so that the argument of β is positive. The geometric phase is

$$\Delta\phi = \int_{-\infty}^{+\infty} (\dot{\phi} - \nu)\,\mathrm{d}t = -6\frac{\nu}{\beta}\,.$$

A schematic picture of this geometric phase is shown in Figure 5. As time progresses the phase variable $\phi(t)$ increases linearly with t, but the geometric phase is a shift which remains for all time. It only shows up after the lift, so the significance of the geometric phase is an open question.

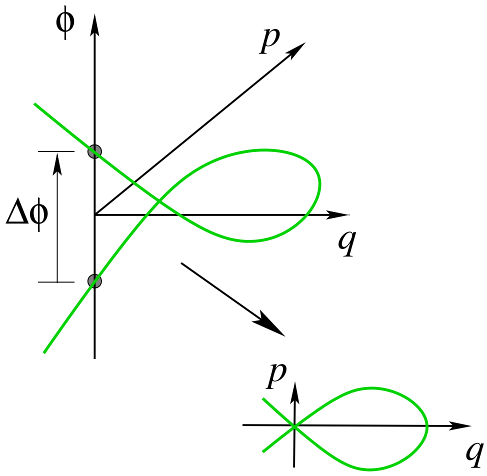

Fig. 5. Schematic picture of the geometric phase in the lifted system.

7. Concluding remarks

The geometric lift used in this paper originated in the study of degenerate periodic orbits [Bridges & Donaldson, 2005] and degenerate relative equilibria [Bridges, 2008]. There the parameter structure is a part of the theory of relative equilibria. The result here is more surprising in that the parameter structure is not apparent in the original problem, and arises from the lift.

If the Hamiltonian function depends on more parameters, say k parameters $\alpha_1, \ldots, \alpha_k$ with $k > 1$, the generalization of (1.2) and (1.5) is obtained by looking at critical points of (1.16) with the "conjugate parameters" $c_1, \ldots, c_k$ defined by

$$c_j = \frac{\partial H}{\partial \alpha_j}\,, \quad j = 1, \ldots, k\,.$$

This problem has been considered by [Sorkin, 1982] from the point of view of critical point theory. This problem can also be lifted to a Hamiltonian system of higher order by introducing k phase variables

$$\begin{aligned} -\dot{\alpha}_j &= 0,\,, \quad j = 1, \ldots, k \\ \dot{\phi}_j &= \tfrac{\partial H}{\partial \alpha_j}\,, \quad j = 1, \ldots, k \\ -\dot{p}_j &= \tfrac{\partial H}{\partial q_j}\,, \quad j = 1, \ldots, n \\ \dot{q}_j &= \tfrac{\partial H}{\partial p_j}\,, \quad j = 1, \ldots, n\,. \end{aligned} \tag{7.1}$$

This is a Hamiltonian system on $\mathbb{R}^{2(n+k)}$ with a $k-$dimensional symmetry group. The $k-$parameter family of equilibria is lifted to a $k-$parameter family of relative equilibria. With modest additional hypotheses, the theory in [Bridges, 2008] applies. It can be shown that a saddle-center transition occurs if the mapping $\mathbf{c} \mapsto \boldsymbol{\alpha}(\mathbf{c})$ has singular Jacobian and the coefficient in the nonlinear normal form κ is related to the intrinsic second derivative of $\boldsymbol{\alpha}(\mathbf{c})$. On the other hand, when $n > 1$ in (7.1) one also has to contend with the fact that the saddle-center transition may be accompanied by purely imaginary eigenvalues in the spectrum which changes the linear and normal form.

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