Computing the Maslov index of solitary waves. Part 1: Hamiltonian systems on a $4-$dimensional phase space

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Abstract

When solitary waves are characterized as homoclinic orbits of a finite-dimensional Hamiltonian system, they have an integer-valued topological invariant, the Maslov index. We are interested in developing a robust numerical algorithm to compute the Maslov index, to understand its properties, and to study the implications for the stability of solitary waves. The algorithms reported here are developed in the exterior algebra representation, which leads to a robust and fast algorithm with some novel properties. We use two different representations for the Maslov index, one based on an intersection index and one based on approximating the homoclinic orbit by a sequence of periodic orbits. New results on the Maslov index for solitary wave solutions of reaction-diffusion equations, the fifth-order Korteweg-De Vries equation, and the longwave-shortwave resonance equations are presented. Part 1 considers the case of four-dimensional phase space, and Part 2 considers the case of $2n-$dimensional phase space with $n > 2$.

1 Introduction

Hamiltonian evolution equations in one space dimension, such as the nonlinear Schrödinger (NLS) equation, fifth-order Korteweg-De Vries (KdV) equation, longwave-shortwave resonance (LW-SW) equations, have the property that their steady part is a finite-dimensional Hamiltonian system. For such systems, solitary wave solutions can be characterized as homoclinic orbits of the Hamiltonian ordinary differential equation (ODE). The spectral problem associated with the linearization about a given homoclinic orbit, in the time-dependent equations, then leads to a parameter-dependent family of linear Hamiltonian systems. The advantage of these Hamiltonian structures is that the linear and nonlinear Hamiltonian systems have global geometric properties that aid in proving existence of the basic solitary wave and in understanding its stability as a solution of the time-dependent equation. Our interest in this paper is in a particular geometric invariant – the Maslov index of homoclinic orbits.

The study of the stability of solitary waves using the Maslov index was pioneered in the papers by Jones [27] and Bose & Jones [6]. The linear stability of standing wave solutions of a spatially-dependent NLS equation is studied in [27]. The linearization about a steady solution results in a linear $\lambda-$dependent Hamiltonian system with two degrees of freedom of the form (1.2) and λ a spectral parameter. Geometric methods are then used to determine the Maslov index, and it is used to prove an instability result. Gradient parabolic PDEs of the form

$$\begin{aligned} u_t &= d_1 u_{xx} + f_u(u,v) \\ v_t &= d_2 v_{xx} + f_v(u,v) \end{aligned} \tag{1.1}$$

are considered in [6], where d_1 and d_2 are positive parameters, $f(u,v)$ is a given smooth function with gradient (f_u, f_v). Linearizing about a steady solution $(\widehat{u}(x), \widehat{v}(x))$, and introducing a spectral parameter leads to a coupled pair of linear second-order ODEs which can be put into the standard form (1.2), with the asymptotic property (1.5) and λ the spectral parameter. Since the PDE is a gradient system it is sufficient to restrict the spectral parameter to be real. Singular perturbation methods are then used to determine the Maslov index, which in turn is related to stability. A key feature of this work is the analysis of the induced system on the exterior algebra space $\bigwedge^2(\mathbb{R}^4)$.

Many of the most interesting solitary waves are only known numerically and therefore a numerical approach to the Maslov index is of interest. It is the aim of this paper to develop a numerical framework for computing the Maslov index of homoclinic orbits. Once the solitary wave solution is known, analytically or numerically, it is the linearization about that solitary wave which encodes the Maslov index. Therefore, the starting point for developing the theory is the following class of parameter-dependent Hamiltonian systems

$$\mathbf{J}\mathbf{u}_x = \mathbf{B}(x,\lambda)\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{R}^4\,, \quad x \in \mathbb{R}\,, \quad \lambda \in \mathbb{R}\,, \tag{1.2}$$

where $\mathbf{J}$ is the standard symplectic operator on $\mathbb{R}^4$

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{1.3}$$

and $\mathbf{B}(x,\lambda)$ is a symmetric matrix depending smoothly on x and λ. Let

$$\mathbf{A}(x,\lambda) = \mathbf{J}^{-1}\mathbf{B}(x,\lambda)\,. \tag{1.4}$$

The fact that $\mathbf{A}(x,\lambda)$ is obtained from the linearization about a solitary wave suggests the following asymptotic property. It is assumed throughout the paper that

$$\mathbf{A}_\infty(\lambda) = \lim_{x\to\pm\infty} \mathbf{A}(x,\lambda)\,, \tag{1.5}$$

and that $\mathbf{A}_\infty(\lambda)$ is strictly hyperbolic for an open set of λ values that includes 0.

The theory applies to linear Hamiltonian systems on a phase space of dimension $2n$ with n any finite natural number. In Part 1 attention is restricted to the case of 4−dimensional phase space which simplifies formulae, and general aspects of the case $n > 2$ are given in Part 2 [18].

The Maslov index is a winding number associated with paths of solutions of (1.2), in particular paths of Lagrangian planes. A Lagrangian plane in $\mathbb{R}^4$ is a 2−dimensional subspace of $\mathbb{R}^4$, say span$\{\mathbf{z}_1, \mathbf{z}_2\}$, satisfying $\langle \mathbf{J}\mathbf{z}_1, \mathbf{z}_2\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is a standard inner product on $\mathbb{R}^4$.

Suppose λ is fixed and on the interval $a \le x \le b$ consider a path of Lagrangian planes

$$[a,b] \mapsto \ \mathbf{Z}(x,\lambda) = [\mathbf{z}_1(x,\lambda)\,|\,\mathbf{z}_2(x,\lambda)] \in \mathbb{R}^{4\times 2}\,,$$

satisfying $\mathbf{Z}_x = \mathbf{A}(x,\lambda)\mathbf{Z}$ for $a \le x \le b$. The Maslov index of this path is a count of the number of times this path of Lagrangian planes has a non-trivial intersection with a fixed reference Lagrangian plane. A precise definition is given in §2.

A byproduct of the present theory is some new observations about the properties of the Maslov index, which in turn are useful in computation. In the numerics, the exterior algebra formulation is also advantageous. We give formulas for different representations of the Maslov index for Lagrangian planes on $\bigwedge^2(\mathbb{R}^4)$ (and for any n in Part 2 [18]), and present a general algorithm that works – in principle – for any dimension n. However, the dimension of $\bigwedge^n(\mathbb{R}^{2n})$ increases rapidly with n and so the algorithm is most effective for low dimensional systems. The algorithm is constructed so that the manifold of Lagrangian planes is attracting. Numerical results are presented in this paper for the cases of $\mathbb{R}^2$ and $\mathbb{R}^4$.

In order to develop a numerical framework for the Maslov index, one of our first difficulties is defining the Maslov index! Although it is easy to give a rough definition, making it precise depends greatly on the context and a surprising number of special difficulties and cases arise.

We will appeal to two constructions of the Maslov index. The first is based on an intersection index between the Lagrangian path and a reference plane. This definition was used in Maslov's original work, and was developed further by Arnold [3] and Duistermaat [25]. It is this definition that is used by Jones [27] and Bose & Jones [6], taking the Lagrangian path to be a path of unstable subspaces and taking the reference plane to be the stable subspace at infinity.

Independently, Chen & Hu [19] give two constructions of the Maslov index of a homoclinic orbit. Their first definition is based on an intersection index and is equivalent to the definition in [27,6]. Their second definition is based on a Fredholm index of (1.2) viewed as an operator in a function space on the real line. However, this latter definition, although equivalent to the definition based on an intersection index, is not useful for numerics.

In this paper, the definition of the Maslov index based on an intersection index is extended by introducing an explicit and computable formula for the crossing form. This theory is developed in §6.

Our second method for computing the Maslov index is to approximate the homoclinic orbit by a sequence of periodic orbits, apply the Maslov index for periodic orbits, and then take limits. There does not appear to be any loss of generality in using periodic approximates. Vanderbauwhede & Fiedler [44] prove that homoclinic orbits in Hamiltonian systems (as well as reversible systems) can be approximated as the limit of a sequence of periodic orbits.

The Maslov index for periodic orbits has been widely developed because of its interest in semi-classical quantization (e.g. [26,21,34,42,39,38] and references therein). In [16] a new numerical scheme is developed to compute the Maslov index of *hyperbolic* periodic orbits, and Chardard [14] proves under suitable hypotheses that if the periodic orbit is asymptotic to a homoclinic orbit, the Maslov index converges to the Maslov index of the limiting homoclinic orbit. This approach ties in with existing schemes for computing the basic solitary wave, where the solitary wave is approximated by a periodic orbit and then a spectral method is used for computation.

The computational framework for the Maslov index is illustrated by application to four examples. The first is a tutorial example on $\mathbb{R}^2$, where the details can be given explicitly. It is a scalar-reaction diffusion equation with an explicit localized solution. The second example is a coupled reaction-diffusion equation which also has an explicit solution. The third example is solitary wave solutions of KdV5. The fourth example is the LW-SW resonance equation which arises in fluid mechanics and consists of a NLS equation coupled to a KdV equation. This latter example has two new interesting features: the spectral problem is on a six-dimensional phase space, and for appropriate parameter values it has a Maslov index which is a non-monotone function of λ.

2 Linear Hamiltonian systems and Lagrangian subspaces

A Lagrangian subspace can be represented by a *Lagrangian frame*: a 4×2 matrix of rank 2

$$\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}, \tag{2.1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are 2×2 matrices satisfying

$$\mathbf{Y}^T\mathbf{X} = \mathbf{X}^T\mathbf{Y}\,. \tag{2.2}$$

There is a correspondence between elements of the unitary group U(2) and Lagrangian subspaces. When $\mathbf{X}^T\mathbf{X} + \mathbf{Y}^T\mathbf{Y} = \mathbf{I}$ then $\mathbf{X} \pm \mathrm{i}\mathbf{Y}$ are unitary matrices. The determinant of a unitary matrix lies on the unit circle. This property suggests defining the angle

$$e^{i\kappa} = \frac{\det[\mathbf{X} - i\mathbf{Y}]}{\det[\mathbf{X} + i\mathbf{Y}]} . \tag{2.3}$$

Along a path of Lagrangian subspaces, this angle will change, and the winding of this angle is the basis of the Maslov index.

Let $\mathbf{Z}(x)$, $a \le x \le b$ be any smooth path of Lagrangian subspaces. If the path is a loop: $\mathbf{Z}(b) = \mathbf{Z}(a)$, then there is an integer associated with the path: the number of times the induced path on the unit circle, represented by $e^{i\kappa(x)}$, encircles the origin. Define the angle associated with the path by

$$e^{i\kappa(x)} = \frac{\det[\mathbf{X}(x) - i\mathbf{Y}(x)]}{\det[\mathbf{X}(x) + i\mathbf{Y}(x)]} . \tag{2.4}$$

Then the Maslov index of the path is the integer

$$\mathsf{Maslov}(\mathbf{Z}, \kappa) := \frac{\kappa(b) - \kappa(a)}{2\pi} , \tag{2.5}$$

where κ here is viewed as the lift from S^1 to the covering space.

The Maslov index can also be defined for an arbitrary path of Lagrangian subspaces by introducing the idea of an intersection form. This approach to the Maslov index goes back to Maslov and Arnold [3]. The key to the definition in this case is the use of an *intersection form* or *crossing form*. Here we will follow the construction in Robbins [42,43] and Robbin & Salamon [40]. Modulo a choice of orientation these definitions are equivalent and lead to the following mapping.

Let V be a fixed reference plane, represented by a Lagrangian frame. For example a typical choice for the reference plane is

$$V = \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} .$$

In the case of homoclinic orbits a natural choice for the reference plane is the stable or unstable manifold at some value of x.

Suppose, for some value of x, denoted x_0, there is a simple intersection between the reference plane V and the path: that is, $\mathbf{Z}(x_0) \cap V$ is one dimensional. The intersection index at x_0 is determined by the sign of Γ, the crossing form, defined by

$$\Gamma(\mathbf{Z}, V, x_0) = \langle \mathbf{J}\mathbf{Z}'(x_0)\beta, \mathbf{Z}(x_0)\beta \rangle \,\mathsf{vol} . \tag{2.6}$$

Here and throughout the paper vol is taken to be the standard volume form on $\mathbb{R}^4$,

$$\mathsf{vol} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 .$$

In the formula (2.6),

$$\mathbf{Z}'(x_0) := \left.\frac{d}{dx}\right|_{x=x_0} \mathbf{Z}(x),$$

and $\beta \in \mathbb{RP}^1$. Real projective space $\mathbb{RP}^1$ is the space of all unoriented lines through the origin of $\mathbb{R}^2$,

$$\mathbb{RP}^1 = \{ \ \beta \in \mathbb{R}^2 \ : \ \beta \neq 0 \,, \quad \beta \sim c\,\beta \,, \quad c \in \mathbb{R} \setminus \{0\} \ \} \,.$$

β determines the linear combination of the columns of $\mathbf{Z}(x_0)$ which span the intersection subspace

$$\mathbf{Z}(x_0) \cap V = \mathrm{span}\{\xi\} \,, \quad \xi := \beta_1 \mathbf{z}_1 + \beta_2 \mathbf{z}_2 \,,$$

where $\mathbf{z}_1, \mathbf{z}_2$ are the columns of $\mathbf{Z}(x_0)$. At each simple intersection between V and the path $\mathbf{Z}(x)$ the sign of the intersection form is ± 1. Adding the intersection indices over the path gives the Maslov index

$$\mathsf{Maslov}(\mathbf{Z}, V) = \sum_{a < x_0 < b} \mathrm{sign}\, \Gamma(\mathbf{Z}, V, x_0) \,. \tag{2.7}$$

This formula assumes that $\mathbf{Z}(a) \cap V = \{0\}$ and $\mathbf{Z}(b) \cap V = \{0\}$ and that there is only a finite number of x_0 where $\mathbf{Z}(x_0) \cap V \neq \{0\}$. When the endpoints have non-trivial intersection the formula can be modified to contribute a half-integer for each end intersection (see page 831 of [40]). Non-regular intersections are not generic in the one-parameter family, and so can be eliminated by perturbing λ.

3 The Evans function for (1.2)

In order to compare the number of eigenvalues of (1.2) with the Maslov index, we will use the Evans function to determine eigenvalues based on the setup in Alexander, Gardner & Jones [1], adapted to the symplectic setting in Bridges & Derks [8], restricted to the case of $\mathbb{R}^4$.

Consider the linear system of ODEs,

$$\mathbf{u}_x = \mathbf{A}(x, \lambda)\mathbf{u} \,, \quad \mathbf{u} \in \mathbb{R}^4 \,, \tag{3.1}$$

where $\mathbf{A}(x, \lambda) = \mathbf{J}^{-1}\mathbf{B}(x, \lambda)$ and $\mathbf{B}(x, \lambda)$ is symmetric and depends smoothly on x and λ. In general λ can be complex but in this paper it will be restricted to be real. Assume that $\mathbf{A}(x, \lambda)$ tends exponentially fast to a matrix $\mathbf{A}_\infty(\lambda)$ when $x \to \pm\infty$.

Define the stable and unstable subspaces of $\mathbf{A}_\infty(\lambda)$ by

$$\mathrm{E}^s_\infty(\lambda) := \{\mathbf{u} \in \mathbb{R}^4 \ : \ \lim_{x \to +\infty} \mathrm{e}^{\mathbf{A}_\infty(\lambda) x} \mathbf{u} = 0\}$$

and

$$\mathrm{E}^u_\infty(\lambda) := \{\mathbf{u} \in \mathbb{R}^4 \ : \ \lim_{x \to -\infty} \mathrm{e}^{\mathbf{A}_\infty(\lambda) x} \mathbf{u} = 0\}$$

$\mathrm{E}^s(\lambda)$ ($\mathrm{E}^u_\infty(\lambda)$) is the direct sum of the generalized eigenspaces associated with the eigenvalues of $\mathbf{A}_\infty(\lambda)$ with negative (positive) real part. The matrix $\mathbf{A}_\infty(\lambda)$ is said to be hyperbolic if $\mathbb{R}^4 = \mathrm{E}^u_\infty(\lambda) \oplus \mathrm{E}^s_\infty(\lambda)$; equivalently if $\mathbf{A}_\infty(\lambda)$ has no purely imaginary eigenvalues. Purely imaginary eigenvalues of $\mathbf{A}_\infty(\lambda)$ are associated with the essential spectrum. The essential spectrum is

$$\begin{aligned}\sigma_{\text{ess}} &= \{\lambda \in \mathbb{C} \mid \mathbf{A}_\infty(\lambda) \text{ is not hyperbolic}\} \\ &= \{\ \lambda \in \mathbb{C}\ :\ \det[\mathbf{A}_\infty(\lambda) - \mathrm{i}\kappa\mathbf{I}] = 0 \quad \text{for some } \kappa \in \mathbb{R}\ \}\,.\end{aligned} \tag{3.2}$$

We will assume throughout that $\lambda \notin \sigma_{\text{ess}}$. Then the Hamiltonian symmetry of $\mathbf{A}_\infty(\lambda)$ gives that $\dim \mathrm{E}^u_\infty(\lambda) = \dim \mathrm{E}^s_\infty(\lambda) = 2$.

Let $\bigwedge^2(\mathbb{R}^4)$ be the vector space of 2−vectors in $\mathbb{R}^4$. There is an induced system from (3.1)

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\mathbf{U}\,, \quad \mathbf{U} \in \textstyle\bigwedge^2(\mathbb{R}^4)\,. \tag{3.3}$$

Let $\sigma_+(\lambda)$ be the sum of the eigenvalues of $\mathbf{A}_\infty(\lambda)$ with positive real part, and let $\sigma_-(\lambda)$ be the sum of the eigenvalues with negative real part. Then there are solutions $\mathbf{U}^+(x,\lambda)$ and $\mathbf{U}^-(x,\lambda)$ of (3.3) with maximal decay as x goes to $-\infty$ and $+\infty$ respectively satisfying

$$\lim_{x\to-\infty} \mathrm{e}^{-\sigma_+(\lambda)x}\mathbf{U}^+(x,\lambda) = \zeta^+(\lambda) \in \textstyle\bigwedge^2(\mathbb{R}^4)\,, \tag{3.4}$$

and

$$\lim_{x\to+\infty} \mathrm{e}^{-\sigma_-(\lambda)x}\mathbf{U}^-(x,\lambda) = \zeta^-(\lambda) \in \textstyle\bigwedge^2(\mathbb{R}^4)\,, \tag{3.5}$$

where $\zeta^\pm(\lambda)$ are eigenvectors

$$\mathbf{A}^{(2)}_\infty(\lambda)\zeta^\pm(\lambda) = \sigma_\pm(\lambda)\zeta^\pm(\lambda)\,. \tag{3.6}$$

The eigenvalues $\sigma_\pm(\lambda)$ are analytic functions of λ and so the eigenvectors $\zeta^\pm(\lambda)$ can be chosen to be analytic as well.

A value $\lambda \in \mathbb{R} \setminus \sigma_{ess}$ is called an eigenvalue if the stable solutions $\mathbf{U}^-(x,\lambda)$ and unstable solutions $\mathbf{U}^+(x,\lambda)$ have nontrivial intersection. Eigenvalues are detected by the *Evans function* [1] which is defined by

$$D(\lambda)\,\mathsf{vol} = \mathbf{U}^-(x,\lambda) \wedge \mathbf{U}^+(x,\lambda) \in \textstyle\bigwedge^4(\mathbb{R}^4)\,. \tag{3.7}$$

The Evans function is independent of x and is an analytic function of λ [1]. Analyticity assures that the zeros of $D(\lambda)$ are isolated. In the definition (3.7), the property of (1.2),

$$\text{Trace}(\mathbf{A}(x,\lambda)) = 0\,, \tag{3.8}$$

has been used. This property follows since $\mathbf{A} = \mathbf{J}^{-1}\mathbf{B}$ with $\mathbf{J}$ skew-symmetric and $\mathbf{B}$ symmetric.

Because of the Hamiltonian structure, the Evans function is invariant under exponential scaling of the following form. Let

$$\widehat{\mathbf{U}}^{\pm}(x,\lambda) = \mathrm{e}^{-\sigma_{\pm}(\lambda)x}\mathbf{U}^{\pm}(x,\lambda)\,,$$

Then the scaled functions satisfy

$$\widehat{\mathbf{U}}^{\pm}_x = [\mathbf{A}^{(2)}(x,\lambda) - \sigma_{\pm}(\lambda)\mathbf{I}]\widehat{\mathbf{U}}^{\pm}\,,$$

but the Evans function becomes

$$D(\lambda)\,\mathsf{vol} = \mathrm{e}^{(\sigma_-(\lambda)+\sigma_+(\lambda))x}\widehat{\mathbf{U}}^{-}(x,\lambda)\wedge\widehat{\mathbf{U}}^{+}(x,\lambda) = \widehat{\mathbf{U}}^{-}(x,\lambda)\wedge\widehat{\mathbf{U}}^{+}(x,\lambda)\,,$$

since $\sigma_-(\lambda)+\sigma_+(\lambda) = \mathrm{Trace}(\mathbf{A}_\infty(\lambda)) = 0$.
An explicit expression for the entries of $\mathbf{A}^{(2)}$ as a function of the entries of $\mathbf{A}$, when $n=2$, is given in Appendix A. It is natural to ask whether there is an induced symplectic structure on $\bigwedge^2(\mathbb{R}^4)$. For example, can the induced system be written in the form

$$\mathbf{J}^{(2)}\mathbf{U}_x = \mathbf{B}^{(2)}(x,\lambda)\mathbf{U}\,,$$

where $\mathbf{J}^{(2)}$ is the induced matrix from $\mathbf{J}$ on $\bigwedge^2(\mathbb{R}^4)$. However, this is not the case. The most significant obstacle is the fact that $\mathbf{J}^{(2)}$ is not invertible. The precise relation between $\mathbf{J}^{(2)}$, $\mathbf{B}^{(2)}$ and $\mathbf{A}^{(2)}$ is given in Appendix B.

4 $\mathbf{U}^{\pm}(x,\lambda)$ represent paths of Lagrangian planes

The paths of stable and unstable subspaces $\mathbf{U}^{\pm}(x,\lambda)$ (or their scaled versions $\widehat{\mathbf{U}}^{\pm}(x,\lambda)$) are paths of Lagrangian subspaces.
Let $\Phi(x,s,\lambda)$ be a fundamental solution matrix for (1.2), that is,

$$\mathbf{J}\Phi_x = \mathbf{B}(x,\lambda)\Phi\,,\quad \Phi(s,s,\lambda) = \mathbf{I}\,,$$

and $\Phi(x,s,\lambda)\Phi(s,t,\lambda) = \Phi(x,t,\lambda)$. Define the stable and unstable subspaces for each $x_0\in\mathbb{R}$ [41],

$$\mathrm{E}^s(x_0,\lambda) = \{\mathbf{u}\in\mathbb{R}^4 \;:\; \lim_{x\to+\infty}\Phi(x,x_0)\mathbf{u} = 0\} = \mathrm{span}\left\{\mathrm{col}(\mathbf{U}^-(x_0,\lambda))\right\}\,,$$

and

$$\mathrm{E}^u(x_0,\lambda) = \{\mathbf{u}\in\mathbb{R}^4 \;:\; \lim_{x\to-\infty}\Phi(x,x_0)\mathbf{u} = 0\} = \mathrm{span}\left\{\mathrm{col}(\mathbf{U}^+(x_0,\lambda))\right\}\,.$$

Both subspaces define invariant vector bundles over $\mathbb{R}$. This means that

$$\mathrm{E}^s(x,\lambda) = \Phi(x,s,\lambda)\mathrm{E}^s(s,\lambda) \quad\text{and}\quad \mathrm{E}^u(x,\lambda) = \Phi(x,s,\lambda)\mathrm{E}^u(s,\lambda)\,. \tag{4.1}$$

Moreover

$$\lim_{x\to+\infty} \mathrm{E}^s(x,\lambda) = \mathrm{E}^s_\infty(\lambda) \quad \text{and} \quad \lim_{x\to-\infty} \mathrm{E}^u(x,\lambda) = \mathrm{E}^u_\infty(\lambda)\,.$$

If $\mathrm{E}^u(x,\lambda)$ is Lagrangian for some x then it is Lagrangian for all x. This observation is implicit in (4.1) but a direct proof can be given as follows. When $\mathbf{u},\mathbf{v} \in \mathrm{E}^u(x,\lambda)$ are solutions of (1.2),

$$\frac{d}{dx}\langle \mathbf{J}\mathbf{u},\mathbf{v}\rangle = \langle \mathbf{J}\mathbf{u}_x,\mathbf{v}\rangle + \langle \mathbf{J}\mathbf{u},\mathbf{v}_x\rangle = \langle \mathbf{B}(x,\lambda)\mathbf{u},\mathbf{v}\rangle - \langle \mathbf{u},\mathbf{B}(x,\lambda)\mathbf{v}\rangle = 0\,,$$

using symmetry of $\mathbf{B}(x,\lambda)$. Hence the value of $\langle \mathbf{J}\mathbf{u},\mathbf{v}\rangle$ is an invariant of (1.2) for any pair of vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^4$:

$$\langle \mathbf{J}\mathbf{u}(x,\lambda),\mathbf{v}(x,\lambda)\rangle = \langle \mathbf{J}\mathbf{u}(x_0,\lambda),\mathbf{v}(x_0,\lambda)\rangle\,, \quad \forall x\,.$$

But $\mathbf{u},\mathbf{v}\in \mathrm{E}^u(x,\lambda)$ and so $\lim_{x\to-\infty}\mathbf{u}(x,\lambda) = \lim_{x\to-\infty}\mathbf{v}(x,\lambda) = 0$, therefore:

$$\lim_{x\to-\infty}\langle \mathbf{J}\mathbf{u}(x,\lambda),\mathbf{v}(x,\lambda)\rangle = 0 \quad \Rightarrow \quad \langle \mathbf{J}\mathbf{u}(x_0,\lambda),\mathbf{v}(x_0,\lambda)\rangle = 0\,.$$

Therefore, $\mathrm{E}^u(x,\lambda)$ is a Lagrangian subspace for any x. A similar proof confirms the result for $\mathrm{E}^s(x,\lambda)$. Another proof is to use Montaldi's Theorem [37] on Lagrangian planes in $\mathbb{R}^4$ and a sketch is given in Appendix C.

5 An example in $\mathbb{R}^2$

Before proceeding to the full definition and properties of the Maslov index for paths of Lagrangian subspaces which are also solutions of (1.2) it will be useful to consider the simplest possible context for the Maslov index, linear systems on $\mathbb{R}^2$. To illustrate the role of λ, a stability problem for a reaction-diffusion equation is used. It is a simplified version of the class of nonlinear parabolic PDEs studied in [6].

Consider the nonlinear parabolic PDE

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} - \phi + \phi^2\,, \quad x\in\mathbb{R}\,, \tag{5.1}$$

for the scalar-valued function $\phi(x,t)$. There is a basic steady solitary wave solution

$$\widehat{\phi}(x) = \tfrac{3}{2}\,\mathrm{sech}^2\left(\tfrac{1}{2}x\right)\,, \tag{5.2}$$

which satisfies $\widehat{\phi}_{xx} - \widehat{\phi} + \widehat{\phi}^2 = 0$. Linearizing (5.1) about the basic state $\widehat{\phi}$ and looking for solutions proportional to $\mathrm{e}^{\lambda t}$ leads to the spectral problem

$$\mathscr{L}\phi = \lambda\phi\,, \quad \text{with} \quad \mathscr{L}\phi := \frac{d^2\phi}{dx^2} - \phi + 2\widehat{\phi}(x)\phi\,. \tag{5.3}$$

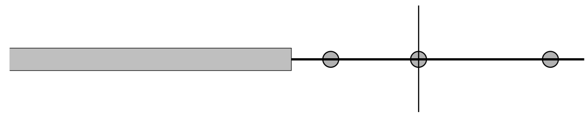

Fig. 1. Plot of the spectrum of $\mathscr{L}$.

The basic state (5.2) is said to be (spectrally) unstable if any part of the spectrum of $\mathscr{L}$ is positive. The spectrum of $\mathscr{L}$ can be explicitly constructed. It consists of a branch of essential spectra and a point spectrum

$$\sigma(\mathscr{L}) = \sigma_{ess}(\mathscr{L}) \cup \sigma_p(\mathscr{L}) \,,$$

with $\sigma_{ess}(\mathscr{L}) = \{\lambda \in \mathbb{R} \;:\; \lambda \leq -1\}$ and $\sigma_p(\mathscr{L}) = \left\{-\frac{3}{4}\,,\,0\,,\,\frac{5}{4}\right\}$. The spectrum is illustrated in Figure 1.

The point spectrum can be verified by constructing the Evans function. First reformulate (5.3) as a first-order system. Let

$$\mathbf{u}(x,\lambda) = \begin{pmatrix} \phi(x,\lambda) \\ \phi_x(x,\lambda) \end{pmatrix} ,$$

then

$$\mathbf{J}\mathbf{u}_x = \mathbf{B}(x,\lambda)\mathbf{u} \,, \quad \mathbf{u} \in \mathbb{R}^2 \,, \quad \lambda \in \mathbb{R} \,, \tag{5.4}$$

with

$$\mathbf{B}(x,\lambda) = \begin{bmatrix} \lambda + 1 - 3\mathrm{sech}^2(\frac{1}{2}x) & 0 \\ 0 & 1 \end{bmatrix} .$$

The eigenvalues of $\mathbf{A}_\infty(\lambda)$ are real and hyperbolic when $\lambda + 1 > 0$. In this formulation the stable ($\mathbf{u}^-$) and unstable ($\mathbf{u}^+$) subspaces are represented by

$$\mathbf{u}^\pm(x,\lambda) = \mathrm{e}^{\pm\gamma s} \begin{pmatrix} h^\pm \\ \frac{1}{2}(h_s^\pm \pm \gamma h^\pm) \end{pmatrix} ,$$

where $s = \frac{1}{2}x$, $\gamma = 2\sqrt{\lambda+1}$,

$$h^\pm(s,\lambda) = \pm a_0 + a_1 \tanh(s) \pm a_2 \tanh^2(s) + a_3 \tanh^3(s) \,,$$

and

$$a_0 = \frac{\gamma}{15}(4-\gamma^2)\, a_3 \,, \quad a_1 = \frac{1}{5}(2\gamma^2 - 3)\, a_3 \,, \quad a_2 = -\gamma\, a_3 \,, \tag{5.5}$$

and a_3 is an arbitrary nonzero real number. The Evans function is then

$$D(\lambda) = \mathbf{u}^-(x,\lambda) \wedge \mathbf{u}^+(x,\lambda) \,.$$

Evaluating at $x = 0$, a straightforward calculation leads to

$$D(\lambda) = -2\sqrt{\lambda+1}\left(\frac{2a_3}{15}\right)^2 \lambda\,(4\lambda+3)(4\lambda-5)\,.$$

The zeros of $D(\lambda)$ are the eigenvalues, confirming the point spectrum $\left\{-\frac{3}{4}, 0, +\frac{5}{4}\right\}$.

For linear Hamiltonian systems on $\mathbb{R}^2$ Lagrangian subspaces are just one-dimensional subspaces. The path of unstable subspaces $\mathbf{u}^+(x,\lambda)$ is used to define the Maslov index. The natural one-dimensional subspace to choose for the reference space is $\mathrm{E}^s_\infty(\lambda)$,

$$\mathrm{E}^s_\infty(\lambda) = \mathrm{span}\left\{\begin{pmatrix} 2 \\ -\gamma \end{pmatrix}\right\}.$$

Then, assume simple intersections between $\mathrm{E}^s_\infty(\lambda)$ and $\mathbf{u}^+(x,\lambda)$ – which can be confirmed a posteriori for the example (5.4) – and assume that

$$\lim_{x\to\pm\infty} \mathbf{u}^+(x,\lambda) \bigcap \mathrm{E}^s_\infty(\lambda) = \{0\}\,.$$

This latter assumption is equivalent to assuming that λ is not an eigenvalue. The Maslov index for this case is

$$\mathsf{Maslov}(\mathbf{u}^+, \mathrm{E}^s_\infty) = \sum_{x_0} \mathrm{sign}\,\langle \mathbf{J}\mathbf{u}^+_x, \mathbf{u}^+\rangle\, \mathsf{vol}\,,$$

with x_0 the points at which $\mathbf{u}^+(x,\lambda) \cap \mathrm{E}^s_\infty$ is non-trivial, and the volume form can be taken to be $\mathsf{vol} = \mathbf{e}_1 \wedge \mathbf{e}_2$. This expression for the Maslov index is the one-dimensional version of (2.7).

The path of unstable subspaces is

$$\mathbf{u}^+(x,\lambda) = \tfrac{1}{2}\mathrm{e}^{\gamma s}\begin{pmatrix} 2h^+ \\ h^+_s + \gamma h^+ \end{pmatrix}. \tag{5.6}$$

The intersection form in this case is

$$\begin{aligned}
\Gamma(\mathbf{u}^+, \mathrm{E}^s_\infty, x_0) &= \langle \mathbf{J}\mathbf{u}^+_x, \mathbf{u}^+\rangle\Big|_{x=x_0} \mathsf{vol}\,,\\
&= (-u^+_1 \dot{u}^+_2 + u^+_2 \dot{u}^+_1)\Big|_{x=x_0} \mathsf{vol}\\
&= \left[((u^+_2)^2 - \lambda - 1 + 12\,\mathrm{sech}^2 s)(u^+_1)^2\right]\Big|_{x=x_0} \mathsf{vol}\,.
\end{aligned}$$

However, at a point x_0 where $\mathbf{u}^+$ intersects E^s_λ, $u^+_2 = -\frac{1}{2}\gamma u^+_1$ and so

$$\Gamma(\mathbf{u}^+, \mathrm{E}^s_\infty, x_0) = 12\,\mathrm{sech}^2 \tfrac{1}{2}x_0\,(u^+_1)^2\, \mathsf{vol}\,.$$

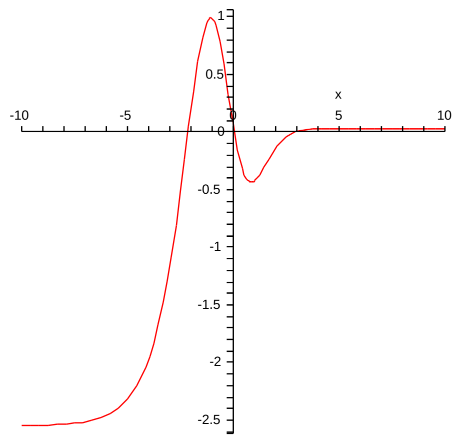

Fig. 2. Plot of $\xi^s \wedge \mathbf{u}^+(x,\lambda)$ for the case $\lambda = -0.8$.

Hence $\Gamma(\mathbf{u}^+, \mathrm{E_\infty}^s) > 0$ at each intersection, and the Maslov index is just the sum of the intersections. An intersection occurs when

$$\xi^s \wedge \mathbf{u}^+ = 0\,, \quad \text{where} \quad \xi^s = \begin{pmatrix} 2 \\ -\gamma \end{pmatrix}.$$

Now

$$\xi^s \wedge \mathbf{u}^+ = 2u_2^+ + \gamma u_1^+\,.$$

The factor $\mathrm{e}^{\gamma s}$ is not important and so can be divided out, giving

$$\xi^s \wedge \mathbf{u}^+ \sim \frac{dh^+}{ds} + 2\gamma h^+\,.$$

This function has $0, 1, 2$ or 3 zeros depending on the value of λ. Each zero corresponds to an intersection between the unstable subspace with $\mathrm{E}_\infty^s(\lambda)$. The function $\xi^s \wedge \mathbf{u}^+$ is illustrated in Figure 2 for the case $\lambda = -0.8$ where $\xi^s \wedge \mathbf{u}^+$ has three zeros indicating three intersections. A summary of the Maslov index in each region is tabulated below.

λ	$-1 < \lambda < -\frac{3}{4}$	$-\frac{3}{4} < \lambda < 0$	$0 < \lambda < \frac{5}{4}$	$\lambda > \frac{5}{4}$
Maslov$(\mathbf{u}^+, \mathrm{E}_\infty^s)$	3	2	1	0

Note that the Maslov index jumps by one at each eigenvalue. Let λ_0 be any fixed real value of λ such that $\lambda_0 > -1$ and λ_0 is not an eigenvalue, then the value of the Maslov index equals the number of eigenvalues of $\mathscr{L}$ in the set $\lambda > \lambda_0$.

5.1 *The Maslov angle in* $\bigwedge^1(\mathbb{R}^2)$

Another way to count intersections between the path $\mathbf{u}^+$ and some reference plane is to use the Maslov angle (2.4). In this case the angle $\kappa(x,\lambda)$ is just the angle determined by a polar representation of $\mathbf{u}^+$

$$\mathrm{e}^{\mathrm{i}\kappa(x,\lambda)} := \frac{u_1^+(x,\lambda) - \mathrm{i}u_2^+(x,\lambda)}{u_1^+(x,\lambda) + \mathrm{i}u_2^+(x,\lambda)} \,.$$

As $x \to \pm\infty$

$$\lim_{x\to\pm\infty} \mathrm{e}^{\mathrm{i}\kappa(x,\lambda)} = \frac{2-\mathrm{i}\gamma}{2+\mathrm{i}\gamma} \,.$$

The Maslov index is then the count of the number of times that κ crosses some reference angle, such as the angle associated with the stable subspace. This version of the Maslov index is equivalent to the definition based on intersection index.

6 The Maslov index for paths

In this section we look at some of the properties of paths of Lagrangian subspaces that are also solutions of (1.2), and bring in the exterior algebra representation.

Here and throughout, let $\mathbf{e}_1,\ldots,\mathbf{e}_4$ be the canonical basis for $\mathbb{R}^4$. The symplectic form in standard form is then

$$\boldsymbol{\omega} = \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4 \,. \tag{6.1}$$

The symplectic form is related to the symplectic operator $\mathbf{J}$ in (1.3) by

$$\mathbf{u} \,\lrcorner\, \boldsymbol{\omega} = \mathbf{J}\mathbf{u} \,, \quad \forall\, \mathbf{u} \in \mathbb{R}^4 \,,$$

where the interior product is defined by

$$\langle \mathbf{u} \,\lrcorner\, \boldsymbol{\omega}, \mathbf{v}\rangle = [\![\boldsymbol{\omega}, \mathbf{u}\wedge\mathbf{v}]\!]_2 \,, \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbb{R}^{2n} \,.$$

Here and throughout $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^4$ and $[\![\cdot,\cdot]\!]_k$ is the induced inner product on $\bigwedge^k(\mathbb{R}^4)$. We will sometimes write $\langle\cdot,\cdot\rangle_d$ on $\mathbb{R}^d$ when the associated dimension is not clear. The equivalence between the induced inner product $[\![\cdot,\cdot]\!]_k$ and $\langle\cdot,\cdot\rangle_d$ when $d = \dim\left(\bigwedge^k(\mathbb{R}^{2n}\right)$ is established in Appendix D. The basic properties of symplectic exterior algebra can be found in [31].

The set of all Lagrangian subspaces of $\mathbb{R}^4$ associated with the standard symplectic operator $\boldsymbol{\omega}$ will be denoted by $\Lambda(2)$. $\Lambda(2)$ is a manifold of dimension 3. It is a submanifold of $\mathrm{G}_2(\mathbb{R}^4)$, the Grassmannian of all 2−dimensional subspaces of $\mathbb{R}^4$ [31].

A path of unstable subspaces will be represented by $\mathbf{U}^+(x,\lambda) \in \bigwedge^2(\mathbb{R}^4)$, which satisfies the equation (3.3) and (3.4). When λ is not an eigenvalue then $\mathbf{U}^+(x,\lambda) \to \mathrm{E}^u_\infty(\lambda)$ as $x \to \pm\infty$ and so in $\mathbb{P}(\mathrm{E}^u(x,\lambda))$, projective space based on $\mathrm{E}^u(x,\lambda)$, a loop of Lagrangian subspaces is obtained. The Maslov index is then taken to be the sum of the weighted intersections of $\mathbf{U}^+(x,\lambda)$ with $\mathrm{E}^s_\infty(\lambda)$.

$\mathrm{E}^s_\infty(\lambda)$ is represented by the 2−form $\zeta^-(\lambda)$ defined in (3.6). An analytic basis can always be constructed for $\mathrm{E}^s_\infty(\lambda)$ [8]. Denote this basis by

$$\mathrm{E}^s_\infty(\lambda) = \mathrm{span}\{\xi^s_1(\lambda), \xi^s_2(\lambda)\}\,.$$

Consider the 3−form

$$\mathbf{U}^+(x,\lambda) \wedge (\alpha_1\xi^s_1(\lambda) + \alpha_2\xi^s_2(\lambda))\,.$$

If for any fixed x and λ,

$$\mathbf{U}^+(x,\lambda) \wedge (\alpha_1\xi^s_1(\lambda) + \alpha_2\xi^s_2(\lambda)) = 0 \quad \Rightarrow \quad \alpha = 0\,,$$

then we say that $\mathrm{E}^s_\infty(\lambda)$ is transverse to $\mathrm{E}^u(x,\lambda) := \mathrm{image}(\mathbf{U}^+(x,\lambda))$ at that value of (x,λ).

We say that $\mathrm{E}^s_\infty(\lambda)$ and $\mathrm{E}^u(x,\lambda)$ have a simple intersection (or regular intersection) if the intersection is one dimensional; there exists $\alpha \in \mathbb{RP}^1$ such that

$$\mathbf{U}(x,\lambda) \wedge \xi = 0 \quad \text{with} \quad \xi = \alpha_1\xi^s_1(\lambda) + \alpha_2\xi^s_2(\lambda)\,. \tag{6.2}$$

We will assume that non-trivial intersections are regular. It is proved in [3] that non-regular intersections can be eliminated by perturbation. This property can also be proved using the homotopy equivalence property of the Maslov index, and a nice proof of this is given in §3.4 of [38].

Now, let $\mathbf{u}^+_1(x,\lambda), \mathbf{u}^+_2(x,\lambda)$ be a basis for $\mathrm{E}^u(x,\lambda)$ such that

$$\mathbf{U}^+(x,\lambda) = \mathbf{u}^+_1(x,\lambda) \wedge \mathbf{u}^+_2(x,\lambda)\,.$$

Then (6.2) implies that there exists $(\beta_1,\beta_2) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$\xi = \beta_1\mathbf{u}^+_1(x,\lambda) + \beta_2\mathbf{u}^+_2(x,\lambda) := \mathbf{Z}^+(x,\lambda)\beta\,. \tag{6.3}$$

The Maslov index is a count of how many times the path $\mathbf{U}^+(x,\lambda)$ crosses the reference space $\mathrm{E}^s_\infty(\lambda)$, weighted by the intersection form. Starting with (2.6), and using the equation (1.2) and the equivalence (6.3) it is

$$\Gamma(\Lambda, V, x_0) = \langle \mathbf{B}(x_0,\lambda)\xi, \xi\rangle\, \mathsf{vol}\,. \tag{6.4}$$

Note that the $x-$dependent path $\mathbf{U}^+(x,\lambda)$ drops out. For fixed λ, once we have found a point x_0 corresponding to a regular intersection, the intersection form can be evaluated using eigenvectors spanning $\mathrm{E}^s_\infty(\lambda)$ and the known symmetric matrix $\mathbf{B}(x_0,\lambda)$. However there is the hidden calculation of determining $\alpha \in \mathbb{RP}^1$ and an algorithm for this is developed in §13.

Suppose λ is not an eigenvalue. Then the Maslov index of the path $\mathbf{U}^+$ relative to E^s_∞ is

$$\mathsf{Maslov}(\mathbf{U}^+, \mathrm{E}^s_\infty) = \sum_{x_0} \operatorname{sign} \langle \xi, \mathbf{B}(x_0, \lambda)\xi \rangle \,\mathsf{vol}\,, \tag{6.5}$$

where the sum is over all points x_0 of intersection in the interval $-\infty < x_0 < +\infty$.

This expression will serve as a definition for the Maslov index of a solitary wave at λ:

$$\mathsf{Maslov}(\lambda) = \mathsf{Maslov}(\mathbf{U}^+(\cdot, \lambda), \mathrm{E}^s_\infty(\lambda)). \tag{6.6}$$

Proposition 1 *Suppose λ is not an eigenvalue. Then under the given hypotheses on (1.2) the Maslov index of a solitary wave is finite.*

Proof. Introduce a metric $\mathsf{dist}(\cdot, \cdot)$ on the manifold of 2−dimensional subspaces of $\mathbb{R}^4$. For example this can be the standard metric on the Grassmannian $\mathrm{G}_2(\mathbb{R}^4)$ [46].

Let $\lambda \in \mathbb{R} \setminus \sigma_{ess}$ which is not an eigenvalue. Since $\mathrm{E}^s_\infty(\lambda)$ and $\mathrm{E}^u_\infty(\lambda)$ are transverse, by a suitable scaling of the stable subspace $\mathrm{E}^s_\infty(\lambda)$ we can take

$$\mathsf{dist}((\mathrm{E}^s_\infty(\lambda)), \mathrm{E}^u_\infty(\lambda)) > 1\,,$$

where $(\mathrm{E}^s_\infty(\lambda))$ is the closed set of planes which are not transverse to $(\mathrm{E}^s_\infty(\lambda))$. To simplify notation, the argument in $\mathsf{dist}(\cdot, \cdot)$ should be interpreted as the representation of $\mathrm{E}^{s,u}$ on the Lagrangian Grassmanian.

When λ is not an eigenvalue we have that $\mathrm{E}^u(x, \lambda) \to \mathrm{E}^u_\infty(\lambda)$ as $x \to \pm\infty$. Therefore, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathsf{dist}(\mathrm{E}^u(x, \lambda), \mathrm{E}^u_\infty(\lambda)) < \varepsilon \quad \text{for} \quad |x| > \frac{1}{\delta}\,.$$

Now use the triangle inequality

$$1 < \mathsf{dist}((\mathrm{E}^s_\infty(\lambda)), \mathrm{E}^u_\infty(\lambda)) \le \mathsf{dist}((\mathrm{E}^s_\infty(\lambda)), \mathrm{E}^u(x, \lambda)) + \mathsf{dist}(\mathrm{E}^u(x, \lambda), \mathrm{E}^u_\infty(\lambda))\,,$$

or

$$\mathsf{dist}(\mathrm{E}^u(x, \lambda), (\mathrm{E}^s_\infty(\lambda))) > 1 - \varepsilon \quad \text{for} \quad |x| > \frac{1}{\delta}\,.$$

Hence there exists $x_* > 0$ such that for $|x| > x_*$ $\mathrm{E}^u(x, \lambda)$ and $\mathrm{E}^s_\infty(\lambda)$ are transverse. Intersections are therefore limited to the finite interval $-x_* < x < x_*$. Since intersections are generically isolated their number is finite. ■

7 The Maslov index on $\bigwedge^2(\mathbb{R}^4)$

The vector space $\bigwedge^2(\mathbb{R}^4)$ is six-dimensional, and the orthonormal basis induced from the basis of $\mathbb{R}^4$ is

$$\begin{array}{lll} \mathbf{E}_1 = \mathbf{e}_1 \wedge \mathbf{e}_2\,, & \mathbf{E}_2 = \mathbf{e}_1 \wedge \mathbf{e}_3\,, & \mathbf{E}_3 = \mathbf{e}_1 \wedge \mathbf{e}_4\,, \\[2mm] \mathbf{E}_4 = \mathbf{e}_2 \wedge \mathbf{e}_3\,, & \mathbf{E}_5 = \mathbf{e}_2 \wedge \mathbf{e}_4\,, & \mathbf{E}_6 = \mathbf{e}_3 \wedge \mathbf{e}_4\,. \end{array} \tag{7.1}$$

Any $\mathbf{U} \in \bigwedge^2(\mathbb{R}^4)$ can be represented in the form

$$\mathbf{U} = \sum_{j=1}^{6} U_j\,\mathbf{E}_j\,. \tag{7.2}$$

The Grassmannian $G_2(\mathbb{R}^4)$ is a subset of $\bigwedge^2(\mathbb{R}^4)$ defined by

$$0 = \mathbf{U} \wedge \mathbf{U} = I_1\,\mathsf{vol}\,, \quad I_1 := U_1U_6 - U_2U_5 + U_3U_4\,. \tag{7.3}$$

The Lagrangian-Grassmannian is the subset defined by

$$0 = \boldsymbol{\omega} \wedge \mathbf{U} = I_2\,\mathsf{vol}\,, \quad I_2 := U_2 + U_5\,. \tag{7.4}$$

The Lagrangian-Grassmannian $\Lambda(2)$ is the three dimensional submanifold of $\mathbb{P}\left(\bigwedge^2(\mathbb{R}^4)\right)$ defined by $I_1 = I_2 = 0$.

Let $V \in \bigwedge^2(\mathbb{R}^4)$ be a fixed Lagrangian plane. Then

$$\Lambda^1(2) = \{\ \mathbf{U} \in \textstyle\bigwedge^2(\mathbb{R}^4) \cap \Lambda(2)\ :\ \mathbf{U} \wedge V = 0\ \}\,,$$

is a codimension one submanifold of $\Lambda(2)$ [3]. We have a sequence of manifolds

Manifold	$\bigwedge^2(\mathbb{R}^4)$	$\mathbb{RP}^5$	$\mathrm{G}_2(\mathbb{R}^4)$	$\Lambda(2)$	$\Lambda^1(2)$
Dimension	6	5	4	3	2

In this table $\mathbb{RP}^5$ represents $\mathbb{P}\left(\bigwedge^2(\mathbb{R}^4)\right)$.

Consider the class of linear Hamiltonian systems (1.2) with $\mathbf{B}(x,\lambda)$ satisfying the asymptotic properties (1.4)-(1.5).

Proposition 2 $\Lambda(2)$ *is an invariant manifold of (3.3).*

Proof.

$$\begin{aligned}
\tfrac{d}{dx}\mathbf{U}\wedge\mathbf{U} &= \mathbf{U}_x\wedge\mathbf{U}+\mathbf{U}\wedge\mathbf{U}_x\\
&= \mathbf{A}^{(2)}\mathbf{U}\wedge\mathbf{U}+\mathbf{U}\wedge\mathbf{A}^{(2)}\mathbf{U}\\
&= \mathrm{Trace}(\mathbf{A})\,\mathbf{U}\wedge\mathbf{U}\\
&= 0
\end{aligned}$$

since $\mathrm{Trace}(\mathbf{A}) = 0$, also using the property [2]

$$\mathbf{A}^{(2)}\mathbf{U}\wedge\mathbf{U}+\mathbf{U}\wedge\mathbf{A}^{(2)}\mathbf{U} = \mathrm{Trace}(\mathbf{A})\mathbf{U}\wedge\mathbf{U}\,.$$

This proves that $\mathbf{U}\wedge\mathbf{U}$ is a constant along solutions. Similarly,

$$\begin{aligned}
\tfrac{d}{dx}\boldsymbol{\omega}\wedge\mathbf{U} &= \boldsymbol{\omega}\wedge\mathbf{U}_x\\
&= \boldsymbol{\omega}\wedge\mathbf{A}^{(2)}\mathbf{U}\\
&= \boldsymbol{\omega}\wedge\mathbf{A}^{(2)}\mathbf{U}+\mathbf{A}^{(2)}\boldsymbol{\omega}\wedge\mathbf{U}-\mathbf{A}^{(2)}\boldsymbol{\omega}\wedge\mathbf{U}\\
&= \mathrm{Trace}(\mathbf{A})\boldsymbol{\omega}\wedge\mathbf{U}-\mathbf{A}^{(2)}\boldsymbol{\omega}\wedge\mathbf{U}\\
&= -\mathbf{A}^{(2)}\boldsymbol{\omega}\wedge\mathbf{U}\\
&= 0\,,
\end{aligned}$$

since $\boldsymbol{\omega}$ is in the kernel of $\mathbf{A}^{(2)}$, a property which is proved in Appendix A. This proves that $\mathbf{U}\wedge\mathbf{U}$ and $\boldsymbol{\omega}\wedge\mathbf{U}$ are constant along solutions. Hence the special case $\mathbf{U}\wedge\mathbf{U} = \boldsymbol{\omega}\wedge\mathbf{U} = 0$ completes the proof. ■

Let $V = \mathrm{span}\{\xi_1,\xi_2\}$ be a fixed Lagrangian plane; that is ξ_1 and ξ_2 are linearly independent and $\langle\mathbf{J}\xi_1,\xi_2\rangle = 0$. The reference subspace V is represented by the form

$$\mathbf{V} = \xi_1\wedge\xi_2\,.$$

The intersection between a path of Lagrangian subspaces $\mathbf{U}(x,\lambda)$ and $\mathbf{V}$ can be described as follows. For each fixed λ define

$$\mathscr{A}(x) = \Big\{\alpha\in\mathbb{R}^2\ :\ \mathbf{U}(x,\lambda)\wedge(\alpha_1\xi_1+\alpha_2\xi_2) = 0\Big\}\,.$$

Then there are three cases

- If $\mathscr{A}(x_0) = \{0\}$ then $\mathbf{U}$ and V are transverse at $x = x_0$.
- If $\mathscr{A}(x_0)$ is one dimensional then $\mathbf{U}$ and V intersect in a one-dimensional subspace at $x = x_0$ (the case of regular crossing).
- If $\mathscr{A}(x_0) = \mathbb{R}^2$ then $\mathbf{U}$ and V intersect in a two-dimensional subspace at $x = x_0$ (this case is sometimes referred to as an intersection between $\mathbf{U}$ and the *vertex* of V).

The Maslov index for a path of Lagrangian subspaces is given by (6.5). However, in the case of $\bigwedge^2(\mathbb{R}^4)$ a new representation of the intersection form can be obtained. Suppose that a regular crossing occurs

$$\mathbf{U} \cap \mathbf{V} = \mathrm{span}\{\xi\}\,,$$

at $x = x_0$, then the crossing form is

$$\Gamma(\mathbf{U}, V, x_0) = \boldsymbol{\omega} \wedge \xi \wedge \mathbf{A}\xi\,, \tag{7.5}$$

To verify this formula, note that

$$\boldsymbol{\omega} \wedge \mathbf{a} \wedge \mathbf{J}^{-1}\mathbf{c} = \langle \mathbf{a}, \mathbf{c} \rangle\, \mathsf{vol}\,, \quad \text{for any } \mathbf{a}, \mathbf{c} \in \mathbb{R}^4\,.$$

Hence

$$\boldsymbol{\omega} \wedge \xi \wedge \mathbf{A}\xi = \boldsymbol{\omega} \wedge \xi \wedge \mathbf{J}^{-1}\mathbf{B}\xi = \langle \xi, \mathbf{B}\xi \rangle\, \mathsf{vol}\,,$$

recovering the expression in (6.4). There is an interesting geometric interpretation of (7.5). At a regular intersection the two-plane $\xi \wedge \mathbf{A}\xi$ is *not* a Lagrangian plane. It is an element of $\mathrm{G}_2(\mathbb{R}^4)$ but not an element of $\Lambda(2)$. Since $\Lambda(2)$ is a codimension one submanifold of $\mathrm{G}_2(\mathbb{R}^4)$, the sign of $\boldsymbol{\omega} \wedge \xi \wedge \mathbf{A}\xi$ determines which side of $\Lambda(2)$ in $\mathrm{G}_2(\mathbb{R}^4)$ it lies. See Appendix E for further discussion of this case.

Hence the Maslov index of a path $\mathbf{U}(x, \lambda)$ relative to V is

$$\mathsf{Maslov}(\mathbf{U}, V) = \sum_{x_0} \mathrm{sign}(\boldsymbol{\omega} \wedge \xi \wedge \mathbf{A}\xi)\,,$$

where the sum is over all interior intersections.

8 The Maslov angle on $\bigwedge^2(\mathbb{R}^4)$

In $\mathbb{R}^2$, the Maslov angle is just the angle associated with the polar representation of a vector in $\mathbb{R}^2$ as shown in §5.1. For a Lagrangian frame of the form (2.1) the Maslov angle is defined as in (2.3) and (2.4). In this section a new formula for the Maslov angle is given for the exterior algebra representation of a Lagrangian plane. Here, the result for 4D phase space is given and in Part 2 [18], the general result for $2n-$dimensional phase space is given.

A Lagrangian frame can be partitioned into two 2×2 blocks as in (2.1) and it can be represented in terms of its columns

$$\mathbf{Z} = [\,\mathbf{z}_1 \,|\, \mathbf{z}_2\,]\,, \quad \text{with} \quad \langle \mathbf{J}\mathbf{z}_1, \mathbf{z}_2 \rangle = 0\,.$$

Denote the exterior algebra representation of the Lagrangian plane by

$$\mathbf{U} = \mathbf{z}_1 \wedge \mathbf{z}_2\,.$$

Proposition 3 *There exists a* $2-$*form* $\mathbf{C}$*,*

$$\mathbf{C} = \mathbf{C}_1 + \mathrm{i}\mathbf{C}_2\,, \quad \textit{with} \quad \mathbf{C}_1, \mathbf{C}_2 \in \textstyle\bigwedge^2(\mathbb{R}^4)\,,$$

such that

$$\det[\mathbf{X} - \mathrm{i}\mathbf{Y}]\,\mathit{vol} = \mathbf{C} \wedge \mathbf{U}\,. \tag{8.1}$$

It follows from this proposition that there exists a scalar complex-valued function K such that $\mathbf{C} \wedge \mathbf{U} = K\mathsf{vol}$. A formula for the Maslov angle is then immediate:

Proposition 4

$$\mathrm{e}^{\mathrm{i}\kappa} = K/\overline{K}\,.$$

It remains to prove Proposition 3. The proof is by explicit construction. Let

$$\mathbf{c}_j = \mathbf{e}_j - \mathrm{i}\mathbf{J}\mathbf{e}_j\,, \quad j = 1, 2\,.$$

Then

$$\mathbf{X} - \mathrm{i}\mathbf{Y} = \begin{pmatrix} \mathbf{I} \\ -\mathrm{i}\mathbf{I} \end{pmatrix}^T \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = [\mathbf{c}_1 \,|\, \mathbf{c}_2]^T [\,\mathbf{z}_1 \,|\, \mathbf{z}_2\,] = \begin{pmatrix} \langle \mathbf{c}_1, \mathbf{z}_1 \rangle & \langle \mathbf{c}_1, \mathbf{z}_2 \rangle \\ \langle \mathbf{c}_2, \mathbf{z}_1 \rangle & \langle \mathbf{c}_2, \mathbf{z}_2 \rangle \end{pmatrix}, \tag{8.2}$$

and so, using the induced inner product [1] on $\bigwedge^2(\mathbb{R}^4)$ (see Appendix D)

$$\det[\mathbf{X} - \mathrm{i}\mathbf{Y}]\mathsf{vol} = \det \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{z}_1 \rangle & \langle \mathbf{c}_1, \mathbf{z}_2 \rangle \\ \langle \mathbf{c}_2, \mathbf{z}_1 \rangle & \langle \mathbf{c}_2, \mathbf{z}_2 \rangle \end{bmatrix} \mathsf{vol} = [\![\mathbf{c}_1 \wedge \mathbf{c}_2, \mathbf{U}]\!]_2 \mathsf{vol}\,.$$

This gives a formula for K,

$$K = [\![\mathbf{c}_1 \wedge \mathbf{c}_2, \mathbf{U}]\!]_2\,.$$

It is not necessary to give an expression for $\mathbf{C}$ since in computation it is K that is needed. However, for completeness it is given. Let $\mathbf{C}$ be an $2-$form satisfying

$$\mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \overline{\mathbf{C}} = [\![\mathbf{c}_1 \wedge \mathbf{c}_2, \overline{\mathbf{c}_1 \wedge \mathbf{c}_2}]\!]_2\,\mathsf{vol}\,. \tag{8.3}$$

Then

$$\det[\mathbf{X} - \mathrm{i}\mathbf{Y}]\mathsf{vol} = \mathbf{C} \wedge \mathbf{U}\,.$$

The $2-$form $\mathbf{C}$ is in fact the Hodge star of $\mathbf{c}_1 \wedge \mathbf{c}_2$ although the details of that characterization are not needed.

Now compute the formula in coordinates on $\bigwedge^2(\mathbb{R}^4)$. On $\mathbb{R}^4$ with the standard basis,

$$\begin{aligned} \mathbf{c}_1 \wedge \mathbf{c}_2 &= (\mathbf{e}_1 - \mathrm{i}\mathbf{J}\mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathrm{i}\mathbf{J}\mathbf{e}_2) \\ &= (\mathbf{e}_1 - \mathrm{i}\mathbf{e}_3) \wedge (\mathbf{e}_2 - \mathrm{i}\mathbf{e}_4) \\ &= \mathbf{e}_1 \wedge \mathbf{e}_2 - \mathrm{i}\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathrm{i}\mathbf{e}_2 \wedge \mathbf{e}_3 - \mathbf{e}_3 \wedge \mathbf{e}_4\,, \end{aligned}$$

and so, if $\mathbf{U} = \sum_{j=1}^{6} U_j \mathbf{E}_j$, with $\mathbf{E}_1, \ldots, \mathbf{E}_6$ the standard basis on $\bigwedge^2(\mathbb{R}^4)$,

$$K = [\![\mathbf{c}_1 \wedge \mathbf{c}_2, \mathbf{U}]\!]_2 = U_1 - \mathrm{i}U_3 + \mathrm{i}U_4 - U_6\,,$$

[1] A real inner product is used throughout the paper. Complexification is used so rarely, a hermitian inner product is not necessary. One just needs to keep track of the complex conjugations.

and so the expression for the Maslov angle is

$$e^{i\kappa} = \frac{U_1 - U_6 - \mathrm{i}U_3 + \mathrm{i}U_4}{U_1 - U_6 + \mathrm{i}U_3 - \mathrm{i}U_4}\,. \tag{8.4}$$

This expression is equivalent to the formula derived in equation (22) of [6].

The two-form $\mathbf{C}$ in (8.1) is computed to be

$$\mathbf{C} = -\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4 + \mathrm{i}(\mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3)\,.$$

9 Further decomposition of the Maslov angle using the eigenvalues of Q

Let $\mathbf{Z} \in \mathbb{R}^{4\times 2}$ be a Lagrangian frame on $\mathbb{R}^4$ of the form (2.1) satisfying (2.2). Then the matrix

$$\mathbf{Q} = (\mathbf{X} - \mathrm{i}\mathbf{Y})(\mathbf{X} + \mathrm{i}\mathbf{Y})^{-1}\,,$$

is a unitary and symmetric (but not hermitian) matrix.

The Maslov angle for closed path is defined using the determinant of $\mathbf{Q}$ (2.3). However, $\mathbf{Q}$ has 2 eigenvalues of unit modulus. Denote these eigenvalues by $\mathrm{e}^{\mathrm{i}\kappa_j}$, $j = 1, 2$ with κ_j real. Then

$$\mathrm{e}^{\mathrm{i}\kappa} = \mathrm{e}^{\mathrm{i}\kappa_1}\,\mathrm{e}^{\mathrm{i}\kappa_2} \quad \Rightarrow \quad \kappa = \kappa_1 + \kappa_2 \ (\mathrm{mod}\ 2\pi)\,.$$

These eigenvalues are independent of the choice of $\mathbf{Z} \in \mathbb{R}^{4\times 2}$ as a representative of a Lagrangian space: choosing another representation leads to similar matrix.

These eigenvalues can also be used to give another formula for the sign of each intersection. Fix the reference angle to be 0 (mod 2π). There is a 1-dimensional intersection at x_0 with a reference plane if and only if there exists $\mathrm{e}^{\mathrm{i}\kappa_r(x_0)} = 1$ with $r = 1$ *or* $r = 2$. There is a 2-dimensional intersection at x_0 with the reference plane if and only if $\mathrm{e}^{\mathrm{i}\kappa_r(x_0)} = 1$ for $r = 1$ *and* $r = 2$. If $\mathrm{e}^{\mathrm{i}\kappa_r(x_0)} \neq 1$ for $r = 1$ and $r = 2$ then the intersection is *transverse.*

When the intersection is regular, the sign of the intersection is given by:

$$\lim_{x\to x_0^+} \#\{r \in S | \kappa_r(x_0) \in \ (0,\pi) + 2\pi\mathbb{Z}\} - \#\{r \in S | \kappa_r(x_0) \in \ (-\pi, 0) + 2\pi\mathbb{Z}\}\,.$$

Thus, it is possible to determine the Maslov index, defined with intersections by simply tracking the crossings of the angles κ_i with $2\pi\mathbb{Z}$.

9.1 The angles κ_j in the exterior algebra representation

The eigenvalues $e^{\mathrm{i}\kappa_j}$ are the roots of the following polynomial:

$$P(\lambda) = \det((\mathbf{X} - \mathrm{i}\mathbf{Y}) - \mu(\mathbf{X} + \mathrm{i}\mathbf{Y}))\,, \quad \mu \in S^1\,.$$

The coefficients of P are antisymmetric multi-linear functions of $\mathbf{Z}$. As a consequence, they can always be expressed as a linear combination of the minors of $\mathbf{Z}$.

In the case of two angles, they satisfy

$$\det(\mu\mathbf{I} - \mathbf{Q}) = \mu^2 - \mathrm{Trace}(\mathbf{Q})\mu + \det(\mathbf{Q}) = 0\,, \tag{9.1}$$

with μ of unit modulus and $\mu_r = \mathrm{e}^{\mathrm{i}\kappa_r}$, $r = 1, 2$. Both the trace and determinant can be expressed in terms of the exterior algebra representation.

For the determinant, as shown in §8,

$$\det[\mathbf{X} - \mathrm{i}\mathbf{Y}] = K := [\![\mathbf{c}_1 \wedge \mathbf{c}_2, \mathbf{U}]\!]_2 = U_1 - \mathrm{i}U_3 + \mathrm{i}U_4 - U_6\,,$$

for $\bigwedge^2(\mathbb{R}^4) \ni \mathbf{U} = \sum_{j=1}^6 U_j \mathbf{E}_j$. Hence

$$\det(\mathbf{Q}) = K/\overline{K}\,. \tag{9.2}$$

It remains to express the Trace of $\mathbf{Q}$ in terms on the exterior algebra representation

Proposition 5

$$\mathrm{Trace}(\mathbf{Q}) = \frac{2}{\overline{K}}(U_1 + U_6)\,. \tag{9.3}$$

To prove this proposition, use (8.2) to relate the columns of $\mathbf{Z}$ to the $\mathbf{X}-\mathbf{Y}$ decomposition

$$\mathbf{X} + \mathrm{i}\mathbf{Y} = \begin{bmatrix} \langle \overline{\mathbf{c}_1}, \mathbf{z}_1 \rangle & \langle \overline{\mathbf{c}_1}, \mathbf{z}_2 \rangle \\ \langle \overline{\mathbf{c}_2}, \mathbf{z}_1 \rangle & \langle \overline{\mathbf{c}_2}, \mathbf{z}_2 \rangle \end{bmatrix}.$$

Hence

$$\mathbf{Q} = \frac{1}{\overline{K}} \begin{pmatrix} \langle \mathbf{c}_1, \mathbf{z}_1 \rangle & \langle \mathbf{c}_1, \mathbf{z}_2 \rangle \\ \langle \mathbf{c}_2, \mathbf{z}_1 \rangle & \langle \mathbf{c}_2, \mathbf{z}_2 \rangle \end{pmatrix} \begin{bmatrix} \langle \overline{\mathbf{c}_2}, \mathbf{z}_2 \rangle & -\langle \overline{\mathbf{c}_1}, \mathbf{z}_2 \rangle \\ -\langle \overline{\mathbf{c}_2}, \mathbf{z}_1 \rangle & \langle \overline{\mathbf{c}_1}, \mathbf{z}_1 \rangle \end{bmatrix},$$

and so

$$\begin{aligned}
\mathrm{Trace}(\mathbf{Q}) &= \tfrac{1}{\overline{K}} \left(\langle \mathbf{c}_1, \mathbf{z}_1 \rangle \langle \overline{\mathbf{c}_2}, \mathbf{z}_2 \rangle - \langle \mathbf{c}_1, \mathbf{z}_2 \rangle \langle \overline{\mathbf{c}_2}, \mathbf{z}_1 \rangle - \langle \mathbf{c}_2, \mathbf{z}_1 \rangle \langle \overline{\mathbf{c}_1}, \mathbf{z}_2 \rangle + \langle \mathbf{c}_2, \mathbf{z}_2 \rangle \langle \overline{\mathbf{c}_1}, \mathbf{z}_1 \rangle \right)\,, \\
&= \tfrac{1}{\overline{K}} \left(\det \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{z}_1 \rangle & \langle \mathbf{c}_1, \mathbf{z}_2 \rangle \\ \langle \overline{\mathbf{c}_2}, \mathbf{z}_1 \rangle & \langle \overline{\mathbf{c}_2}, \mathbf{z}_2 \rangle \end{bmatrix} + \det \begin{bmatrix} \langle \overline{\mathbf{c}_1}, \mathbf{z}_1 \rangle & \langle \overline{\mathbf{c}_1}, \mathbf{z}_2 \rangle \\ \langle \overline{\mathbf{c}_2}, \mathbf{z}_1 \rangle & \langle \overline{\mathbf{c}_2}, \mathbf{z}_2 \rangle \end{bmatrix} \right)\,, \\
&= \tfrac{1}{\overline{K}} \left([\![\mathbf{c}_1 \wedge \overline{\mathbf{c}_2}, \mathbf{z}_1 \wedge \mathbf{z}_2]\!]_2 + [\![\overline{\mathbf{c}_1} \wedge \mathbf{c}_2, \mathbf{z}_1 \wedge \mathbf{z}_2]\!]_2 \right) \\
&= \tfrac{1}{\overline{K}} \left([\![\mathbf{c}_1 \wedge \overline{\mathbf{c}_2} + \overline{\mathbf{c}_1} \wedge \mathbf{c}_2, \mathbf{z}_1 \wedge \mathbf{z}_2]\!]_2 \right) \\
. &= \tfrac{2}{\overline{K}} \left([\![\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4, \mathbf{z}_1 \wedge \mathbf{z}_2]\!]_2 \right) \\
&= \tfrac{2}{\overline{K}}(U_1 + U_6)\,,
\end{aligned}$$

using

$$\mathrm{Re}\,(\mathbf{c}_1 \wedge \overline{\mathbf{c}_2}) = \mathrm{Re}\,((\mathbf{e}_1 - \mathrm{i}\mathbf{e}_3) \wedge (\mathbf{e}_2 + \mathrm{i}\mathbf{e}_4)) = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4\,,$$

proving (9.3).

Given a path $\mathbf{U} \in \bigwedge^2(\mathbb{R}^4)$ the eigenvalues of $\mathbf{Q}$ can be computed by substituting (9.2) and (9.3) in (9.1) leading to

$$\mu_{1,2} := \mathrm{e}^{\mathrm{i}\kappa_{1,2}} = \frac{U_1 + U_6 \pm \sqrt{4U_1U_6 + 2U_3U_4 - U_3^2 - U_4^2}}{U_1 + \mathrm{i}U_3 - \mathrm{i}U_4 - U_6}\,.$$

Using the properties of a Lagrangian plane, $U_2 + U_5 = 0$ and $U_1U_6 - U_2U_5 + U_3U_4 = 0$, this formula reduces to

$$\mu_{1,2} := \mathrm{e}^{\mathrm{i}\kappa_{1,2}} = \frac{U_1 + U_6 \pm \mathrm{i}\sqrt{4U_5^2 + (U_3 + U_4)^2}}{U_1 + \mathrm{i}U_3 - \mathrm{i}U_4 - U_6}\,.$$

The general formula for the decomposition of the Maslov angles in any dimension is given in Part 2 [18].

10 λ dependence of the Maslov index

Let $\mathbf{B}(x,\lambda)$ be as defined in (1.2) with the asymptotic property (1.5). When the $\lambda-$dependence of $\mathbf{B}(x,\lambda)$ takes a simple form one can say more about the $\lambda-$dependence of the Maslov index. For example, in §5 the matrix $\mathbf{B}(x,\lambda)$ in (5.4) satisfies

$$\frac{\partial}{\partial\lambda}\mathbf{B}(x,\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\,,$$

that is, it is positive semi-definite. In the example in §11 the matrix $\mathbf{B}(x,\lambda) = \mathbf{J}\mathbf{A}(x,\lambda)$ with $\mathbf{A}(x,\lambda)$ defined in (11.3) has the property

$$\frac{\partial}{\partial\lambda}\mathbf{B}(x,\lambda) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\,,$$

which is negative semi-definite.

When $\partial_\lambda\mathbf{B}(x,\lambda)$ is semi-definite, the Maslov index is a monotone function of λ. For the case of gradient systems (as in §5 and §11) this property is proved in Lemmas 3.3 and 3.7 of Bose & Jones [6]. Related results are proved by Arnold [4] and generalizations of these results are proved by Chardard [15]. These results are summarized in

Lemma 6 *Assume that:*

- $\mathbf{B}(x,\lambda)$ *is a smooth function with respect to x and analytic with respect to λ.*

- *There exists* $\mathbf{B}_\infty(\lambda)$, $\gamma > 0$ *and* $F > 0$ *such that* $\forall x, \lambda \quad \|\mathbf{B}(x,\lambda) - \mathbf{B}_\infty(\lambda)\| \leq F e^{-\gamma|x|}$.
- *The open set* $\mathbb{X} = \mathbb{R} - \sigma_{ess}$ *of real numbers is not empty.*
- $\partial_\lambda \mathbf{B}(x,\lambda)$ *is semi-definite symmetric matrix.*

If $[\lambda_1, \lambda_2] \cap \sigma_{ess} = \emptyset$ *and* $\lambda_1, \lambda_2 \notin \sigma$, *then* $\mathsf{Maslov}(\lambda_2) - \mathsf{Maslov}(\lambda_1)$ *is equal to the number of eigenvalues with multiplicity in* $[\lambda_1, \lambda_2]$.

The first three assumptions are the usual hypotheses made to prove the analyticity of the Evans function and the theorems linking eigenvalues and the zeros of the Evans function. Thanks to this lemma, it is possible to define the following invariant for the homoclinic orbit:

Definition 7 $\mathsf{Maslov}^{\text{homoclinic}}$ *is defined as* $\lim_{\lambda\to 0^+} \mathsf{Maslov}(\lambda)$.

If we make the hypotheses of Lemma 6, $\mathsf{Maslov}^{\text{homoclinic}}$ is only dependent on $\mathbf{A}(x,0)$, and hence on the linearization of the ODE satisfied by the homoclinic orbit. In fact, it is possible to define $\mathsf{Maslov}^{\text{homoclinic}}$ without any reference to a parameter λ (see [6,19,15]). However, these constructions are not convenient for numerical computations.

10.1 The Maslov index for large values of $|\lambda|$

When $\lambda \to +\infty$ (or $\lambda \to -\infty$ if the essential spectrum extends to minus infinity) we expect the Maslov index to converge to some finite value. This property is similar to the property of the Evans function for large λ. The hypotheses are based on the analogous result of [45], adapted to the setting of the Maslov index.

Hypothesis 8 *Suppose*

- *that there exists* $\lambda_0 \in \mathbb{R}$ *such that* σ_{ess} *is empty for all* $\lambda > \lambda_0$*;*
- *For large enough* λ, $\mathbf{A}_\infty(\lambda)$ *has no purely imaginary eigenvalues;*
- *Let* $\mathbf{V}(\lambda)$ *be a symplectic* 4×4 *matrix depending analytically on* λ *whose first* 2 *columns are a basis for* $\mathrm{E}^u_\infty(\lambda)$ *and whose last* 2 *columns are a basis for* $\mathrm{E}^s_\infty(\lambda)$ *Define*

$$\mathbf{F}(x,\lambda) = \mathbf{V}^{-1}(\lambda)(\mathbf{A}(x,\lambda) - \mathbf{A}_\infty(\lambda))\mathbf{V}(\lambda)\,.$$

- *Suppose that, for large enough* λ*:*

$$\textstyle\int_{\mathbb{R}} |\mathbf{F}(x,\lambda)|\mathrm{d}x \quad \textit{is bounded, uniformly in } \lambda$$

$$\textstyle\int_{|x|>x_0} |\mathbf{F}(x,\lambda)|\mathrm{d}x \quad \textit{tends to } 0 \textit{ when } x_0 \to \infty\textit{, uniformly in } \lambda$$

$$\textstyle\int_{\mathbb{R}} |\mathbf{F}^{(2)}(x,\lambda)\mathbf{e}_1|\mathrm{d}x \quad \textit{tends to } 0\,.$$

Remark. If there exists λ_0 such that σ_{ess} is empty for all $\lambda < \lambda_0$ the above hypotheses can be modified accordingly.

Proposition 9 *Assume that hypothesis 8 is met by* $\mathbf{A}(x,\lambda)$, *then*

$$\lim_{\lambda\to+\infty} D(\lambda) = 1\,, \quad \textit{and} \quad \lim_{\lambda\to+\infty} \mathsf{Maslov}(\cdot;\lambda) = 0\,.$$

The proof is obtained by following the argument in Proposition 1.17 in Pego & Weinstein [45] and the Appendix of Bridges & Derks [8]. One proves that $(V^{[2]}(\lambda))^{-1}Y(.,\lambda)$ converges, uniformly in x, to the constant vector $\mathbf{e}_1$ when $\lambda \to -\infty$. Then, for large enough λ, $(V^{[2]}(\lambda))^{-1}Y(.,\lambda)$ has a null Maslov index and so does $Y(.,\lambda)$. See [15] for a detailed proof.

10.2 Defining a Maslov index at $\lambda = 0$ when the basic state is approximated by hyperbolic periodic solutions

When $\lambda = 0$, system (1.2) often satisfies the following hypothesis: $\mathbf{B}(x,0) = D^2H(\widehat{\phi(x)})$ where $H : \mathbb{R}^4 \to \mathbb{R}$ is the Hamiltonian function and $\widehat{\phi}$ is the basic homoclinic solution of the autonomous system

$$\mathbf{J}\mathbf{u}_x = \nabla H(\mathbf{u})\,. \tag{10.1}$$

Suppose that $\partial_\lambda \mathbf{B}$ is semi-definite near $\lambda = 0$ and that the dimension of the space of square integrable solutions of $\mathbf{J}\mathbf{u}_x = \mathbf{B}(x,0)\mathbf{u}$ is one. Furthermore, suppose $\widehat{\phi}$ is approximated by hyperbolic $\frac{2\pi}{k}$-periodic solutions $\widehat{\phi}_k$, for which the Maslov index is well-defined [16,14]. If the Maslov index at $\lambda = 0$ is defined as the limit of the Maslov index of the periodic solutions $\widehat{\phi}_k$, when $k \to 0$, then, as shown in [14] under natural hypotheses, the Maslov index at 0 is the value of limit of the Maslov indices of the periodic orbits when λ is close to 0 and it has the sign of $f'(k)$ near $k = 0$, where $f(k) := H(\widehat{\phi}_k)$.

11 A coupled reaction-diffusion equation with explicit Maslov index

Consider the system of reaction-diffusion equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - 4u + 6u^2 - c(u-v) \\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} - 4v + 6v^2 + c(u-v)\,, \end{aligned} \tag{11.1}$$

where c, the coupling constant, is a non-zero real parameter, restricted to the values $c > -2$. When $c > -2$, it is straightforward to show that the trivial solution is stable in the time dependent problem, and the trivial solution of the steady equation is hyperbolic. This system has the exact steady solitary-wave solution

$$u = v := \widehat{u}(x) = \operatorname{sech}^2(x)\,.$$

Linearizing (11.1) about the basic state $\widehat{u}$ and taking perturbations of the form

$$e^{\lambda t}(u(x,\lambda), v(x,\lambda))\,,$$

leads to the coupled ODE eigenvalue problem

$$\begin{aligned} u_{xx} &= (\lambda + 4 + c - 12\widehat{u}(x))\, u - c\, v \\ v_{xx} &= -c\, u + (\lambda + 4 + c - 12\widehat{u}(x))\, v \,. \end{aligned} \tag{11.2}$$

This eigenvalue problem can be written in the standard form

$$\mathbf{u}_x = \mathbf{A}(x, \lambda)\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{R}^4\,, \tag{11.3}$$

with $\mathbf{u} = (u, v, u_x, v_x)$ and

$$\mathbf{A}(x, \lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f(x, \lambda) & -c & 0 & 0 \\ -c & f(x, \lambda) & 0 & 0 \end{pmatrix}, \quad \text{with} \quad f(x, \lambda) = \lambda + 4 + c - 12\,\mathrm{sech}^2(x)\,.$$

The system (11.3) is Hamiltonian: $\mathbf{JA}$ is symmetric.

The spectral problem (11.3) can be solved explicitly. Write the second-order problem in the form,

$$\begin{pmatrix} u \\ v \end{pmatrix}_{xx} = f(x, \lambda) \begin{pmatrix} u \\ v \end{pmatrix} - c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{11.4}$$

Let

$$\begin{aligned} u &= \widetilde{u} - \widetilde{v} \\ v &= \widetilde{u} + \widetilde{v}\,. \end{aligned}$$

Then substitution into (11.4) leads to the decoupled system

$$\begin{aligned} \widetilde{u}_{xx} + 12\,\mathrm{sech}^2(x)\, \widetilde{u} &= (\lambda + 4)\widetilde{u} \\ \widetilde{v}_{xx} + 12\,\mathrm{sech}^2(x)\, \widetilde{v} &= (\lambda + 4 + 2c)\widetilde{v}\,. \end{aligned} \tag{11.5}$$

These two systems have explicit solutions (cf. Appendix F), and using these results one finds that there are exactly six eigenvalues for the spectral problem (11.2):

$$\begin{aligned} &\lambda_1 = -3 - 2c\,, \quad \lambda_2 = -3\,, \quad \lambda_3 = -2c\,, \\ &\lambda_4 = 0\,, \quad \lambda_5 = 5 - 2c\,, \quad \lambda_6 = 5\,. \end{aligned}$$

The essential spectrum is

$$\sigma_{\text{ess}} = \{\lambda \in \mathbb{R} \;:\; \lambda \le -4\} \cup \{\lambda \in \mathbb{R} \;:\; \lambda \le -4 - 2c\}\,.$$

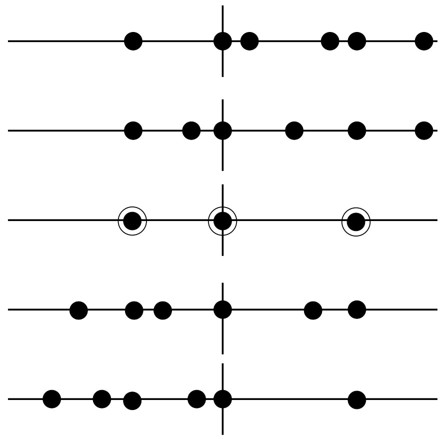

Fig. 3. Plot of the spectrum for $c = -2$ (starting at the top), $c = -1$, $c = 0$, $c = 1$ and $c = 3$.

When $c = 0$ then there are three double eigenvalues:

$$\lambda_1 = \lambda_2 = -3\,, \quad \lambda_3 = \lambda_4 = 0\,, \quad \lambda_5 = \lambda_6 = 5\,.$$

For c nonzero and small the eigenvalues λ_1, λ_3 and λ_5 are perturbed to the left (when $c > 0$) and to the right (when $c < 0$). Hence positive coupling is stabilizing and negative coupling is destabilizing.

When $c = -2$ (the lower bound on c for stability of the zero state and hyperbolicity) there are four positive eigenvalues, which is the maximum number of positive eigenvalues. At $c = -\frac{3}{2}$ one of the positive eigenvalues passes through zero, leaving 3 positive eigenvalues. Then when $c = 0$ another eigenvalue arrives at zero leaving two positive eigenvalues. Then when $c = \frac{5}{2}$ a third eigenvalue passes through zero. For all $c > \frac{5}{2}$ there is one positive eigenvalue, one zero eigenvalue and four negative eigenvalues. The configuration of the eigenvalues as a function of c is shown in Figure 3.

When the system decomposes into two subsystems, the Maslov index is the sum of the Maslov indices of the subsystems

$$\mathsf{Maslov}^{2D} \oplus \mathsf{Maslov}^{2D} = \mathsf{Maslov}^{4D}\,. \tag{11.6}$$

This property is obvious in the present example, since the system decouples. The Maslov index for the first 2D system in (11.5) is tabulated below.

λ	$-4 < \lambda < -3$	$-3 < \lambda < 0$	$0 < \lambda < 5$	$\lambda > 5$
Maslov_1^{2D}	3	2	1	0

The Maslov index for the second 2D system in (11.5) is tabulated here.

λ	$-4-2c<\lambda<-3-2c$	$-3-2c<\lambda<-2c$	$-2c<\lambda<5-2c$	$\lambda>5-2c$
Maslov_2^{2D}	3	2	1	0

The Maslov index for the full 4D system for any λ is then obtained by fixing c and then applying the sum formula (11.6). For example, fix $c=-1$ and compute $\mathsf{Maslov}^{\text{homoclinic}}$,

$$\mathsf{Maslov}^{\text{homoclinic}}\Big|_{c=-1} = \lim_{\lambda\to 0^+}\left[\mathsf{Maslov}_1^{2D}\oplus\mathsf{Maslov}_2^{2D}\right]\Big|_{c=-1} = 3\,.$$

A summary of the Maslov index of the homoclinic orbit as a function of c is given in the following table.

λ	$-2<c<-\frac{3}{2}$	$-\frac{3}{2}<c<0$	$0<c<\frac{5}{2}$	$c>\frac{5}{2}$
$\mathsf{Maslov}^{\text{homoclinic}}$	4	3	2	1

12 Numerical implementation – approximation by periodic orbits

When the solitary wave is approximated by a hyperbolic periodic orbit, the Maslov index of the periodic orbit is computed using the Maslov angle (2.4)-(2.5) in the exterior algebra representation (e.g. equation (8.4)). An algorithm for this case has been proposed in [16], and a proof of convergence of the Maslov index in the limit as the periodic solution converges to the solitary wave is given in [14].

Fix λ and a basic periodic solution which approximates a solitary wave. The steps in the algorithm are as follows.

(1) Choose a large enough interval $[-L,L]$.
(2) Compute the eigenvalue with largest real part of $\mathbf{A}_\infty^{(2)}(\lambda)$, denoted by $\sigma_+(\lambda)$, and its associated eigenvector $\zeta^+(\lambda)$.
(3) Integrate equation
$$\mathbf{U}_x^+ = [\mathbf{A}^{(2)}(x,\lambda)-\sigma_+(\lambda)\mathbf{I}]\mathbf{U}^+\,, \tag{12.1}$$
on $[-L,L]$, taking $\zeta^+(\lambda)$ as initial condition at $x=-L$, using any standard numerical integration scheme. The justification for the arbitrariness in choice of numerical scheme is given in Appendix G.
(4) $\mathbf{U}^+(L,\lambda)$ and $\mathbf{U}^+(-L,\lambda)$ are nearly collinear, and an approximation to the Evans function is determined from their constant of proportionality
$$\mathbf{U}^+(L,\lambda) = D(\lambda)\mathbf{U}^+(-L,\lambda) + \mathsf{Error}\,,$$
where the error is generally of the order of machine precision.

(5) Compute $e^{i\kappa(x)}$ using equation (8.4) or analogous representation.
(6) Compute a lift of $\kappa(x)$ and choose the stepsize Δx so that $|\kappa(x+\Delta x)-\kappa(x)|<\pi$.
(7) Compute the Maslov index using (2.5).

There are a number of sources of error in the algorithm. There is an approximation error due to the fact that the solitary wave is approximated by a periodic orbit. Two parameters have to be chosen: L and the step size Δx. The choice of step size is a familiar source of error. The consistency error of the numerical integration scheme will be of the form $C\,\Delta x^p$, for some natural number p, at each step, where C is a constant depending on the derivatives of $\mathbf{A}$. The choice of numerical scheme will also impose some stability condition.

Since the Maslov index is an integer, the proposed scheme will give the Maslov index if the relative error on $\mathbf{U}^+(\cdot,\lambda)$ is small enough. However, if $\sup_{\mathbb{R}}\|\mathbf{U}^+(\cdot,\lambda)\|$ is very small (for example when the Evans function is small), the relative error may be too big and lead to a miscomputed Maslov index. This is the case when λ is an eigenvalue or near an eigenvalue, since there is an integer-valued jump in the Maslov index at eigenvalues.

13 Numerical implementation – intersection index algorithm

The numerical algorithm based on the intersection index is similar to the algorithm in §12 except that the computation of the angle $\kappa(x)$ is replaced by the computation of the angles κ_1 and κ_2. The previous algorithm can be modified as follows:

(1) Choose a large interval $-L\le x\le L$. Initialize Maslov to 0.
(2) Construct a symplectic matrix $\mathbf{K}(\lambda)$ such that

$$\mathbf{K}(\lambda)\begin{pmatrix} I \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}(\lambda)\begin{pmatrix} 0 \\ I \end{pmatrix},$$

represent the stable and unstable spaces of $\mathbf{A}_\infty(\lambda)$. $\mathbf{K}(\lambda)$ defines a symplectic change of coordinates in which the coordinates of stable and unstable spaces in the exterior algebra are respectively

$$\mathbf{U}_0=\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{V}_0=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Let $\mathcal{K}(\lambda)$ be the matrix whose entries are the 2×2 minors of $\mathbf{K}(\lambda)$. In terms of bialternate product, $\mathcal{K}(\lambda)=\mathbf{K}(\lambda)\odot\mathbf{K}(\lambda)$.
(3) Compute the eigenvalue with largest real part, $\sigma_+(\lambda)$, of $\mathbf{A}_\infty^{(2)}(\lambda)$.

(4) Define $\widetilde{\mathbf{U}^{+}}(x,\lambda) = \mathcal{K}(\lambda)^{-1}\mathbf{U}^{+}(x,\lambda)$ and integrate the equation for $\widetilde{\mathbf{U}^{+}}$,

$$\frac{d}{dx}\widetilde{\mathbf{U}^{+}} = [\mathcal{K}(\lambda)^{-1}\mathbf{A}^{(2)}(x,\lambda)\mathcal{K}(\lambda) - \sigma_{+}(\lambda)\mathbf{I}])\widetilde{\mathbf{U}^{+}}\,, \tag{13.1}$$

on $[-L, L]$, taking $\mathbf{U}_0$ as initial condition for $\widetilde{\mathbf{U}^{+}}$ at $x = -L$, using any standard numerical integration scheme.

(5) Compute the angles (κ_1 and κ_2 corresponding to $\widetilde{\mathbf{U}^{+}}$ over $[-L, L]$. If an angle κ_i crosses $2\pi\mathbb{Z}$ between x and $x + \Delta x$, update the value of the Maslov index to:

$$\mathsf{Maslov} \mapsto \mathsf{Maslov} + \text{sign}\,(\kappa_i(x+\Delta x) - \kappa_i(x))\;.$$

(6) Return $\widetilde{\mathbf{U}^{+}}(L,\lambda) \wedge \mathbf{U}_0$ as an approximation to the Evans function.

(7) At $x = +L$, return the value of the Maslov index.

14 The Maslov index of solitary wave solutions of KdV5

In this section we study the Maslov index as a function of λ for the ODE eigenvalue problem

$$\phi_{xxxx} - P\phi_{xx} + a(x)\phi = \lambda\phi\,, \tag{14.1}$$

where $\phi(x,\lambda)$ is scalar valued, P is a real parameter and $a(x)$ is a localized function which satisfies $a(x) \to a_\infty$ as $x \to \pm\infty$, with exponential decay of $a(x)$ at infinity. For definiteness it is assumed that $a_\infty > 0$. The ODE (14.1) can be put in the form (1.2) with

$$\mathbf{u} = \begin{pmatrix} \phi \\ \phi_{xx} \\ \phi_{xxx} - P\phi_x \\ \phi_x \end{pmatrix} \quad \text{and} \quad \mathbf{B}(x,\lambda) = \begin{bmatrix} a(x)-\lambda & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & P \end{bmatrix}\,. \tag{14.2}$$

The spectrum of the system at infinity $\mathbf{A}_\infty(\lambda) = \mathbf{J}^{-1}\mathbf{B}_\infty(\lambda)$ has the characteristic polynomial

$$\det[\mathbf{A}_\infty(\lambda) - \mu\mathbf{I}] = \mu^4 - P\mu^2 + a_\infty - \lambda\,, \tag{14.3}$$

With $a_\infty > 0$ and $\lambda = 0$ the four roots are hyperbolic for all P such that

$$P + 2\sqrt{a_\infty} > 0\,,$$

which is assumed to be satisfied henceforth. When $\lambda \neq 0$ the essential spectrum will form the boundary of the hyperbolic region. The essential spectrum is

$$\sigma_{\text{ess}} = \{\lambda \in \mathbb{R}\ :\ \lambda = a_\infty + Ps^2 + s^4\,,\ s \in \mathbb{R}\}\,.$$

When

$$\lambda < \lambda^{\text{edge}} = a_\infty - \frac{1}{8}P(P - |P|)\,,$$

the spectrum of $\mathbf{A}_\infty(\lambda)$ is hyperbolic. Hence, all the hypotheses for the existence of the Evans function and the Maslov index are satisfied. We will apply this theory to determine the Maslov index of a class of homoclinic orbits.

The eigenvalue problem (1.2) appears in the linearization about a solitary wave solution of the fifth-order Korteweg-de Vries equation (KdV5). KdV5 appears for example as a model equation in plasma physics, and in the study of capillary–gravity water waves [28,30,22,13,7,23,9].

To see the role of (1.2) in the linearization of KdV5, consider the following form of the fifth-order KdV equation relative to a moving frame, moving at speed c,

$$\frac{\partial \phi}{\partial t} - c\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\left(\phi^{q+1}\right) + P\frac{\partial^3 \phi}{\partial x^3} - \frac{\partial^5 \phi}{\partial x^5} = 0\,, \quad q \geq 1\,. \tag{14.4}$$

A further scaling can be introduced so that $c = 1$, but including c is useful for comparing with results in the literature on KdV5. Effectively, q is a third parameter, but its value is restricted to natural numbers.

Steady solutions of (14.4), that decay to zero as $x \to \pm\infty$ satisfy the 4^{th}−order ordinary differential equation

$$\phi_{xxxx} - P\phi_{xx} + c\phi - \phi^{q+1} = 0. \tag{14.5}$$

The system (14.4) linearized about a solitary wave $\widehat{\phi}(x)$ solution of (14.5) takes the form

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(\mathscr{L}\phi\right)\,,$$

with

$$\mathscr{L}\phi := \phi_{xxxx} - P\,\phi_{xx} + c\phi - (q+1)\widehat{\phi}(x)^q\,\phi\,. \tag{14.6}$$

In this case, $a_\infty = c$. There are two spectral problems:

$$\mathscr{L}\phi = \lambda\phi \quad \text{and} \quad \mathbf{L}\phi = \widehat{\lambda}\phi\,, \quad \mathbf{L}\phi := \frac{d}{dx}\mathscr{L}\,. \tag{14.7}$$

The operator $\mathscr{L}$ is self-adjoint (in a suitably-chosen Hilbert space) and so $\lambda \in \mathbb{R}$ whereas $\mathbf{L}$ is not self-adjoint and $\widehat{\lambda}$ – which is the stability exponent – can in general be complex. The relationship between these two eigenvalue problems is discussed in §15. First the Maslov index of the spectral problem $\mathscr{L}\phi = \lambda\phi$, which can be put in the form (1.2), is studied.

The ODE (14.5) has been extensively studied and many solitary wave solutions have been found; a classification is given in [11]. There are some special cases where explicit solitary wave solutions can be constructed. An example is the explicit solution $\widehat{\phi}(x) = \frac{35}{24}\text{sech}^4\left(\frac{x}{2\sqrt{6}}\right)$ which exists when $q = 1$, $c = 1$ and $P = \frac{13}{6}$. However, the interesting solutions of (14.5) need to be computed numerically. They can be computed using a spectral method (approximate the solitary wave by a periodic function of large wavelength and then use Fourier series to represent it), or in the case of symmetric solitary waves a shooting algorithm can be used. We used both methods to compute solitary waves. An

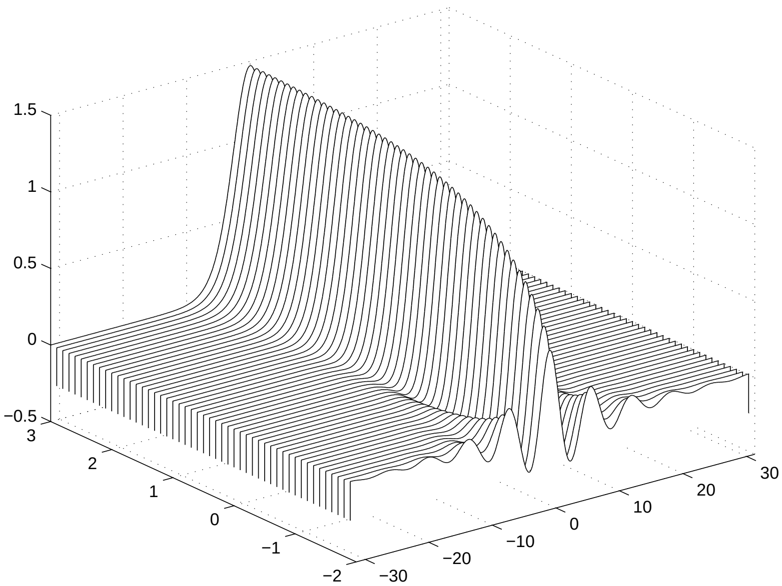

Fig. 4. Numerically computed solitary waves for the KdV 5 equation for the case $q = 1$, $c = 1$ ($a_\infty = 1$) and $-2 < P < 2$.

example of the family of one-mode solitary waves as a function of P, computed using a spectral method, is shown in Figure 14. Although these solitary waves are solutions of the model ODE, they are representative of solutions of the full water-wave problem. Dias, Menasce & Vanden-Broeck [24] have found large-amplitude branches of these solutions in the full water-wave equations.

Symmetric solutions are computed numerically using a shooting method: the starting point is an element of the tangent space of the unstable manifold and the ending point is a symmetric point. The time step typically used is $\frac{1}{1000}$ and the integrator chosen was the fourth-order Runge-Kutta method.

The ODE (14.5) can be characterized as a Hamiltonian system on $\mathbb{R}^4$ and the Hamiltonian in the original coordinates is

$$E(\phi) = \tfrac{1}{2}\phi_{xx}^2 + \tfrac{1}{2}P\phi_x^2 - \tfrac{1}{2}c\phi^2 + \frac{1}{q+2}\phi^{q+2} - \phi_x\phi_{xxx}\,. \tag{14.8}$$

The function $E(\phi)$ is constant along solutions of (14.5) (i.e. $\frac{dE}{dx} = 0$). Physically, for equations like KdV5, this quantity is associated with momentum flux. For simplicity, we will just refer to it as energy. The energy of the periodic approximants gives some information about the nature of the limiting homoclinic orbit.

The energy can be plotted as a function of wavenumber k along a branch of periodic solutions as the wavelength tends to infinity ($k \to 0$) as a function of q and P. First the case $P = \frac{13}{6}$, $c = 1$ and $q = 1$ is considered and it is shown in Figure 14. In this

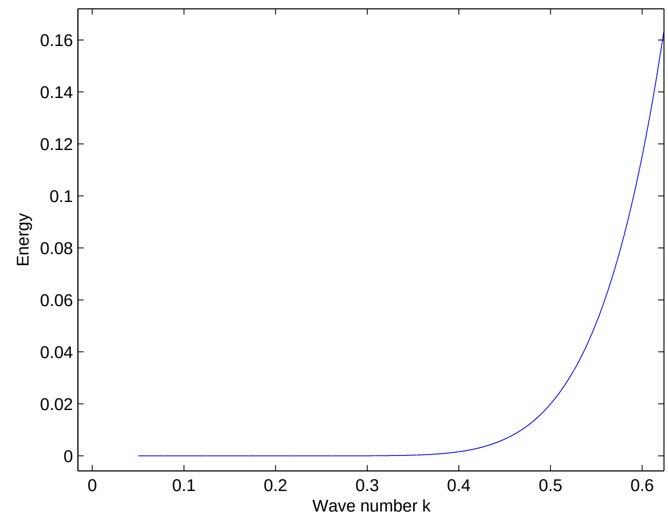

Fig. 5. Energy of the $\frac{2\pi}{k}$-periodic solutions as function of k for $q = 1$, $c = 1$ and $P = \frac{13}{6}$.

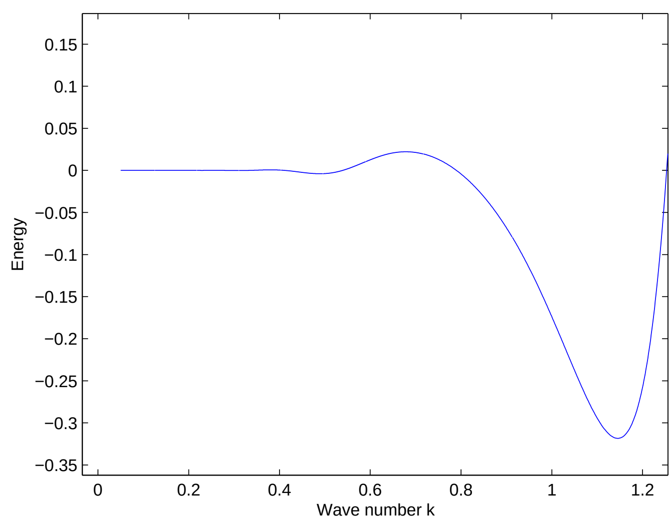

Fig. 6. Energy of the $\frac{2\pi}{k}$-periodic solutions as a function of k for $q = 1$, $c = 1$ and $P = -1$.

case the energy is a monotone function of wavenumber and the convergence $k \to 0$ is rapid. Keeping $q = 1$ and $c = 1$ but decreasing P to $P = -1$ begins to show oscillations indicative of a Shilnikov-type bifurcation as shown in Figure 14. Decreasing P further to $P = -1.9$ shows more dramatically the Shilnikov-type oscillations, as shown in Figure 14. In Figure 14 a sequence of bifurcations occurs along the branch. Each point on the energy-wavenumber diagram where $E'(k) = 0$ corresponds to a saddle-centre bifurcation of Floquet multipliers. There are always two Floquet multipliers at $+1$ due to the fact that (14.5) is autonomous. When $E'(k) = 0$ an additional pair of Floquet multipliers coalesces at $+1$. Each one of these saddle-centre bifurcations of the branch of periodic orbits leads

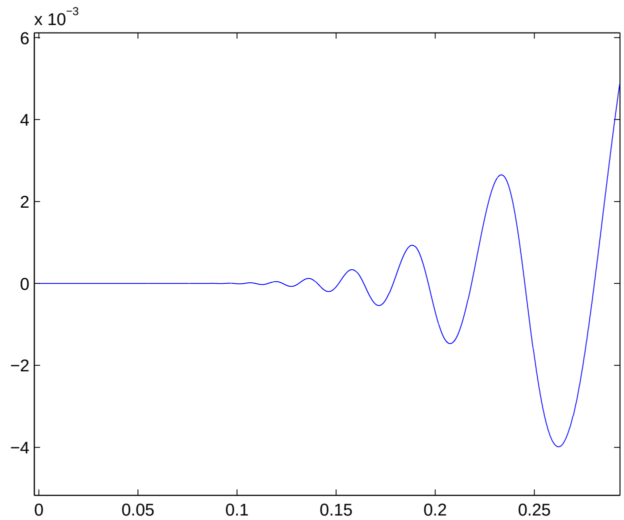

Fig. 7. Enlargement of the low wavenumber region of the Energy-wavenumber plot for the parameter values $q = 1$, $c = 1$ and $P = -1.9$.

to a secondary homoclinic bifurcation [10]. So, in addition to the limiting homoclinic orbit that we are principally interested in, there is a countable number of other orbits generated along the branch, which are homoclinic to the branch of periodic orbits. Although there is an infinite number of bifurcations along the branch our numerical results show that the Maslov index of the limiting homoclinic orbit is finite.

14.1 Computing the Maslov index as a function of λ

First consider the case $P = \frac{13}{6}$, $q = 1$ and $c = 1$ where the unimodal solitary wave is given explicitly. The lifts $\kappa(x)$ of the Maslov angle for this system are plotted as a function of x in Figure 8 for various values of λ. In Figure 9, the corresponding Maslov indices have been plotted as a function of λ. The Evans function shows that $\mathscr{L}$ has exactly three eigenvalues in this case. Denote these eigenvalues by

$$\lambda_1 < \lambda_2 = 0 < \lambda_3 \,.$$

The qualitative behaviour of the Maslov index in this case is similar to the example on $\mathbb{R}^2$ in §5. The values of the Maslov are shown in the table below. The Maslov index in this case is computed using the Maslov angle, and this Maslov index is denoted by $\mathsf{Maslov}(\kappa, \lambda)$.

λ	$\lambda < \lambda_1$	$\lambda_1 < \lambda < \lambda_2$	$\lambda_2 < \lambda < \lambda_3$	$\lambda > \lambda_3$
$\mathsf{Maslov}(\kappa, \lambda)$	0	1	2	3

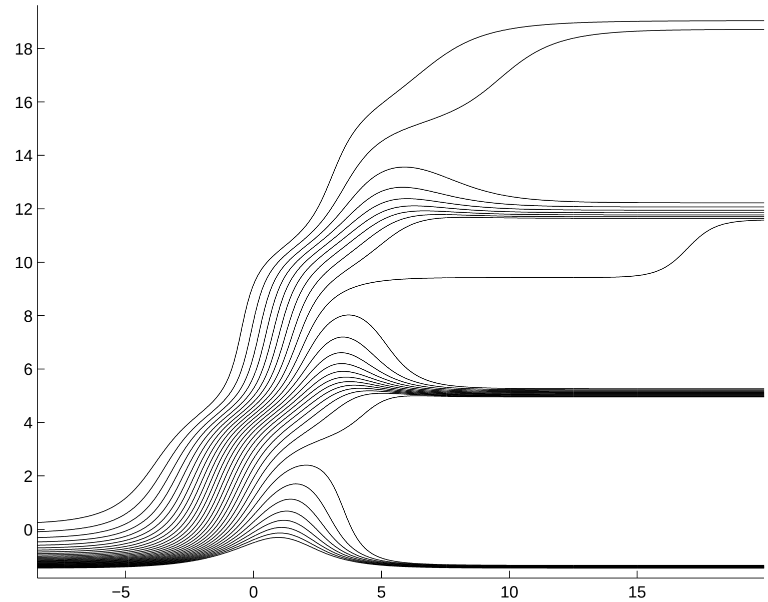

Fig. 8. κ as a function of x for the following values of λ:$-2, -1.9, \ldots, 0.9$. κ is λ-growing. The parameter values are $P = \frac{13}{6}$, $c = 1$ and $q = 1$.

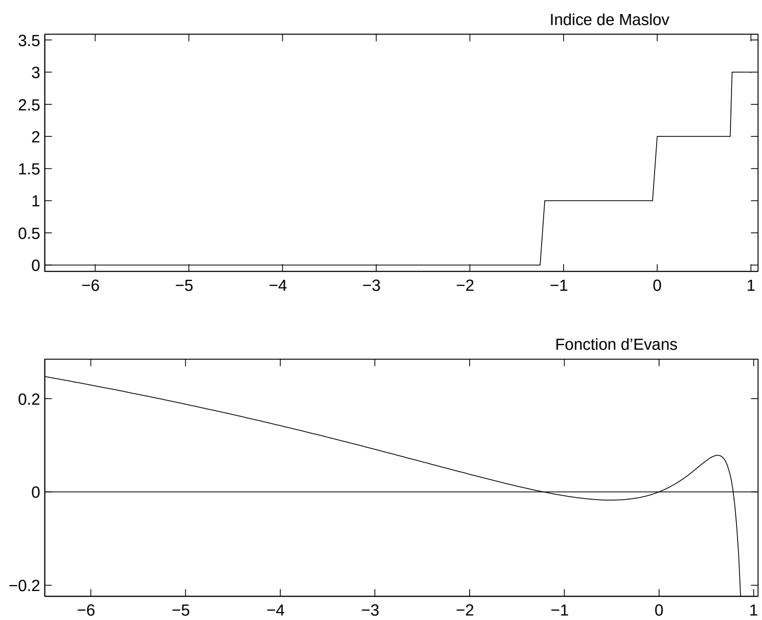

Fig. 9. Evans function and Maslov index as a function of λ for the explicit unimodal solitary wave solution when $P = \frac{13}{6}$, $c = 1$ and $q = 1$

Note that the Maslov index in each region predicts the number of eigenvalues of $\mathscr{L}$ in each λ interval.

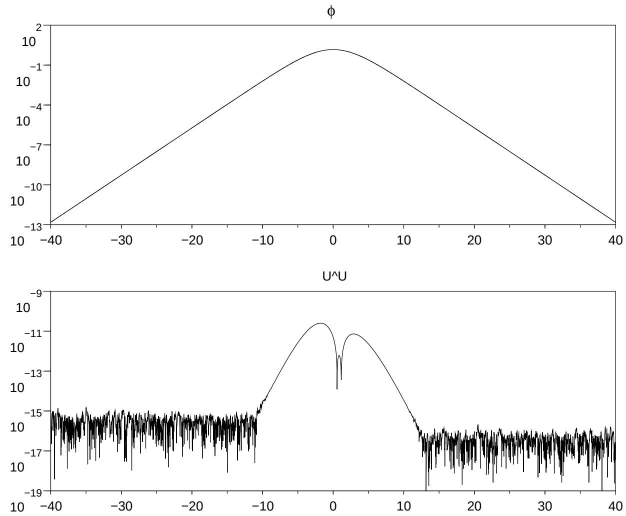

Fig. 10. The decomposability of a 2-form is equivalent to $I_1 = 0$ where I_1 is defined in (14.9). The upper figure shows the logarithm of the difference between $a(x)$ and its limit a_∞, and the lower figure shows the logarithm of the value of I_1 when $q = 1$, $P = \frac{13}{6}$, $\Delta x = .01$ and $\lambda = -10$ The fast oscillations are associated with round-off errors.

λ region	$\lambda < \lambda_1$	$\lambda < \lambda_2$	$\lambda < \lambda_3$	$\lambda < \lambda^{\text{edge}}$
# $\mathsf{Eigs}(\mathscr{L})$	0	1	2	3

It is immediate from this table that

$$\mathsf{Maslov}^{\text{homoclinic}} = \lim_{\lambda \to 0^+} \mathsf{Maslov}(\kappa, \lambda) = 2\,.$$

The operator $\mathscr{L}$ has exactly one negative eigenvalue in this case. Our calculations indicate that this is the case for all the unimodal homoclinic orbits. It is easy to show analytically that the Maslov index of a unimodal homoclinic orbit is greater than or equal to 2. An elementary proof in given in Appendix H. This result has implications for stability of the solitary waves as solutions of KdV5 and it is discussed in §15.

To test how accurately the Lagrangian Grassmannian is preserved by the numerical scheme, the values of

$$I_1 = \mathbf{U} \wedge \mathbf{U} \quad \text{and} \quad I_2 = \boldsymbol{\omega} \wedge \mathbf{U}\,, \tag{14.9}$$

are computed as a function of x. In these calculations the standard explicit fourth-order Runge-Kutta algorithm is used. The value of I_1 is shown in Figure 10 and shows that the error is of order of the machine accuracy, except for a small region around zero, but the error there is still exceptionally small. Concerning I_2, it is in fact exactly preserved, even numerically: if the value of $\boldsymbol{\omega} \wedge \mathbf{U}$ is the machine zero, it remains at the machine zero.

Using Proposition 9 and the proof in Appendix I, it follows that the KdV5 system satisfies hypothesis 8 and therefore $D(\lambda) \to 1$ and the Maslov index tends to 0 as $\lambda \to -\infty$.

15 Spectrum of $\mathscr{L}$ and the stability of solitary waves for KdV5

One of the intriguing properties of the Maslov index is the connection between the number of eigenvalues in subsets of the λ space, the Maslov index and the stability of solitary waves.

For KdV5 this connection is not obvious. For unimodal solitary-wave solutions of KdV5, Kodama & Pelinovsky [30] have studied this connection and they show the following result. Suppose the following integral exists

$$N(c,P) = \int_{\mathbb{R}} \widehat{\phi}(x,c,P)^2 \mathrm{d}x \,,$$

and is a differentiable function of c, and define

$$r = \begin{cases} 0 & \text{if } \frac{\partial N}{\partial c}(1,P) < 0 \\ 1 & \text{if } \frac{\partial N}{\partial c}(1,P) > 0 \end{cases} .$$

The functional $N(c,P)$ is sometimes called the momentum of the solitary wave. If $q = 1$, it is proved by Lewandosky [33] that $r = 1$ for all admissable P.

In [30] it is argued (see proposition 3.8 there) that a unimodal solitary wave is stable if $r = +1$ and $\#\mathscr{L}^- = 1$, where $\#\mathscr{L}^-$ is the number of negative eigenvalues of $\mathscr{L}$. This result assumes that there are no pure imaginary eigenvalues of $\mathbf{L}$. Hence – under these assumptions – the stability of the solitary wave is determined by the sign of $\frac{dN}{dc}$. This observation is consistent with the theory of [7] where a instability results for a class of unimodal solitary waves were obtained.

15.1 Stability of two-pulse solitary waves

More refined results on stability of two-pulse solitary waves were obtained by Pelinovsky & Chugunova [20]. Suppose that $\mathbf{L}$ has only simple eigenvalues except a double eigenvalue at 0 and suppose $\frac{\partial N}{\partial c}(1,P) \neq 0$. In [20] it is proved that

$$N_{unst} = \#\mathscr{L}^- - r - N^-_{imag} \,,$$

where N_{unst} is the number of eigenvalues with strictly positive real part of $\mathbf{L}$ and N^-_{imag} is the number of pure imaginary eigenvalues of $\mathbf{L}$ with negative Krein signature. Using the formula $\#\mathscr{L}^- = \mathsf{Maslov}^{\text{homoclinic}} - 1$ gives

$$N_{unst} = \mathsf{Maslov}^{\text{homoclinic}} - 1 - r - N^-_{imag} \,.$$

It is immediate from this formula that if $N^-_{imag} = 0$ and $\mathsf{Maslov}^{\text{homoclinic}} = 2$ then the basic state is stable if $\frac{dN}{dc} > 0$ and unstable if $\frac{dN}{dc} < 0$.

Using the classification of Buffoni, Champneys & Toland [11], a two-pulse solitary wave has the classification $\mathbf{2}(\ell)$ where ℓ is a natural number. In [17], it is found numerically that two-pulse solutions have the following formula for the Maslov index

$$\mathsf{Maslov}^{\text{homoclinic}} = \begin{cases} 3 & \text{if } \ell \text{ is even} \\ 4 & \text{if } \ell \text{ is odd} \end{cases} . \tag{15.1}$$

We can make several observations using this formula for the Maslov index of two-pulse homoclinic orbits. Suppose $r = +1$, then

$$N_{unst} = \mathsf{Maslov}^{\text{homoclinic}} - 2 - N^-_{imag} = \begin{cases} 1 - N^-_{imag} & \text{if } \ell \text{ is even} \\ 2 - N^-_{imag} & \text{if } \ell \text{ is odd} \end{cases} ,$$

We have the immediate observation that a necessary condition for a $2-$pulse homoclinic orbit to be stable is $N^-_{imag} > 0$. From the parity of N^-_{imag}, we have $N_{unst} = 1$ and $N^-_{imag} = 0$ when l is even and the solitary wave is unstable. When l is odd, the parity of N^-_{imag} is not sufficient to determine the stability of the solitary wave.

Buryak & Champneys [12] used a completely different method to study stability and they found that $2-$pulse solitary waves are stable if ℓ is odd and unstable if ℓ is even, assuming that $r = 1$. This is consistent with the value we found for the Maslov index. When l is odd, the stability of the solitary wave is equivalent to $N^-_{imag} = 2$.

The Maslov index does not give any information about the purely imaginary eigenvalues, and so to determine their number, a calculation of the spectral problem is necessary. Some results on this are reported by Chardard [15]. There it is found that when l is odd, there are eigenvalues with non-zero imaginary part, but they appear to have very small real parts. Further results are necessary to be certain about the spectral stability of $2-$pulse solitary waves when l is odd.

One way to check whether the real part of a complex eigenvalue is nonzero is to use the formula (for the case $q = 1$)

$$\text{Re}(\lambda) = -\frac{1}{\|u\|^2} \int_{-\infty}^{+\infty} \widehat{\phi}_x |u(x)|^2 \, \mathrm{d}x \, . \tag{15.2}$$

Here λ is the complex eigenvalue associated with the stability exponent and u is the associated eigenmode:

$$\frac{d}{dx} \left(u_{xxxx} - P u_{xx} + cu - 2\widehat{\phi}\, u \right) = \lambda u \, . \tag{15.3}$$

The formula (15.2) is derived by multiplying (15.3) by the complex conjugate of $u(\cdot)$ and integrating over $\mathbb{R}$.

In spite of the simplicity of the formula (15.2) there is not much that one can say in general. If $\widehat{\phi}(x)$ is an even function then $\widehat{\phi}_x$ is an odd function. Then it is immediate that

$|u(x)|^2$ even implies that $\mathrm{Re}(\lambda) = 0$. However, this is a highly special case.

16 A model PDE for long-wave short-wave resonance

In this section the Maslov index is computed for a class of solitary waves which arise in a model PDE for long-wave short-wave (LW-SW) resonance (cf. Kawahara et al. [29], Ma [35], Benilov & Burtsev [5], Latifi & Leon [32]). The LW-SW equations are a coupled system with one equation of nonlinear Schrödinger type and the other of KdV type. A typical form is

$$\begin{aligned} E_t &= \mathrm{i}(E_{xx} + \rho E - \nu E) \\ \rho_t &= \partial_x(\rho_{xx} - c\rho + 3\rho^2 + |E|^2)\,, \end{aligned} \tag{16.1}$$

were $\rho(x,t)$ is real valued, $E(x,t)$ is complex valued, and c, ν are considered to be positive real parameters. In real coordinates, $E = u + \mathrm{i}v$ and $\rho = w$, the above equations can be written:

$$\begin{aligned} u_t &= -v_{xx} - vw + \nu v \\ v_t &= u_{xx} + uw - \nu u \\ w_t &= w_{xxx} - cw_x + 6ww_x + 2uu_x + 2vv_x\,. \end{aligned} \tag{16.2}$$

This system can be expressed as a Hamiltonian system in the time direction. However, we will not emphasize this property since it is the spatial Hamiltonian structure that is associated with the Maslov index (see Appendix J for the temporal Hamiltonian structure). Solitary waves satisfy the steady equations

$$\begin{aligned} -2u_{xx} - 2uw + 2\nu u &= 0 \\ -2v_{xx} - 2vw + 2\nu v &= 0 \\ -w_{xx} + cw - 3w^2 - u^2 - v^2 &= \mathsf{constant}\,, \end{aligned} \tag{16.3}$$

where the signs and coefficients are modified to ensure that they are the Euler-Lagrange equation associated with the Hamiltonian function $H(Z)$ in Appendix J. Exact solutions of this problem are known [35]; for example,

$$u(x) = A\,\mathrm{sech}(\sqrt{\nu}\,x)\,,\quad v(x) = 0 \quad \text{and} \quad w(x) = 2\nu\,\mathrm{sech}^2(\sqrt{\nu}\,x)\,, \tag{16.4}$$

with $\mathsf{constant} = 0$ and $A^2 = 2\nu\,(c - 4\nu)$, and the existence condition $c - 4\nu > 0$.

To study the Maslov index of these solutions, linearize the steady equations about the basic solitary wave and introduce a spectral parameter: $\mathbf{L}Z = \lambda Z$ with $\mathbf{L} = D^2H(\widehat{Z})$.

Written out, this equation is

$$\begin{aligned} -2u_{xx} - 2\widehat{w}u - 2\widehat{u}w + 2\nu u &= \lambda u \\ -2v_{xx} - 2\widehat{w}v - 2\widehat{v}w + 2\nu v &= \lambda v \\ -w_{xx} + cw - 6\widehat{w}w - 2\widehat{u}u - 2\widehat{v}v &= \lambda w \end{aligned} \tag{16.5}$$

When $\widehat{v} = 0$ this system decouples into a second order equation for v, and a fourth order coupled system for u, w,

$$\begin{aligned} -2u_{xx} - 2\widehat{w}u - 2\widehat{u}w + 2\nu u &= \lambda u \\ -w_{xx} + cw - 6\widehat{w}w - 2\widehat{u}u &= \lambda w \end{aligned} \tag{16.6}$$

The decoupled equation for v is then

$$-2v_{xx} - 2\widehat{w}v + 2\nu v = \lambda v \tag{16.7}$$

This latter system can be analyzed completely and the result in given in Appendix K.

The fourth-order system for u, w (16.6) can be written as a standard Hamiltonian ODE in the form (1.2) with $n = 2$ by taking

$$\mathbf{u}(x,\lambda) = \begin{pmatrix} u \\ w \\ 2u_x \\ w_x \end{pmatrix}, \quad \mathbf{B}(x,\lambda) = \begin{pmatrix} \lambda - 2\nu + 2\hat{w}(x) & 2\hat{u}(x) & 0 & 0 \\ 2\hat{u} & \lambda - c + 6\hat{w}(x) & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The essential spectrum for this equation is

$$\sigma_{ess} = \{\ \lambda \in \mathbb{R}\ :\ \lambda \geq 2\nu \text{ and } \quad \lambda \geq c\}\,.$$

Adding the condition that $c > 4\nu$, the system at infinity is hyperbolic for all $\lambda \in \mathbb{R}$ such that $\lambda < 2\nu$.

The Maslov index is computed for the case $c = 1$ and $\nu = 0.2$ and the results are shown, along with the Evans function, in Figure 11 and tabulated in the table below, where $\lambda_1 < \lambda_2 = 0 < \lambda_3$ are the three roots of the Evans function.

λ	$\lambda < \lambda_1$	$\lambda_1 < \lambda < \lambda_2$	$\lambda_2 < \lambda < \lambda_3$	$\lambda > \lambda_3$
Maslov$(\mathbf{U}^+, \mathrm{E}^s_\infty)$	0	-1	-2	-3

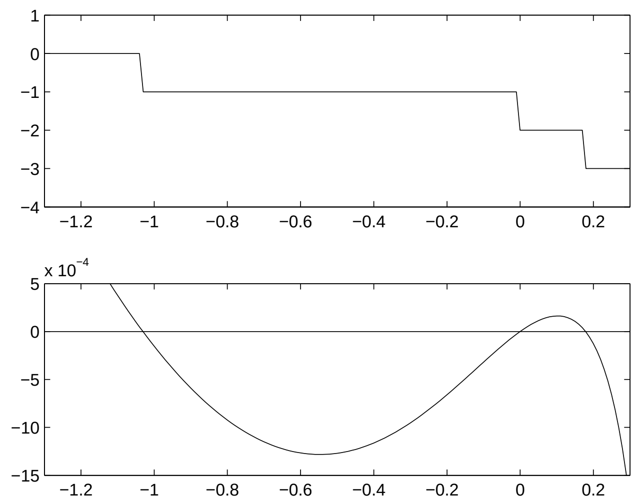

Fig. 11. Longwave-Shortwave problem for the following parameters $c = 1$, $\nu = 0.2$. Top, the Maslov index. Bottom: the Evans function.

17 A non-monotone Maslov index.

In the LW-SW system and the KdV5 equation, the Maslov index is a monotone function of λ. (Note however that it is not a monotone function of x.) Here we show an example where the Maslov index is not a monotone function of λ. It is a slight modification of the LW-SW resonance equations. In this case the correlation between the number of roots of the Evans function and the value of the Maslov index is no longer apparent. Look at the eigenvalue problem

$$\mathbf{L}\begin{pmatrix} u \\ w \end{pmatrix} = \lambda \begin{pmatrix} u \\ w \end{pmatrix}, \quad \text{with} \quad \mathbf{L}\begin{pmatrix} u \\ w \end{pmatrix} := \begin{pmatrix} -2u_{xx} - 2\widehat{w}(x)u + 2\widehat{u}(x)w + 2\nu u \\ w_{xx} - cw + 6\widehat{w}(x)w + 2\widehat{u}(x)u \end{pmatrix}, \tag{17.1}$$

with $c > 4\nu > 0$,

$$\widehat{u}(x) = A\mathrm{sech}(\sqrt{\nu}\,x) \quad \text{and} \quad \widehat{w}(x) = 2\nu\,\mathrm{sech}^2(\sqrt{\nu}\,x)$$

with $A^2 = 2\nu(c - 4\nu)$, and the requirement $c > 4\nu > 0$.

The spectral problem associated to this operator can be expressed in the form (1.2) with

$n = 2$,

$$\mathbf{u}(x) = \begin{pmatrix} u \\ w \\ 2u_x \\ w_x \end{pmatrix}, \quad \text{and} \quad \mathbf{B}(x,\lambda) = \begin{pmatrix} \lambda - 2\nu + 2\hat{w}(x) & 2\hat{u}(x) & 0 & 0 \\ 2\hat{u}(x) & -\lambda - c + 6\hat{w}(x) & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The essential spectrum of $\mathbf{L}$ consists of

$$\{\, \lambda \ : \ -\infty < \lambda < -c \ \cup \ 2\nu < \lambda < +\infty \,\}.$$

The essential spectrum is unbounded from above and below, hence a *Morse index* can not be defined for $\mathbf{L}$. However, we will still be able to compute a Maslov index. The key property that leads to non-monotonicity is that the matrix $\partial_\lambda \mathbf{B}(x,\lambda)$ is not semi-definite: the matrix $\partial_\lambda \mathbf{B}(x,\lambda)$ has eigenvalues $\{0, 0, -1, +1\}$ and so is not semi-definite.

Results for the case $c = 1$ and $\nu = 0.21$ are tabulated below and shown in Figure 12. In this case there are 5 eigenvalues, but there is no longer a correlation between the Maslov index and the number of eigenvalues in a subset of λ.

λ	$-c$				0				2ν
$D(\lambda)$	$+\infty$	$+$	$-$	$+$	0	$-$	$+$	$-$	$-\infty$
Maslov(λ)		-4	-3	-2		-1	-2	-3	

18 Concluding remarks

We have only just scratched the surface of the implications of the Maslov index for homoclinic orbits and solitary waves. Other important questions for Hamiltonian systems on four-dimensional phase space are (a) the connection between transversality of the homoclinic orbit and the parity of the Maslov index, (b) the jump of the Maslov index at bifurcations, (c) whether the angles κ_1 and κ_2 in the decomposition $\kappa = \kappa_1 + \kappa_2$ contain other useful information, (d) the role of purely imaginary eigenvalues in the stability of solitary waves in KdV5, and (e) the Maslov index of multi-pulse homoclinic orbits. The latter question is addressed in the paper [17].

The extension of the Maslov index of homoclinic orbits to phase space with dimension greater than four is straightforward in principle but there are some differences in detail. The dimension of the basic manfolds jumps considerably when n goes from 2 to 3.

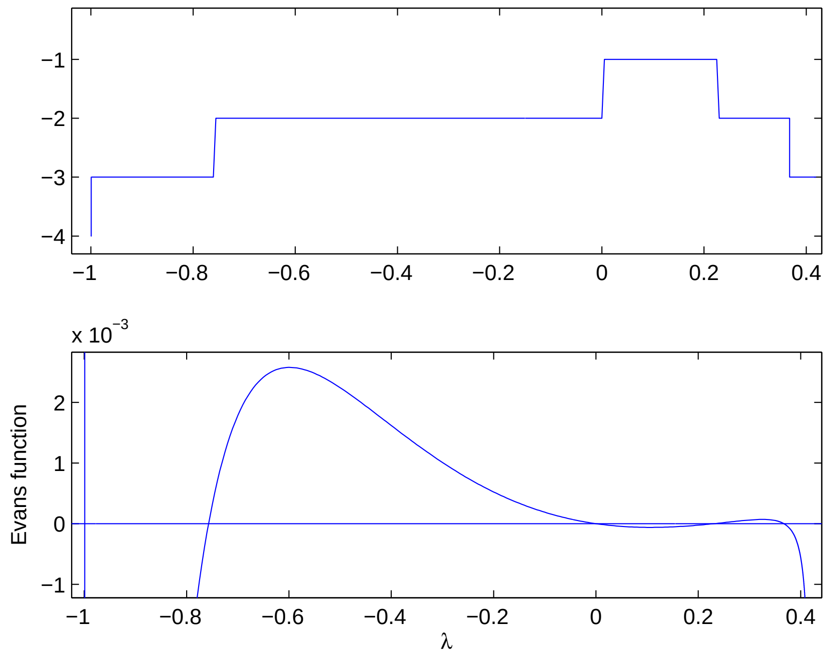

Fig. 12. Plot of the Maslov index for the non-monotone example (17.1) for the case $c = 1$ and $\nu = 0.21$. Upper figure shows the Maslov index and the lower figure the Evans function.

Manifold	$\bigwedge^3(\mathbb{R}^6)$	$\mathbb{RP}^{19}$	$G_3(\mathbb{R}^6)$	$\Lambda(3)$	$\Lambda^1(3)$
Dimension	20	19	9	6	5

In this table, $\mathbb{RP}^{19}$ is the projectification of $\bigwedge^3(\mathbb{R}^6)$. The biggest change in the numerics is the difficulty due to the jump in dimension of the Lagrangian Grassmanian. Whereas it is 3−dimensional in the case $n = 2$, it jumps to double that dimension when $n = 3$. The details of the theory and numerics for $n \geq 3$ are given in Part 2 [18].

— Appendix —

A Kernel of $\mathbf{A}^{(2)}$ on $\bigwedge^2(\mathbb{R}^4)$

Let $\mathbf{A}$ be an arbitrary 4×4 matrix with entries a_{ij}. Then, with respect to the standard basis (7.1) on $\bigwedge^2(\mathbb{R}^4)$ the induced matrix is

$$\mathbf{A}^{(2)} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}, \tag{A-1}$$

A constructive proof is given in §2 of [2].

Proposition 10 *The induced matrix* $\mathbf{A}^{(2)}$ *satisfies* $\mathbf{A}^{(2)}\boldsymbol{\omega} = 0$, *where* $\boldsymbol{\omega}$ *is defined in (6.1), if and only if* $\mathbf{JA}$ *is symmetric.*

Proof. An explicit calculation gives

$$\mathbf{A}^{(2)}\boldsymbol{\omega} = \begin{pmatrix} a_{23}-a_{14} \\ a_{11}+a_{33} \\ a_{43}+a_{12} \\ a_{21}+a_{34} \\ a_{22}+a_{44} \\ -a_{41}+a_{32} \end{pmatrix}.$$

On the other hand

$$\mathbf{JA} = \begin{bmatrix} -a_{31} & -a_{32} & -a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -a_{43} & -a_{44} \\ a_{11} & a_{12} & a_{31} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}.$$

In order for $\mathbf{JA}$ to be symmetric we require

$$a_{41} = a_{32}\,, \quad a_{11} = -a_{33}\,, \quad a_{12} = -a_{43}\,, \quad a_{21} = -a_{34}\,, \quad a_{22} = -a_{44}\,, \quad a_{23} = a_{14}\,.$$

These conditions are satisfied if and only if $\mathbf{A}^{(2)}\boldsymbol{\omega} = 0$. ∎

B The connection between $\mathbf{A}^{(2)}$, $\mathbf{J}^{(2)}$ and $\mathbf{B}^{(2)}$

In this appendix the role of the induced symplectic operator $\mathbf{J}^{(2)}$ on $\bigwedge^2(\mathbb{R}^4)$ is explored. Using the standard formula for the induced matrix (A-1), the induced form of the symplectic operator is

$$\mathbf{J}^{(2)} = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

Note that $\mathbf{J}^{(2)}\boldsymbol{\omega} = 0$ and $\boldsymbol{\omega}^T\mathbf{J}^{(2)} = 0$ using $\boldsymbol{\omega}$ defined in (6.1). In fact, the kernel of $\mathbf{J}^{(2)}$ is four dimensional $\mathrm{Kernel}(\mathbf{J}^{(2)}) = \mathrm{span}\{\mathbf{E}_2, \mathbf{E}_5, \mathbf{E}_1 + \mathbf{E}_6, \mathbf{E}_3 + \mathbf{E}_4\}$, where $\mathbf{E}_1, \ldots, \mathbf{E}_6$ is the standard basis for $\bigwedge^2(\mathbb{R}^4)$.

Now suppose $\mathbf{B}$ is a symmetric matrix and

$$\mathbf{A} = \mathbf{J}^{-1}\mathbf{B} = -\mathbf{J}\mathbf{B}\,.$$

then

$$\mathbf{A}^{(2)} = \begin{bmatrix} b_{13} + b_{24} & b_{34} & b_{44} & -b_{33} & -b_{34} & 0 \\ -b_{12} & 0 & -b_{14} & b_{23} & 0 & -b_{34} \\ -b_{22} & -b_{23} & b_{13} - b_{24} & 0 & b_{23} & b_{33} \\ b_{11} & b_{14} & 0 & b_{24} - b_{13} & -b_{14} & -b_{44} \\ b_{12} & 0 & b_{14} & -b_{23} & 0 & b_{34} \\ 0 & b_{12} & -b_{11} & b_{22} & -b_{12} & -b_{13} - b_{24} \end{bmatrix}.$$

The induced matrix $\mathbf{A}^{(2)}$ does not equal the product of the induced matrices for $\mathbf{J}$ and $\mathbf{B}$ but it has the following form

$$\mathbf{A}^{(2)} = -\mathbf{J}^{(2)}\mathbf{B}^{(2)} + \mathbf{S}\,,$$

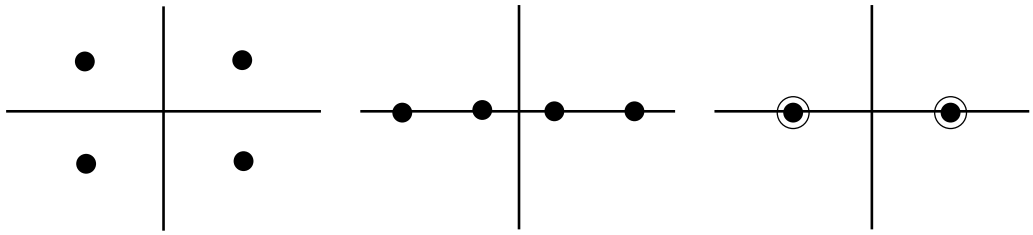

Fig. C.1. The three cases of hyperbolic spectra for constant coefficient Hamiltonian systems on $\mathbb{R}^4$.

where $\mathbf{S}$ is the skew-symmetric matrix

$$\mathbf{S} = \begin{bmatrix} 0 & b_{12} & -b_{11} & b_{22} & -b_{12} & -b_{24}-b_{13} \\ -b_{12} & 0 & -b_{14} & b_{23} & 0 & -b_{34} \\ b_{11} & b_{14} & 0 & b_{24}-b_{13} & -b_{14} & -b_{44} \\ -b_{22} & -b_{23} & b_{13}-b_{24} & 0 & b_{23} & b_{33} \\ b_{12} & 0 & b_{14} & -b_{23} & 0 & b_{34} \\ b_{13}+b_{24} & b_{34} & b_{44} & -b_{33} & -b_{34} & 0 \end{bmatrix} .$$

The skew-symmetric matrix $\mathbf{S}$ has the properties $\mathbf{S}\boldsymbol{\omega} = 0$, $\boldsymbol{\omega}^T\mathbf{S} = 0$, and $\mathbf{B}^{(2)}$ has the properties

$$\mathbf{B}^{(2)}\boldsymbol{\omega} \neq 0 \quad \text{but} \quad \mathbf{B}^{(2)}\boldsymbol{\omega} \quad \text{is in the kernel of} \quad \mathbf{J}^{(2)} .$$

Hence the property $\mathbf{A}^{(2)}\boldsymbol{\omega} = 0$ is recovered. Moreover, since $\boldsymbol{\omega}^T\mathbf{J}^{(2)} = 0$, we also have $\boldsymbol{\omega}^T\mathbf{A}^{(2)} = 0$.

C Hyperbolic subspaces and Lagrangian planes

For a linear constant-coefficient Hamiltonian system on $\mathbb{R}^4$, $\mathbf{u}_x = \mathbf{A}\mathbf{u}$, there are three cases where the spectrum of $\mathbf{A}$ is strictly hyperbolic and they are shown in Figure C. The unstable subspace in each case is a Lagrangian plane. This observation is a special case of a result of Montaldi [37]. Here we sketch a proof based on eigenvectors.

Take the first case. The unstable eigenvalues are of the form $\mu = \nu \pm \mathrm{i}\tau$ with $\nu > 0$ and $\tau > 0$, with eigenvectors $\xi_1 \pm \mathrm{i}\xi_2$ where

$$\mathbf{B}_\infty(\xi_1 + \mathrm{i}\xi_2) = (\nu + \mathrm{i}\tau)\mathbf{J}(\xi_1 + \mathrm{i}\xi_2) ,$$

or

$$\mathbf{B}_\infty\xi_1 = \nu\mathbf{J}\xi_1 - \tau\mathbf{J}\xi_2 , \quad \mathbf{B}_\infty\xi_2 = \nu\mathbf{J}\xi_2 + \tau\mathbf{J}\xi_1 .$$

Taking the inner product of the first equation with ξ_2 and the second with ξ_1 and subtracting,

$$0 = \xi_2^T \mathbf{B}_\infty \xi_1 - \xi_1^T \mathbf{B}_\infty \xi_2 = \nu \xi_2^T \mathbf{J} \xi_1 - \nu \xi_1^T \mathbf{J} \xi_2 = 2\nu \xi_2^T \mathbf{J} \xi_1 \,.$$

Now since $\nu > 0$ it follows that $\xi_2^T \mathbf{J} \xi_1 = 0$ and so $\text{span}\{\xi_1, \xi_2\}$ is a Lagrangian subspace.

A similar calculation verifies the other two cases, with the third requiring the introduction of generalized eigenspaces.

Now, use the fact that Lagrangian subspaces are invariant for the $x-$dependent system, as shown in §4, to conclude that hyperbolic sets are Lagrangian manifolds.

D The induced inner product on $\bigwedge^k(\mathbb{R}^{2n})$

In this appendix the equivalence between the induced inner product $[\![\cdot,\cdot]\!]_k$ on $\bigwedge^k(\mathbb{R}^{2n})$, which has dimension d, and the standard inner product on $\mathbb{R}^d$ is established. Here the general case of $2n-$dimensional phase space is considered, which will be required in Part 2 [18].

With the standard orthonormal basis for $\mathbb{R}^{2n}$, $\{\mathbf{e}_1, \ldots, \mathbf{e}_{2n}\}$, the nonzero and distinct members of the set

$$\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \;:\; i_1, \ldots, i_k = 1, \ldots, 2n\,\} \tag{D-1}$$

form a basis for the vector space $\bigwedge^k(\mathbb{R}^{2n})$, with exactly $d = \frac{n!}{(n-k)!k!}$ distinct elements.

Choose an ordering such as a standard lexical ordering and label the nonzero distinct elements in the set (D-1) by $\mathbf{E}_1, \ldots, \mathbf{E}_d$. Then, any element $\mathbf{U} \in \bigwedge^k(\mathbb{R}^{2n})$ can be represented as $\mathbf{U} = \sum_{j=1}^d U_j \, \mathbf{E}_j$. The inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$ induces an inner product on each vector space $\bigwedge^k(\mathbb{R}^{2n})$ as follows. Let

$$\mathbf{U} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \quad \text{and} \quad \mathbf{V} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k \,, \quad \mathbf{u}_i, \mathbf{v}_j \in \mathbb{R}^{2n} \,, \quad \forall \, i, j = 1, \ldots, k \,,$$

be any decomposable k-forms. A $k-$form is decomposable if it can be written as a pure form: a wedge product between k linearly independent vectors in $\mathbb{R}^{2n}$. The inner product of $\mathbf{U}$ and $\mathbf{V}$ is defined by

$$[\![\mathbf{U}, \mathbf{V}]\!]_k := \det \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_1, \mathbf{v}_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_k, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_k, \mathbf{v}_k \rangle \end{bmatrix} \,, \quad \mathbf{U}, \mathbf{V} \in \textstyle\bigwedge^k(\mathbb{R}^{2n}) \,. \tag{D-2}$$

Since every element in $\bigwedge^k(\mathbb{R}^{2n})$ is a sum of decomposable elements, this definition extends by (bi)-linearity to any k-form. Using the orthonormality of the induced basis

$$[\![\mathbf{E}_i, \mathbf{E}_j]\!]_k = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \,,$$

the inner product between two elements $\mathbf{U} = \sum_{i=1}^{d} U_i \mathbf{E}_i$ and $\mathbf{V} = \sum_{j=1}^{d} V_j \mathbf{E}_j$ is

$$
\begin{aligned}
[\![\mathbf{U}, \mathbf{V}]\!]_k &= \left[\!\!\left[\textstyle\sum_{i=1}^{d} U_i \mathbf{E}_i, \sum_{k=1}^{d} V_j \mathbf{E}_j \right]\!\!\right]_k = \textstyle\sum_{i=1}^{d} \sum_{j=1}^{d} U_i V_j [\![\mathbf{E}_i, \mathbf{E}_j]\!]_k \\
&= \textstyle\sum_{i=1}^{d} U_i V_i := \langle \mathbf{U}, \mathbf{V} \rangle_d \,,
\end{aligned}
$$

yielding the equivalent representation

$$
[\![\mathbf{U}, \mathbf{V}]\!]_k = \langle \mathbf{U}, \mathbf{V} \rangle_d \,, \quad \mathbf{U}, \mathbf{V} \in \textstyle\bigwedge^k(\mathbb{R}^{2n}) \,. \tag{D-3}
$$

E Plus and minus subspaces in $\bigwedge^2(\mathbb{R}^4)$

Consider $\mathbb{R}^4$ with the standard symplectic and volume forms. Let $V = \text{span}\{\xi_1, \xi_2\}$ be a two-dimensional *oriented* subspace of $\mathbb{R}^4$.

An oriented subspace is defined as follows. A subspace is an equivalence class of bases; that is, $\text{span}\{\xi_1, \xi_2\}$ and $\text{span}\{\eta_1, \eta_2\}$ represent the same subspace if and only if there is an invertible 2×2 matrix $\mathbf{m}$ such that $[\xi_1|\xi_2] = [\eta_1|\eta_2]\mathbf{m}$. A subspace is oriented if $\mathbf{m}$ is restricted to have positive determinant. The oriented subspace $\text{span}\{\xi_1, \xi_2\}$ has one of three types

$$
\begin{array}{rl}
\textsf{plus subspace} & \textsf{if } \omega \wedge \xi_1 \wedge \xi_2 > 0 \\[1em]
\textsf{Lagrangian subspace} & \textsf{if } \omega \wedge \xi_1 \wedge \xi_2 = 0 \\[1em]
\textsf{minus subspace} & \textsf{if } \omega \wedge \xi_1 \wedge \xi_2 < 0, .
\end{array}
$$

Proposition 11 *The sign of $\omega \wedge \mathbf{U}$ is an invariant of (3.3).*

Use Proposition 2 to conclude that

$$
\boldsymbol{\omega} \wedge \mathbf{U}(x) = \textsf{constant} \,,
$$

along solutions of (3.3).

There is an interesting connection between Krein signature and the above classification of oriented subspaces. Krein signature is a sign which is associated with purely imaginary eigenvalues (in the linearization about an equilibrium) or Floquet multipliers (in the linearization about a periodic orbit). Consider the case of a simple purely imaginary eigenvalue $\mathrm{i}\nu$, $\nu > 0$. Its complex eigenvector $\zeta = \xi_1 + \mathrm{i}\xi_2$ satisfies

$$
\mathbf{B}\zeta = \mathrm{i}\nu \mathbf{J}\zeta \,.
$$

The Krein signature is defined as the sign of

$$
\mathrm{i}\langle \mathbf{J}\overline{\zeta}, \zeta \rangle = 2\langle \mathbf{J}\xi_2, \xi_1 \rangle \,.
$$

Now use the identity $\mathbf{J}\xi_2 = \xi_2 \lrcorner\, \boldsymbol{\omega}$ to obtain

$$\mathrm{i}\langle \mathbf{J}\overline{\zeta}, \zeta\rangle \mathsf{vol} = 2\langle \xi_2 \lrcorner\, \boldsymbol{\omega}, \xi_1\rangle \mathsf{vol} = [\![\boldsymbol{\omega}, \xi_2 \wedge \xi_1]\!]_2 \mathsf{vol} = -[\![\boldsymbol{\omega}, \xi_1 \wedge \xi_2]\!]_2 \mathsf{vol} = \boldsymbol{\omega} \wedge \xi_1 \wedge \xi_2 \,.$$

This observation also emphasizes the fact that a choice of orientation underlies the definition of Krein signature.

F A spectral problem with sech^2 potential

This appendix establishes the basic properties of the ODE eigenvalue problem

$$\phi_{xx} + 12\,\mathrm{sech}^2 x\,\phi = \kappa\,\phi\,, \tag{F-1}$$

in the set $\mathscr{K} := \{\kappa \in \mathbb{R} \mid \kappa > 0\}$. The solutions of this ODE can be determined explicitly. The eigenvalues are $\kappa = 1, 4, 9$. For all $\kappa \in \mathscr{K} \setminus \{1, 4, 9\}$, the two functions

$$\phi^{\pm}(x,\kappa) = \mathrm{e}^{\pm\sqrt{\kappa}\,x}\left(\pm a_0 + a_1 \tanh(x) \pm a_2 \tanh^2(x) + \tanh^3(x)\right)\,, \tag{F-2}$$

are linearly independent, where

$$a_0 = \frac{\sqrt{\kappa}}{15}(4-\kappa)\,, \quad a_1 = \frac{1}{5}(2\kappa - 3)\,, \quad a_2 = -\sqrt{\kappa}\,.$$

The eigenvalues can be verified by explicit calculation. That $\kappa = 1, 4, 9$ are the only eigenvalues in $\mathscr{K}$, and that $\phi^{\pm}$ are linearly independent is verified by computing the Wronskian

$$W(x,\kappa) = \det\begin{bmatrix} \phi^+ & \phi^- \\ \phi^+_x & \phi^-_x \end{bmatrix}\,.$$

It is easily verified that $W_x = 0$ and so $W(x,\kappa)$ is independent of x. Evaluate at $x = 0$

$$W(0,\kappa) = \det\begin{bmatrix} a_0 & -a_0 \\ a_1 + a_0\sqrt{\kappa} & a_1 + a_0\sqrt{\kappa} \end{bmatrix} = 2a_0(a_1 + a_0\sqrt{\kappa})\,.$$

Substituting for a_0 and a_1,

$$W(0,\kappa) = \frac{2}{225}\sqrt{\kappa}(\kappa-1)(\kappa-4)(\kappa-9)\,.$$

Hence $\phi^{\pm}$ are linearly independent for all $\kappa \in \mathscr{K} \setminus \{1, 4, 9\}$.

The eigenfunctions are

$$\phi(x,1) = \text{sech}(x)(4 - 5\,\text{sech}^2(x))\,, \quad \text{when} \quad \kappa = 1$$

$$\phi(x,4) = \tanh(x)\,\text{sech}^2(x)\,, \quad \text{when} \quad \kappa = 4$$

$$\phi(x,9) = \text{sech}^3(x)\,, \quad \text{when} \quad \kappa = 9\,.$$

modulo an arbitrary multiplicative constant.

G Attractivity of the Lagrangian Grassmannian $\Lambda(2)$

One of the advantages of subtracting off the growth rate at infinity in the equations on $\bigwedge^2(\mathbb{R}^4)$, as in (12.1), is that the Lagrangian Grassmannian becomes an attracting invariant manifold. When $\Lambda(2)$ is attractive, one has greater freedom in choosing the numerical integration scheme.

To prove attractivity, consider the integration of the 2−form representing the unstable subspace $\mathbf{U}^+(x,\lambda)$

$$\frac{d}{dx}\mathbf{U}^+ = \mathbf{A}^{(2)}(x,\lambda)\mathbf{U}^+ \quad \mathbf{U}^+ \in \textstyle\bigwedge^2(\mathbb{R}^4) \quad -L < x < +L\,.$$

Introduce the transformation

$$\mathbf{U}^+(x,\lambda) = \mathrm{e}^{\sigma_+(\lambda)x}\,\widehat{\mathbf{U}}^+(x,\lambda)$$

where $\sigma_+(\lambda)$ is the sum of the eigenvalues of $\mathbf{A}_\infty(\lambda)$ with positive real part. Then $\widehat{\mathbf{U}}^+$ satisfies

$$\frac{d}{dx}\widehat{\mathbf{U}}^+ = [\mathbf{A}^{(2)}(x,\lambda) - \sigma_+(\lambda)\mathbf{I}]\widehat{\mathbf{U}}^+ \qquad -L < x < +L \tag{G-1}$$

The Lagrangian Grassmannian is the set

$$\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+ = 0 \quad \text{and} \quad \boldsymbol{\omega} \wedge \widehat{\mathbf{U}}^+ = 0\,.$$

When evaluated on the differential equation (G-1) these invariants satisfy

$$\begin{aligned}
\tfrac{d}{dx}\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+ &= \tfrac{d}{dx}\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+ + \widehat{\mathbf{U}}^+ \wedge \tfrac{d}{dx}\widehat{\mathbf{U}}^+ \\
&= \mathbf{A}^{(2)}\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+ + \widehat{\mathbf{U}}^+ \wedge \mathbf{A}^{(2)}\widehat{\mathbf{U}}^+ - 2\sigma_+\,\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+ \\
&= \text{Trace}(\mathbf{A})\,\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+ - 2\sigma_+\,\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+ \\
&= -2\sigma_+\,\widehat{\mathbf{U}}^+ \wedge \widehat{\mathbf{U}}^+\,,
\end{aligned}$$

since Trace($\mathbf{A}$) = 0. A similar calculation with $\boldsymbol{\omega}\wedge\widehat{\mathbf{U}}^+$ yields

$$\frac{d}{dx}\boldsymbol{\omega}\wedge\widehat{\mathbf{U}}^+ = \boldsymbol{\omega}\wedge(\mathbf{A}^{(2)}-\sigma_+\mathbf{I})\mathbf{U}^+ = -\mathbf{A}^{(2)}\boldsymbol{\omega}\wedge\mathbf{U}^+ - \sigma_+\,\boldsymbol{\omega}\wedge\widehat{\mathbf{U}}^+ = -\sigma_+\,\boldsymbol{\omega}\wedge\widehat{\mathbf{U}}^+\,,$$

using the fact that $\mathbf{A}^{(2)}\boldsymbol{\omega}=0$, which is proved in Appendix A., This proves that

$$\widehat{\mathbf{U}}^+\wedge\widehat{\mathbf{U}}^+(x) = \mathrm{e}^{-2\sigma_+x}\widehat{\mathbf{U}}^+\wedge\widehat{\mathbf{U}}^+\Big|_{x=-L} \quad\text{and}\quad \boldsymbol{\omega}\wedge\widehat{\mathbf{U}}^+(x) = \mathrm{e}^{-\sigma_+x}\boldsymbol{\omega}\wedge\widehat{\mathbf{U}}^+\Big|_{x=-L}\,,\quad\text{for}\quad x>-L\,.$$

The eigenvalue σ_+ is real and positive. Hence when integrating the unstable subspace $\mathbf{U}^+$ along the Lagrangian Grassmannian, both $\widehat{\mathbf{U}}^+\wedge\widehat{\mathbf{U}}^+$ and $\boldsymbol{\omega}\wedge\widehat{\mathbf{U}}^+(x)$ are exponentially attracted to the zero set. Therefore a special integrator is not required for maintaining the constraints; a standard Runge-Kutta algorithm is quite satisfactory.

H The existence of at least one negative eigenvalue

Consider the linear operator

$$\mathscr{L}\phi := \phi_{xxxx} - P\phi_{xx} + a(x)\phi\,, \tag{H-1}$$

introduced in (14.1) with $a(x) = c-(q+1)\widehat{\phi}(x)^q$ and $\widehat{\phi}(x)$ satisfying (14.5). Assume

$$P+2c\ge 0 \quad\text{and}\quad 0<c\le 1 \quad\text{or}\quad P>0 \quad\text{and}\quad c>0\,. \tag{H-2}$$

The essential spectrum for this problem is non-negative. Here it is proved that $\mathscr{L}$ has at least one negative eigenvalue in the point spectrum.

Multiply (14.5) by the basic state $\widehat{\phi}(x)$,

$$\begin{aligned}\widehat{\phi}^{q+2} &= c\widehat{\phi}^2 - P\,\widehat{\phi}\widehat{\phi}_{xx} + \widehat{\phi}\widehat{\phi}_{xxxx}\\ &= c\widehat{\phi}^2 - (P+2c)\,\widehat{\phi}\widehat{\phi}_{xx} + 2c\widehat{\phi}\widehat{\phi}_{xx} + \widehat{\phi}\widehat{\phi}_{xxxx}\\ &= c(\widehat{\phi}+\widehat{\phi}_{xx})^2 - (P+2c)\,\widehat{\phi}\widehat{\phi}_{xx} - c\widehat{\phi}_{xx}^2 + \widehat{\phi}\widehat{\phi}_{xxxx}\,.\end{aligned}$$

Hence integrating, using the fact that $\widehat{\phi}$ and its derivatives decay exponentially as $x\to\pm\infty$, and the hypotheses (H-2)

$$\int_{-\infty}^{\infty}\widehat{\phi}^{q+2}\mathrm{d}x = \int_{-\infty}^{\infty}c(\widehat{\phi}+\widehat{\phi}_{xx})^2\mathrm{d}x + (P+2c)\int_{-\infty}^{\infty}\widehat{\phi}_x^2\mathrm{d}x + (1-c)\int_{-\infty}^{+\infty}\widehat{\phi}_{xx}^2\,\mathrm{d}x > 0\,. \tag{H-3}$$

or if $P>0$ and $c>0$,

$$\int_{-\infty}^{\infty}\widehat{\phi}^{q+2}\mathrm{d}x = \int_{-\infty}^{\infty}(c\widehat{\phi}^2 + P\widehat{\phi}_x^2 + \widehat{\phi}_{xx}^2)\,\mathrm{d}x > 0\,. \tag{H-4}$$

To prove that (H-1) has a negative eigenvalue, we will show that the quadratic form $\langle u, \mathscr{L} u\rangle$ is negative when $u = \widehat{\phi}$ where $\langle u, \mathscr{L} u\rangle := \int_{-\infty}^{\infty} u\,\mathscr{L} u\,\mathrm{d}x$. Now

$$\begin{aligned}
\langle u, \mathscr{L} u\rangle\Big|_{u=\widehat{\phi}} &= \textstyle\int_{-\infty}^{\infty}(\widehat{\phi}(\widehat{\phi}_{xxxx} - P\widehat{\phi}_{xx} + a(x)\widehat{\phi}))\,\mathrm{d}x \\
&= \textstyle\int_{-\infty}^{\infty}(\widehat{\phi}_{xx}^2 + P\,\widehat{\phi}_x^2 + c\widehat{\phi}^2)\,\mathrm{d}x - (q+1)\int_{-\infty}^{\infty}\widehat{\phi}^{q+2}\mathrm{d}x \\
&= -q\textstyle\int_{-\infty}^{\infty}\widehat{\phi}^{q+2}\mathrm{d}x\,,
\end{aligned}$$

using (H-4) in the last line. It follows from (H-3) or (H-4) that $\langle\widehat{\phi}, \mathscr{L}\widehat{\phi}\rangle < 0$.

I Check of hypothesis 8 for the KdV5 system

In this appendix, the details are given of the proof that $\mathbf{A}(x,\lambda) = \mathbf{J}^{-1}\mathbf{B}(x,\lambda)$ for KdV5, with $\mathbf{B}(x,\lambda)$ defined in (14.2), satisfies Hypothesis 8.

First set

$$s = \frac{1}{(1-\lambda)^{\frac{1}{4}}}\,.$$

When λ is large and negative, s is a small parameter. This parameter will be used to obtain series expansions of the eigenvalues and eigenvectors.

The characteristic polynomial of $\mathbf{A}_\infty(\lambda)$ is

$$0 = \det[X\mathbf{I} - \mathbf{A}_\infty(\lambda)] = X^4 - PX^2 + \frac{1}{s^4}\,.$$

This polynomial is a biquadratic and for s small it has four complex roots, one in each quadrant. Let $\theta(s)$ be the eigenvalue in the right-upper quadrant. Its Taylor expansion is:

$$\theta(s) = \frac{1}{s\sqrt{2}}\left(1 - \frac{1}{4}\,\mathrm{i}Ps^2 - \frac{1}{32}P^2s^4 - \frac{1}{128}\,\mathrm{i}P^3s^6 + O\left(s^8\right)\right)\,.$$

The other eigenvalues are $\overline{\theta(s)}$, $-\theta(s)$,$-\overline{\theta(s)}$. The eigenvector associated with $\theta(s)$ is :

$$\mathbf{v}(s) = \frac{1}{s}\begin{pmatrix} -\frac{1}{s^4\theta} \\ \theta \\ 1 \\ \theta^2 \end{pmatrix}\,.$$

A Taylor expansion of this eigenvector is:

$$\mathbf{v}(s) = s\mathbf{v}_1 + \mathrm{i}s\mathbf{v}_2 + O\left(s^6\right)\,.$$

with

$$\mathbf{v}_1(s) = \begin{pmatrix} -\left(1/2\sqrt{2} + \frac{1}{8}\sqrt{2}Ps^2 - \frac{1}{64}\sqrt{2}P^2s^4 + \frac{1}{256}\sqrt{2}P^3s^6\right)s^{-3} \\ \frac{1}{256}\sqrt{2}P^3s^5 - \frac{1}{64}\sqrt{2}P^2s^3 + \frac{1}{8}\sqrt{2}Ps + \frac{\sqrt{2}}{2s} \\ 1 \\ \frac{1}{2}P \end{pmatrix}$$

and

$$\mathbf{v}_2(s) = \begin{pmatrix} -\left(-1/2\sqrt{2} + \frac{1}{8}\sqrt{2}Ps^2 + \frac{1}{64}\sqrt{2}P^2s^4 + \frac{1}{256}\sqrt{2}P^3s^6\right)s^{-3} \\ -\frac{1}{256}\sqrt{2}P^3s^5 - \frac{1}{64}\sqrt{2}P^2s^3 - \frac{1}{8}\sqrt{2}Ps + \frac{\sqrt{2}}{2s} \\ 0 \\ s^{-2} - \frac{1}{8}s^2P^2 \end{pmatrix}.$$

$(\Re v(s), \Im v(s))$ is a basis of the unstable space. Let V_{unst} the matrix whose columns are $\Re v(s)$ and $\Im v(s)$).

The eigenvector associated to $-\theta(s)$ is:

$$\mathbf{w}(s) = s\begin{pmatrix} \frac{1}{s^4\theta} \\ -\theta \\ 1 \\ \theta^2 \end{pmatrix}$$

$(\Re\mathbf{w}(s), \Im\mathbf{w}(s))$ is a basis of the unstable space. Let U_{st} the matrix whose columns are $\Re\mathbf{w}(s)$ and $\Im\mathbf{w}(s)$).

The matrix $\left(V_{unst}|U_{st}\right)$ is not a symplectic matrix but $V(s) = \left(V_{unst}|V_{st}\right)$, with $V_{st} = -U_{st}(V_{unst}^T J U_{st})^{-1}$, is. Besides, we have:

$$V_{st} = \begin{pmatrix} \frac{1}{2s} & \frac{-\frac{1}{4}s^2P - \frac{1}{32}P^3s^6}{s} \\ 0 & \frac{\frac{1}{2}s^2 + \frac{1}{16}s^6P^2}{s} \\ \frac{\frac{1}{4}\sqrt{2}s^3 - \frac{1}{16}\sqrt{2}Ps^5}{s} & \frac{-\frac{1}{4}\sqrt{2}s^3 - \frac{1}{16}\sqrt{2}Ps^5}{s} \\ \frac{-\frac{1}{4}\sqrt{2}s + \frac{1}{16}\sqrt{2}Ps^3 - \frac{3}{128}\sqrt{2}P^2s^5}{s} & \frac{-\frac{1}{4}\sqrt{2}s - \frac{1}{16}\sqrt{2}Ps^3 - \frac{3}{128}\sqrt{2}P^2s^5}{s} \end{pmatrix} + O(s^6)$$

We also have $V^{-1} = -J({}^TV)J$ since V is a symplectic matrix.

Let

$$\mathbf{B} = \begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix}.$$

We are now able to evaluate: $V^{-1}BV$: $V^{-1}BV = O\left(s^2\right)$ but also

$$V^{-1}BV \begin{pmatrix} 1\,0 \\ 0\,1 \\ 0\,0 \\ 0\,0 \end{pmatrix} = VBV_{unst} = O\left(s^2\right).$$

Therefore, $(V^{-1}BV)^{(2)}e_1 = O\left(s^2\right)$. Hence, as $R(x,\lambda) = A(x,\lambda) - A_\infty(\lambda) = (1-a(x))B$ and as $|1-a(x)| \leq C_1 \mathrm{e}^{-C_2|x|}$, this proves that Hypothesis 8 of Proposition 9 is satisfied.

J Hamiltonian evolution equation for LW-SW equations

The LW-SW equations (16.2) can be expressed in Hamiltonian form as follows. Let

$$\mathbf{K} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\partial_x \end{bmatrix},$$

$$H(Z) = \int_{-\infty}^{+\infty} \left(u_x^2 + v_x^2 + \frac{1}{2} w_x^2 - w(w^2 + u^2 + v^2) + \frac{1}{2} c w^2 + \nu (u^2 + v^2) \right) \mathrm{d}x,$$

with $Z = (u, v, w)$. Then the system becomes

$$Z_t = \mathbf{K} \nabla H(Z),$$

since, with respect to an $L_2(\mathbb{R})$ inner product,

$$\nabla H(Z) = \begin{pmatrix} H_u \\ H_v \\ H_w \end{pmatrix} = \begin{pmatrix} -2u_{xx} - 2uw + 2\nu u \\ -2v_{xx} - 2vw + 2\nu v \\ -w_{xx} + cw - 3w^2 - u^2 - v^2 \end{pmatrix}$$

K The reduced eigenvalue problem associated with LW-SW equations

The two-dimensional ODE (16.7) that arises in the reduced problem for LW-SW resonance can be written in the form

$$v_{xx} + 2\nu\,\mathrm{sech}^2(\sqrt{\nu}x)\,v = \left(\nu - \frac{1}{2}\lambda\right) v\,.$$

ODEs of this type can be solved explicitly as noted in Appendix F. The essential spectrum is the semi-infinite interval $\sigma_{ess}(L) = [2\nu, +\infty)$. Now suppose that $\lambda < 2\nu$. Then the system at infinity is hyperbolic and one can explicitly construct the solutions (v^+, v^-) which give the solutions for the stable and unstable subspace

$$v^{\pm}(x;\lambda) = e^{\pm\mu\sqrt{\nu}x}(\mp\mu + \tanh(\sqrt{\nu}x))\,, \quad \mu = \sqrt{1 - \frac{\lambda}{2\nu}}\,.$$

The Evans function can be obtained from

$$D(\lambda) = \det \begin{bmatrix} v^+(x;\lambda) & v^-(x;\lambda) \\ v^+_x(x;\lambda) & v^-_x(x;\lambda) \end{bmatrix}\Bigg|_{x=0} = 2\mu\sqrt{\nu}(\mu^2 - 1) = -\frac{\lambda}{\sqrt{\nu}}\sqrt{1 - \frac{\lambda}{2\nu}}\,.$$

The Maslov index is

$$\mathsf{Maslov}(\lambda) = \begin{cases} 1 & \text{if } \lambda < 0 \\ 0 & \text{if } 0 < \lambda < 2\nu \end{cases}\,.$$

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