

# Secondary criticality of water waves. Part 1. Definition, bifurcation and solitary waves

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A generalization of criticality – called secondary criticality – is introduced and applied to finite-amplitude Stokes waves. The theory shows that secondary criticality signals a bifurcation to a class of *steady dark solitary waves* which are biasymptotic to a Stokes wave with a phase jump in between, and synchronized with the Stokes wave. We find that the bifurcation to these new solitary waves – from Stokes gravity waves in shallow water – is pervasive, even at low amplitude. The theory proceeds by generalizing concepts from hydraulics: three additional functionals are introduced which represent non-uniformity and extend the familiar mass flux, total head and flow force, the most important of which is the wave action flux. The theory works because the hydraulic quantities can be related to the governing equations in a precise way using the multi-symplectic Hamiltonian formulation of water waves. In this setting, uniform flows and Stokes waves coupled to a uniform flow are relative equilibria which have an attendant geometric theory using symmetry and conservation laws. A flow is then ‘critical’ if the relative equilibrium representation is degenerate. By characterizing successively non-uniform flows and unsteady flows as relative equilibria, a generalization of criticality is immediate. Recent results on the local nonlinear behaviour near a degenerate relative equilibrium are used to predict all the qualitative properties of the bifurcating dark solitary waves, including the phase shift. The theory of secondary criticality provides new insight into unsteady waves in shallow water as well. A new interpretation of the Benjamin–Feir instability from the viewpoint of hydraulics, and the connection with the creation of unsteady dark solitary waves, is given in Part 2.

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## 1. Introduction

Criticality, uniform flows and bulk quantities such as mass flux, total head and the flow force are at the heart of the subject of open-channel hydraulics in one space dimension (Henderson 1966; Abbott 1979). The ideal theory can be deduced from the shallow-water equations for a constant-density fluid

$$h_t + (uh)_x = 0, \quad u_t + \left(gh + \frac{1}{2}u^2\right)_x = 0, \quad (1.1)$$

where  $h(x, t)$  is the depth,  $u(x, t)$  is the depth-averaged horizontal velocity and  $g$  is the gravitational constant. Accompanying these two equations is conservation of momentum,

$$(uh)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0. \quad (1.2)$$

Clearly steady flows – uniform flows – which satisfy all three of these equations realize constant values of the mass flux ( $Q$ ), the total head ( $R$ ) and the flow force ( $S$ ) where

$$R = gh + \frac{1}{2}u^2, \quad Q = uh, \quad S = hu^2 + \frac{1}{2}gh^2. \quad (1.3)$$

Many of the fundamental properties of open channel flows can be deduced from these three equations.

For the shallow-water equations, there are many ways to define criticality, all of which lead to the ‘Froude number unity’ condition, where the Froude number is defined by  $F^2 = u^2/gh$ . However, not all these equivalent definitions generalize to non-trivial flows. For example, a widely used concept of criticality is that based on whether or not a flow is faster or slower than the maximum linear wave speed, but this definition is based on a single parameter (the flow speed) and therefore does not generalize to non-trivial flows.

The most useful definition of criticality for uniform flows is the determinant condition

$$\left| \frac{\partial(R, Q)}{\partial(h, u)} \right| = 0. \quad (1.4)$$

Using (1.3), it is clear that this condition is equivalent to  $F^2 = 1$ . This definition combines the two most well-known definitions of critical uniform flow: the flow which maximizes  $Q$  for fixed  $R$ , equivalently minimizes  $R$  for fixed  $Q$ . The most important feature of (1.4) is that it is the form of the criticality definition which extends easily to more general flows, including  $x$ -dependent steady states.

Motivated by open questions in the theory of water waves, there are three issues which emerge from (1.3) and (1.4) which we would like to generalize: move from uniform flows to general  $x$ -dependent steady states, generalize the concept of criticality of uniform flows to non-trivial  $x$ -dependent steady states, and develop a theory for the implications of criticality for both the steady and unsteady problem.

The only attempts to generalize criticality to  $x$ -dependent states in the literature restrict attention to either a quasi-static approximation, or treat the flow as slowly varying in the  $x$ -direction (cf. Gill 1977; Killworth 1992; Johnson & Clarke 2001; and references therein.) One of our central observations is that an exact theory of criticality for finite-amplitude states is possible, when the flow is periodic in the  $x$ -direction, even when the basic state has non-trivial  $z$ -dependence (where  $z$  represents the vertical direction). We call this generalization *secondary criticality* and the secondary criticality of Stokes waves is our primary example.

The algebraic definition of uniform flows (inverting (1.3) for fixed values of  $R$  and  $Q$ ) is unsatisfactory for generalization. However, another way to characterize uniform flows is to seek values of  $(h, u)$  that make the flow force stationary while  $Q$  and  $R$  are held fixed. This principle is apparently well known in the hydraulics literature, and Benjamin (1971) gives an argument to show that this is a general principle for parallel ( $x$ -independent) flows (cf. Benjamin 1971, §3.5 and Appendix 2), and it is this observation which is the starting point for our generalization of criticality to non-parallel ( $x$ -dependent) flows.

In the case of the uniform flows satisfying (1.3), the principle of stationary flow force is trivial and does not provide any new information: it amounts to solving

$$\nabla L = 0 \quad \text{where} \quad L(h, u, a, b) = S(h, u) - aR(h, u) - bQ(h, u), \quad (1.5)$$

where  $(a, b)$  are Lagrange multipliers, and this functional is subject to the side conditions  $gh + \frac{1}{2}u^2 = \mathcal{R}$  and  $uh = \mathcal{Q}$  with  $\mathcal{R} > 0$  and  $\mathcal{Q}$  given real numbers. (Script symbols represent the value of the function or functional.) This variational principle, when non-degenerate, provides uniform flows  $(h(\mathcal{R}, \mathcal{Q}), u(\mathcal{R}, \mathcal{Q}))$ . Degeneracy occurs when the determinant (1.4) vanishes (in this example, the Lagrange multipliers  $(a, b)$  turn out to be  $(h, u)$ ): degeneracy implies criticality.

One of the main results of this paper is to show that this characterization of parallel flows has a natural generalization to steady flows with non-trivial  $x$ -dependence. This result is applied to steady periodic water waves coupled to a uniform flow, but the basic idea should extend to a range of fluid flows. For the water-wave problem, we find that there exists – in addition to  $S, Q$  and  $R$  – a fourth functional  $B$  such that the  $x$ -dependent coupled states satisfy the variational principle

$$\nabla L = 0 \quad \text{where} \quad L(Z, h_0, u_0, k) = S(Z) - h_0 R(Z) - u_0 Q(Z) - k B(Z), \quad (1.6)$$

where  $(h_0, u_0, k)$  are Lagrange multipliers. These Lagrange multipliers are the mean depth ( $h_0$ ), the mean velocity ( $u_0$ ) and the wavenumber associated with the periodic  $x$ -variation ( $k$ ). This variational principle is subject to the side conditions  $R(Z) = \mathcal{R}$ ,  $Q(Z) = \mathcal{Q}$  and  $B(Z) = \mathcal{B}$ , with  $\mathcal{Q}, \mathcal{R}, \mathcal{B}$  given, where  $Z$  is a set of coordinates representing water-wave variables. The functionals  $S, R, Q$  and  $B$  represent the physical quantities of flow force, total head, mass flux and a fourth functional  $B$  which can be interpreted as the *wave action flux*. This constrained variational principle is non-degenerate precisely when

$$\det(\mathbf{C}(\mathbf{p})) \neq 0 \quad \text{where} \quad \mathbf{C}(\mathbf{p}) = \frac{\partial(R, Q, B)}{\partial(h_0, u_0, k)}, \quad \mathbf{p} = (h_0, u_0, k). \quad (1.7)$$

Hereinafter, the matrix  $\mathbf{C}(\mathbf{p})$  is called the *criticality matrix*. Criticality of the basic state is defined by the vanishing of the determinant

$$\det[\mathbf{C}(\mathbf{p})] = 0. \quad (1.8)$$

As far as we are aware, this is the first attempt to give an exact definition of criticality (and its implications) for fully  $x$ -dependent steady states.

Why is this generalization of criticality interesting? One of the fundamental implications of criticality of uniform flows is the bifurcation of solitary waves. In the context of shallow-water flows, this solitary wave is the Russell solitary wave. (A ‘Russell solitary wave’ is the classical solitary wave which decays monotonically to the zero state at infinity; also called the KdV solitary wave or Boussinesq solitary wave. It is to be distinguished from envelope solitary waves (familiar as solutions of NLS in deep water) which decay to zero at infinity but also oscillate, and the dark solitary wave which is localized and oscillatory but is asymptotic to a non-trivial periodic state at infinity.) At a point of *secondary criticality*, where the determinant (1.8) vanishes, a pair of Floquet multipliers in the linearization about the basic state coalesce at +1, and generate in the nonlinear problem a ‘homoclinic torus bifurcation’. The connection between (1.8) and movement of Floquet multipliers is summarized in Appendices A and B. The homoclinic bifurcation is in space and so represents the bifurcation of a family of dark solitary waves (DSWs). The implication of secondary criticality for water waves is a new class of steady solitary water waves which are biasymptotic as  $x \rightarrow \pm\infty$  to Stokes waves, and synchronized with the Stokes wave.

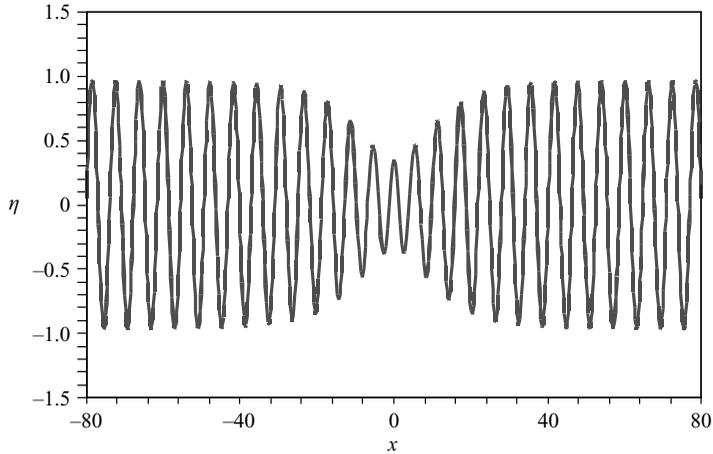


FIGURE 1. Schematic of the free-surface elevation associated with a dark solitary wave.

These waves look like Stokes waves for large  $|x|$ , but are deformed from periodicity for small  $|x|$ . An example is shown in figure 1.

In order to show that the functionals  $SQR$  and  $B$  do indeed have relevance for criticality there is a second crucial step to the theory: it is necessary to relate the hydraulic quantities  $SQR$  and  $B$  to the governing equations in a precise way.

This issue is resolved by using a Hamiltonian formulation for water waves. However, usual formulations for water waves such as the Lagrangian formulation of Luke (1967), the Zakharov Hamiltonian formulation for the time-dependent problem (Zakharov 1968), and the spatial dynamics formulations (e.g. Benjamin 1984; Mielke 1991; Baesens & MacKay 1992; Bridges 1992; Groves & Toland 1997) all have shortcomings. The multi-symplectic Hamiltonian formulation of water waves (e.g. Bridges 1996, 1997, 2001, 2006a) is a framework where all the hydraulic quantities are on the same footing and can be related to the governing equations in a precise way. In addition, a new choice of coordinates for the multi-symplectic formulation of water waves is introduced which provides an interpretation of the Bernoulli equation as a conservation law for total head. It is surprising that the importance of Bernoulli's equation as a conservation law in the Hamiltonian setting has not been noticed before. For example, Bernoulli's equation is not identified as a conservation law in Benjamin & Olver (1982). More importantly, Bernoulli's equation is related to a symmetry here, and it is this latter observation that makes the characterization of Bernoulli's equation as a conservation law useful. It arises in a central way in the theory of secondary criticality.

In the Hamiltonian setting, uniform flows and Stokes waves turn out to have a natural representation as relative equilibria, and it is the geometry of relative equilibria which leads to a new characterization of criticality for uniform flows and it extends easily to non-uniform flows and unsteady flows. Relative equilibria are solutions which travel along a symmetry group at a constant rate (cf. Marsden 1992).

An outline of Part 1 of this study is as follows. Throughout this paper, a Boussinesq model for water waves is used for illustration, and its Hamiltonian formulation and the associated structure for the  $SQR + B$  theory is given in §2. A theory for characterizing uniform flows as relative equilibria is given in §3, and the connection between

degenerate relative equilibria, classical criticality and solitary wave bifurcation – the bifurcation of Russell-type solitary waves – is presented. In §4, the Hamiltonian structure and  $SQR + B$  structure for water waves are introduced.

In §5, the theory of criticality for non-trivial  $x$ -dependent steady states is presented. The implications of this concept of criticality, principally the homoclinic torus bifurcation manifested as DSWs, are applied to a Boussinesq model in §6 and to the full water-wave problem in §7. Some of the mathematical tools required for development of the theory are summarized in the Appendices, namely Jordan chain theory for generalized eigenvectors, in a Hamiltonian context, and the relation between criticality and eigenvalues.

## 2. A Boussinesq model for analysing secondary criticality

The simplest class of shallow-water equations with dispersion, that generalize the system (1.1) and model the water-wave problem, is the Boussinesq class (cf. Dingemans 1997; Craig & Groves 1994). An exemplar of that class which is sufficient for the present purposes is the model proposed by Zufiria (1987),

$$\left. \begin{aligned} h_t + uh_x + hu_x + \frac{1}{3}u_{xxx} + \frac{2}{15}u_{xxxx} &= 0, \\ u_t + uu_x + h_x &= 0. \end{aligned} \right\} \tag{2.1}$$

Steady solutions of (2.1) satisfy

$$\left. \begin{aligned} uh + \frac{1}{3}u_{xx} + \frac{2}{15}u_{xxxx} &= q, \quad q_x = 0, \\ \frac{1}{2}u^2 + h &= r, \quad r_x = 0. \end{aligned} \right\} \tag{2.2}$$

If the fourth derivative term (the fifth-order dispersion term in (2.1)) is dropped, giving the usual form for a Boussinesq equation, the steady equation reduces to a planar system which cannot have secondary bifurcations. Therefore to capture the bifurcation of DSWs from the steady periodic state it is essential to retain the fourth-order derivative term.

By introducing new coordinates  $Z = (\gamma, \phi, w_1, u, r, q, v, w_2)$  defined by

$$\phi_x = u, \quad \gamma_x = r - \frac{1}{2}u^2, \quad v = u_x, \quad w_1 = -\frac{2}{15}v_x, \quad w_2 = -\frac{1}{3}v - \frac{2}{15}v_{xx},$$

and a Hamiltonian function

$$S(Z) = \frac{1}{2}r^2 + qu + \frac{15}{4}w_1^2 + w_2v + \frac{1}{6}v^2 - \frac{1}{2}ru^2 + \frac{1}{8}u^4, \tag{2.3}$$

the steady Boussinesq model (2.1) has the Hamiltonian formulation,

$$\left. \begin{aligned} -r_x = \partial S / \partial \gamma &= 0, & \gamma_x &= \partial S / \partial r = r - \frac{1}{2}u^2, \\ -q_x = \partial S / \partial \phi &= 0, & \phi_x &= \partial S / \partial q = u, \\ -v_x = \partial S / \partial w_1 &= \frac{15}{2}w_1, & w_{1x} &= \partial S / \partial v = w_2 + \frac{1}{3}v, \\ -w_{2x} = \partial S / \partial u &= q - ru + \frac{1}{2}u^3, & u_x &= \partial S / \partial w_2 = v. \end{aligned} \right\} \tag{2.4}$$

This system can be written in standard form for a Hamiltonian ODE

$$\mathbf{J}Z_x = \nabla S(Z), \quad Z \in \mathbb{R}^8, \tag{2.5}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and  $S(Z)$  is given by (2.3).

Note that a potential function has been introduced for both the velocity field and the surface elevation:

$$u = \phi_x, \quad h = \gamma_x. \quad (2.6)$$

The idea of using a potential for the velocity is a familiar one. Surprisingly, the use of a potential for the surface elevation also has a long history. According to Dooge (1987), Boussinesq first proposed the idea in pages 271–274 of his 680 page paper (Boussinesq 1877) for the shallow-water equations (1.1). It was later followed up by Deymie (1935). See the historical essay of Dooge (1987, p. 221) for discussion. Our use of the potential was inspired by the paper of Nutku (1983) where it was also used for the shallow-water equations (1.1).

One way to motivate the use of a potential for the free surface is to note that it generates a symmetry. An arbitrary constant can be added to  $\gamma$  without affecting  $h(x)$ . The conservation law generated by this symmetry is the ‘conservation of total head’, which represents a conservation form of Bernoulli’s equation. In other words, the introduction of the potential  $\gamma$  gives a way to introduce the Bernoulli function (labelled  $r$ ) explicitly into the equations. This property is reminiscent of how the introduction of a velocity potential leads to a symmetry (addition of an arbitrary constant to  $\phi(x)$ ), which is then linked with the mass conservation law.

The symmetry associated with the potentials (2.6) provides an important link between solutions and conservation laws. To define the potential symmetries in a precise way for the system in the form (2.5), let  $\mathbf{g}_1 = \mathbf{e}_1$  and  $\mathbf{g}_2 = \mathbf{e}_2$  where  $\mathbf{e}_j$  is the standard  $j$ th unit vector in  $\mathbb{R}^8$ . Then, the fact that  $S(Z)$  is independent of  $\gamma$  and  $\phi$  can be represented by the statement

$$S(Z + \theta_1 \mathbf{g}_1 + \theta_2 \mathbf{g}_2) = S(Z), \quad \forall (\theta_1, \theta_2) \in \mathbb{R}^2.$$

More generally, these symmetries can be characterized as a two-parameter group of *affine translations of the phase space*  $\mathbb{R}^8$ . The action of this group on an element  $Z \in \mathbb{R}^8$  is

$$\mathbf{G}(\boldsymbol{\theta})Z = Z + \theta_1 \mathbf{g}_1 + \theta_2 \mathbf{g}_2, \quad \forall \boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2. \quad (2.7)$$

In terms of this group, the symmetry of (2.5) is represented by the requirement

$$S(\mathbf{G}(\boldsymbol{\theta})Z) = S(Z), \quad \forall \boldsymbol{\theta} \in \mathbb{R}^2. \quad (2.8)$$

A consequence of this property is that  $\mathbf{G}(\boldsymbol{\theta})Z(x)$  is a solution for any  $\boldsymbol{\theta} \in \mathbb{R}^2$  whenever  $Z(x)$  is a solution.

In the Hamiltonian setting, symmetry is related to conservation laws. The above two-parameter group of symmetries can be related to the conservation of mass and Bernoulli’s equation as follows. Let  $\langle \cdot, \cdot \rangle$  be a standard inner product on  $\mathbb{R}^8$ , and

define

$$R(Z) = \langle \mathbf{Jg}_1, Z \rangle = \langle \mathbf{e}_5, Z \rangle = r, \quad Q(Z) = \langle \mathbf{Jg}_2, Z \rangle = \langle \mathbf{e}_6, Z \rangle = q. \quad (2.9)$$

Then

$$\frac{\partial}{\partial x} R(Z) = \langle \mathbf{Jg}_1, Z_x \rangle = -\langle \mathbf{g}_1, \mathbf{JZ}_x \rangle = -\langle \mathbf{g}_1, \nabla S(Z) \rangle = -\frac{\partial}{\partial \theta_1} S(G(\theta)Z)|_{\theta=0} = 0;$$

resulting in Bernoulli’s equation in conservation form. A geometric formulation of *conservation of mass* can be derived similarly leading to  $Q_x = 0$ . Conservation of impulse can be derived using a similar geometric approach, but will not be explicitly required.

There is a fourth conservation law which arises in the later analysis of  $x$ -dependent states. Let  $Z(x, s)$  be a closed curve (an ensemble) of solutions of (2.5), parameterized by  $s$ ; that is,  $Z(x, s + 2\pi) = Z(x, s)$  and  $Z(x, s)$  satisfies (2.5) for each  $s$ . Define

$$B(Z) = \frac{1}{2} \oint \langle \mathbf{JZ}_s, Z \rangle ds, \quad \oint \cdot ds := \frac{1}{2\pi} \int_0^{2\pi} \cdot ds.$$

Then it is straightforward to verify that  $B_x = 0$ . This conservation law is in fact the geometric formulation of the *conservation of wave action* (Bridges 1997b), restricted to one space dimension:  $B(Z)$  is the wave action flux. Note that in this conservation law,  $Z(x, s)$  is not necessarily periodic in  $x$ . It is periodic only in the ensemble parameter  $s$ .

### 3. Critical uniform flows, degenerate relative equilibria and solitary waves

In this section it is shown how the association between critical uniform flows and degenerate relative equilibria leads to a nonlinear theory for the bifurcation of solitary waves. This theory is a special case of the secondary criticality theory and shows how the theory works in a simpler setting. It also shows that the mechanism for the bifurcation of dark solitary waves and the mechanism for bifurcation of the Russell solitary wave are the same.

To illustrate, it is sufficient to use the steady Boussinesq model (2.2) simplified further by dropping the fourth-order term,

$$uh + \frac{1}{3}u_{xx} = q, \quad q_x = 0,$$

$$\frac{1}{2}u^2 + h = r, \quad r_x = 0.$$

By letting  $h = \gamma_x$ ,  $u = \phi_x$  and  $u_x = -3v$ , a Hamiltonian formulation of this model is

$$\left. \begin{aligned} -r_x &= 0, & \gamma_x &= r - \frac{1}{2}u^2, \\ -q_x &= 0, & \phi_x &= u, \\ -v_x &= q - ru + \frac{1}{2}u^3, & u_x &= -3v. \end{aligned} \right\} \quad (3.1)$$

This system is of the form

$$\mathbf{JZ}_x = \nabla S(Z) \quad \text{with} \quad Z = (\gamma, \phi, u, r, q, v), \quad (3.2)$$

and standard  $\mathbf{J}$ . The system (3.1) can be completely solved explicitly. The uniform flows correspond to  $u = u_0$  and  $r - u_0^2/2 = h_0$ . Perturbing about this solution and reducing the system leads to the planar ODE

$$\left. \begin{aligned} -v_x &= q - u_0r + \frac{3}{2}u_0u^2 + \dots, \\ u_x &= -3v, \end{aligned} \right\} \quad (3.3)$$

which can be explicitly solved to find the bifurcating solitary wave solutions that bifurcate near criticality:  $h_0 - u_0^2 \approx 0$ .

This result is recovered by the theory of relative equilibria as follows. The uniform flows correspond to relative equilibria associated with the two-dimensional affine translation group introduced in (2.7). This relative equilibrium is degenerate precisely when the uniform flow is critical. There is a universal nonlinear normal form near degenerate relative equilibria (Bridges & Donaldson 2005; Bridges 2006*b*) and this normal form predicts the bifurcation of solitary waves.

Relative equilibria of (3.2) take the form

$$Z(x) = G(\theta(x))\widehat{Z} = \widehat{Z} + \theta_1(x)\mathfrak{g}_1 + \theta_2(x)\mathfrak{g}_2 \quad \text{with} \quad \ddot{\theta}_1 = \ddot{\theta}_2 = 0, \tag{3.4}$$

where  $\mathfrak{g}_j = \mathbf{e}_j$ ,  $j = 1, 2$ , and  $\widehat{Z} \in \mathbb{R}^6$  is an  $x$ -independent vector. With the above requirements

$$\theta_1(x) = h_0x + \theta_1^0, \quad \theta_2(x) = u_0x + \theta_2^0, \tag{3.5}$$

where  $\theta_1^0$  and  $\theta_2^0$  are arbitrary (phase) constants, and  $h_0$  and  $u_0$  are constants which represent rate of change along the group orbit.

To determine  $(h_0, u_0, \widehat{Z})$ , substitute (3.4) into (3.2) and use the equivariance properties (2.8) of the system to obtain

$$\nabla S(\widehat{Z}) = h_0\nabla R(\widehat{Z}) + u_0\nabla Q(\widehat{Z}) \quad \text{with} \quad R(\widehat{Z}) = \mathcal{R}, \quad Q(\widehat{Z}) = \mathcal{Q}. \tag{3.6}$$

The vector  $\widehat{Z}$  can be characterized as a critical point of  $S$  subject to the constraints of constant  $R$  and  $Q$ , with  $h_0$  and  $u_0$  then appearing as Lagrange multipliers. It follows from standard Lagrange multiplier theory that

$$h_0 = \frac{\partial S}{\partial \mathcal{R}}, \quad u_0 = \frac{\partial S}{\partial \mathcal{Q}}. \tag{3.7}$$

Carrying through the calculation associated with the first equation of (3.6) results in

$$\widehat{Z}(h_0, u_0) = (0, 0, u_0, h_0 + \frac{1}{2}u_0^2, h_0u_0, 0).$$

The Lagrange multipliers are then determined by solving the two constraint equations in (3.6) which are solvable when

$$0 \neq \left| \frac{\partial(R, Q)}{\partial(h_0, u_0)} \right| = \det \begin{bmatrix} 1 & u_0 \\ u_0 & h_0 \end{bmatrix} = h_0 - u_0^2 = h_0(1 - F^2).$$

The relative equilibrium (uniform flow) is degenerate when the determinant vanishes. An important consequence of degeneracy is that the linearization about the degenerate relative equilibrium has an eigenvalue zero of algebraic multiplicity 6 but geometric multiplicity two. The connection between degeneracy and eigenvalue movement is demonstrated in Appendices A and B.

To study the nonlinear problem in the neighbourhood of a degenerate relative equilibrium let

$$Z(x) = G(\theta(x))[\widehat{Z} + W(x)] := \widehat{Z} + W(x) + \theta_1(x)\mathfrak{g}_1 + \theta_2(x)\mathfrak{g}_2, \tag{3.8}$$

then the nonlinear problem for  $W(x)$  is

$$\mathbf{J}W_x = \mathbf{L}(h_0, u_0)W + \frac{1}{2}D^3S(\widehat{Z})(W, W) + \dots \tag{3.9}$$

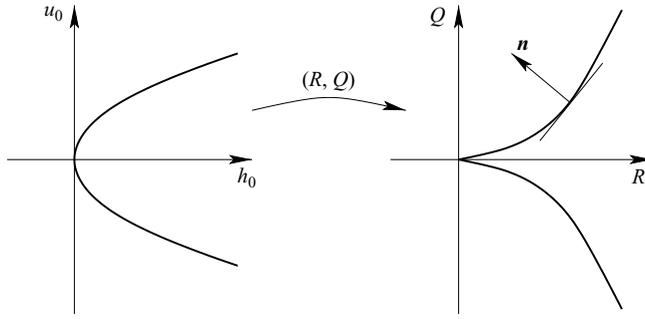


FIGURE 2. Geometry of the mapping from  $(h_0, u_0)$ -space to  $(R, Q)$ -space of the criticality curve  $h_0 - u_0^2 = 0$ .

where the dots indicate terms of degree three and higher in  $\|W\|$ , and  $D^3S(\widehat{Z})(W, W)$  is the third derivative of  $S(Z)$  at the point  $\widehat{Z}$  and

$$\mathbf{L}(h_0, u_0) = D^2S(\widehat{Z}) - h_0D^2R(\widehat{Z}) - u_0D^2Q(\widehat{Z}).$$

There exists a set of six constant vectors  $\{\xi_1, \dots, \xi_6\}$ , which are constructed using the generalized eigenvectors (see Appendices A and B), such that the transformation

$$W(x) = \widetilde{\phi}_1(x)\xi_1 + \widetilde{\phi}_2(x)\xi_2 + \widetilde{u}(x)\xi_3 - s_1\widetilde{I}_1(x)\xi_6 + s_2\widetilde{I}_2(x)\xi_4 + s_1\widetilde{v}(x)\xi_5 + \dots \quad (3.10)$$

results in the reduced system of nonlinear ODEs

$$\left. \begin{aligned} -\dot{\widetilde{I}}_1 &= 0, & \widetilde{\phi}_1 &= \widetilde{u} + \dots, \\ -\dot{\widetilde{I}}_2 &= 0, & \widetilde{\phi}_2 &= s_2\widetilde{I}_2 + \dots, \\ -\dot{\widetilde{v}} &= \widetilde{I}_1 + \delta\widetilde{u} - \frac{1}{2}\kappa\widetilde{u}^2 + \dots, & \dot{\widetilde{u}} &= s_1\widetilde{v} + \dots. \end{aligned} \right\} \quad (3.11)$$

This system is universal in the sense that it arises near points of degeneracy for any two-parameter family of relative equilibria (Bridges & Donaldson 2005; Bridges 2006b).

The sign  $s_1$  is a property of the Jordan normal form structure and  $s_1 = -1$  in this case. The sign  $s_2$  is the sign of the non-zero eigenvalue of  $\mathbf{C}(\mathbf{p})$  at criticality (clearly  $s_2 = +1$  in this case). The parameter  $\delta$  is an unfolding parameter, proportional to the value of  $\det(\mathbf{C}(\mathbf{p})) = h_0 - u_0^2$ , and is not qualitatively significant. The important parameters are  $\kappa$  and  $\widetilde{I}_1$ . The formula for  $\kappa$  is

$$\kappa = a_0^3 \mathbf{n}^T \text{Hess}(\mathcal{P})\mathbf{n},$$

where  $a_0$  is an explicitly computable constant ( $a_0 = \sqrt{3}$ , in the above example), and  $\mathcal{P} = n_1\mathcal{R} + n_2\mathcal{Q}$ , with  $\mathbf{n}$  defined below. It is remarkable that the coefficient  $\kappa$  of the nonlinear term in the normal form (3.11) is completely determined by properties of the criticality matrix  $\mathbf{C}(\mathbf{p})$ .

The normal vector  $\mathbf{n}$  is defined by

$$\mathbf{C}(\mathbf{p})\mathbf{n} = \mathbf{0} \quad \Rightarrow \quad \mathbf{n} = \text{span} \left\{ \begin{pmatrix} -u_0 \\ 1 \end{pmatrix} \right\} \quad \text{since} \quad \mathbf{C}(\mathbf{p}) = \begin{bmatrix} 1 & u_0 \\ u_0 & h_0 \end{bmatrix}.$$

For definiteness, take  $\mathbf{n} = (-u_0, 1)$  (there is no need to normalize the length to unity).

The role of  $\mathbf{n}$  as a normal vector can be seen as follows. Criticality is defined by the determinant condition (1.4). Setting this determinant to zero defines a curve in the  $(h_0, u_0)$ -plane. For the case of uniform flows, the curve is a parabola (figure 2). The

image of the curve of criticality in the  $(R, Q)$ -plane is the curve with cusp as shown in figure 2. (The cusp in figure 2 is yet another view of the cusp in Benjamin & Lighthill (1954); but note that the Benjamin–Lighthill cusp is viewed in the  $(R, S)$ -plane with  $Q$  fixed.) Let  $(\dot{h}_0, \dot{u}_0)$  be a tangent vector to the criticality curve in the  $(h_0, u_0)$ -plane. A tangent vector in the  $(R, Q)$ -plane then satisfies

$$\begin{pmatrix} \dot{R} \\ \dot{Q} \end{pmatrix} = \begin{bmatrix} R_{h_0} & R_{u_0} \\ Q_{h_0} & Q_{u_0} \end{bmatrix} \begin{pmatrix} \dot{h}_0 \\ \dot{u}_0 \end{pmatrix}.$$

If  $\mathbf{n}$  is in the kernel of the criticality matrix then clearly

$$\mathbf{n} \cdot \begin{pmatrix} \dot{R} \\ \dot{Q} \end{pmatrix} = 0,$$

and hence is a normal vector as shown in figure 2. The normal vector  $\mathbf{n}$  shows which direction in parameter space is normal to the curve of criticality. The bifurcating solitary wave exists on one side of the curve of criticality only and the nonlinear theory is used to predict which side of the criticality curve the solitary wave exists. For the case of figure 2, the homoclinic bifurcation exists inside the curve of criticality in the  $(R, Q)$ -plane; that is, given a point  $(\mathcal{R}_0, \mathcal{Q}_0)$  on the curve of criticality with  $\mathcal{Q}_0 > 0$ , then the homoclinic orbit exists in the region

$$-u_0(\mathcal{R} - \mathcal{R}_0) + (\mathcal{Q} - \mathcal{Q}_0) < 0.$$

The bifurcation is in the direction of  $-\mathbf{n}$  on the upper branch shown in figure 2.

Using the above definition of  $\mathbf{n}$ ,

$$\text{Hess}(\mathcal{P}) = n_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + n_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -u_0 \end{bmatrix},$$

and so

$$\kappa = -9\sqrt{3}u_0.$$

Another interesting feature of the normal form is that the coordinate  $\tilde{I}_1$  is a local normal coordinate in parameter space near the surface of criticality. To leading order,

$$\tilde{I}_1 = a_0(n_1(\mathcal{R} - \mathcal{R}_0) + n_2(\mathcal{Q} - \mathcal{Q}_0)) = a_0(-u_0(\mathcal{R} - \mathcal{R}_0) + (\mathcal{Q} - \mathcal{Q}_0)),$$

where  $(\mathcal{R}_0, \mathcal{Q}_0)$  is any point on the curve of criticality in figure 2 away from the cusp point. The Russell solitary wave is then determined to leading order from the solution of

$$\tilde{u}_{xx} = \tilde{I}_1 - \frac{1}{2}\kappa\tilde{u}^2,$$

with both  $\tilde{I}_1$  and  $\kappa$  predicted by properties of the criticality matrix. Note that there is a geometric phase shift given by

$$\Delta\tilde{\phi}_1 = \int_{-\infty}^{+\infty} (\tilde{u} - u_\infty) dx, \quad u_\infty^2 = \frac{2\tilde{I}_1}{\kappa}.$$

Mathematically, this phase shift is identical to the phase jump of dark solitary waves (see equation (8.8) in §8). Here the phase shift is in the potentials  $\gamma(x)$  and  $\phi(x)$ .

The bifurcating solitary wave predicted by the normal form (3.11) agrees with that found by the explicit calculation using (3.3). However, we now have a theory that will work in more general settings: degenerate relative equilibrium implies solitary wave bifurcation.

#### 4. Multi-symplecticity and the $SQR + B$ theory of water waves

Consider the classical two-dimensional water-wave problem (inviscid, irrotational fluid with constant density) in finite depth with a gravitational force field (with gravitational constant  $g$ ), in the following form

$$u_x + \phi_{zz} = 0 \quad \text{for } 0 < z < \eta(x, t), \quad x \in \mathbb{R}, \quad (4.1)$$

where  $\phi(x, z, t)$  is the velocity potential and  $u(x, z, t) = \phi_x(x, z, t)$ . The bottom boundary condition is

$$\phi_z = 0 \quad \text{at } z = 0, \quad x \in \mathbb{R}, \quad (4.2)$$

and the conditions at the free surface are

$$\eta_t + u\eta_x - \phi_z = 0, \quad r_x = 0 \quad \text{at } z = \eta(x, t), \quad x \in \mathbb{R}. \quad (4.3)$$

Define  $\Phi(x, t) = \phi(x, z, t)|_{z=\eta(x,t)}$ ; then

$$r(t) := \Phi_t + u\Phi_x + g\eta - \frac{1}{2}(u^2 + \phi_z^2), \quad \text{with } u \text{ and } \phi_z \text{ evaluated at } z = \eta. \quad (4.4)$$

##### 4.1. Bernoulli equation and conservation of total head

The main non-standard feature of this formulation is the appearance of the dependent variable  $r$ , and the subtle distinction between the definition of  $r$  (4.4) and the boundary condition  $r_x = 0$ . The variable  $r(t)$  can be – and usually is – incorporated into the potential; but, retaining the Bernoulli function  $r(t)$  explicitly turns out to have interesting consequences. In the Hamiltonian setting,  $r$  is the conjugate variable associated with  $\gamma$ .

To see that  $r_x = 0$  is the right boundary condition, use  $r$  in standard coordinates,

$$r := \Phi_t + u\Phi_x + g\eta - \frac{1}{2}(u^2 + \phi_z^2) = [\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta]_{z=\eta},$$

and so

$$\begin{aligned} r_x &= (\phi_{xt} + \phi_{tz}\eta_x + \phi_x\phi_{xx} + \phi_x\phi_{xz}\eta_x + \phi_z\phi_{zx} + \phi_z\phi_{zz}\eta_x + g\eta_x)|_{z=\eta} \\ &= (u_t + uu_x + wu_z)|_{z=\eta} + \eta_x(w_t + uw_x + ww_z + g)|_{z=\eta} \\ &= -[p_x + \eta_x p_z]|_{z=\eta} \\ &= -\sqrt{1 + \eta_x^2} \nabla p \cdot \mathbf{t}|_{z=\eta}, \end{aligned}$$

where  $\mathbf{t} = (1 + \eta_x^2)^{-1/2}(1, \eta_x)$  is the unit vector tangent to the free surface. Therefore, the condition  $r_x = 0$  is equivalent to the condition of constant pressure along the free surface.

Taking a hint from the Hamiltonian formulation of the Boussinesq model (2.5), we introduce a potential function for the free surface

$$\eta(x, t) = \frac{\partial}{\partial x} \gamma(x, t).$$

In terms of the above coordinates, the four main functionals for steady water waves are

$$\left. \begin{aligned} S(Z) &= \frac{1}{2} \int_0^\eta (u^2 - \phi_z^2) dz - \frac{1}{2} g \eta^2 + r \eta, \\ R(Z) &= r, \\ Q(Z) &= \int_0^\eta u dz, \\ B(Z) &= \int_0^\eta u \phi_x dz + r \gamma_x, \end{aligned} \right\} \quad (4.5)$$

where

$$Z = \begin{pmatrix} \Phi(x, t) \\ \eta(x, t) \\ \gamma(x, t) \\ \phi(x, z, t) \\ u(x, z, t) \\ r(t) \end{pmatrix} \quad \text{with} \quad \begin{cases} \Phi(x, t) = \phi(x, z, t)|_{z=\eta(x,t)}, \\ u(x, z, t) = \partial_x \phi(x, z, t), \\ \eta(x, t) = \gamma_x(x, t), \end{cases}$$

and  $r(t)$  is defined in (4.4).

The approach in this paper to finite-amplitude periodic waves on a uniform flow is to be contrasted with the *SQR* theory for water waves initiated by Benjamin & Lighthill (1954) and developed further by Keady & Norbury (1975), Doole & Norbury (1995), Doole (1997) and references therein. In that theory, both uniform flows and the periodic states are studied in the *SQR* space. Our theory shows that the non-trivial  $x$ -dependent steady states should be studied in the *SQR + B* parameter space. The introduction of the function  $B$  has two consequences: without  $B$  there is no obvious generalization of criticality and its implications, and secondly, there is no obvious way to relate the *SQR* properties to the stability problem without additional functionals (the latter issue is considered in Part 2, Bridges & Donaldson 2006).

#### 4.2. *SQR + B gradient structure and symmetry*

It is in terms of the set of variables  $Z$  that the gradient of each of the functionals has the right structure. An appropriate inner product is required in order to define the gradient.

Let  $\mathbb{H}$  represent the set of functions of the type  $Z$ : six-component vector-valued functions where only the fourth and fifth components depend on the cross-section. Then a natural inner product on  $\mathbb{H}$  is

$$\langle W, Z \rangle_\eta = W_1 Z_1 + W_2 Z_2 + W_3 Z_3 + \int_0^\eta (W_4 Z_4 + W_5 Z_5) dz + W_6 Z_6, \quad Z, W \in \mathbb{H}. \quad (4.6)$$

Using the above inner product, and including integration over  $x$  and  $t$ , the governing equations for water waves can be written in multi-symplectic form

$$\mathbf{M}(Z)Z_t + \mathbf{J}(Z)Z_x = \nabla S(Z), \quad Z \in \mathbb{H}. \quad (4.7)$$

This multi-symplectic formulation is a generalization of the formulation in Bridges (1996, 2001). The new feature here is the introduction of  $\gamma$  and  $r$  as dependent variables. The specifics of the system (4.7) are not required in the sequel, as the functionals  $S$ ,  $R$ ,  $Q$  and  $B$  will form the basis for the analysis.

However, the symmetry of the equations is important. The analogue of the affine translation group (2.7) for water waves is

$$G(\boldsymbol{\theta})Z = Z + \theta_1 \mathbf{g}_1 + \theta_2 \mathbf{g}_2, \quad \forall \boldsymbol{\theta} \in \mathbb{R}^2, \quad Z \in \mathbf{H}, \quad (4.8)$$

with

$$\mathbf{g}_1 = (0, 0, 1, 0, 0, 0), \quad \mathbf{g}_2 = (1, 0, 0, 1, 0, 0),$$

(using the same notation as in §3, which should not cause confusion). These symmetries are related to the invariants  $R$  and  $Q$  by  $\nabla R(Z) = \mathbf{J}(Z)\mathbf{g}_1$  and  $\nabla Q(Z) = \mathbf{J}(Z)\mathbf{g}_2$ .

### 4.3. Uniform flows for the full water-wave problem

We are now in a position to show that uniform flows of the water-wave problem correspond to relative equilibria associated with the group  $G(\boldsymbol{\theta})$  in (4.8). First, note that the only true equilibria – i.e. critical points of the flow force  $S(Z)$  – are the trivial solutions  $\phi = \text{constant}$  and  $\gamma = \text{constant}$ . Consider steady solutions of the water-wave problem in the form of relative equilibria (3.4),

$$Z(x, z) = G(\boldsymbol{\theta}(x))\widehat{Z}(z) = \widehat{Z}(z) + \theta_1(x)\mathbf{g}_1 + \theta_2(x)\mathbf{g}_2,$$

with

$$\theta_1(x) = h_0x + \theta_1^0, \quad \theta_2(x) = u_0x + \theta_2^0,$$

where we have anticipated that the parameters  $h_0$  and  $u_0$  represent uniform flows. Substitution into the governing equations leads to the familiar equation

$$\nabla S(\widehat{Z}) = h_0 \nabla R(\widehat{Z}) + u_0 \nabla Q(\widehat{Z}),$$

which is easily solved explicitly,

$$\widehat{Z} = (0, h_0, 0, 0, u_0, gh_0 + \frac{1}{2}u_0^2).$$

Now, substituting this result into  $R$  and  $Q$ ,

$$R(\widehat{Z}) = r = gh_0 + \frac{1}{2}u_0^2, \quad Q(\widehat{Z}) = \int_0^\eta u \, dz = u_0h_0.$$

Hence the relative equilibria (uniform flows) are non-degenerate precisely when

$$0 \neq \left| \frac{\partial(R, Q)}{\partial(h_0, u_0)} \right| = \det \begin{bmatrix} g & u_0 \\ u_0 & h_0 \end{bmatrix} = gh_0(1 - F^2),$$

as anticipated.

The homoclinic bifurcation associated with criticality of uniform flows for the full water-wave problem results in the well-known Russell solitary wave.

## 5. Secondary criticality

In this section, the concept of criticality is extended to steady periodic waves coupled to a uniform flow. The theory is based on the representation of these states as relative equilibria associated with a three-parameter group. With this representation, waves of wavelength  $2\pi/k$  coupled to a uniform flow can be characterized as solutions of a constrained variational problem with

$$\nabla S(\widehat{Z}) = h_0 \nabla R(\widehat{Z}) + u_0 \nabla Q(\widehat{Z}) + k \nabla B(\widehat{Z}), \quad (5.1)$$

and non-degeneracy condition

$$\det [\mathbf{C}(\mathbf{p})] \neq 0 \quad \text{where} \quad \mathbf{C}(\mathbf{p}) = \frac{\partial(R, Q, B)}{\partial(h_0, u_0, k)}, \tag{5.2}$$

and the point at which this non-degeneracy condition fails corresponds to criticality.

To simplify the discussion, the details of the theory for the Boussinesq model (2.1) are presented, and then results of the theory for the full water-wave problem are given.

Take the Hamiltonian ODE (2.5) as a starting point. A periodic wave coupled to a uniform flow can be characterized by letting  $\widehat{Z}$  in (3.4) depend periodically on  $x$ ,

$$Z(x) = \mathbf{G}(\boldsymbol{\theta}(x))\widehat{Z} := \widehat{Z}(\theta_3) + \theta_1(x)\mathbf{g}_1 + \theta_2(x)\mathbf{g}_2, \tag{5.3}$$

where  $\widehat{Z}(\theta_3 + 2\pi) = \widehat{Z}(\theta_3)$ ,

$$\theta_1(x) = h_0x + \theta_1^0, \quad \theta_2(x) = u_0x + \theta_2^0, \quad \theta_3(x) = kx + \theta_3^0, \tag{5.4}$$

and, as before,  $\theta_j^0$  are arbitrary phase constants. Substitution of this form into (2.5) results in

$$k\mathbf{J}\frac{\partial\widehat{Z}}{\partial\theta_3} + h_0\mathbf{J}\mathbf{g}_1 + u_0\mathbf{J}\mathbf{g}_2 = \nabla S(\widehat{Z}),$$

which is equivalent to (5.1) by taking

$$R(\widehat{Z}) = \oint \langle \mathbf{J}\mathbf{g}_1, \widehat{Z} \rangle d\theta_3, \quad Q(\widehat{Z}) = \oint \langle \mathbf{J}\mathbf{g}_2, \widehat{Z} \rangle d\theta_3, \quad B(\widehat{Z}) = \oint \frac{1}{2} \left\langle \mathbf{J}\frac{\partial\widehat{Z}}{\partial\theta_3}, \widehat{Z} \right\rangle d\theta_3, \tag{5.5}$$

with

$$\oint f(\theta_3) d\theta_3 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta_3) d\theta_3.$$

The equation for  $\widehat{Z}$  in (5.1) is now an ordinary differential equation, on a space of periodic functions, rather than an algebraic equation as in (3.6), and the inner product includes averaging over  $\theta_3$ .

The constraints associated with (5.1) are  $R(\widehat{Z}) = \mathcal{R}$ ,  $Q(\widehat{Z}) = \mathcal{Q}$  and  $B(\widehat{Z}) = \mathcal{B}$ , and the non-degeneracy condition is (5.2). The generalization of the pair of derivative conditions (3.7) is

$$\left. \begin{aligned} h_0 &= \frac{\partial S}{\partial \mathcal{R}}, & \mathcal{Q}, \mathcal{B} \text{ fixed,} \\ u_0 &= \frac{\partial S}{\partial \mathcal{Q}}, & \mathcal{R}, \mathcal{B} \text{ fixed,} \\ k &= \frac{\partial S}{\partial \mathcal{B}}, & \mathcal{R}, \mathcal{Q} \text{ fixed.} \end{aligned} \right\} \tag{5.6}$$

A given state of the form (5.3) is defined to be critical when the non-degeneracy condition (5.2) fails. This definition is justified in Appendix B by showing that the consequences of criticality for these states is the same as the consequence for uniform flows: the coalescence of a pair of eigenvalues and a (secondary) bifurcation of solitary waves.

Before showing that this concept of criticality is indeed a generalization of uniform flow criticality to non-trivial steady states, the weakly nonlinear case is developed and applied to the Boussinesq model and to water waves.

5.1. Secondary criticality of weakly nonlinear periodic waves

There is a universal form for the criticality matrix when the periodic function  $\widehat{Z}(\theta_3)$  is expanded in a weakly nonlinear Fourier series. Let

$$\widehat{Z}(\theta_3) = A_1 \xi_1 e^{i\theta_3} + A_2 \xi_2 e^{2i\theta_3} + \text{c.c.} + \dots, \tag{5.7}$$

where  $\xi_1$  is an eigenvector of the linear problem and the  $\dots$  represent terms of higher order in  $|A_1|$ . Substituting this Fourier expansion into the functional

$$\mathcal{F}(\widehat{Z}, \mathbf{p}) = \oint [S(\widehat{Z}) - h_0 R(\widehat{Z}) - u_0 Q(\widehat{Z}) - kB(\widehat{Z})] d\theta_3, \tag{5.8}$$

and eliminating the Fourier coefficients  $A_2, A_3, \dots$  successively, leads to the *reduced functional*

$$\widehat{\mathcal{F}}(|A_1|^2, \mathbf{p}) = \mathcal{F}_0(\mathbf{p}) + D(\mathbf{p})|A_1|^2 + \frac{1}{2}\varrho(\mathbf{p})|A_1|^4 + \dots \tag{5.9}$$

where  $D(\mathbf{p})$  is the dispersion relation and  $\varrho(\mathbf{p})$  is the coefficient of the leading-order nonlinear term. The relation between  $A_1$  and the parameters is determined from the derivative of  $\widehat{\mathcal{F}}$  with respect to  $A_1$ ,

$$D(\mathbf{p}) + \varrho(\mathbf{p})|A_1|^2 + \dots = 0. \tag{5.10}$$

To determine the entries of the criticality matrix, the leading-order expressions for the functionals  $R, Q$  and  $B$  are required. Comparing (5.8) with (5.9) shows that

$$\frac{\partial \mathcal{F}}{\partial h_0} := -R = \frac{\partial \widehat{\mathcal{F}}_0}{\partial h_0} + \frac{\partial D}{\partial h_0} |A_1|^2 + \dots := \frac{\partial \widehat{\mathcal{F}}}{\partial h_0}.$$

Anticipating the case of water waves, take

$$\widehat{\mathcal{F}}_0 = -\frac{1}{2}gh_0^2 - h_0u_0^2, \tag{5.11}$$

and compute the derivatives of  $\widehat{\mathcal{F}}$  with respect to  $u_0$  and  $k$ ,

$$\left. \begin{aligned} R &= gh_0 + \frac{1}{2}u_0^2 - \frac{\partial D}{\partial h_0} |A_1|^2 + \dots, \\ Q &= h_0u_0 - \frac{\partial D}{\partial u_0} |A_1|^2 + \dots, \\ B &= -\frac{\partial D}{\partial k} |A_1|^2 + \dots, \end{aligned} \right\} \tag{5.12}$$

with  $|A_1|^2$  determined as a function of  $\mathbf{p}$  from (5.10). The principal observation here is that secondary criticality in the weakly nonlinear limit is determined from the dispersion relation  $D(\mathbf{p})$  and the weakly nonlinear correction to the dispersion relation  $\varrho(\mathbf{p})$ .

The determinant of the Jacobian, with  $\widehat{\mathcal{F}}_0$  of the form (5.11), is

$$\det[\mathbf{C}(\mathbf{p})] = \left| \frac{\partial(R, Q, B)}{\partial(h_0, u_0, k)} \right| = \frac{D_k^2}{\varrho} (gh_0 - u_0^2) + C_1(\mathbf{p})|A_1|^2 + \dots, \tag{5.13}$$

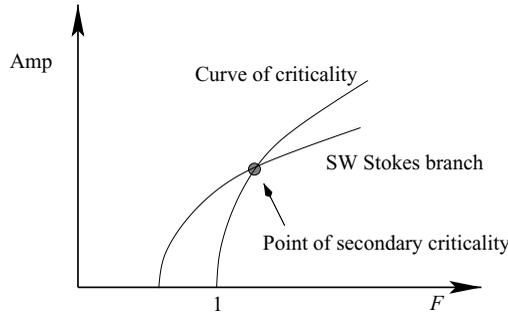


FIGURE 3. Typical weakly nonlinear curve of waves and curve of criticality in the Froude number versus amplitude plane.

where the subscripts on  $u_0$  and  $h_0$  in the Jacobian are suppressed for brevity, and

$$C_1(\mathbf{p}) = -\text{Trace}(\mathbf{E}(\mathbf{p})^{\text{adj}}\text{Hess}_{\mathbf{p}}(D)), \tag{5.14}$$

where

$$\mathbf{E}(\mathbf{p}) = \begin{pmatrix} g + \frac{D_h^2}{\varrho} & u_0 + \frac{D_h D_u}{\varrho} & \frac{D_h D_k}{\varrho} \\ u_0 + \frac{D_u D_h}{\varrho} & h_0 + \frac{D_u^2}{\varrho} & \frac{D_u D_k}{\varrho} \\ \frac{D_k D_h}{\varrho} & \frac{D_k D_u}{\varrho} & \frac{D_k^2}{\varrho} \end{pmatrix}, \quad \text{Hess}_{\mathbf{p}}(D) = \begin{pmatrix} \frac{\partial^2 D}{\partial h \partial h} & \frac{\partial^2 D}{\partial h \partial u} & \frac{\partial^2 D}{\partial h \partial k} \\ \frac{\partial^2 D}{\partial u \partial h} & \frac{\partial^2 D}{\partial u \partial u} & \frac{\partial^2 D}{\partial u \partial k} \\ \frac{\partial^2 D}{\partial k \partial h} & \frac{\partial^2 D}{\partial k \partial u} & \frac{\partial^2 D}{\partial k \partial k} \end{pmatrix}.$$

For symmetric matrices, the adjugate,  $\mathbf{E}^{\text{adj}}$  is the cofactor matrix, and when  $\mathbf{E}$  is invertible,  $\mathbf{E}^{\text{adj}} = \det(\mathbf{E})\mathbf{E}^{-1}$ . In the limit as  $|A_1| \rightarrow 0$ , there are two factors in the criticality determinant (5.13). It clearly vanishes when  $u_0^2 = gh_0$ , but also when  $D_k = 0$ . The significance of the latter term is discussed in Appendix C.

A schematic of secondary criticality curves is shown in figure 3. The axes are the Froude number based on mean velocity and mean elevation,  $F^2 = u_0^2/gh_0$ , and the vertical axis is the normalized amplitude of the wave height, denoted by Amp. The curve emanating from  $F = 1$  (and labelled ‘curve of criticality’ above the intersection point) is the new curve of secondary criticality, and the other curve is the curve along which the basic wave exists. When the two curves in figure 3 intersect, the criticality theory predicts that the linearization about the wave has a pair of Floquet multipliers that coalesce at +1, and is the starting point for the nonlinear bifurcation of dark solitary waves. Computed cases of these curves for the Boussinesq model and the water wave problem are shown in §§ 6 and 7.1, respectively.

### 6. Computing secondary criticality using a Boussinesq model

The dispersion relation for the linearized Boussinesq model is a truncated Taylor expansion of the exact dispersion relation. With  $h_0$  normalized to unity, it is

$$F^2 = 1 - \frac{1}{3}k^2 + \frac{2}{15}k^4.$$

This dispersion relation is plotted in figure 4. Boussinesq models are valid for small wavenumber, but how small the wavenumber depends on the particular Boussinesq model. Clearly the dispersion relation in figure 4 becomes non-physical for

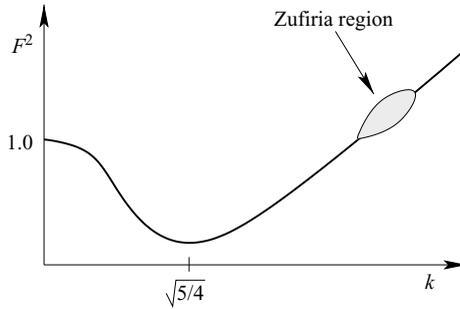


FIGURE 4. The dispersion relation for Zufiria’s Boussinesq model linearized with  $h_0 = 1$  and  $F^2 = u_0^2$ .

wavenumbers greater than  $\sqrt{5/4}$ . The region marked ‘Zufiria region’ will be discussed later when comparing to the secondary bifurcation theory of Zufiria.

The change in slope of the dispersion relation can be corrected by replacing higher space derivatives by time derivatives (e.g. § 5.4 of Dingemans 1997; Gobbi, Kirby & Wei 2000; Madsen, Bingham & Liu 2002). In addition to improved regularity, the dispersion relation of the linear problem becomes closer to the exact dispersion relation of water waves. However, these models are not Galilean invariant. Without Galilean invariance, the duality between steady solutions relative to a fixed frame, and steady solutions in a moving frame is lost. These models will not give the correct physics of travelling waves coupled to a mean flow. See Christov (2001) for numerical experiments showing how the lack of Galilean symmetry in Boussinesq models can also affect the dynamics of solitary waves.

To apply the theory of secondary criticality for weakly nonlinear waves of the steady Boussinesq model, expand  $\hat{Z}(\theta_3)$  in a Fourier series. (Note that the steady Boussinesq model (2.2) has an exact periodic solution in terms of cnoidal functions (Bridges & Fan 2004), but this exact solution does not give any more useful information than the weakly nonlinear theory.) By expanding  $\gamma(\theta_3)$  and  $\phi(\theta_3)$  the other elements in  $\hat{Z}(\theta_3)$  can be obtained and

$$\begin{aligned} \hat{\gamma}(\theta_3) &= A_1 e^{i\theta_3} + A_2 e^{2i\theta_3} + \text{c.c.} + \dots, \\ \hat{\phi}(\theta_3) &= B_1 e^{i\theta_3} + B_2 e^{2i\theta_3} + \text{c.c.} + \dots. \end{aligned}$$

Substituting into (5.8) results in the reduced functional

$$\widehat{\mathcal{F}} = -\frac{1}{2}h_0^2 - \frac{1}{2}h_0 u_0^2 + D(\mathbf{p})|A_1|^2 + \frac{1}{2}\varrho(\mathbf{p})|A_1|^4 + \dots,$$

with

$$D(\mathbf{p}) = \frac{k^2(15u_0^2 - 15h_0 + 5k^2 - 2k^4)}{15h_0 + 2k^4 - 5k^2},$$

where it is assumed that the denominator is non-vanishing, and

$$\varrho(\mathbf{p}) = \frac{45}{2} \frac{k^2(16k^4 - 20k^2 + 45h_0)}{(2k^2 - 1)(15h_0 + 2k^4 - 5k^2)^2}, \tag{6.1}$$

assuming  $2k^2 - 1 \neq 0$ . (The singularity  $k^2 = 1/2$  is an anomaly of the dispersion relation. When  $h_0 = 1$  and  $u_0^2 = 13/15$  both  $2k$  and  $k$  are roots of the dispersion relation (see figure 4). This resonance is similar to a Wilton resonance, but is anomalous here as the model (2.1) is derived for pure gravity waves.) Hence, the relation between  $|A_1|$

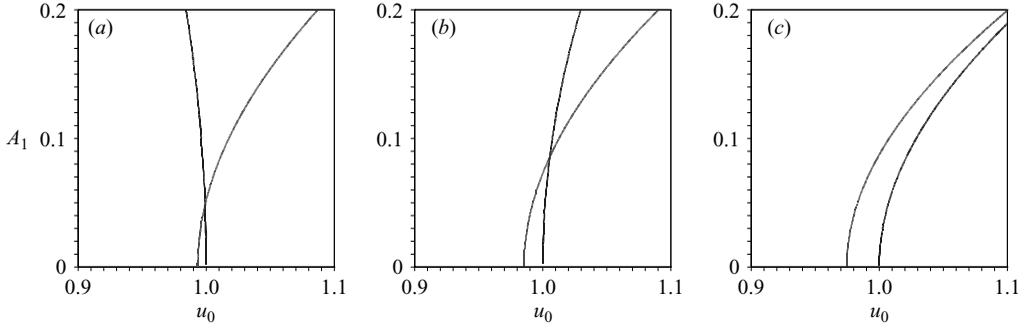


FIGURE 5. Plots of the curve of secondary criticality and the curve of periodic waves of the Boussinesq model when  $h_0 = 1$  and with (a)  $k = 0.2$ ; (b)  $k = 0.3$ ; (c)  $k = 0.4$ .

and the parameters is

$$D(\mathbf{p}) + \varrho(\mathbf{p})|A_1|^2 + \dots = 0. \tag{6.2}$$

Using the theory for secondary criticality of weakly nonlinear waves in § 5.1,

$$R = h_0 + \frac{1}{2}u_0^2 + \left( \frac{15k^2}{15h_0 + 2k^4 - 5k^2} - 15k^2 \frac{(2k^4 - 5k^2 + 15h_0 - 15u_0^2)}{(15h_0 + 2k^4 - 5k^2)^2} \right) |A_1|^2 + \dots,$$

$$Q = h_0 u_0 - \frac{30k^2 u_0}{15h_0 + 2k^4 - 5k^2} |A_1|^2 + \dots,$$

$$B = \left( 2k \frac{(2k^4 - 5k^2 + 15h_0 - 15u_0^2)}{15h_0 + 2k^4 - 5k^2} + k^2 \frac{8k^3 - 10k}{15h_0 + 2k^4 - 5k^2} - k^2 \frac{(2k^4 - 5k^2 + 15h_0 - 15u_0^2)(8k^3 - 10k)}{(15h_0 + 2k^4 - 5k^2)^2} \right) |A_1|^2 + \dots.$$

The determinant of the criticality matrix is then

$$\det[\mathbf{C}(\mathbf{p})] = D_k^2 \frac{(h_0 - u_0^2)}{\varrho} + C_1(\mathbf{p}) |A_1|^2 + \dots,$$

with

$$C_1(\mathbf{p}) = \frac{2k^4}{15} \frac{\widetilde{C}_1}{(16k^4 - 20k^2 + 45h_0)(15h_0 + 2k^4 - 5k^2)^2},$$

and

$$\begin{aligned} \widetilde{C}_1 = & 5376k^{12} - 31360k^{10} + 90480h_0k^8 + 54800k^8 - 297300h_0k^6 - 26000k^6 \\ & + 156000h_0k^4 + 2500k^4 + 448200h_0^2k^4 - 5625h_0k^2 - 317250h_0^2k^2 + 16875h_0^2. \end{aligned}$$

A point of secondary criticality occurs when  $\det[\mathbf{C}(\mathbf{p})] = 0$  along a branch of waves. The solutions of (6.2) and the curves of  $\det[\mathbf{C}(\mathbf{p})] = 0$  are shown plotted in figure 5 for  $k = 0.2$ ,  $k = 0.3$  and  $k = 0.4$  with  $h_0 = 1$ . With this normalization, the abscissa in the figures can be interpreted as the nonlinear Froude number. The ordinate is the amplitude. For all  $k$  sufficiently small (less than about 0.35) there is a point of secondary criticality in the weakly nonlinear approximation.

It is shown in Appendix B that secondary criticality is associated with a pair of spatial Floquet multipliers passing through +1. In Zufiria (1987) a branch of periodic waves is followed numerically and it is found that a pair of Floquet multipliers coalesces at +1 (see figure 3 on p. 380 of Zufiria 1987). Since Floquet multipliers can coalesce at +1 if and only if there is a point of secondary criticality (generically), there should be a point of secondary criticality associated with this secondary bifurcation. Indeed, taking  $h_0 = 1$  and  $k = 2\pi/3.8$ , it is confirmed that the curve (6.2) and the curve  $\det[\mathbf{C}(\mathbf{p})] = 0$  have an intersection point.

Note, however, that  $k = 2\pi/3.8 \approx 1.6$  is quite large and outside the reasonable range of validity of the model. Hence, the secondary bifurcations in Zufiria, in the region noted in figure 4, are of no physical significance.

However, the results of the theory of secondary criticality applied in the region of validity of the Boussinesq model show that secondary bifurcation is to be expected for shallow-water waves. In the next section it will be confirmed that these secondary bifurcations are also found by applying the theory directly to the full water-wave problem.

### 7. Periodic water waves coupled to a uniform flow

The above formulation goes through for the full water-wave problem, leading to the following coupled problem for the periodic wave coupled to a mean flow

$$\left. \begin{aligned} \nabla S(\widehat{Z}) &= h_0 \nabla R(\widehat{Z}) + u_0 \nabla Q(\widehat{Z}) + k \nabla B(\widehat{Z}), \\ R(\widehat{Z}) &= \mathcal{R}, \\ Q(\widehat{Z}) &= \mathcal{Q}, \\ B(\widehat{Z}) &= \mathcal{B}. \end{aligned} \right\} \quad (7.1)$$

These four equations are solved for  $(\widehat{Z}, h_0, u_0, k)$  as functions of  $\mathcal{R}$ ,  $\mathcal{Q}$  and  $\mathcal{B}$ .

It is not essential to solve (7.1) in the  $\widehat{Z}$  coordinates. Once the structure is established, the equations can be put in any coordinates that are convenient, say for numerical computation. For example, suppose we want to solve (7.1) in the standard coordinates of velocity potential and wave height.

To ease notation let  $\theta_3 := \theta$  for the remainder of this subsection, and define  $\widehat{\eta}(\theta) = h_0 + \widehat{\Gamma}(\theta)$ . Then the system (7.1) takes the form

$$k^2 \widehat{\phi}_{\theta\theta} + \widehat{\phi}_{zz} = 0 \quad \text{for } 0 < z < h_0 + \widehat{\Gamma}(\theta),$$

subject to the boundary conditions,

$$\widehat{\phi}_z = 0 \quad \text{at } z = 0, \quad \widehat{\phi}_z = ku_0 \widehat{\Gamma}_\theta + k^2 \widehat{\phi}_\theta \widehat{\Gamma}_\theta \quad \text{at } z = h_0 + \widehat{\Gamma}(\theta),$$

and

$$ku_0(\widehat{\phi}_\theta - \overline{\widehat{\phi}_\theta}) + \frac{1}{2}k^2 \widehat{\phi}_\theta^2 + \frac{1}{2}\widehat{\phi}_z^2 + g\widehat{\Gamma} = \overline{\frac{1}{2}\widehat{\phi}_z^2 + \frac{1}{2}k^2 \widehat{\phi}_\theta^2} \quad \text{at } z = h_0 + \widehat{\Gamma}(\theta).$$

The surface elevation  $\widehat{\eta}(\theta)$  is split into a mean value  $h_0$  and a fluctuating part  $\widehat{\Gamma}(\theta)$ . The unknowns in these equations are therefore  $(\widehat{\phi}(\theta, z), \widehat{\Gamma}(\theta), h_0, u_0, k)$  where  $\widehat{\phi}$  and  $\widehat{\Gamma}$  are  $2\pi$ -periodic functions of  $\theta$  and have mean value zero. The overbar denotes an average over  $\theta$ :  $\overline{(\cdot)} = (1/2\pi) \oint (\cdot) d\theta$ . The system is completed by adding the

constraints in the coordinates  $\widehat{\phi}$  and  $\widehat{\Gamma}$ ,

$$\begin{aligned} \mathcal{R} &= gh_0 + \frac{1}{2}u_0^2 + ku_0\overline{\widehat{\phi}_\theta} + \frac{1}{2}\overline{\widehat{\phi}_z^2} + \frac{1}{2}k^2\overline{\widehat{\phi}_\theta^2} \quad \text{at } z = h_0 + \widehat{\Gamma}(\theta), \\ \mathcal{Q} &= h_0u_0 + k\int_0^{\widehat{\eta}}\widehat{\phi}_\theta \, dz \quad \text{where } \widehat{\eta}(\theta) = h_0 + \widehat{\Gamma}(\theta), \\ \mathcal{B} &= u_0\int_0^{\widehat{\eta}}\widehat{\phi}_\theta \, dz + k\int_0^{\widehat{\eta}}\widehat{\phi}_\theta^2 \, dz. \end{aligned}$$

This problem is set up for fixed  $(\mathcal{R}, \mathcal{Q}, \mathcal{B})$ , but alternative choices of parameters are possible. The three functionals are paired with co-parameters (the Lagrange multipliers)

$$(h_0, \mathcal{R}), \quad (u_0, \mathcal{Q}), \quad (k, \mathcal{B}),$$

and in each of these three sets, any one of each pair can be fixed. For example,  $(h_0, u_0, k)$  could be fixed, and then  $(\mathcal{R}, \mathcal{Q}, \mathcal{B})$  computed. This latter ordering is certainly the simplest approach (the constraints would decouple) and would also be naturally conducive to computing the derivatives in  $\mathbf{C}(\mathbf{p})$  numerically.

As far as we are aware, shallow-water periodic waves have never been considered in this generality in the literature. Invariably most of the parameters are fixed, and a slice through the parameter space is studied. Representative examples are the calculations in Doole & Norbury (1995) where  $h_0, \mathcal{Q}$  and  $k$  are fixed with  $\mathcal{R}, u_0$  and (in principle)  $\mathcal{B}$  varying, and the calculations in §3 of McLean (1982) where  $\mathcal{R}, \mathcal{Q}$  and  $k$  are fixed, with  $h_0, u_0$  and (in principle)  $\mathcal{B}$  varying. The only work to consider the use of wave action flux in parameterizing shallow-water waves is Stiassne & Peregrine (1980). Indeed, they parameterize the waves in terms of wave action flux, mass flux and Bernoulli constant as here, considered as functions of mean velocity, mean depth and wavenumber, but ultimately set the mass flux to zero.

In general, the decision about which parameters to fix depends on the physical problem of interest. For example, in computation, it is customary to fix the mean velocity and elevation and solve for the mass flux and total head. However, this would be near impossible in an experiment, where it is natural to input the mass flux and total head and let them determine the mean velocity and mean elevation.

### 7.1. Criticality of weakly nonlinear periodic water waves

In this section, the above characterization of periodic waves is illustrated by computing the criticality matrix for weakly nonlinear water waves.

According to (5.1), we need to construct

$$\widehat{\mathbf{Z}}(\theta_3) = (\widehat{\Phi}(\theta_3), \widehat{\eta}(\theta_3), \widehat{\gamma}(\theta_3), \widehat{\phi}(\theta_3, z), \widehat{u}(\theta_3, z), \widehat{r}(\theta_3)),$$

with each element a  $2\pi$ -periodic function of  $\theta_3$ . Let

$$\left. \begin{aligned} \widehat{\gamma}(\theta_3) &= A_1 e^{i\theta_3} - \overline{A_1} e^{-i\theta_3} + A_2 e^{2i\theta_3} - \overline{A_2} e^{-2i\theta_3} + \dots, \\ \widehat{\phi}(\theta_3, z) &= (B_1 e^{i\theta_3} + \overline{B_1} e^{-i\theta_3}) \frac{\cosh kz}{\cosh kh_0} + (B_2 e^{2i\theta_3} + \overline{B_2} e^{-2i\theta_3}) \frac{\cosh 2kz}{\cosh 2kh_0} + \dots, \end{aligned} \right\} \quad (7.2)$$

where  $A_1, A_2, B_1$  and  $B_2$  are complex amplitudes. From these expressions, the other elements in  $\widehat{\mathbf{Z}}$  are determined from

$$\widehat{\eta}(\theta_3) = h_0 + k \frac{\partial \widehat{\gamma}}{\partial \theta_3}, \quad \widehat{u}(\theta_3, z) = u_0 + k \frac{\partial \widehat{\phi}}{\partial \theta_3},$$

and the equations. Substitute these expressions into the functional (5.8), and eliminate the higher-order Fourier coefficients to obtain the reduced functional

$$\widehat{\mathcal{F}}(|A_1|^2, \mathbf{p}) = -\frac{1}{2}gh_0^2 - \frac{1}{2}h_0u_0^2 + D(\mathbf{p})|A_1|^2 + \frac{1}{2}\varrho(\mathbf{p})|A_1|^4 + \dots,$$

where

$$D(\mathbf{p}) = \frac{k^3u_0^2}{\tanh(kh_0)} - gk^2, \tag{7.3}$$

and

$$\varrho(\mathbf{p}) = -\frac{1}{2}gk^6 \left( \frac{9 - 10 \tanh^2(kh_0) + 9 \tanh^4(kh_0)}{\tanh^4(kh_0)} \right). \tag{7.4}$$

The relationship between  $|A_1|$  and the parameters is then found from the derivative of  $\widehat{\mathcal{F}}$  with respect to  $A_1$ ,

$$D(\mathbf{p}) + \varrho(\mathbf{p})|A_1|^2 + \dots = 0.$$

This expression recovers a familiar result for weakly nonlinear water waves in finite depth

$$u_0^2 = \frac{g}{k} \tanh(kh_0) + \frac{1}{2}gk^3 \left( \frac{9 - 10 \tanh^2(kh_0) + 9 \tanh^4(kh_0)}{\tanh^3(kh_0)} \right) |A_1|^2 + \dots. \tag{7.5}$$

Now evaluate the criticality matrix on this solution. Substituting the weakly nonlinear expressions into  $(R, Q, B)$  we find

$$\left. \begin{aligned} R &= gh_0 + \frac{1}{2}u_0^2 - \frac{\partial D}{\partial h_0}|A_1|^2 + \dots, \\ Q &= h_0u_0 - \frac{\partial D}{\partial u_0}|A_1|^2 + \dots, \\ B &= -\frac{\partial D}{\partial k}|A_1|^2 + \dots, \\ S &= h_0u_0^2 + \frac{1}{2}gh_0^2 - \left( h_0 \frac{\partial D}{\partial h_0} + u_0 \frac{\partial D}{\partial u_0} + k \frac{\partial D}{\partial k} \right) |A_1|^2 + \dots. \end{aligned} \right\} \tag{7.6}$$

Explicit expressions for the derivatives of  $D$  are easily obtained using (7.3). The expressions for  $R$ ,  $Q$  and  $B$  give the most general perturbation of the hydraulic quantities due to a periodic wave on the uniform flow. Parts of these expressions agree with special cases in the literature. The result for  $R$  in (7.6) agrees with Whitham's calculation (in equation (16.81) of Whitham 1974), and the expression for  $Q$  in (7.6) agrees with Whitham's result (in equation (16.84) of Whitham 1974). However, the weakly nonlinear  $R$  above differs from the weakly nonlinear  $R$  in Doole & Norbury (1995) because  $Q$  is held fixed there.

The terms proportional to  $|A_1|^2$  provide the correction to the total head, mass flux and action flux due to the periodic wave. The most important quantity is the determinant of the criticality matrix. A straightforward but lengthy computation leads to

$$\det[\mathbf{C}(\mathbf{p})] = \left| \frac{\partial(R, Q, B)}{\partial(h_0, u_0, k)} \right| = \frac{D_k^2}{\varrho} (gh_0 - u_0^2) + C_1(\mathbf{p})|A_1|^2 + \dots, \tag{7.7}$$

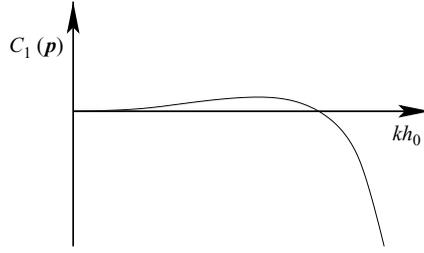


FIGURE 6. A plot of the wave-generated criticality function  $C_1(\mathbf{p})$  plotted against  $kh_0$ . The change of sign occurs at  $kh_0 = (kh_0)^{point\ 1} \approx 0.588$ .

where the subscripts to  $u_0$  and  $h_0$  in the Jacobian are suppressed for brevity. Using the formula for  $C_1(\mathbf{p})$  in (5.14),

$$C_1(\mathbf{p}) = -\frac{1}{\varrho} (C_2(\mathbf{p}) + (gh_0 - u_0^2)\varrho D_{kk}), \tag{7.8}$$

with

$$C_2(\mathbf{p}) = g(D_{u_0}D_{u_0}D_{kk} - 2D_{u_0}D_kD_{u_0k} + D_k^2D_{u_0u_0}) + h_0D_k(D_kD_{h_0h_0} - 2D_{h_0}D_{h_0k}) + h_0D_h^2D_{kk} + 2u_0(D_{h_0}D_kD_{u_0k} - D_{h_0}D_{u_0}D_{kk} - D_k^2D_{h_0u_0} + D_{u_0}D_kD_{h_0k}).$$

Now expand each of these terms and use  $D(\mathbf{p})$  for gravity waves,

$$C_1(\mathbf{p}) = -\frac{g^3k^5}{\varrho\sigma^6} (18\sigma^3 - 34\sigma^5 + 26\sigma^7 + kh_0(-45\sigma^2 + 87\sigma^4 - 81\sigma^6 + 23\sigma^8) + k^2h_0^2(36\sigma - 72\sigma^3 + 74\sigma^5 - 36\sigma^7 - 2\sigma^9) + k^3h_0^3(-9 + 19\sigma^2 - 19\sigma^4 + 9\sigma^6))$$

where  $\sigma = \tanh(kh_0)$ . This expression provides the leading-order term of the ‘wave-generated criticality’, and does not appear to have been noticed before in the literature. When  $|A_1| = 0$  the determinant (7.7) is negative when  $gh_0 - u_0^2 > 0$  since  $\varrho < 0$  (assuming  $D_k \neq 0$ ). So, for  $|A_1|^2$  sufficiently small, the determinant of the criticality matrix is strictly negative. However, it can vanish at finite amplitude. Setting  $\det[\mathbf{C}(\mathbf{p})] = 0$  leads to

$$F^2 = 1 + \frac{\varrho h_0 C_1(\mathbf{p})}{gk^2 D_k^2} |\text{Amp}|^2 + \dots \tag{7.9}$$

where

$$\text{Amp} = \frac{k|A_1|}{h_0}, \quad F^2 = \frac{u_0^2}{gh_0}.$$

The curve (7.9) is fundamental to shallow-water Stokes waves, but does not appear anywhere in the literature heretofore. It is the curve of secondary criticality.

For gravity waves,  $D_k \neq 0$  and  $\varrho < 0$ . Therefore the sign of the weakly nonlinear (wave-generated) term in (7.9) depends on the sign of  $C_1$ , and a plot of  $C_1$  is shown in figure 6. A significant feature is the change of sign occurring at  $(kh_0)^{point\ 1}$ .

Now, compare this with the existence branch (7.5) expressed in terms of the mean Froude number,

$$F^2 = \frac{u_0^2}{gh_0} = \frac{\tanh(kh_0)}{kh_0} + \frac{kh_0}{2} \left( \frac{9 - 10 \tanh^2(kh_0) + 9 \tanh^4(kh_0)}{\tanh^3(kh_0)} \right) |\text{Amp}|^2 + \dots \tag{7.10}$$

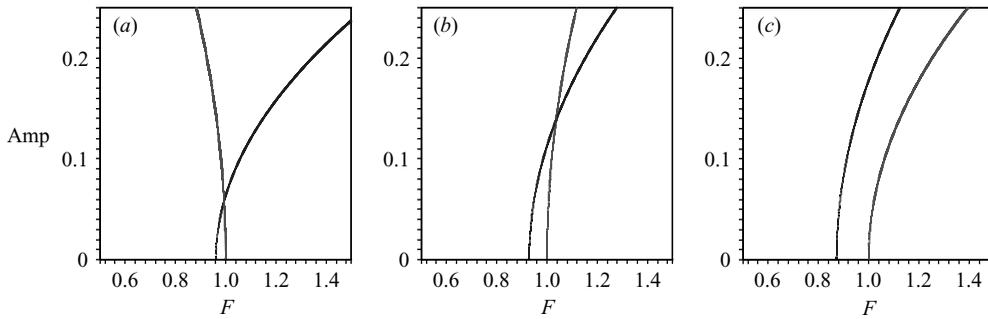


FIGURE 7. Effect of criticality curve on shallow-water Stokes waves for different values of  $kh_0$ : (a)  $kh_0 = 0.5 < (kh_0)^{point 1}$ ; (b)  $(kh_0)^{point 1} < kh_0 = 0.7 < (kh_0)^{point 2}$ ; and (c)  $kh_0 = 1.0 > (kh_0)^{point 2}$ .

For any fixed  $kh_0$ , secondary criticality occurs along a weakly nonlinear branch if these two curves (7.10) and (7.9), considered in the  $(F, \text{Amp})$ -plane, intersect. Setting these two expressions for  $F^2$  equal to each other leads to the following condition for secondary criticality to occur in the weakly nonlinear case

$$-\frac{\varrho h_0 C_1(\mathbf{p})}{gk^2 D_k^2} + \frac{kh_0}{2} \left( \frac{9 - 10 \tanh^2(kh_0) + 9 \tanh^4(kh_0)}{\tanh^3(kh_0)} \right) > 0.$$

Evaluating this expression numerically, shows that it is satisfied for all  $kh_0 < (kh_0)^{point 2} \approx 0.85$ . Therefore, there are three regions of interest and plots of the Froude number amplitude plane for each are shown in figure 7. These figures agree qualitatively with the weakly nonlinear secondary criticality occurring in the Boussinesq model in figure 5. Note however that the values of  $kh_0$  at which secondary criticality occurs are shifted to lower wavenumbers in the Boussinesq model. This is probably because the nonlinear dispersion relation for the Boussinesq model has an anomolous singularity at  $k^2 = 1/2$  (see equation (6.2) and the definition of  $\varrho$  in (6.1)). Hence the shallow-water behaviour shown in figure 7 appears to be compressed into lower wavenumbers in figure 5.

Although periodic Stokes waves exist in shallow water for all  $kh_0 > 0$  (see Cokelet 1977; Stiassne & Peregrine 1980; Amick & Toland 1981 for details), the two-term approximation to the periodic Stokes wave will not be accurate for low values of  $kh_0$  (high Ursell number (the Ursell number in the notation of this paper is  $\text{Amp} (kh_0)^{-2}$ ), and so quantitative estimation of points of secondary criticality at low values of  $kh_0$  will require significantly more terms in the expansion for the wave height, or other numerical calculation. The elementary analytical calculations are, however, sufficient to capture the qualitative properties of secondary criticality.

The main conclusion from this analysis is that weakly nonlinear gravity waves in shallow water have a point of secondary criticality at low amplitude for all  $kh_0$  less than about 0.85. It is highly likely that for  $kh_0 > (kh_0)^{point 2}$  there is still a point of secondary criticality, but at higher amplitude.

If we argue in reverse: when a pair of spatial Floquet multipliers passes through +1 in the linearization about Stokes waves, it corresponds to a point of secondary criticality, there is evidence for secondary criticality occurring at finite amplitude for Stokes waves in finite depth in the numerical work of Vandebroek (1983) (and Zufria (1987) as discussed in §6). Vandebroek computes secondary bifurcations,

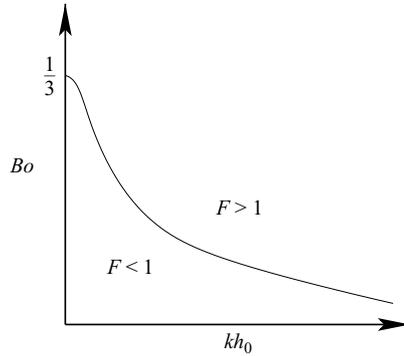


FIGURE 8. Curve of Froude number unity in the Bond number versus  $kh_0$ -plane.

and a necessary condition for the secondary bifurcations there is the appearance of Floquet multipliers on the unit circle.

The theory of this paper does not apply to water waves in infinite depth, but there is a form of criticality arising there, in the sense that spatial Floquet multipliers passing through +1 have been observed. For Stokes waves in infinite depth, this bifurcation of Floquet multipliers was first discovered by Baesens & MacKay (1992). They used the flow force for water waves to show that a fold point occurs precisely when a pair of Floquet multipliers passes through +1. Remarkably, Buffoni, Dancer & Toland (2000) have proved that there is an infinite number of such fold points along a branch of Stokes waves in deep water. However, in deep water, these points occur at very large amplitude.

7.2. Implications of surface tension for criticality of water waves

When surface tension is included, the secondary criticality of periodic capillary-gravity waves on shallow water is even more pervasive than for gravity waves. For capillary-gravity waves the existence curve (7.10) is modified to

$$F^2 = (1 + Bo(kh_0)^2) \frac{\tanh(kh_0)}{kh_0} + \tilde{q}(Bo)|\text{Amp}|^2 + \dots, \tag{7.11}$$

where  $\tilde{q}$  reduces to the expression in (7.10) when the Bond number  $Bo$  is zero. The expression for the criticality curve (7.9) will have the same general form when surface tension is present, and when  $D_k \neq 0$ , the leading-order part depends only on the sign of  $qC_1(\mathbf{p})$  (modified by surface tension). The possibilities for sign changes are much richer here (and there is the additional singularity at  $D_k = 0$  discussed in Appendix C). But, explicit expressions are not required for any of these coefficients, as rather general qualitative behaviour can be deduced from the form alone.

When  $\text{Amp} = 0$  and  $D_k \neq 0$  the criticality curve always starts at  $F^2 = 1$ , with or without surface tension. When surface tension vanishes ( $Bo = 0$ ), the existence curve (7.10) always starts with  $F^2 < 1$ ; i.e. to the left of the criticality curve. However, when  $Bo > 0$  the existence curve can start at  $F^2 < 1$  or  $F^2 > 1$ , depending on the value of  $kh_0$ .

For each  $Bo \in (0, 1/3)$  there is a value of  $kh_0$  at which  $F^2 = 1$ :

$$Bo = -\frac{1}{kh_0^2} + \frac{1}{kh_0 \tanh kh_0} \quad \text{with} \quad \lim_{kh_0 \rightarrow 0} Bo = \frac{1}{3}.$$

In figure 8, this curve of Froude number unity is shown. For any fixed  $Bo \in (0, 1/3)$ ,

as  $kh_0$  is increased and the  $F^2 = 1$  curve is crossed, then generically (as long as the coefficient of  $|A_1|^2$  remains finite) the existence and criticality curves cross at low amplitude. Therefore, for each  $Bo \in (0, 1/3)$ , there always (generically) exist values of  $kh_0$  where secondary criticality occurs in the weakly nonlinear limit. These points of secondary criticality will then be bifurcation points for branches of steady dark (capillary–gravity) solitary waves.

**8. Nonlinearity, secondary criticality and steady dark solitary waves**

At points of secondary criticality, that is points on the surface defined by  $\det[\mathbf{C}(\mathbf{p})] = 0$ , the theory of Bridges (2006*b*) can be applied to determine the properties of the bifurcating homoclinic orbits. In the present context, the homoclinic orbit is biasymptotic to the periodic Stokes waves coupled to a mean flow. This homoclinic orbit in the spatial setting is a steady DSW. There is a three-parameter family of bifurcating DSWs. They can be parameterized by the Bernoulli constant, mass flux and wave action flux, and hence they occupy a large region of physical parameter space. A sketch of how this theory applies to the Boussinesq model is given, and then results for water waves will be summarized.

Consider the steady Boussinesq equation (2.5) perturbed about a family of relative equilibria near a point where  $\det[\mathbf{C}(\mathbf{p})] = 0$ . Taking a solution of the form

$$Z(x) = G(\theta(x))[\widehat{Z} + W(x)] := \widehat{Z}(\theta_3) + W(\theta_3, x) + \theta_1(x)\mathbf{g}_1 + \theta_2(x)\mathbf{g}_2, \tag{8.1}$$

the nonlinear problem for  $W(\theta_3, x)$  is

$$\mathbf{J}W_x = \mathbf{L}(\mathbf{p})W + \frac{1}{2}D^3S(\widehat{Z})(W, W) + \dots \tag{8.2}$$

where the dots indicate terms of degree three and higher in  $\|W\|$ , and  $D^3S(\widehat{Z})(W, W)$  is the third derivative of  $S(Z)$  at the point  $\widehat{Z}$ , and

$$\mathbf{L}(\mathbf{p}) = D^2S(\widehat{Z}) - h_0 D^2R(\widehat{Z}) - u_0 D^2Q(\widehat{Z}) - k D^2B(\widehat{Z}).$$

There exists a set of eight vectors  $\{\xi_1(\theta_3), \dots, \xi_8(\theta_3)\}$ , which are constructed using the generalized eigenvectors (see Appendix B), such that the transformation

$$\left. \begin{aligned} W(x, \theta_3) = & \widetilde{\phi}_1(x)\xi_1(\theta_3) + \widetilde{\phi}_2(x)\xi_2(\theta_3) + \widetilde{\phi}_3(x)\xi_3(\theta_3) + \widetilde{u}(x)\xi_4(\theta_3) \\ & -s_1\widetilde{I}_1(x)\xi_8(\theta_3) + s_2\widetilde{I}_2(x)\xi_5(\theta_3) + s_3\widetilde{I}_3(x)\xi_6(\theta_3) + s_1\widetilde{v}(x)\xi_7(\theta_3) + \dots \end{aligned} \right\} \tag{8.3}$$

(note the ordering of the generalized eigenvectors in this expression) results in the reduced system of nonlinear ODEs

$$\left. \begin{aligned} -\widetilde{I}_1 &= 0, & \widetilde{\phi}_1 &= \widetilde{u} + \dots, \\ -\widetilde{I}_2 &= 0, & \widetilde{\phi}_2 &= s_2\widetilde{I}_2 + \dots, \\ -\widetilde{I}_3 &= 0, & \widetilde{\phi}_3 &= s_3\widetilde{I}_3 + \dots, \\ -\widetilde{v} &= \widetilde{I}_1 + \delta\widetilde{u} - \frac{1}{2}\kappa\widetilde{u}^2 + \dots, & \widetilde{u} &= s_1\widetilde{v} + \dots. \end{aligned} \right\} \tag{8.4}$$

The parameter  $\delta$  is an unfolding parameter; it is proportional the value of  $\det(\mathbf{C}(\mathbf{p}))$  and is not qualitatively significant. The signs  $s_2 = \pm 1$  and  $s_3 = \pm 1$  are the signs of the non-zero eigenvalues of the criticality matrix  $\mathbf{C}(\mathbf{p})$ . They determine information about direction of drift along the phases. The sign  $s_1 = \pm 1$  is related to the Jordan structure and has no qualitative impact on the bifurcating solitary waves. The important parameters are  $\kappa$  and  $\widetilde{I}_1$ .

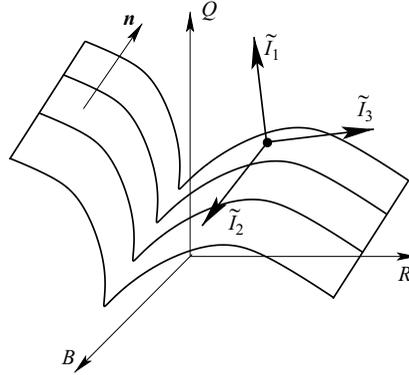


FIGURE 9. Illustration of the local coordinates  $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$  on the surface of criticality.

The coordinates  $\tilde{I}_j$  are a translation and rotation of the coordinates  $(R, Q, B)$  to a point on the criticality surface (see figure 9). (The surface in  $(R, Q, B)$ -space with a cusp along an edge in figure 9 is a generalization of the curve with a cusp in the  $(R, Q)$ -plane that is central to the Benjamin–Lighthill (1954) theory. It is also a generalization of the cusp for uniform flows shown in figure 2 in this paper.) The coordinates are oriented so that  $\tilde{I}_2$  and  $\tilde{I}_3$  give local coordinates tangent to the surface and  $\tilde{I}_1$  is a local normal coordinate given to leading order in terms of  $(R, Q, B)$  by

$$\tilde{I}_1 = a_0(\mathcal{P}(\mathbf{p}) - \mathcal{P}(\mathbf{p}_{crit})) + \dots \quad \text{with} \quad \mathcal{P} := n_1\mathcal{R} + n_2\mathcal{Q} + n_3\mathcal{B}, \quad (8.5)$$

where  $a_0$  is a positive constant.

The normal vector  $\mathbf{n} = (n_1, n_2, n_3)$  is defined by  $\mathbf{C}(\mathbf{p})\mathbf{n} = 0$ , and  $\mathbf{n}$  can be taken to be a column of the adjugate matrix, for example,

$$\mathbf{n} = \begin{pmatrix} R_u Q_k - R_k Q_u \\ R_k Q_h - R_h Q_k \\ R_h Q_u - R_u Q_h \end{pmatrix} \Rightarrow \mathbf{C}(\mathbf{p})\mathbf{n} = \det[\mathbf{C}(\mathbf{p})] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (8.6)$$

Hence,  $\mathbf{n}$  is an eigenvector of  $\mathbf{C}(\mathbf{p})$  corresponding to the zero eigenvalue when  $\mathbf{p} = \mathbf{p}_{crit}$ .

The most important parameter is  $\kappa$ , the coefficient of the nonlinear term in the normal form, and the explicit formula for it is

$$\kappa = a_0^3 \mathbf{n}^T \text{Hess}_p(\mathcal{P})\mathbf{n}, \quad (8.7)$$

where  $a_0$  is a positive constant (the same constant as in (8.5)) and

$$\text{Hess}_p(\mathcal{P}) = n_1 \text{Hess}_p(\mathcal{R}) + n_2 \text{Hess}_p(\mathcal{Q}) + n_3 \text{Hess}_p(\mathcal{B}) \quad \text{evaluated at } \mathbf{p} = \mathbf{p}_{crit}.$$

$\text{Hess}_p$  is the matrix of second derivatives with respect to the parameters  $\mathbf{p} = (h_0, u_0, k)$ .

The formula (8.7) is remarkable. It shows that the critical coefficient in the nonlinear normal form is determined by curvature information in the parameter space  $(R, Q, B)$ . A special case of this result was first announced in Bridges & Donaldson (2005a). In the present context, it enables complete qualitative information about the bifurcating DSWs to be determined from the criticality matrix.

The solitary wave solutions are determined from the reduced equation

$$\begin{aligned} -\tilde{v} &= \tilde{I}_1 + \delta\tilde{u} - \frac{1}{2}\kappa\tilde{u}^2 + \dots, \\ \tilde{u} &= s_1\tilde{v} + \dots. \end{aligned}$$

Note that the degree of difficulty of solving for the DSWs using this equation is the same as the degree of difficulty of solving for the Russell solitary wave in §3. The simplification is due to the universality of the nonlinear normal form (8.4). The oscillation of the solitary wave is encoded in the vectors  $\tilde{\xi}_j(\theta_3)$ ,  $j = 1, \dots, 8$ .

The side of the surface where the DSWs lie is determined by the values of  $\tilde{I}_1$  and  $\kappa$  such that the reduced system has non-zero fixed points,

$$\tilde{I}_1 + \delta\tilde{u} - \frac{1}{2}\kappa\tilde{u}^2 + \dots = 0.$$

For  $\delta$  small, and  $\tilde{u}$  small, it is clear that fixed points exist for  $\kappa\tilde{I}_1 > 0$ . Hence, if  $\kappa > 0$ , then the direction of bifurcation of DSWs is aligned with the chosen normal direction and vice versa.

When  $\kappa\tilde{I}_1 > 0$ , solitary waves exist and are given locally by

$$\tilde{u}(x) = v - 3 \left( v - \frac{\delta}{\kappa} \right) \operatorname{sech}^2(\mu x), \quad \mu^2 = \frac{1}{4}(s_1\kappa v - s_1\delta),$$

where  $v$  is the root of the quadratic

$$\kappa v^2 - 2\delta v - 2\tilde{I}_1 = 0,$$

chosen so that  $s_1\kappa v > 0$ . Note that such a choice is always possible when  $\delta$  is sufficiently small.

An important property of a dark solitary wave is the geometric phase. The total phase is obtained by solving the equations for  $\tilde{\phi}_1, \dots, \tilde{\phi}_3$ ,

$$\begin{aligned} \tilde{\phi}_1(x) &= vx - \frac{3}{\mu} \left( v - \frac{\delta}{\kappa} \right) \tanh(\mu x) + \tilde{\phi}_1^0, \\ \tilde{\phi}_2(x) &= s_2\tilde{I}_2x + \tilde{\phi}_2^0, \\ \tilde{\phi}_3(x) &= s_3\tilde{I}_3x + \tilde{\phi}_3^0, \end{aligned}$$

where  $\tilde{\phi}_j^0$ ,  $j = 1, 2, 3$  is an arbitrary phase shift. The interesting part of the phase shift is the geometric phase

$$\Delta\tilde{\phi}_1 := -\frac{3}{\mu} \left( v - \frac{\delta}{\kappa} \right) \tanh(\mu x) \Big|_{x=-\infty}^{x=+\infty} = -\frac{6}{\mu} \left( v - \frac{\delta}{\kappa} \right). \tag{8.8}$$

The geometric phase gives the phase jump in going from the Stokes wave at  $x = -\infty$  to the Stokes wave at  $x = +\infty$ .

Substituting the solutions of (8.4) into  $W(x, \theta_3)$  in (8.3) brings in the  $\theta_3$  dependence and the full form of the steady dark solitary wave, including the oscillatory part.

### 9. Steady dark solitary waves of the water-wave problem

The theory of §8 is applied to the water-wave problem to determine the properties of the bifurcating dark solitary waves. The mechanism for the bifurcation of these waves is different from other oscillatory solitary waves that have been found in the water wave problem. Therefore it is useful to first review the literature on known oscillatory solitary waves and dark solitary waves found in water wave models.

This bifurcation of steady DSWs is mathematically similar to – but physically different from – the DSWs that appear in the defocusing nonlinear Schrödinger (NLS) equation for water waves (e.g. Peregrine 1983, 1985). The DSWs in this paper are steady in an absolute frame of reference, whereas the DSW solutions of the NLS

equation for gravity water waves in finite depth are unsteady in any reference frame. Consider the NLS model for modulation of Stokes waves in finite depth

$$iA_\tau + \mu A_{\xi\xi} + \tilde{v}|A|^2 A = 0. \quad (9.1)$$

For gravity waves  $\mu < 0$  and for  $kh_0$  less than the critical value of approximately 1.36, the coefficient  $\tilde{v}$  is positive. Unsteady DSWs exist when  $\mu\tilde{v} < 0$  with explicit expression

$$A(\xi, \tau) = e^{i(\tilde{k}\xi - \tilde{\omega}\tau)} \left( \frac{2\mu^2}{|\mu\tilde{v}|} \right)^{1/2} (\tilde{k} + i\chi \tanh(\chi\xi)), \quad \chi^2 = \frac{1}{2\mu^2}(\mu\tilde{\omega} - 3\mu^2\tilde{k}^2), \quad (9.2)$$

and the requirement  $\mu\tilde{\omega} > 3a^2\tilde{k}^2$ . The full unsteadiness of these waves becomes apparent when viewed as solutions of the water-wave problem: the leading-order expression for the waveheight is

$$\eta(x, t) = \varepsilon \operatorname{Re}(A(\xi, \tau)e^{i(k_0x - \omega_0t)}), \quad \xi = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t.$$

There are two frame speeds: the phase speed of the Stokes wave, and the frame speed of the DSW which is the group velocity, and further unsteadiness due to the  $e^{-i\tilde{\omega}\tau}$  term. In fact, the NLS type modulation equations for Stokes waves in finite depth are intrinsically unsteady (cf. Bridges 2005), and therefore cannot capture steady DSWs that are synchronized with the Stokes wave. Unsteady DSWs are generated by a different mechanism: the transition from Benjamin–Feir unstable waves to stable waves. These unsteady DSWs will come back into the story when unsteady aspects of criticality are considered in Bridges & Donaldson (2006).

In the full two-dimensional water-wave problem, steady solitary waves with oscillatory tails have been found previously, but only when surface tension is present. There are surface-tension dominated solitary waves which are biasymptotic to the trivial state (or to an exponentially small state) associated with the minimum of the dispersion relation; see Dias & Iooss (2003) for a recent review of these waves with extensive references. Dias & Iooss (1994) also find steady DSWs at the interface between two fluids when surface tension is present. The bifurcation of DSWs is similar mathematically to the nonlinear behaviour near a transition to superharmonic instability, but in that context the homoclinic orbit is in time, and not related to a solitary wave (cf. Bridges 2004).

In three-dimensions, steady gravity-driven DSWs have been found in water-wave problems when the variation is in the transverse direction. Examples are given in Peregrine (1983, 1985 §8). Roberts & Peregrine (1983) compute transverse DSWs up to sixth order for the full water-wave problem.

For constant density fluids, solitary waves with oscillatory tails which represent finite-amplitude DSWs, have been found previously in models for water waves when surface tension is non-zero. When the Bond number is approximately 1/3 and the Froude number is greater than one, the existence of these surface-tension dominated waves was first proved by Amick & Toland (1992) for a model ODE (see also Grimshaw & Joshi 1995). The analytical results show that these waves are biasymptotic to periodic states with exponentially small amplitude. However, numerical continuation of these waves by Champneys & Lord (1997) show that they have finite-amplitude periodic solutions at infinity when continued to finite amplitude.

### 9.1. Gravity-driven steady dark solitary waves

It was shown in §7.1 that there exists a critical surface satisfying  $\det[\mathbf{C}(\mathbf{p})] = 0$  for gravity waves in shallow water. The local nonlinear normal form for bifurcating

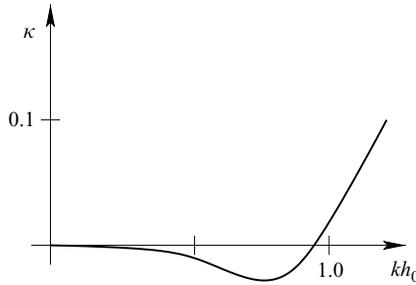


FIGURE 10. The curvature coefficient  $\kappa$  as a function of  $kh_0$ .

solitary waves is given by (8.4). By computing  $\kappa$ , all the local nonlinear behaviour can be predicted. The formula for the coefficient  $\kappa$  is given in (8.7) with the formula for  $\mathbf{n}$  given in (8.6). Given the entries of the criticality matrix computed in §7 for weakly nonlinear water waves,  $\kappa$  can be explicitly computed and it is plotted in figure 10 as a function of  $kh_0$ . Since the calculation in figure 10 is based on the two-term Stokes expansion, the values for very small  $kh_0$  should be interpreted with caution.

In §7.1, it was shown that secondary criticality occurs at low amplitude (weakly nonlinear theory) for all  $kh_0 < (kh_0)^{point2} \approx 0.85$ . According to figure 10,  $\kappa < 0$  for all  $kh_0$  in this range (the zero-crossing of the curve in figure 10 is greater than 0.95). Hence the bifurcating solitary waves exist for  $\tilde{I}_1 < 0$ ; that is, the solitary waves exist on the side of the criticality surface in the direction opposite to the chosen normal vector. To express this region in terms of the Bernoulli constant, mass flux and wave action flux, let  $\mathbf{p}_{crit}$  be any point satisfying  $\det[\mathbf{C}(\mathbf{p})] = 0$ , and let

$$(\mathcal{R}_0, \mathcal{Q}_0, \mathcal{B}_0) := (\mathcal{R}(\mathbf{p}_{crit}), \mathcal{Q}(\mathbf{p}_{crit}), \mathcal{B}(\mathbf{p}_{crit})).$$

Then dark solitary waves arise locally in parameter space with

$$n_1(\mathcal{R} - \mathcal{R}_0) + n_2(\mathcal{Q} - \mathcal{Q}_0) + n_3(\mathcal{B} - \mathcal{B}_0) < 0 \quad \text{with} \quad |\tilde{I}_1| \ll 1. \quad (9.3)$$

This is a very large region of parameter space, suggesting that steady dark solitary waves are just as plentiful as Stokes periodic waves. However, whether these waves are stable or not is an open question. If they are stable then they should be experimentally verifiable.

The inequality (9.3) is independent of the sign of  $\mathbf{n}$  – even though the sign of  $\mathbf{n}$  can be chosen arbitrarily. To see this, suppose  $\mathbf{n}$  is replaced by  $-\mathbf{n}$ . It would appear that the inequality in (9.3) is reversed; but the sign change also affects the sign of  $\mathcal{P}(\mathbf{p})$  since it depends cubically on  $\mathbf{n}$ . In other words, the sign of the product  $\kappa \tilde{I}_1$  is independent of the choice of normal vector. The inequality (9.3) is unambiguous in the following sense: it signals whether the direction of bifurcation is aligned with or opposite to the direction of the chosen normal vector. In the present case, with  $\mathbf{n}$  chosen using the formula (8.6) the solitary waves exist in the  $-\mathbf{n}$  direction.

We have computed the signs of the non-zero eigenvalues of  $\mathbf{C}(\mathbf{p})$  for  $kh_0 = 0.5$  and  $kh_0 = 0.7$  which correspond to the first two graphs in figure 7, and in both cases the eigenvalues have opposite sign. Choosing an appropriate ordering we can take  $s_2 = -1$  and  $s_3 = +1$ . However, the only impact of these signs is to give direction information about the drift along the phase directions  $\tilde{\phi}_2$  and  $\tilde{\phi}_3$ . The important phase information is the geometric phase, and the explicit formula is given in (8.8).

For water waves the geometric phase is given to leading order by

$$\Delta\tilde{\phi}_1 = \pm \frac{12}{|\kappa|} (2\kappa\tilde{I}_1)^{1/4}, \quad \tilde{I}_1 < 0.$$

The theory of §8 assumes a finite-dimensional phase space, whereas the water-wave problem has an infinite-dimensional phase space. However, we can reduce the infinite dimensional phase space to a finite dimensional phase space using spatial centre-manifold theory (cf. Mielke 1991; Bridges & Mielke 1995). Then the theory applies as before to the finite-dimensional system. The important point is that the existence of steady dark solitary waves can be precisely justified for the full water-wave problem. In contrast there is no known theory for justifying the existence of unsteady DSWs in the full water-wave problem.

## 10. Concluding remarks

This paper showed that there is a generalization of criticality to finite-amplitude Stokes waves in finite depth, and the principal consequence of this ‘secondary criticality’ is the bifurcation of steady DSWs. The theory required three principal steps. The determinant condition for primary criticality (1.4) was generalized to a determinant condition for secondary criticality (1.8). The second step was to show the implication of this condition for the linearization about Stokes waves coupled to a mean flow: a pair of spatial Floquet multipliers coalesces at +1. Although spatial Floquet multipliers associated with the linearization about waves have been computed and reported in the literature, the connection with criticality is established here for the first time. The third step, which involved recent developments in dynamical systems, is to show that the nonlinear problem near secondary criticality always has a bifurcation of solitary waves and the physical properties of the bifurcating steady DSWs are dictated by the properties of the criticality matrix. By analysing Boussinesq models for shallow-water waves and the full water-wave problem, it was shown that secondary criticality and the bifurcation of steady DSWs is pervasive in shallow water. In Part 2, it is shown that the concepts discovered for the steady problem can be generalized to unsteady waves to give new insight into the Benjamin–Feir instability in finite depth.

Relative equilibria can be extended to the time-dependent problem, and the concept of the degenerate relative equilibrium is well-defined. This leads to a naive definition of criticality for a class of unsteady flows (time periodic in a moving frame); but what is the implication of this definition of unsteady criticality? The main result of Part 2 of this study is to show how the hydraulic properties of the uniform flow coupled to a spatially periodic wave and the criticality matrix  $\mathbf{C}(\mathbf{p})$  enter into the linear stability of the periodic wave. A sketch of the result is as follows. A periodic wave in shallow water, with specified total head, mass flux and wave action flux is linearly unstable if the determinant of the deformed criticality matrix,  $\mathbf{N}(\Omega)$ , has a root  $\Omega$  with non-zero imaginary part, where

$$\mathbf{N}(\Omega) = \frac{\partial(R, Q, B)}{\partial(h_0, u_0, k)} + \Omega \begin{bmatrix} 0 & 1 & \star \\ 1 & 0 & \star \\ \star & \star & \star \end{bmatrix} + \Omega^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \star \end{bmatrix}, \quad (10.1)$$

where  $\star$  represents entries which are determined by the theory for unsteady criticality.

The instability condition based on (10.1) is new, and connects bulk quantities familiar from hydraulics to the Benjamin–Feir instability: the criticality matrix is the central feature in (10.1).

The authors are grateful to Howell Peregrine whose probing questions and suggestions led to several new dimensions in this paper.

**Appendix A. Algebraic multiplicity, Jordan chain theory and criticality**

In this Appendix and Appendix B, it is shown how Jordan chain theory for generalized eigenvectors enters the analysis of criticality. Let  $\mathbf{K}$  be a non-selfadjoint linear operator acting on a finite-dimensional vector space. (Comments on differential operators are at the end of this Appendix.)

Suppose zero is an eigenvalue of  $\mathbf{K}$  of *geometric multiplicity* one; that is, there exist an eigenvector  $\xi_1$  and adjoint eigenvector  $\zeta_1$  such that

$$\mathbf{K}\xi_1 = 0, \quad \mathbf{K}^T \zeta_1 = 0.$$

The geometric multiplicity is the number of linearly independent vectors in the kernel of  $\mathbf{K}$ . Now suppose zero is an eigenvalue of *algebraic multiplicity* of at least two. In finite dimensions, the algebraic multiplicity of an eigenvalue equals the multiplicity of it as a root of the characteristic polynomial. For differential operators, where the concept of a characteristic polynomial does not apply, the algebraic multiplicity is defined using Jordan chain theory (cf. Lancaster & Tismenetsky 1985, chap. 6). If zero is an eigenvalue of  $\mathbf{K}$  of algebraic multiplicity greater than or equal to two, then there exists a vector  $\xi_2$  satisfying

$$\mathbf{K}\xi_2 = \xi_1,$$

and the algebraic multiplicity is greater than two if and only if there exists a vector  $\xi_3$  with

$$\mathbf{K}\xi_3 = \xi_2.$$

But this equation is solvable if and only if the right-hand side is in the range of  $\mathbf{K}$ ,

$$\langle \zeta_1, \xi_2 \rangle = 0.$$

In summary, if  $\langle \zeta_1, \xi_2 \rangle \neq 0$  then the algebraic multiplicity of the zero eigenvalue is exactly two, and if  $\langle \zeta_1, \xi_2 \rangle = 0$  the algebraic multiplicity is greater than or equal to three.

The generalization of this result, suitable for the theory of criticality, is as follows. Suppose zero is an eigenvalue of geometric multiplicity  $m$ : there exist  $m$  linearly independent eigenvectors  $\xi_1, \dots, \xi_m$  and  $m$  linearly independent adjoint eigenvectors  $\zeta_1, \dots, \zeta_m$  satisfying

$$\mathbf{K}\xi_j = 0, \quad \mathbf{K}^T \zeta_j = 0 \quad \text{for } j = 1, \dots, m.$$

Suppose that each of the eigenvectors is associated with a Jordan chain of length two

$$\mathbf{K}\xi_{m+j} = \xi_j, \quad j = 1, \dots, m.$$

It follows from Jordan chain theory that the algebraic multiplicity of the eigenvalue zero is greater than or equal to  $2m$ .

For the algebraic multiplicity to be strictly greater than  $2m$ , at least one of the Jordan chains must exceed two in length: there exists a non-zero  $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{R}^m$

such that the following equation is solvable

$$\mathbf{K}\xi_{2m+1} = n_1\xi_{n+1} + n_2\xi_{m+2} + \cdots + n_n\xi_{2m}.$$

This equation is solvable if and only if the right-hand side is in the range of  $\mathbf{K}$ ,

$$\begin{bmatrix} \langle \zeta_1, \xi_{m+1} \rangle & \cdots & \langle \zeta_1, \xi_{2m} \rangle \\ \vdots & \ddots & \vdots \\ \langle \zeta_m, \xi_{m+1} \rangle & \cdots & \langle \zeta_m, \xi_{2m} \rangle \end{bmatrix} \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

which in turn has a non-trivial solution if and only if

$$\det \begin{bmatrix} \langle \zeta_1, \xi_{m+1} \rangle & \cdots & \langle \zeta_1, \xi_{2m} \rangle \\ \vdots & \ddots & \vdots \\ \langle \zeta_m, \xi_{m+1} \rangle & \cdots & \langle \zeta_m, \xi_{2m} \rangle \end{bmatrix} = 0. \quad (\text{A } 1)$$

This theory simplifies considerably when  $\mathbf{K}$  is a Hamiltonian operator. A Hamiltonian operator is the product of a skew-symmetric operator and a symmetric operator,

$$\mathbf{K} = \mathbf{J}^{-1}\mathbf{L}, \quad \text{with } \mathbf{J}^T = -\mathbf{J}, \quad \mathbf{L}^T = \mathbf{L}.$$

When the linear operator  $\mathbf{K}$  is decomposed in this form the adjoint eigenvectors are related to the eigenvectors by  $\mathbf{J}$ : suppose  $\xi_0$  is an eigenvector associated with a zero eigenvalue:  $\mathbf{K}\xi_0 = 0$ . In the Hamiltonian context, this is equivalent to  $\mathbf{L}\xi_0 = 0$ . The adjoint eigenvector is  $\zeta_0 = a_0\mathbf{J}\xi_0$  for some constant  $a_0$ , since

$$\mathbf{K}^T \zeta_0 = a_0(\mathbf{J}^{-1}\mathbf{L})^T \mathbf{J}\xi_0 = -a_0\mathbf{L}\mathbf{J}^{-1}\mathbf{J}\xi_0 = -a_0\mathbf{L}\xi_0 = 0.$$

The solvability condition (A 1) can now be expressed purely in terms of the vectors  $\xi_1, \dots, \xi_{2m}$ ,

$$\det \begin{bmatrix} \langle \mathbf{J}\xi_1, \xi_{m+1} \rangle & \cdots & \langle \mathbf{J}\xi_1, \xi_{2m} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{J}\xi_m, \xi_{m+1} \rangle & \cdots & \langle \mathbf{J}\xi_m, \xi_{2m} \rangle \end{bmatrix} = 0. \quad (\text{A } 2)$$

In the context of criticality, there is additional structure which relates this solvability condition to hydraulic properties. Mathematically, this structure arises because there exists a vector  $\widehat{Z}(\theta, \mathbf{a})$  with  $\theta = (\theta_1, \dots, \theta_m)$  and  $\mathbf{a} = (a_1, \dots, a_m)$  and

$$\xi_j = \frac{\partial \widehat{Z}}{\partial \theta_j} \Big|_{\theta=0} = \mathbf{J}^{-1}\nabla P_j(\widehat{Z}), \quad \xi_{n+j} = \frac{\partial \widehat{Z}}{\partial a_j}, \quad j = 1, \dots, m,$$

since  $\nabla P_j(\widehat{Z}) = \mathbf{J}\xi_j$ . Then the solvability condition can be expressed in terms of the Jacobian

$$0 = \det \begin{bmatrix} \langle \nabla P_1, \xi_{m+1} \rangle & \cdots & \langle \nabla P_1, \xi_{2m} \rangle \\ \vdots & \ddots & \vdots \\ \langle \nabla P_m, \xi_{m+1} \rangle & \cdots & \langle \nabla P_m, \xi_{2m} \rangle \end{bmatrix} = \left| \frac{\partial(P_1, \dots, P_m)}{\partial(a_1, \dots, a_m)} \right|.$$

To summarize: when the linear operator  $\mathbf{K}$  is associated with the linearization about an  $m$ -parameter family of symplectic relative equilibria and the eigenvalue zero has geometric multiplicity  $m$ , then the algebraic multiplicity is greater than  $2m$  if and

only if

$$\left| \frac{\partial(P_1, \dots, P_m)}{\partial(a_1, \dots, a_m)} \right| = 0.$$

The extension to linear differential operators follows the same lines when the operators are Fredholm with index zero (dimension of the kernel of  $\mathbf{K}$  equals the dimension of the kernel of the adjoint of  $\mathbf{K}$ ). Similarly, for differential operators with periodic coefficients there is a variant of Jordan chain theory which is combined with Floquet theory (cf. Iooss & Adelmeyer 1992, chap. 3). The Hamiltonian version of this theory then follows the same lines as above.

### Appendix B. Criticality and eigenvalues

The theory of Appendix A is combined with the theory for secondary criticality, based on degeneracy of the criticality matrix  $\mathbf{C}(\mathbf{p})$ , to show that there is an additional pair of zero Floquet exponents at degeneracy. The details of the argument are presented for the Boussinesq model in the form (2.5).

Consider the steady equation (2.5) near the family of solutions (5.3), of the form (8.1). Substitution into (2.5) and use of the symmetry properties of the equation leads to the linearized system

$$\mathbf{J}W_x = \mathbf{L}(\mathbf{p})W \quad \text{with} \quad \mathbf{L}(\mathbf{p}) = \mathbf{D}^2S(\widehat{Z}) - k\mathbf{D}^2B(\widehat{Z}). \quad (\text{B } 1)$$

In contrast to the linearization about uniform flows, this equation is a PDE – even for the Boussinesq model, but with coefficients that are periodic functions of  $\theta_3$ . Taking a Floquet form of solutions, the spectral problem is obtained by taking

$$W(\theta_3, x) = e^{\mu x} V(\theta_3), \quad V(\theta_3 + 2\pi) = V(\theta_3), \quad \text{leading to the ODE} \quad \mathbf{L}(\mathbf{p})V = \mu\mathbf{J}V. \quad (\text{B } 2)$$

This spectral problem has a zero eigenvalue of (at least) geometric multiplicity three and (at least) algebraic multiplicity six. If the geometric multiplicity is exactly three, the algebraic multiplicity of the zero eigenvalue is greater than six if and only if the basic state (the uniform flow coupled to a periodic wave) is critical.

These statements are confirmed as follows. The kernel of  $\mathbf{J}^{-1}\mathbf{L}(\mathbf{p})$  is spanned by the generators of the group of symmetries

$$\mathbf{L}(\mathbf{p})\mathbf{g}_1 = \mathbf{L}(\mathbf{p})\mathbf{g}_2 = \mathbf{L}(\mathbf{p})\mathbf{g}_3 = 0 \quad \text{with} \quad \mathbf{g}_3 = \frac{\partial \widehat{Z}}{\partial \theta_3}. \quad (\text{B } 3)$$

Linear independence of  $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$  then ensures that the geometric multiplicity is at least three. Assume that the geometric multiplicity is exactly three.

The structure of the relative equilibria generates three generalized eigenvectors,

$$\left. \begin{aligned} \mathbf{L}(\mathbf{p}) \left( \frac{\partial \widehat{Z}}{\partial h_0} \right) &= \nabla R(\widehat{Z}) = \mathbf{J}\mathbf{g}_1, \\ \mathbf{L}(\mathbf{p}) \left( \frac{\partial \widehat{Z}}{\partial u_0} \right) &= \nabla Q(\widehat{Z}) = \mathbf{J}\mathbf{g}_2, \\ \mathbf{L}(\mathbf{p}) \left( \frac{\partial \widehat{Z}}{\partial k} \right) &= \nabla B(\widehat{Z}) = \mathbf{J}\mathbf{g}_3. \end{aligned} \right\} \quad (\text{B } 4)$$

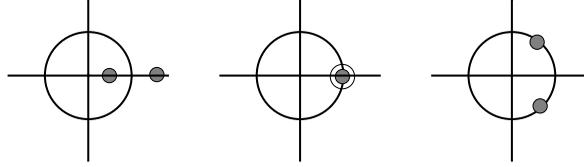


FIGURE 11. Transition of Floquet multipliers due to secondary criticality.

The three pairs  $(\mathbf{g}_1, \partial\widehat{Z}/\partial h_0)$ ,  $(\mathbf{g}_2, \partial\widehat{Z}/\partial u_0)$  and  $(\mathbf{g}_3, \partial\widehat{Z}/\partial k)$  each form a Jordan chain of length two. Applying Jordan chain theory for linear ODEs with periodic coefficients (see Appendix A), zero is an eigenvalue of algebraic multiplicity at least six, and it is greater than six if and only if there exists a non-zero  $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$  such that the equation

$$\mathbf{L}(\mathbf{p}) = \mathbf{J} \left( n_1 \frac{\partial\widehat{Z}}{\partial h_0} + n_2 \frac{\partial\widehat{Z}}{\partial u_0} + n_3 \frac{\partial\widehat{Z}}{\partial k} \right)$$

is solvable. However, this equation is solvable if and only if the right-hand side is in the range of  $\mathbf{L}$ , that is, if

$$\left\langle \mathbf{J}\mathbf{g}_j, \left( n_1 \frac{\partial\widehat{Z}}{\partial h_0} + n_2 \frac{\partial\widehat{Z}}{\partial u_0} + n_3 \frac{\partial\widehat{Z}}{\partial k} \right) \right\rangle = 0,$$

for  $j = 1, 2, 3$ , which is equivalent to  $\det(\mathbf{C}(\mathbf{p})) = 0$ .

To summarize, if the geometric multiplicity of zero is exactly three, then the algebraic multiplicity is greater than six if and only if the basic state – a spatially periodic wave coupled to two-component mean flow – is critical. Passing through criticality is therefore a saddle-centre bifurcation of Floquet exponents, and in Floquet multiplier space a pair of Floquet multipliers coalesces at +1 as shown in figure 11. In addition to the moving pair of Floquet multipliers there are six multipliers at +1 in this scenario.

At criticality, the number of generalized eigenvectors increases, but the number of eigenvectors remains the same. Therefore criticality is a sufficient – but not necessary – condition for the multiplicity of the zero eigenvalue to increase. The multiplicity could also increase owing to an increase in geometric multiplicity (i.e. the appearance of an additional independent eigenvector in the kernel of  $\mathbf{K}$ ). This latter case does not arise for parameter regions associated with water waves studied in this paper and is therefore not considered.

**Appendix C. Criticality of uniform flows and ‘group velocity’**

Classical uniform flows are critical if and only if the Froude number is unity. However, a curiosity appears in §7.1 in the leading-order term in the expression for the determinant of the criticality matrix. In the limit as  $|A_1| \rightarrow 0$  in (7.7), the condition for secondary criticality reduces to

$$\lim_{|A_1| \rightarrow 0} \det[\mathbf{C}(\mathbf{p})] = \frac{gh_0}{\varrho} D_k^2 (1 - F^2).$$

Therefore if  $D_k \neq 0$ , the Froude number unity condition for criticality is recovered. But what happens when  $D_k = 0$ ? Does it also signify a form of criticality?

The classical group velocity for a Stokes gravity wave is

$$c_g = \frac{1}{2}u_0 \left( 1 + \frac{kh_0}{\sigma}(1 - \sigma^2) \right).$$

It is important to note that the association with group velocity here is by analogy. There is no concept of group velocity for a steady wave without the introduction of time. On the other hand,  $D_k = gk(1 - (kh_0/\sigma)(1 - \sigma^2))$  and so

$$D_k = \frac{2gk}{u_0}(u_0 - c_g).$$

The singularity  $D_k = 0$  corresponds to a point where  $u_0$  is equal to the ‘group velocity’. This singularity does occur in the water-wave problem, but only when surface tension is present. It corresponds to the minimum of the dispersion relation when the phase speed is plotted as a function of  $kh_0$ . The point where  $c_g = u_0$  is a well-known point of bifurcation for solitary waves with oscillatory tails, but the tails decay to zero at infinity. See Dias & Iooss (2003) for a review of results of solitary waves near this point. Points where  $c_g = u_0$  are also called *stopping velocities* (cf. Bridges, Christodoulides & Dias 1995).

In summary, the theory shows that the point  $D_k = 0$  can also be interpreted as a point of criticality. However, it is found by taking the limit as the amplitude of the Stokes wave goes to zero in the secondary criticality condition.

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