ROGUE WAVES

F. Dias
School of Mathematical Sciences
University College Dublin, Dublin, Ireland
frederic.dias@ucd.ie

T. J. Bridges
Departement of Mathematics
University of Surrey, Guildford, UK
t.bridges@surrey.ac.uk

J. M. Dudley
Institut FEMTO-ST
University of Franche-Comté, Besançon, France
john.dudley@univ-fcomte.fr

Rogue waves are fascinating: once part of the folklore, they now make the news each time an observation is made. At the time of printing, the last example is that of the Louis Majesty cruise ship that was hit by an abnormal wave in March 2010, off the coast of Catalonia in the Mediterranean. The wave impact was immortalized by several videos taken by tourists on board the ship. Fortunately for travellers but unfortunately for scientists, rogue waves do not occur very often and their origin remains a mystery, even if the state of the art in the understanding of rogue waves has witnessed some unprecedented progress in the last five years. Recently similar phenomena were observed in different fields of physics, in particular in optics. Is there hope to learn more on rogue waves from other fields or are these extreme events disconnected phenomena? This chapter provides a review on rogue waves, with an emphasis on the modulational instability and the absolute or convective character of this instability.

1. Introduction

The study of rogue waves is still relatively recent, even if this mysterious phenomenon has been known in various environments such as ocean waves for centuries. Undoubtedly rogue waves have practical consequences and are not simply a theoretical subject. Views on freak waves are sometimes controversial and even contradicting. Even the definition of a rogue wave is not so easy. The standard approach is to call a wave a rogue wave whenever the wave height H (distance from trough to crest) exceeds a certain threshold related to the sea state. More precisely the common criteria states that a wave is a rogue wave when

$$H/H_s > 2 \tag{1.1}$$

where H_s is the significant wave height, here defined as four times the standard deviation of the surface elevation. Was this criteria satisfied during the Louis Majesty incident? The answer is no. Indeed, the wave height (in fact, there were three large waves) has been estimated to be 8 m, while the significant wave height was 5 m. Nevertheless, the waves were powerful enough to kill two tourists and to do quite a bit of damage.

Rogue waves arise in arbitrary water depth (in deep as well as shallow water), with or without currents. The observed probability of occurrence of freak waves in deep and shallow waters is approximately the same. It is important to remember that one is dealing here with rare events and consequently scientists have only few data available. However the understanding of rogue waves is witnessing regular progress. Freak waves may have the shape of a solitary wave or correspond to a group of several waves. Various mechanisms have been proposed for rogue wave formation, either linear or nonlinear. Assuming that wind waves, at least in the framework of linear theory, can be considered as the sum of a large number of independent monochromatic waves with different frequencies and directions, a freak wave may arise in the process of spatial wave focusing (geometrical focusing) and spatio-temporal focusing (dispersion enhancement). The interaction between a wave and a counter-propagating current can also be at the origin of large wave events. Because freak waves are large-amplitude steep waves, one would expect nonlinearity to play an important role as well in the formation and the evolution of rogue waves. Nonlinearity modifies the linear focusing mechanisms, but does not destroy them. In fact linear mechanisms are more and more regarded as pre-conditioning for nonlinear focusing. It is now recognized that most focusing mechanisms are also robust with respect to random wave components.

There is one mechanism of freak wave formation which is suggested in the framework of nonlinear theory only: the modulational instability (MI), also referred to as the Benjamin-Feir (BF) instability in the hydrodynamics community.[a] A uniform train of relatively steep waves is unstable to sideband disturbances, that is disturbances whose frequencies deviate slightly from the fundamental frequency of the carrier waves. The BF instability increases the frequency of occurrence of freak waves in comparison with the linear theory. At the same time the randomness of the wave field reduces the BF instability. All the processes mentioned above can be investigated in the framework of weakly nonlinear models like the nonlinear Schrödinger (NLS) equation, the Davey-Stewartson system, the Korteweg-de Vries equation, and the Kadomtsev-Petviashvili equation. An excellent review is given in the recent book by Kharif, Pelinovsky & Slunyaev [14]. An earlier version was given in the review article by Kharif & Pelinovsky [13] (see also [9]). The state of the art on rogue waves can be found in the special issue of the European Journal of Physics which will be published in December 2010.

Since the BF instability is the main focus of this chapter, let us provide a brief review. A full account of the history of the BF instability can be found in Hunt's review article [11]. The discovery of the BF instability of traveling waves was a milestone in the history of water waves. Before 1960, the idea that a Stokes wave could be unstable does not appear to be given much thought. The possibility that the Stokes wave could be unstable was pointed out in the late 1950s, but it was the seminal work of Benjamin and Feir [3] that combined experimental evidence with a weakly nonlinear theory that convinced the scientific community.

Indeed, Benjamin and Feir started their experiments in 1963 assuming that Stokes waves were stable. After several frustrating years watching their waves disintegrate - in spite of equipment and laboratory changes and improvements - they finally came to the conclusion that they were witnessing a new kind of instability. The appearance of "sidebands" in the experiments suggested the form that the perturbations should take. The water wave community will celebrate soon the 50-year anniversary of the discovery of deep water wave instability but it is much more recently that the community was convinced that this effect is able to generate a rogue wave in the real sea. However there is still some controversy. Indeed the BF instability may be suppressed by various unfavourable conditions [21, 6]. Moreover

[a]In order to distinguish between nonlinear optics and hydrodynamics, we will use the MI terminology for optics and the BF terminology for water waves.

it is well-known that in two dimensions the BF instability does not occur in shallow water. This means that the BF instability is not necessarily the dominant mechanism causing rogue waves at least in the coastal zone where wave focusing and blocking due to bathymetry and current effects are important.

The BF instability applies to a plane wave to which a small perturbation is superimposed. Since ocean waves are characterized by a finite width spectrum, the concept of BF instability must be generalized. The Benjamin-Feir Index (BFI) was introduced by Janssen [12]. It measures the ratio between the wave steepness and the spectral bandwidth. Rigorous results for a broad-band spectrum are not straightforward, unless some hypotheses on the statistics (usually a quasi-Gaussian approximation) are introduced. Numerical results and recent experiments [19] show that sea states characterized by steep, long-crested waves are more likely to give rise to rogue waves as opposed to those characterized by a large directional spreading.

Several ship accidents have occurred in crossing sea conditions. The BF instability of a crossing sea was investigated by Onorato et al. [18], who computed the growth rates based on two coupled NLS equations, and by Laine-Pearson [16], who extended the analysis to the full water-wave problem. Tamura et al. [26] investigated the sinking of the *Suwa-Maru* fishing boat east of Japan on 23 June 2008. Their hindcast result for sea-state conditions at the time of the incident indicated that a crossing sea state developed four hours before the accident. However, the wave condition was unimodal at the time of the accident and was favorable for the occurrence of freak waves according to quasi-resonance theory. Thus, for the case of the Suwa-Maru incident, the crossing sea was a "precursor" to the development of the narrow spectrum. Interactions between wind waves and swell took place as the wind speed increased and the sea state rapidly developed into a unimodal freakish state.

In 2007 a paper by Solli et al. [23] led to a fundamental change in scientific thinking about rogue waves: Rogue waves are not restricted to ocean waves. They also occur in optics. More recently they have also been observed in capillary waves [22] and conjectured in the atmosphere [24]. The Solli et al. paper prompted two of us (FD and JMD) to develop a new multidisciplinary project, the MANUREVA project.[b] Can the optics com-

[b]MANUREVA stands for "Mathematical modelling and experiments studying nonlinear instabilities, rogue waves and extreme phenomena".

munity help the hydrodynamics community? A realistic ocean wave theory should be based on reasonable physical principles that can be formulated in mathematical terms and should generate useful predictions about what can happen, based on physically meaningful observations and parameters. It is a big challenge to find a way to satisfy both requirements. This is the purpose of the MANUREVA project. It is obvious that a useful theory that describes the statistics of rogue waves, regardless of how they are defined, must go beyond the linear stochastic Gaussian wave theory based on frequency decompositions. One must look for models that can reproduce steep and asymmetric waves. The MANUREVA project is still under way, but some conclusions have already been reached. They are summarized in the MANUREVA paper of the special issue of the European Journal of Physics mentioned above [7]. Collisions appear to play a central role in the generation of large amplitude waves [10]. Indeed it is possible that the only real waves with the statistics that can be characterized as "rogue" with genuine long tails only arise from collisions. Collisions within NLS systems were proposed as ocean rogue wave generators previously. NLS related dynamics are obvious from a mathematical viewpoint, but actually linking these effects to experiments is not so clear. For example the discovery that Akhmediev breather theory and MI were linked experimentally was made only recently, quite surprisingly [8, 15]. Although the MI dynamics might be well known they can still seed a wide range of different behaviours because one is always in a perturbed NLS system.

In the context of rogue waves in optical fibre systems, Taki et al. [25] provided theoretical and numerical evidence that optical rogue waves originate from convective modulational instabilities. This is an important observation because the BF instability for water waves is not convective as shown by Brevdo and Bridges [5]. This is what we review now.

2. The NLS equation

The celebrated NLS equation usually includes cubic nonlinearity and second-order dispersion, at least in hydrodynamics. Here we consider the cubic NLS equation with additional third-order dispersion,

$$\mathrm{i}a_1 A_t + a_2 A_{xx} + \mathrm{i}a_3 A_{xxx} + a_4 |A|^2 A = 0.$$

Third-order dispersion is often considered in optics [2], together with Raman scattering and self-steepening. By scaling the time t, the space x, and the amplitude A, all the coefficients can be set to plus or minus one except the coefficient of A_{xxx}. Assuming the situation for BF instability, the canonical form of the equation is

$$\mathrm{i}A_t + A_{xx} + \mathrm{i}bA_{xxx} + |A|^2 A = 0, \tag{2.1}$$

where b is a real parameter.

The basic state which represents the Stokes wave is

$$A(x,t) = \xi \mathrm{e}^{\mathrm{i}\Omega t}\,, \quad \Omega = \|\xi\|^2\,. \tag{2.2}$$

Consider the linear stability of the Stokes wave (2.2); let

$$A(x,t) = (\xi + B(x,t))\mathrm{e}^{\mathrm{i}\Omega t}\,.$$

Substituting into (2.1) and linearizing about (2.2) yields

$$\mathrm{i}B_t + B_{xx} + \mathrm{i}bB_{xxx} + \xi^2\overline{B} + \|\xi\|^2 B = 0\,. \tag{2.3}$$

The general solution of (2.3) is

$$B(x,t) = B_1\,\mathrm{e}^{(\lambda t + \mathrm{i}kx)} + B_2\,\mathrm{e}^{(\overline{\lambda} t - \mathrm{i}kx)}\,.$$

Substitution into (2.3) gives

$$\begin{aligned} \mathrm{i}\lambda B_1 - k^2 B_1 + bk^3 B_1 + \xi^2\overline{B_2} + \|\xi\|^2 B_1 &= 0 \\ -\mathrm{i}\lambda\overline{B_2} - k^2\overline{B_2} - bk^3\overline{B_2} + \overline{\xi}^2 B_1 + \|\xi\|^2\overline{B_2} &= 0\,. \end{aligned}$$

Solutions exist if and only if the following condition is satisfied:

$$\det\begin{bmatrix} \mathrm{i}\lambda - k^2 + bk^3 + \|\xi\|^2 & \xi^2 \\ \overline{\xi}^2 & -\mathrm{i}\lambda - k^2 - bk^3 + \|\xi\|^2 \end{bmatrix} = 0\,, \tag{2.4}$$

or

$$\lambda = \mathrm{i}bk^3 \pm \sqrt{2k^2\|\xi\|^2 - k^4}\,.$$

When $b = 0$ we recover the usual plane-wave instability of NLS: when the amplitude $\|\xi\| > k/\sqrt{2}$ there is a real positive eigenvalue giving instability. For small $\|\xi\|$ or large $\|k\|$ the plane wave is stable.

Now, when $b \neq 0$ the main change is that λ becomes complex. Adding in the complex conjugate, there are four roots

$$\lambda = \pm\mathrm{i}bk^3 \pm \sqrt{2k^2\|\xi\|^2 - k^4}\,.$$

When $b \neq 0$, $k \neq 0$ and $2\|\xi\|^2 = k^2$ there is a collision of eigenvalues of opposite signature on the imaginary axis at $\lambda = \pm\mathrm{i}bk^3$.

In summary, with third-order dispersion the nature of the instability is different. The difference is explained in the next section.

3. Absolute and convective instabilities

An instability is absolute if the dispersion relation has an unstable saddle point, and the saddle point satisfies the pinching condition [4]. An instability is convective if it is not absolute!

Saddle points play a central role when looking for absolute instabilities. Let $\lambda = -\mathrm{i}\omega$. Then the dispersion relation (2.4) can be written in the form

$$D(\omega, k) = -\omega^2 - 2bk^3\omega - 2k^2\|\xi\|^2 + k^4 - b^2k^6 \,.$$

Saddle points satisfy

$$D = D_k = 0 \,,$$

where

$$D_k = -6bk^2\omega - 4k\|\xi\|^2 + 4k^3 - 6b^2k^5 \,,$$

and so, when $b \neq 0$ and $k \neq 0$, $D_k = 0$ gives

$$\omega = -\frac{2}{3bk}\|\xi\|^2 + \frac{2}{3b}k - bk^3 \,. \tag{3.1}$$

Back substitution into the dispersion relation gives a polynomial in k. Assuming $k \neq 0$ and $b \neq 0$ this polynomial is

$$D(\omega, k) = k^2(k^2 - 2\|\xi\|^2) - \frac{4}{9b^2k^2}(k^2 - \|\xi\|^2)^2 \,.$$

Simplifying and re-arranging gives $\delta(k) = 0$ with

$$\delta(k) = 4(\|\xi\|^2 - k^2)^2 + 9b^2k^4(2\|\xi\|^2 - k^2) \,.$$

This is a polynomial of degree six in k. However it is a polynomial of degree two in $\|\xi\|^2$. So let us solve for $\|\xi\|$ as a function of k. Assume b is non-zero and denote saddle points by (ω_0, k_0) with k_0 a root of $\delta(k_0) = 0$. Then solving for $\delta(k_0) = 0$ gives

$$\|\xi\|^2 = k_0^2 \frac{\sqrt{1-\theta^2}}{1+\sqrt{1-\theta^2}} \,, \quad \theta = \frac{2}{3bk_0} \,. \tag{3.2}$$

The frequency ω_0 is obtained by substituting (3.2) into (3.1)

$$\omega_0 = \frac{2k_0}{3b}\left(\frac{1}{1+\sqrt{1-\theta^2}}\right) - bk_0^3 \,.$$

The only saddle points (ω_0, k_0) of interest are with $\|\xi\|$ real, since $\|\xi\|$ is the modulus of the amplitude of the Stokes wave. All saddle points giving real $\|\xi\|$ satisfy (3.2). However, note that real roots exist only if $\theta < 1$ or

$$k_0 > \frac{2}{3b} \,.$$

4. The case with only second-order dispersion

This is the case $b = 0$ and it has already been considered in [5]. In this case the dispersion relation simplifies to

$$D(\omega, k) = -\omega^2 - 2k^2\|\xi\|^2 + k^4 .$$

The necessary condition for absolute instability is the existence of a pair (k_0, ω_0), with $\mathrm{Im}(\omega_0) > 0$ satisfying $D = D_k = 0$, that is, the existence of an unstable saddle point of $\omega := \omega(k)$. Now,

$$D_k = -4k\|\xi\|^2 + 4k^3 = 4k(k^2 - \|\xi\|^2) .$$

The point $k = 0$ corresponds to $\omega = 0$ and so is a neutral saddle point. When $k \neq 0$ there are two roots

$$k_\pm = \pm\|\xi\| .$$

The corresponding values of ω are obtained from the dispersion relation

$$-\omega^2 - \|\xi\|^4 = 0 \quad \text{or} \quad \omega = \pm \mathrm{i}\|\xi\|^2 ,$$

and so the unstable saddle point is

$$\omega = \mathrm{i}\|\xi\|^2 .$$

It is shown in [5] that the pinching condition is satisfied in this case. Here is another proof that the pinching condition is satisfied. The pinching condition is defined as follows. Let (ω_0, k_0) be a saddle point. That is

$$D(\omega_0, k_0) = D_k(\omega_0, k_0) = 0 .$$

Let $\omega = \omega_0 + \mathrm{i}y$ with y real and positive. Then look at the roots of $D(\omega_0 + \mathrm{i}y, k(y))$, with $k(0) = k_0$ the double root. The instability is absolute if k_0 splits into two roots $k^-(y)$ and $k^+(y)$ with

$$\mathrm{Im}(k^-(y)) < 0 \quad \text{and} \quad \mathrm{Im}(k^+(y)) > 0 \quad \text{as} \quad y \to \infty .$$

A proof that the pinching condition is satisfied in the case $b = 0$ is as follows. In this case

$$\omega_0 = \mathrm{i}\, k_0^2 \quad \text{with} \quad k_0 = \|\xi\| ,$$

and so

$$D(\omega, k) = -\omega^2 - 2k^2\|\xi\|^2 + k^4 = -(\omega^2 - \omega_0^2) + (k^2 - k_0^2)^2 .$$

Now set $\omega = \omega_0 + \mathrm{i}y$. Then

$$D(\omega, k) = y^2 + 2k_0^2 y + (k^2 - k_0^2)^2 .$$

Setting $D = 0$ then gives

$$k^{\pm}(y) = k_0 \pm \mathrm{i}\frac{\sqrt{y}}{2k_0} + \cdots ,$$

where the $\cdots$ represent terms which go to zero as $y \to \infty$. Clearly in the limit as $y \to \infty$, k_0 splits into two roots with imaginary parts of opposite sign. Hence the pinching condition is satisfied and the instability is absolute in the case $b = 0$.

5. Classifying the instabilities in the presence of third-order dispersion

In the case $b \neq 0$ the instability is convective for some values of b. To see this note that when $\theta^2 > 1$ then the only saddle points are associated with complex values of $\|\xi\|$. Hence there are no physical saddle points. Hence the instability cannot be absolute and is therefore convective.

One can check whether there are any transition points, where the instability goes from convective to absolute (or vice versa). According to [27] (see also the review [17]), a change from absolute to convective instability (or vice versa) occurs when

$$D = D_k = D_{kk} = 0\,;$$

that is,

$$\begin{aligned} D &= -\omega^2 - 2bk^3\omega - 2k^2\|\xi\|^2 + k^4 - b^2k^6 = 0 \\ D_k &= -6bk^2\omega - 4k\|\xi\|^2 + 4k^3 - 6b^2k^5 = 0 \\ D_{kk} &= -12bk\omega - 4\|\xi\|^2 + 12k^2 - 30b^2k^4 = 0\,. \end{aligned}$$

Solving the latter two equations gives

$$\|\xi\|^2 = -k^2 + \frac{9}{2}b^2k^4 \qquad \text{and} \qquad \omega = \frac{4k}{3b} - 4bk^3\,.$$

Substituting these two expressions into D gives

$$D = -k^2\left(18b^2k^4 - 11k^2 + \frac{16}{9b^2}\right)\,.$$

This equation has four complex roots

$$k^2 = \left(\frac{11 \pm \mathrm{i}\sqrt{7}}{36}\right)\frac{1}{b^2}\,.$$

These roots are complex and so it suggests that there is no change from absolute to convective instability, since when one of these values of k is substituted into the expression for $\|\xi\|^2$, it gives non-physical complex values of $\|\xi\|^2$.

6. Summary and conclusions

It appears that the instability is absolute when $b = 0$ (second-order dispersion only) and convective for all $b \neq 0$. This change is not continuous. The discontinuity appears to be due to the fact that $b \neq 0$ is a singular perturbation. The character of the dispersion relation is dramatically changed when $b \neq 0$:

$$D(\omega, k, b) = -\omega^2 - 2k^2\|\xi\|^2 + k^4 - bk^3(2\omega + bk^3)\,.$$

While in nonlinear optics the effects of higher-order dispersion are well understood, their significance is not so clear in hydrodynamics. We have found one major difference between the description of ocean waves and the description of waves in optical fibres. Although in each case the NLS equation seems to be a valid model, what we observe in reality is another matter. In optics one is measuring an averaged intensity, that is the square of the modulus of the envelope. The carrier frequency is usually forgotten. In the ocean one is observing the waves at the carrier frequency. Then an important parameter is the phase difference between the carrier and the envelope. The latter has been extensively discussed in optics when dealing with ultrashort pulses that contain only a few cycles but has not been discussed in the case of ocean waves. Meanwhile, if this parameter is small, one can observe higher amplitudes as opposed to the case when this parameter is close to π. Moreover this parameter may change during wave propagation. Then one may see the specific property of rogue waves that appear from nowhere and disappear without a trace [1]. In the ideal case, the effect would be periodic but in a chaotic wave field this may happen only once.

To conclude, one can state that rogue wave studies are the most mature in environments governed by the NLS equation (or its analogues), where efforts of experts with various scientific cultures have shaped the existing mechanisms and created a coherent picture about potential phenomena. However it is easy to get lost in the mathematical complexity of the problem. It is essential to remain focussed on trying to provide some concrete insight into the formation of rogue waves and prediction must be a priority. Going back to water waves, there is still a lack of laboratory experiments where the two-dimensional surface is measured in time. Moreover the importance of wave breaking in the study of extreme waves is being more and more emphasized [20]. But is there a limiting process equivalent to wave breaking in optics?

Acknowledgments

The content of this note is based on a lecture given at IMS of National University of Singapore in April 2009. FD thanks Professor Keith Moffatt and IMS for providing such a nice opportunity.

We acknowledge support from the French Agence Nationale de la Recherche project MANUREVA ANR-08-SYSC-019 and the 2008 Framework Program for Research, Technological Development and Innovation of the Cyprus Research Promotion Foundation under the Project AΣTI/0308(BE)/05.

References

1. N. Akhmediev, A. Ankiewicz and M. Taki, "Waves that appear from nowhere and disappear without a trace", *Physics Letters A*, **373** (2009), 675–678.
2. N. N. Akhmediev, V. I. Korneev and N. V. Mitskevich, "Modulation instability of a continuous signal in an optical fiber taking into account third-order dispersion", *Izv., Radio*, **33** (1990), 95–100.
3. T.B. Benjamin and J.E. Feir, "The disintegration of wavetrains on deep water. Part 1: Theory", *J. Fluid Mech.*, **27** (1967), 417–430.
4. L. Brevdo, "A study of absolute and convective instabilities with an application to the Eady model", *Geophys. Astro. Fluid Dyn.*, **40** (1988), 1–92.
5. L. Brevdo and T.J. Bridges, "Absolute and convective instabilities of temporally oscillating flows", *Z. angew. Math. Phys.*, **48** (1997), 290–309.
6. T.J. Bridges and F. Dias, "Enhancement of the Benjamin-Feir instability with dissipation", *Physics of Fluids*, **19** (2007), 104104.
7. J. M. Dudley, C. Finot, G. Millot, J. Garnier, G. Genty, D. Agafontsev and F. Dias, "Extreme events in optics: challenges of the MANUREVA project", *The European Physical Journal*, (2010).
8. J. M. Dudley, G. Genty, F. Dias, B. Kibler and N. Akhmediev, "Modulation instability, Akhmediev Breathers and continuous wave supercontinuum generation", *Optics Express*, **17** (2009), 21497–21508.
9. K. Dysthe, H.E. Krogstad and P. Müller, "Oceanic rogue waves", *Annu. Rev. Fluid Mech.*, **40** (2008), 287–310.
10. G. Genty, C. M. de Sterke, O. Bang, F. Dias, N. Akhmediev and J. M. Dudley, "Collisions and turbulence in optical rogue wave formation", *Physics Letters A*, **374** (2010), 989–996.
11. J.C.R. Hunt, "Nonlinear and wave theory contributions of T. Brooke Benjamin (1929-1995)", *Ann. Rev. Fluid Mech.*, **38** (2006), 1–25.

12. P.A.E.M. Janssen, "Nonlinear four-wave interactions and freak waves", *J. Phys. Oceanography*, **33** (2003), 863–884.
13. C. Kharif and E. Pelinovsky, "Physical mechanisms of the rogue wave phenomenon", *Eur. J. Mech. B/Fluids*, **22** (2003), 603–634.
14. C. Kharif, E. Pelinovsky and A. Slunyaev, *Rogue waves in the ocean*, Springer, Berlin, 2009.
15. B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev and J. M. Dudley, "The Peregrine Soliton in nonlinear fibre optics", (2010), to appear.
16. F.E. Laine-Pearson, "Instability growth rates of crossing sea states", *Phys. Rev. E*, **81** (2010), 036316.
17. E. de Langre, "Absolutely unstable waves in inviscid hydroelastic systems", *J. Sound Vib.*, **256** (2002), 299–317.
18. M. Onorato, A.R. Osborne and M. Serio, "Modulational instability in crossing sea states: A possible mechanism for the formation of freak waves", *Physical Review Letters*, **96** (2006), 014503.
19. M. Onorato, T. Waseda, A. Toffoli, L. Cavaleri, O. Gramstad, P.A.E.M. Janssen, T. Kinoshita, J. Monbaliu, N. Mori, A.R. Osborne, M. Serio, C.T. Stansberg, H. Tamura and K. Trulsen, "Statistical properties of directional ocean waves: The role of the modulational instability in the formation of extreme events", *Physical Review Letters*, **102** (2009), 114502.
20. I.A. Papadimitrakis and F. Dias, "Occurrence and breaking of extreme waves in deep water. A stochastic approach revisit", *The Open Ocean Engineering Journal*, (2010).
21. H. Segur, D. Henderson, J. Carter, J. Hammack, C.-M. Li, D. Pheiff and K. Socha, "Stabilizing the Benjamin-Feir instability", *J. Fluid Mech.*, **539** (2005), 229.
22. M. Shats, H. Punzmann and H. Xia, "Capillary Rogue Waves", *Phys. Rev. Lett.*, **104** (2010), 104503.
23. D.R. Solli, C. Ropers, P. Koonath and B. Jalali, "Optical rogue waves", *Nature*, **450** (2007), 1054–1057.
24. L. Stenflo and M. Marklund, "Rogue waves in the atmosphere", *J. Plasma Phys.*, **76** (2010), 293–295.
25. M. Taki, A. Mussot, A. Kudlinski, E. Louvergneaux, M. Kolobov and M. Douay, "Third-order dispersion for generating optical rogue solitons", *Physics Letters A*, **374** (2010), 691–695.
26. H. Tamura, T. Waseda and Y. Miyazawa, "Freakish sea state and swell-windsea coupling: Numerical study of the *Suwa-Maru* incident", *Geophysical Research Letters*, **36** (2009), L01607.
27. G.S. Triantafyllou, "Note on the Kelvin-Helmholtz instability of stratified fluids", *Phys. Fluids*, **6** (1994), 164–171.