Breakdown of the shallow water equations due to growth of the horizontal vorticity

THOMAS J. BRIDGES[1] AND **DAVID J. NEEDHAM**[2]

[1]Department of Mathematics, University of Surrey, Guildford, Surrey GU2 7XH, UK

[2]School of Mathematcs, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

(Received ?? and in revised form ??)

In an oceanographic setting, the shallow water equations are an asymptotic approximation to the full Euler equations, in the limit $\varepsilon = \frac{h_0}{L} \to 0$, with h_0 being the vertical length scale and L a horizontal length scale associated with the fluid layer. However, in arriving at the shallow water equations an additional key step in the derivation is the condition that at some reference time (e.g. $t = 0$) the thin-layer horizontal vorticity field is identically zero, which corresponds to the horizontal fluid velocity field being independent of the vertical coordinate, z, at $t = 0$. With this condition in place, the "thin-layer equations" reduce exactly to the shallow water equations. In this paper we show that this exact condition may be unstable: small, even infinitesimal, perturbations of the thin-layer horizontal vorticity field can grow without bound. When the thin-layer horizontal vorticity grows to be of order one, the shallow water equations are no longer asymptotically valid as a model for shallow water hydrodynamics, and the "thin-layer equations" must be adopted in their place.

Key Words: shallow water equations, thin-layer equations, thin-layer horizontal vorticity, potential vorticity, instability

1. Introduction

The shallow water equations are one of the most widely used models for oceanographic flows, tidal simulations, and coastal hydrodynamics. In terms of a cartesian coordinate system (x, y, z) with z pointing vertically upwards, horizontal velocity field $(u(x, y, t), v(x, y, t))$ and free surface elevation given by $z = h(x, y, t)$ they take the standard form

$$\begin{aligned} \tfrac{Dh}{Dt} + h(u_x + v_y) &= 0\,, \\ \tfrac{Du}{Dt} + gh_x &= fv\,, \\ \tfrac{Dv}{Dt} + gh_y &= -fu\,, \end{aligned} \tag{1.1}$$

where here and throughout $\frac{D}{Dt}$ represents the horizontal material derivative

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\,, \tag{1.2}$$

and f, g are the rotation and gravitational parameters respectively. The shallow water equations conserve potential vorticity (cf. Salmon 1998; McIntyre 2003). Potential vorticity is associated with the vertical vorticity, $\mathcal{P} := (v_x - u_y + f)/h$.

The shallow water equations are an approximation to the Euler equations for three-dimensional inviscid flow with a free surface, and they can be derived using an asymptotic argument (e.g. Dingemans 1997; Johnson 1997; Salmon 1998; Stoker 1958). The small parameter is $\varepsilon = \frac{h_0}{L}$, where L is a representative horizontal length scale, and h_0 is representative of the undisturbed water depth. Scaling the dependent and independent variables in the usual way and taking the limit as $\varepsilon \to 0$ results in a hydrostatic pressure field. However, the horizontal velocity field still depends on the vertical coordinate. The asymptotic argument does not lead to a horizontal velocity field that is independent of the vertical coordinate. An additional assumption is necessary. The usual assumption is that (u_z, v_z) are zero at $t = 0$ and that this property is preserved for all time. In fact this assumption and its consequence are exact. With this asymptotic argument followed by the assumption on the horizontal velocity field, the reduction of the full water-wave problem to the shallow water equations is a precise and rational argument.

However, a question that does not appear to have been addressed heretofore is whether the initial condition $(u_z, v_z) = (0, 0)$ is stable. If (u_z, v_z) is small – even infinitesimal – at $t = 0$ will it remain small for all time? It is the purpose of this paper to show that this property is demonstrably false. First, a simple exact solution of the nonlinear problem will be presented to illustrate this point. Then the general linear instability problem for thin-layer horizontal vorticity will be formulated.

Our conclusion is that it appears, in general, to be very difficult to control the growth of thin-layer horizontal vorticity. Hence in real oceanographic flows, where the solutions can be quite complicated, and perturbations of horizontal vorticity will invariably be present, the shallow water equations as a model will rapidly become invalid.

In §2 the standard shallow water scaling and asymptotic argument are reviewed, and the thin-layer equations derived. In §3 an exact solution is constructed of the thin-layer equations (shallow water hydrodynamics with the assumption of hydrostatic pressure field only) which has unbounded growth of horizontal vorticity. Then in §3 the exact linear stability problem for perturbations of horizontal vorticity is formulated, and its key properties are identified. When the shallow water equations breakdown, the conservation of potential vorticity (PV) is also lost, and the precise effect of non-zero thin-layer horizontal vorticity on PV is presented in §6.

2. Asymptotic derivation of the thin-layer equations

The governing equations for three-dimensional, inviscid, incompressible, water waves in (x, y, z, t) with horizontal coordinates (x, y) and vertical coordinate z, and the free surface represented by $z = h(x, y, t)$, are the usual Euler equations with the dynamic and kinematic free surface boundary conditions. The rotation term is neglected since it does not affect the general argument presented here and can be brought back in as appropriate.

Let $U_0 = \sqrt{gh_0}$ be the representative horizontal velocity scale, and introduce the standard shallow-water scaling (e.g. p. 482 of Dingemans 1997) and (Chapter 2 of Johnson 1997),

$$\begin{aligned} \varepsilon &= \frac{h_0}{L}\,, \quad \widetilde{x} = \frac{x}{L}\,, \quad \widetilde{y} = \frac{y}{L}\,, \quad \widetilde{z} = \frac{z}{h_0} = \frac{z}{\varepsilon L}\,, \quad \widetilde{t} = \frac{tU_0}{L}\,, \\ \widetilde{u} &= \frac{u}{U_0}\,, \quad \widetilde{v} = \frac{v}{U_0}\,, \quad \widetilde{w} = \frac{w}{\varepsilon U_0}\,, \quad \widetilde{h} = \frac{h}{h_0}\,. \end{aligned} \tag{2.1}$$

Introducing the scalings (2.1) into the governing incompressible Euler equations we arrive at

$$\begin{aligned} \widetilde{u}_{\widetilde{x}} + \widetilde{v}_{\widetilde{y}} + \widetilde{w}_{\widetilde{z}} &= 0\,, \\ \widetilde{u}_{\widetilde{t}} + \widetilde{u}\widetilde{u}_{\widetilde{x}} + \widetilde{v}\widetilde{u}_{\widetilde{y}} + \widetilde{w}\widetilde{u}_{\widetilde{z}} + \widetilde{p}_{\widetilde{x}} &= 0\,, \\ \widetilde{v}_{\widetilde{t}} + \widetilde{u}\widetilde{v}_{\widetilde{x}} + \widetilde{v}\widetilde{v}_{\widetilde{y}} + \widetilde{w}\widetilde{v}_{\widetilde{z}} + \widetilde{p}_{\widetilde{y}} &= 0\,, \\ \varepsilon^2\left(\widetilde{w}_{\widetilde{t}} + \widetilde{u}\widetilde{w}_{\widetilde{x}} + \widetilde{v}\widetilde{w}_{\widetilde{y}} + \widetilde{w}\widetilde{w}_{\widetilde{z}}\right) + \widetilde{p}_{\widetilde{z}} &= -1\,. \end{aligned} \tag{2.2}$$

with boundary conditions at the free surface

$$\widetilde{p} = 0 \quad \text{and} \quad \widetilde{h}_{\widetilde{t}} + \widetilde{u}\widetilde{h}_{\widetilde{x}} + \widetilde{v}\widetilde{h}_{\widetilde{y}} = \widetilde{w} \quad \text{at} \quad \widetilde{z} = \widetilde{h}(\widetilde{x}, \widetilde{y}, \widetilde{t})\,, \tag{2.3}$$

and

$$\widetilde{w} = 0 \quad \text{at} \quad \widetilde{z} = 0\,. \tag{2.4}$$

The only place that ε appears explicitly is in the vertical momentum equation, and formally taking the limit $\varepsilon \to 0$ results in a hydrostatic pressure field. With a hydrostatic pressure field, the terms $\widetilde{p}_{\widetilde{x}}$ and $\widetilde{p}_{\widetilde{y}}$ can be expressed in terms of $\widetilde{h}$. The resulting equations are as follows (the tildes have been dropped since these equations are really the starting point for the paper)

$$\begin{aligned} u_x + v_y + w_z &= 0\,, \\ u_t + uu_x + vu_y + wu_z + h_x &= 0\,, \\ v_t + uv_x + vv_y + wv_z + h_y &= 0\,, \end{aligned} \tag{2.5}$$

with boundary conditions

$$w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad w = h_t + uh_x + vh_y \quad \text{at} \quad z = h\,. \tag{2.6}$$

The equations (2.5) with boundary conditions (2.6) will be called the *thin-layer equation problem* (TLEP) in order to distinguish these equations from the further reduction to the shallow water equations.

In TLEP the horizontal velocity field (u, v) still, in general, depends on z and the vertical velocity component w is still present in the momentum equations in (2.5). The typical assumption at this point is to assume that (u_z, v_z) is zero at $t = 0$ and that this property is maintained for all time. With this assumption and the boundary conditions (2.6),

$$w(x, y, z, t) = -z(u_x + v_y) \quad \text{and} \quad h_t + uh_x + vh_y = w(x, y, h(x, y, t), t)\,,$$

combine to give the mass equation in the shallow water equations. The reduced shallow water equations are then

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0\,, \\ u_t + uu_x + vu_y + h_x &= 0\,, \\ v_t + uv_x + vv_y + h_y &= 0\,. \end{aligned} \tag{2.7}$$

The reduced shallow water equations (2.7) are called the *shallow water equations problem* (SWEP).

Solutions of SWEP are also ($z-$independent in u and v) exact solutions of TLEP. Of interest is whether these solutions are stable as solutions of TLEP, or whether small $z-$dependent perturbations in u and/or v may generate growing thin-layer horizontal vorticity.

2.1. *The horizontal vorticity field*

The above-mentioned assumption on (u_z, v_z) required to reduce TLEP to SWEP can be made precise by deriving governing equations for the components of the thin-layer horizontal vorticity in TLEP. Differentiating the second and third equations of (2.5) with respect to z gives

$$\begin{aligned} \tfrac{D}{Dt}u_z + wu_{zz} &= v_z u_y - u_z v_y \,, \\ \tfrac{D}{Dt}v_z + wv_{zz} &= u_z v_x - v_z u_x \,, \end{aligned} \tag{2.8}$$

where $\frac{D}{Dt}$ is the horizontal material derivative (1.2). This equation can be given a more illuminating form by noting that $(-v_z, u_z)$ is the thin-layer asymptotic form of the horizontal vorticity as $\varepsilon \to 0$. Define

$$\mathbf{\Omega} := \begin{pmatrix} -v_z \\ u_z \end{pmatrix} . \tag{2.9}$$

Then, via (2.8), $\mathbf{\Omega}$ satisfies,

$$\frac{D}{Dt}\mathbf{\Omega} + w\mathbf{\Omega}_z = \mathbf{D}^T\mathbf{\Omega} \,, \tag{2.10}$$

where

$$\mathbf{D} := \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} . \tag{2.11}$$

It is evident from (2.10) that the assumption

$$\mathbf{\Omega} = \mathbf{0} \quad \text{for all } t > 0 \quad \text{when} \quad \mathbf{\Omega} = \mathbf{0} \text{ at } t = 0$$

is fully justified. It is also apparent however, that stability of this assumption may not in general be assured because of the $\mathbf{D}$ term on the right-hand side of equation (2.10).

3. An exact nonlinear unstable solution of TLEP

In this section an exact solution of TLEP (2.5)-(2.6) is constructed. The solution is a function of α, a real parameter, and when $\alpha = 0$ it is an exact solution of SWEP (2.7). When $\alpha \neq 0$ the thin-layer horizontal vorticity grows algebraically in time.

Introduce the fluid domains

$$\Delta = \{ \ (x, y, t) \in \mathbb{R}^3 \ : \ 0 < x < S(t) \,, \ -\infty < y < +\infty \,, \ t > 0 \ \} \,, \tag{3.1}$$

and for $t \geqslant 0$,

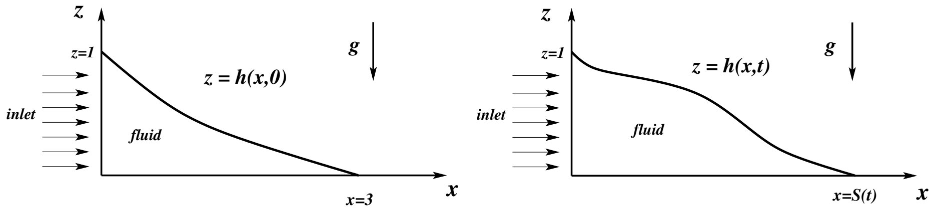
$$D(t) = \{ \ (x, y, z) \in \mathbb{R}^3 \ : \ 0 < x < S(t) \,, \ -\infty < y < +\infty \,, \ 0 < z < h(x, y, t) \ \} \,, \tag{3.2}$$

and

$$\Lambda = \{ \ (x, y, z, t) \ : \ (x, y, z) \in D(t) \,, \quad t > 0 \ \} \,. \tag{3.3}$$

The domain occupied by the fluid is, for $t \geqslant 0$, given by $\overline{D}(t)$. The function $S : [0, \infty) \to \mathbb{R}^+$ represents the edge of an advancing fluid layer at $x = S(t)$, $t \geqslant 0$, whilst $h : \overline{\Delta} \to \mathbb{R}^+ \bigcup \{0\}$ is such that $z = h(x, y, t)$ represents the location of the fluid free surface and $(u, v, w) : \overline{\Lambda} \to \mathbb{R}$ represent the components of the fluid velocity field in the (x, y, z) directions respectively. Consider TLEP in Λ with initial conditions

$$S(0) = 3 \,, \tag{3.4}$$



(a) (b)

FIGURE 1. Schematic of the problem at (a) $t = 0$ and (b) $t > 0$.

$$h(x,y,0) = \frac{1}{9}(3-x^2)\,,\quad (x,y)\in[0,3]\times\mathbb{R}\,, \tag{3.5}$$

$$u(x,y,z,0) = 1+\frac{2}{3}x \quad\text{and}\quad v(x,y,z,0)=\alpha z v_0(x)\,,\quad (x,y,z)\in\overline{D}(0)\,, \tag{3.6}$$

and boundary conditions

$$u(0,y,z,t) = 1\,,\quad (y,z,t)\in\mathbb{R}\times[0,1]\times[0,\infty)\,, \tag{3.7}$$
$$v(0,y,z,t) = \alpha z\,,\quad (y,z,t)\in\mathbb{R}\times[0,1]\times[0,\infty)\,, \tag{3.8}$$
$$w(x,y,0,t) = 0\,,\quad (x,y,t)\in\overline{\Delta}\,, \tag{3.9}$$
$$h(S(t),y,t) = 0\,,\quad (y,t)\in\mathbb{R}\times[0,\infty)\,, \tag{3.10}$$
$$u(S(t),y,0,t) = \dot{S}(t)\,,\quad (y,t)\in\mathbb{R}\times[0,\infty)\,. \tag{3.11}$$

Schematics of the problem at $t = 0$ and $t > 0$ are shown in Figure 1. We consider classical solutions to TLEP, $(u,v,w):\overline{\Lambda}\to\mathbb{R}$, $h:\overline{\Delta}\to\mathbb{R}$, and $S:[0,\infty)\to\mathbb{R}$. The function $v_0:[0,3]\to\mathbb{R}$ is a prescribed continuously differentiable function with $\max_{x\in[0,3]}|v_0(x)| = 1$, and α is a real-valued parameter.

Now, when $\alpha = 0$, then both $u(x,y,z,0)$ and $v(x,y,z,0)$ are independent of z on $\overline{D}(0)$. It then follows from the argument in §2.1 that u, v remain independent of z on $\overline{\Lambda}$, and are thus solutions of the associated SWEP reduction of TLEP. In fact, when $\alpha = 0$, the solution to TLEP (which is reduced to SWEP) is obtained as

$$S(t) = 3(1+t)\,,\quad t\geqslant 0\,, \tag{3.12}$$

$$h(x,y,t) = \frac{1}{9}\left(3-\frac{x}{(1+t)}\right)^2\,,\quad (x,y,t)\in\overline{\Delta}\,, \tag{3.13}$$

$$u(x,y,z,t) = (3-2\sqrt{h(x,y,t)}) = 1+\frac{2}{3}\frac{x}{(1+t)}\,,\quad (x,y,z,t)\in\overline{\Lambda}\,, \tag{3.14}$$

$$v(x,y,z,t) = 0\,,\quad (x,y,z,t)\in\overline{\Lambda}\,, \tag{3.15}$$

$$w(x,y,z,t) = -\frac{2}{3}\frac{z}{(1+t)}\,,\quad (x,y,z,t)\in\overline{\Lambda}\,. \tag{3.16}$$

This solution is readily verified by direct substitution. Before proceeding to considering TLEP when $\alpha\neq 0$, we first consider the thin-layer horizontal vorticity in TLEP. In TLEP the thin-layer horizontal vorticity $\mathbf{\Omega}:\overline{\Lambda}\to\mathbb{R}^2$ is defined in (2.9) and satisfies (2.10) and is subject to the initial conditions

$$\mathbf{\Omega}(x,y,z,0) := \big(-v_z(x,y,z,0), u_z(x,y,z,0)\big) = \big(-\alpha v_0(x), 0\big)\,, \tag{3.17}$$

for $(x,y,z)\in\overline{D}(0)$. Note that when $\alpha = 0$ then $\mathbf{\Omega}(x,y,z,0) = \mathbf{0}$ on $\overline{D}(0)$ and so $\mathbf{\Omega}(x,y,z,t) = \mathbf{0}$ on $\overline{\Lambda}$.

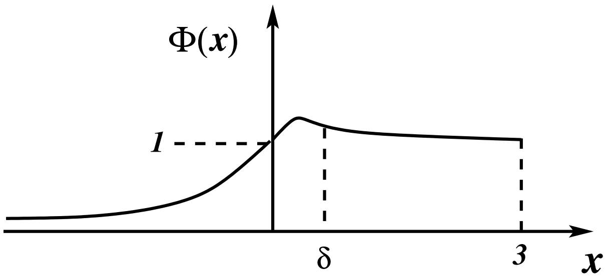


FIGURE 2. Schematic of the function $\Phi(x)$.

3.1. *Nonlinear solution of TLEP when $\alpha \neq 0$*

We now consider TLEP when $\alpha \neq 0$; the case where TLEP does not reduce to SWEP. In fact the exact solution may be obtained as $S(t) = 3(1+t)$, $t \geqslant 0$,

$$h(x,y,t) = \frac{1}{9}\left(3 - \frac{x}{(1+t)}\right)^2, \quad (x,y,t) \in \overline{\Delta}, \tag{3.18}$$

$$u(x,y,z,t) = 1 + \frac{2}{3}\frac{x}{(1+t)}, \quad w(x,y,z,t) = -\frac{2}{3}\frac{z}{(1+t)}, \quad (x,y,z,t) \in \overline{\Lambda}, \tag{3.19}$$

with $v : \overline{\Lambda} \to \mathbb{R}$ being the solution to

$$\frac{\partial v}{\partial t} + \left(1 + \frac{2}{3}\frac{x}{(1+t)}\right)\frac{\partial v}{\partial x} - \left(\frac{2}{3}\frac{z}{(1+t)}\right)\frac{\partial v}{\partial z} = 0, \quad (x,y,z,t) \in \overline{\Lambda},$$

subject to $v(x,y,z,0) = \alpha z v_0(x)$ for $(x,y,z) \in \overline{D}(0)$ and $v(0,y,z,t) = \alpha z$ for $t \geqslant 0$ and $(y,z) \in \mathbb{R} \times [0,1]$. The solution to the problem for v, after using (3.18)-(3.19), is readily obtained as

$$v(x,y,z,t) = \alpha(1+t)^{2/3} z \Phi\left(3 + \left(\frac{x}{(1+t)} - 3\right)(1+t)^{1/3}\right), \quad (x,y,z,t) \in \overline{\Lambda},$$

and with $\Phi : (-\infty, 3] \to \mathbb{R}$ given by

$$\Phi(x) = \begin{cases} \dfrac{9}{(3-x)^2}\,; & x \leqslant 0 \\ v_0(x)\,; & 0 \leqslant x \leqslant 3 \end{cases}.$$

Now choose $v_0(x)$, $x \in [0,3]$, so that Φ has the form shown in Figure 2, with Φ continuously differentiable, $\Phi(0) = 1$ and $\Phi(x) = 1$ for $x \in [\delta, 3]$, with $0 < \delta \ll 1$. The horizontal vorticity field is then $\mathbf{\Omega} = (\Omega_1, 0)$ with

$$\Omega_1(x,y,z,t) = -\alpha(1+t)^{2/3}\Phi\left(3 + \left(\frac{x}{(1+t)} - 3\right)(1+t)^{1/3}\right), \tag{3.20}$$

for $(x,y,x,t) \in \overline{\Lambda}$. From this construction, we conclude that the solution with $\alpha = 0$ to TLEP (which is also a solution to SWEP) is nonlinearly unstable to small perturbations in the thin-layer horizontal vorticity in the $x-$direction. The solution to TLEP when $\alpha \neq 0$ undergoes blow-up in the thin-layer horizontal vorticity (in particular in the $x-$direction) as $t \to \infty$, with the blow-up algebraic in t, like $\alpha t^{2/3}$ as $t \to \infty$. On the other hand, the horizontal *inlet* (at $x = 0$, $t \geqslant 0$) and *initial* (at $t = 0$, $0 \leqslant x \leqslant 3$) thin-layer horizontal vorticity is bounded, of $\mathcal{O}(\alpha)$, in the solution to TLEP.

At this stage it is worth making some additional observations on the exact nonlinear solution (3.18)-(3.20). First, for $t \gg 1$, and $a \in (-\infty, 3]$ we introduce

$$x(t;a) = 3(1+t) - (3-a)(1+t)^{2/3}\,, \tag{3.21}$$

and the closed, cross-sectionally bounded region $\overline{D}(t;a) \subset \mathbb{R}^2$, via

$$\overline{D}(t;a) = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x(t;a) \leqslant x \leqslant 3(1+t)\,,\ 0 \leqslant z \leqslant h(x,y,t)\,,\ -\infty < y < \infty\right\},$$

and the cross section of $\overline{D}(t;a)$, labelled $\overline{A}(t;a) \subset \mathbb{R}^2$, with,

$$\overline{A}(t;a) = \left\{(x,z) \in \mathbb{R}^2 \ : \ (x,0,z) \in \overline{D}(t;a)\right\}. \tag{3.22}$$

It now follows, via (3.18)-(3.20), and the structure of $\Phi : \mathbb{R} \to \mathbb{R}$, that,

$$|\Omega_1(x,y,z,t)| \geqslant |\alpha| \min(\Phi(a),1)(1+t)^{2/3}\,, \tag{3.23}$$

for all $(x,y,z) \in \overline{D}(t;a)$ and $t \geqslant T(a)$ with,

$$T(a) = \max\left(0, \frac{1}{27}(3-a)^3 - 1\right).$$

In addition, we have

$$1 \leqslant |u(x,y,z,t)| \leqslant 3 \tag{3.24}$$

$$0 \leqslant |v(x,y,z,t)| \leqslant \frac{1}{9}|\alpha| \min(\Phi(a),1)(3-a)^2 \tag{3.25}$$

$$0 \leqslant |w(x,y,z,t)| \leqslant \frac{2}{27}\frac{(3-a)^2}{(1+t)^{5/3}}\,, \tag{3.26}$$

$$0 \leqslant h(x,y,t) \leqslant \frac{1}{9}\frac{(3-a)^2}{(1+t)^{2/3}}\,, \tag{3.27}$$

for all $(x,y,z) \in \overline{D}(t;a)$ and $t \geqslant T(a)$. The inequalities (3.23)-(3.27) hold on each spatial region $\overline{D}(t;a)$ for each fixed $a \in (-\infty, 3]$. Moreover, the bounds in the inequalities (3.23)-(3.27) are achieved on $\overline{D}(t;a)$. Finally, it is straightforward to establish that, for each fixed $a \in (-\infty, 3]$, the area of the cross section of $\overline{D}(t;a)$, that is $|\overline{A}(t;a)|$, remains constant for each $t \geqslant T(a)$, so that

$$|\overline{A}(t;a)| = |\overline{A}(T(a);a)|\,,$$

for all $t \geqslant T(a)$. Thus the blow-up of Ω_1, as $t \to \infty$, via (3.23), occurs in the spatial region $\overline{D}(t;a)$, with $\overline{D}(t;a)$ being a cylindrical region (axis parallel to the $y-$axis) which has *finite* cross-sectional area $|\overline{A}(T(a);a)|$ as $t \to \infty$. The inequalities (3.24)-(3.27) establish that u, v, w and h remain bounded on $\overline{D}(t;a)$ as $t \to \infty$. A sketch of the structure of Ω_1 for $t \gg 1$ is shown in Figure 3, where $\sigma(t) := 3(1+t) - (3-\delta)(1+t)^{2/3}$.

4. Linear stability of thin-layer horizontal vorticity

In this section we let $(h_s(x,y,t), u_s(x,y,t), v_s(x,y,t))$ be a solution of SWEP (2.7). This solution is also an exact solution of TLEP (2.5), with zero thin-layer horizontal vorticity. The vertical velocity field is then $w_s(x,y,z,t)$, given by,

$$w_s = -z\left(\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y}\right), \tag{4.1}$$

for x, y, z in the domain occupied by the fluid at each $t \geqslant 0$. We now linearize the equation for the thin-layer horizontal vorticity (2.10) about this solution to SWEP, which becomes

$$\mathbf{\Omega}_t + u_s\mathbf{\Omega}_x + v_s\mathbf{\Omega}_y + w_s\mathbf{\Omega}_z = \mathbf{D}_s^T\mathbf{\Omega}\,, \tag{4.2}$$

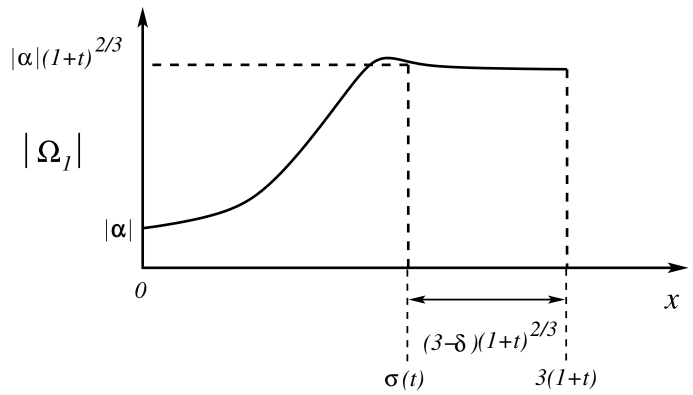


FIGURE 3. A schematic of the structure of Ω_1 versus x for $t \gg 1$.

for $(x, y, z) \in D_s(t)$, $t > 0$, with $D_s(t)$ being the domain occupied by the fluid in the base flow to SWEP and $\mathbf{D}_s$ is $\mathbf{D}$ in (2.11) evaluated on the base flow (u_s, v_s).

When $\mathbf{D}_s \equiv \mathbf{0}$, then (4.2) reveals that the thin-layer horizontal vorticity is simply convected along particle paths of the base flow (u_s, v_s, w_s) and so the thin-layer horizontal vorticity field at any time $t > 0$ is simply a re-arrangement of the thin-layer horizontal vorticity field at $t = 0$. Thus if $\boldsymbol{\Omega}\big|_{t=0}$ is bounded, then $\boldsymbol{\Omega}\big|_{t>0}$ is bounded for $t > 0$. However, when the base flow is such that $\mathbf{D}_s \neq \mathbf{0}$, then this may not be so. In particular, we say that the solution of SWEP given by (h_s, u_s, v_s) is linearly unstable to perturbations in thin-layer horizontal vorticity if the trivial solution $\boldsymbol{\Omega} = \mathbf{0}$ is an unstable solution of (4.2).

4.1. *The Jacobian of the horizontal velocity*

The Jacobian of the horizontal velocity field evaluated on a solution of SWEP, $\mathbf{D}_s$, is also important in determining instability of Lagrangian particle paths in SWEP. Particle paths in SWEP associated with $(h_s(x, y, t), u_s(x, y, t), v_s(x, y, t))$ satisfy

$$\dot{x}_s(x_s^0, y_s^0, t) = u_s(x, y, t) \quad \text{and} \quad \dot{y}_s(x_s^0, y_s^0, t) = v_s(x, y, t)\,, \tag{4.3}$$

where $(x_s^0, x_s^0) \in \mathbb{R}^2$ are coordinates for the initial data in a Lagrangian reference space. It is $\mathbf{D}_s$, the Jacobian of the horizontal velocity field (u_s, v_s) with respect to (x, y), that controls the linearization of the dynamical system (4.3). Hence there is a close connection between unstable Lagrangian pathlines in SWEP and unstable thin-layer horizontal vorticity in TLEP. Unstable pathlines are in abundance in planar flows, and indeed are the key to mixing (e.g. Ottino 1989), and so provide a general mechanism for generating unstable thin-layer horizontal vorticity in TLEP.

An additional point to note is that by differentiating (2.7), a governing equation is obtained for $\mathbf{D}_s$, namely the matrix Riccati equation

$$\frac{D}{Dt}\mathbf{D}_s + \mathbf{D}_s^2 = -\begin{bmatrix} h_{sxx} & h_{sxy} \\ h_{syx} & h_{syy} \end{bmatrix}, \tag{4.4}$$

for (x, y) in the domain occupied by the fluid, and $t \geqslant 0$. This Riccati equation is very similar to the matrix Riccati equation for the full Euler equations, with the Hessian of h_s replaced by the Hessian of the pressure (cf. Ohkitani 2010, and references therein). It may be possible to determine some general results on $\mathbf{D}_s$ by analyzing this equation.

The linearized stability problem for the thin-layer horizontal vorticity has now been reduced to studying the linear problem (4.2), and we next consider the specific case when the base flow solution to SWEP leads to the matrix $\mathbf{D}_s$ being dependent upon t alone.

5. The case when $\mathbf{D}_s$ depends only on time t

When $\mathbf{D}_s$ depends only upon time, the linearized thin-layer horizontal vorticity equation (4.2) can be further simplified. This case is of interest since there is a class of exact solutions of SWEP with a corresponding $\mathbf{D}_s$ depending upon time t alone. These solutions of SWEP are linear functions of x and y, with $t-$dependent coefficients, namely,

$$\begin{aligned} u_s(x,y,t) &= a_{10}(t) + a_{11}(t)x + a_{12}(t)y \\ v_s(x,y,t) &= a_{20}(t) + a_{21}(t)x + a_{22}(t)y\,, \end{aligned} \tag{5.1}$$

with an associated free surface at $z = h_s(x,y,t)$. This form for the velocity field is known to generate an exact solution of SWEP (cf. Ball 1965; Cushman-Roisin 1987; Ripa 1987; Thacker 1981; Young 1986). Moreover, Ripa (1987) has shown that they are stable solutions of SWEP.

However, it does not appear to have been considered heretofore that the class of solutions (5.1) may be unstable to perturbations in the thin-layer horizontal vorticity field in TLEP. Indeed, the fully-nonlinear unstable example studied in §3 is an exemplar of this class of solutions.

Now, for (5.1) the matrix $\mathbf{D}_s$ is just

$$\mathbf{D}_s(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}\,. \tag{5.2}$$

We now write the solution to (4.2) in the form

$$\mathbf{\Omega}(x,y,z,t) = \mathbf{Q}(t)\mathbf{F}(x,y,z,t)\,, \tag{5.3}$$

where the 2×2 matrix $\mathbf{Q}(t)$ is the fundamental matrix of the homogeneous linear system

$$\dot{\mathbf{Q}} = \mathbf{D}_s(t)^T\mathbf{Q}\,, \quad t > 0\,, \tag{5.4}$$

with $\mathbf{Q}(0) = \mathbf{I}$. It then follows that

$$\mathbf{F}_t + u_s\mathbf{F}_x + v_s\mathbf{F}_y + w_s\mathbf{F}_z = 0\,, \tag{5.5}$$

for (x,y,z) in the domain occupied by the fluid in the base flow (h_s, u_s, v_s) at each $t > 0$. It follows from (5.5) that $\mathbf{F}(x,y,z,t)$ is simply a re-arrangement of the initial perturbation in the thin-layer horizontal vorticity, and this remains bounded for all $t > 0$ by the bound on the initial condition.

In particular, let $\overline{D}(0) \subset \mathbb{R}^3$ be a closed, bounded, subregion of the region occupied by the base flow at $t = 0$, and let $\overline{D}(t) \subset \mathbb{R}^3$ be the correspondingly closed bounded subregion of the region occupied by the base flow for $t > 0$, obtained by base flow convection acting in $\overline{D}(0)$. It is readily established that the volume of $\overline{D}(t)$ is equal to the volume of $\overline{D}(0)$ for all $t \geqslant 0$, that is,

$$|\overline{D}(t)| = |\overline{D}(0)| \quad \forall\, t \geqslant 0\,. \tag{5.6}$$

Now suppose that the initial horizontal vorticity field is bounded away from zero on $\overline{D}(0)$, that is,

$$|\mathbf{F}(x,y,z,0)| \geqslant M_{\overline{D}} > 0\,, \quad \text{for all } (x,y,z) \in \overline{D}(0)\,. \tag{5.7}$$

It then follows from (5.3) that,

$$|\mathbf{\Omega}(x,y,z,t)| \geqslant M_{\overline{D}}\|\mathbf{Q}(t)\|\,, \tag{5.8}$$

for all $(x,y,z) \in \overline{D}(t)$ and $t > 0$, with $\|\cdot\|$ representing the usual matrix norm. Thus, it follows from (5.6) and (5.8), that whenever the fundamental matrix $\mathbf{Q}(t)$ of (5.4) has at least one temporally-growing entry, then any non-trivial continuous initial perturbation in the thin-layer horizontal vorticity field will temporally grow in a closed, bounded subregion of the region occupied by the base flow, which has a finite and constant volume for all $t > 0$. In this case, the class of solutions (5.1) to SWEP generates linearly unstable thin-layer horizontal vorticity. To investigate (5.4) further, we write,

$$\mathbf{Q}(t) = \left[\mathbf{w}^{(1)}(t) \,\middle|\, \mathbf{w}^{(2)}(t)\right], \tag{5.9}$$

where $\mathbf{w}^{(1)}(t)$ and $\mathbf{w}^{(2)}(t)$ are the two linearly independent solutions to the ordinary differential equation

$$\dot{\mathbf{w}} = \mathbf{D}_s(t)^T\mathbf{w}, \quad t > 0, \tag{5.10}$$

with $\mathbf{w}^{(1)}(0) = (1,0)^T$ and $\mathbf{w}^{(2)}(0) = (0,1)^T$. We may conclude that perturbations to the thin-layer horizontal vorticity in the class of SWEP flows (5.1) are unstable whenever $\mathbf{w} = \mathbf{0}$ is an unstable solution of (5.10).

An application of the above result is to the case studied in §3, where the full nonlinear problem was solvable. The above linearized theory should predict instability of the SWEP base flow (5.1) when

$$a_{10}(t) = 1, \quad a_{11}(t) = \frac{2}{3(1+t)}, \quad a_{12}(t) = 0 \tag{5.11}$$

and $a_{20}(t) = a_{21}(t) = a_{22}(t) = 0$ for all $t \geqslant 0$. In this case, using (5.2), we obtain, on solving (5.10)

$$\mathbf{w}^{(1)}(t) = \left((1+t)^{2/3}, 0\right)^T \quad \text{and} \quad \mathbf{w}^{(2)}(t) = \left(0, 1\right)^T, \tag{5.12}$$

in $t \geqslant 0$, and conclude that the base flow to SWEP given by (5.1) with (5.11) and $v_s = 0$ is linearly unstable to perturbations in the thin-layer horizontal vorticity, in accord with the fully nonlinear example in §3.

6. Induced instability of the shallow-water potential vorticity

In this section the conservation of PV in the shallow water equations is derived from the viewpoint of TLEP. From this viewpoint the effect of growth of thin-layer horizontal vorticity on the shallow water PV can be analysed. Let

$$\mathcal{P} = \frac{v_x - u_y}{h}.$$

Then differentiating, and using TLEP, the governing equation for $\mathcal{P}$ in TLEP is

$$\frac{D\mathcal{P}}{Dt} = -\frac{\mathcal{P}}{h}\left(\frac{Dh}{Dt} + h(u_x + v_y)\right) - w\mathcal{P}_z + \frac{1}{h}(\Omega_1 w_x + \Omega_2 w_y). \tag{6.1}$$

If (h,u,v) satisfy SWEP then $\boldsymbol{\Omega} \equiv \mathbf{0}$ and the shallow water PV is conserved: $\frac{D\mathcal{P}}{Dt} = 0$. In order to reduce from PV in TLEP, via (6.1), to PV in SWEP three assumptions are required: (a) h must satisfy shallow water conservation of mass, (b) $\mathcal{P}$ must be independent of z and (c) the thin-layer horizontal vorticity must vanish. On the other hand, if the shallow water equations are perturbed, leading to growth of the thin-layer horizontal vorticity, then it is clear from (6.1) that a source term for potential vorticity is created, and this source term can create an instability in the PV field.

7. Concluding remarks

For a thin layer of inviscid, incompressible fluid above a rigid, horizontal boundary, with a free surface, and under the action of gravity, the thin-layer equations (2.5) and (2.6) are the formal limit of the Euler equations and boundary conditions as $\varepsilon \to 0$, where $\varepsilon = h_0/L$. It is shown that when the thin-layer horizontal vorticity field $\mathbf{\Omega}$ – defined in (2.9) – is identically zero, then it remains zero for all subsequent times $t > 0$. When this is so, TLEP reduces exactly to SWEP, which are the usual shallow water equations, (2.7). The question we have addressed here, is whether or not this reduction is stable. That is, if we consider TLEP, when initially $\mathbf{\Omega}$ is small, is it the case that $\mathbf{\Omega}$ remains small for all subsequent times $t > 0$? When this is so, we are justified in using SWEP as a rational and uniform approximation to TLEP, but not otherwise. We have produced a specific solution to TLEP which has $\mathbf{\Omega}$ uniformly small initially and on the inlet boundary, but subsequently $\mathbf{\Omega}$ grows without bound and algebraically in time t as $t \to \infty$. Thus it is the case that there are solutions to SWEP which are unstable, particularly in $\mathbf{\Omega}$, when embedded as solutions to TLEP. This situation is considered in more generic form via a linearized theory, which provides a criterion for the occurence of this instability in base flow solutions to SWEP. The consequences for conservation of potential vorticity in SWEP are significant.

REFERENCES

Ball, F. K. 1965 The effect of rotation on the simpler modes of motion of a liquid in an elliptic paraboloid. *J. Fluid Mech.* **22**, 529–545.

Cushman-Roisin, B. 1987 Exact analytical solutions for elliptical vortices of the shallow-water equations. *J. Fluid Mech.* **39A**, 235–244.

Dingemans, M. W. 1997 *Water Wave Propagation Over Uneven Bottoms. Part 2: Nonlinear Wave Propagation*. Singapore: World Scientific.

Johnson, R. S. 1997 *A Modern Introduction to the Mathematical Theory of Water Waves*. Cambridge: Cambridge University Press.

McIntyre, M.E. 2003 Potential vorticity. In *Encylopedia of Atmospheric Sciences* (ed. J.R. Holton, J.A. Pyle & J.A. Curry), *London*, vol. 2. Academic/Elsevier.

Ohkitani, K. 2010 Dynamics of the Jacobian matrices arising in three-dimensional Euler equations: application of Riccati theory. *Proc. R. Soc. Lond. A* **466**, 1429–1439.

Ottino, J. M. 1989 *The Kinematics of Mixing: Stretching, Chaos, and Transport*. Cambridge: Cambridge University Press.

Ripa, P. 1987 On the stability of elliptical vortex solution of the shallow-water equations. *J. Fluid Mech.* **183**, 343–363.

Salmon, R. 1998 *Lectures on Geophysical Fluid Dynamics*. Oxford: Oxford University Press.

Stoker, J. J. 1958 *Water Waves: The Mathematical Theory with Applications*. New York: John Wiley and Sons.

Thacker, W. C. 1981 Some exact solutions to the nonlinear shallow-water equations. *J. Fluid Mech.* **107**, 499–508.

Young, W. R. 1986 Elliptical vortices in shallow water. *J. Fluid Mech.* **171**, 101–119.