Emergence of dark solitary waves from large-amplitude spatially-periodic waves

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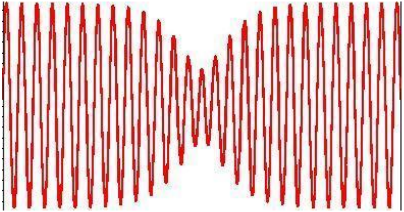


Classical NLS dark solitary waves

Defocussing NLS,

$$\mathrm{i}\mathbf{A}_t + \mathbf{A}_{xx} - |\mathbf{A}|^2\mathbf{A} = 0\,.$$

has an exact DSW solution

$$\mathbf{A}(x,t) = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(kx-t)}\big(k + \mathrm{i}\beta\tanh(\beta x)\big)\,,$$

with $\beta^2 = \frac{1}{2}(1-3k^2)$.
Bi-asymptotic to a periodic state with a phase shift

$$\mathbf{A}(x,t) \to \sqrt{2}\,\mathrm{e}^{\mathrm{i}(kx-t)}\big(k \pm \mathrm{i}\beta\big) \quad \text{as } x \to \pm\infty\,.$$

Mechanism for emergence of DSWs

What is the mechanism for the emergence of DSWs?

Can we generalise the creation of DSWs to non-integrable systems?

Chicken and egg game: what comes first? state at infinity, connecting orbit?

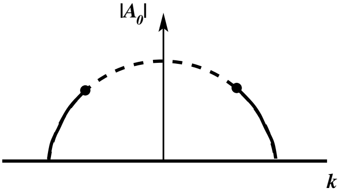
Argument: take an arbitrary periodic pattern, determine conditions for the emergence of a dark solitary wave.

Spatially periodic states of NLS

$$\mathrm{i}\mathbf{A}_t + \mathbf{A}_{xx} + \mathbf{A} - |\mathbf{A}|^2\mathbf{A} = 0\,.$$

Spatially periodic states

$$\mathbf{A}(x) = \mathbf{A}_0\, \mathrm{e}^{\mathrm{i}kx} \quad \text{with} \quad k^2 + |\mathbf{A}_0|^2 = 1\,.$$



Transition from (spatially) elliptic to hyperbolic at $k^2 = \frac{1}{3}$.

KIVSHAR (1990) Physical Review A

Kivshar notes that as $k^2 \to \frac{1}{3}$, the DSW goes to a sech^2 solitary wave. He proposes that the birth of a DSW (*a "low amplitude DSW"*) is governed by the KdV equation.
He proposes a solution of the form

$$\mathbf{A}(x,t) = (\mathbf{A}_0 + B(X,T))\mathrm{e}^{\mathrm{i}(kx+\phi(X,T))} \,,$$

where

$$X = \varepsilon x \quad \text{and} \quad T = \varepsilon^3 t \,,$$

and shows that $B(X,T)$ satisfies (to leading order) a KdV equation

$$c_0 B_T + c_1 B B_X + c_2 B_{XXX} = 0 \,.$$

– *see also* SAUT ET AL (2009), CHIRON & ROUSSET (2010)

Features and generalisations

- NLS $\rightarrow$ KdV: integrable PDE $\rightarrow$ integrable PDE
- KIVSHAR, ANDERSON & LISAK (1993) Physica Scripta: showed a similar bifurcation for the non-integrable NLS

$$\mathrm{i}A_t + A_{xx} + f(|A|^2)A = 0\,.$$

- How can this be generalised?

The change along the periodic orbit is a saddle-centre transition of a periodic orbit of a Hamiltonian system.
Can also be interpreted as a collision between two periodic orbits, one elliptic and one hyperbolic.

(Spatial) energy-wavenumber space

The NLS is Hamiltonian

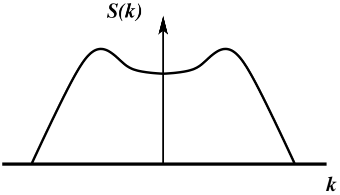
$$\mathrm{i}\mathbf{A}_t = \frac{\delta H}{\delta \overline{\mathbf{A}}} \quad \text{with} \quad H = |\mathbf{A}_x|^2 - |\mathbf{A}|^2 + \tfrac{1}{2}|\mathbf{A}|^4 \,.$$

However, it is the spatial energy that is of importance along the branch of spatially-periodic solutions

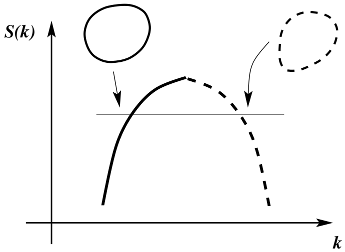
$$S = |\mathbf{A}_x|^2 + |\mathbf{A}|^2 - \tfrac{1}{2}|\mathbf{A}|^4 \,.$$

Evaluated on the branch of periodic solutions

$$S(k) = \tfrac{1}{2}(1 - k^2)(1 + 3k^2) \,.$$



Collision of two periodic orbits



The elliptic periodic orbit has a Krein signature.

This scenario that can be found in general in Hamiltonian systems.

$$\mathbf{J}\mathbf{u}_x = \nabla S(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^{2n}, \quad n \geq 2.$$

Embed in a PDE with Hamiltonian spatial part

Consider PDEs of the form

$$\mathbf{M}\mathbf{u}_t + \mathbf{J}\mathbf{u}_x = \nabla S(\mathbf{u})\,, \quad \mathbf{u} \in \mathbb{R}^{2n}\,, \quad n \geq 2\,.$$

That is PDEs with Hamiltonian spatial part, where **M** is at this point an arbitrary constant $2n \times 2n$ matrix.

Suppose the steady system has a collision of periodic orbits associated with an elliptic-hyperbolic transition. Then nearby dynamics is governed (to leading order) by

$$m'(k) B_T + S''(k) B B_X + \mathscr{K} B_{XXX} = 0\,.$$

A KdV equation with geometrically determined coefficients.

KdV → DSWs

$$m'(k)B_T + S''(k)BB_X + \mathscr{K} B_{XXX} = 0\,.$$

Here $X = \varepsilon x$, $T = \varepsilon^3 t$, $m(k)$ is the momentum, and $S(k)$ is the momentum flux. $\mathscr{K}$ is the Krein signature of the elliptic periodic orbit in the collision.

- The coefficient of the nonlinear term is determined by the curvature of the momentum flux (spatial energy) as a function of wavenumber.
- The sign of the dispersion is determined by the Krein signature of the collision.

Strategy

Given a PDE, with the steady part a Hamiltonian system

$$\mathbf{J}\mathbf{u}_x = \nabla S(\mathbf{u}) , \quad \mathbf{u} \in \mathbb{R}^{2n} , \quad n \geq 2 ,$$

plot families of periodic orbits in the energy-frequency (momentumflux-wavenumber plane in the spatial case) and identify points where

$$S'(k) = 0 .$$

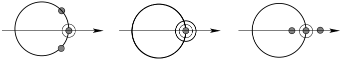
Such points are indications of a saddle centre transition. Extend to the time-dependent case. The nearby dynamics is governed, to leading order, by

$$m'(k)B_T + S''(k)BB_X + \mathscr{K} B_{XXX} = 0 .$$

How do we show this?

Saddle centre transition

At the transition, $+1$ is a (spatial) Floquet multiplier of geometric multiplicity one and algebraic multiplicity four.



Jordan chain ($\theta = kx + \theta_0$)

$$\mathbf{L}\xi_1(\theta) = 0\,, \quad \mathbf{L}\xi_2(\theta) = \xi_1(\theta)\,, \quad \mathbf{L}\xi_3(\theta) = \xi_2(\theta)\,, \quad \mathbf{L}\xi_4(\theta) = \xi_3(\theta)\,.$$

How to scale the Jordan chain?
Propose solution of PDE of the form

$$\mathbf{u} = (\varepsilon\xi_1)\phi + (\varepsilon^2\xi_2)B + \mathscr{K}(\varepsilon^3\xi_3)v - \mathscr{K}(\varepsilon^4\xi_4)I + \varepsilon^5 W(X, T, \theta, \varepsilon)\,.$$

Modulation equation

Project onto the generalised eigenspace,

$$\begin{aligned} \phi_X &= B + \cdots \\ B_X &= \mathscr{K} v + \cdots \\ m_1 \phi_T - v_X &= I - \tfrac{1}{2}\kappa B^2 + \cdots \\ -m_1 B_T - I_X &= 0 + \cdots , \end{aligned}$$

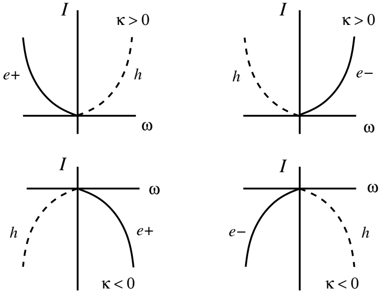
with

$$\kappa = -2\langle\langle \xi_2, \mathcal{N}(\xi_2, \xi_2)\rangle\rangle \quad \text{and} \quad m_1 = \langle\langle \xi_2, \mathbf{M}\xi_1 \rangle\rangle .$$

Can prove that κ is proportional to $S''(k)$ (a few pages!). Combining the above equations gives the KdV equation for $B(X, T)$, with the phase shift incorporated in $\phi(X, T)$.

Four cases

$$m\,B_T + \kappa B B_X + e^{\pm}\,B_{XXX} = 0$$



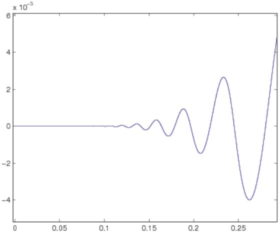
Example – KdV planforms

Consider the elliptic PDE (a variant of the steady Swift-Hohenberg equation)

$$w_{xxxx} - Pw_{xx} - w_{yy} + w - w^2 = 0\,.$$

The $y-$independent part is Hamiltonian with (spatial energy)

$$S = \tfrac{1}{2}w_{xx}^2 + \tfrac{1}{2}Pw_x^2 - \tfrac{1}{2}w^2 + \tfrac{1}{3}w^3 - w_x w_{xxx}\,.$$



Steady SH equation

The steady SH equation

$$w_{xxxx} - Pw_{xx} - w_{yy} + w - w^2 = 0\,.$$

can be written in the form

$$\mathbf{M}\mathbf{u}_y + \mathbf{J}\mathbf{u}_x = \nabla S(\mathbf{u})\,, \quad \mathbf{u} \in \mathbb{R}^4\,,$$

with

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\,.$$

KdV planforms

At transition points, where $S'(k) = 0$, the theory goes through and the modulation equation is KdV – in the (x, y) plane

$$m'(k)B_Y + S''(k)BB_X + \mathscr{K} B_{XXX} = 0\,,$$

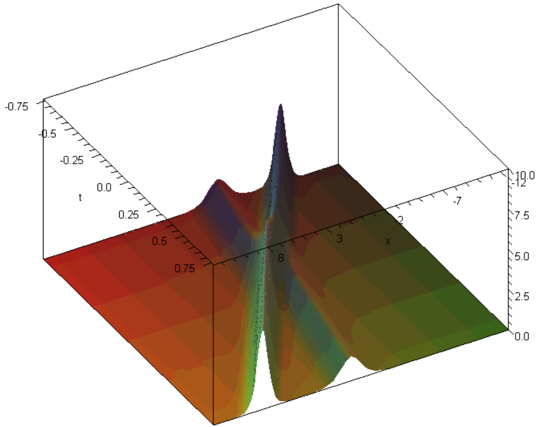
with coordinates

$$X = \varepsilon x \quad \text{and} \quad Y = \varepsilon^3 y\,.$$

KdV has a whole range of $N-$soliton solutions which become $N-$soliton planforms in this case.

Two-soliton solution of KdV

The KdV equation has n-soliton solutions. In this case they generate $n-$soliton planforms. Example: 2-soliton



Water waves

Look at the steady water wave problem along branches of spatially periodic waves – with c fixed and the wavenumber varying along a branch. Does a saddle-centre transition of Floquet multipliers occur? Yes!

- In infinite depth: BAESENS & MACKAY (1992) J Fluid Mech
- In finite depth: VANDEN-BROECK (1983) Physics of Fluids
- At low amplitude coupled to mean flow: TJB & DONALDSON (2006) J Fluid Mech.

The theory proposed here suggests that near each of these points, the appropriate model is the KdV equation.
In the first case, the KdV is a model equation for water waves in infinite depth!

DSWs in the water wave problem

The appearance of DSWs in the water wave problem is not so surprising in finite depth since the NLS model in finite depth is the de-focussing NLS, which has unsteady DSW solutions.

To find these DSWs at finite amplitude, look for saddle centre transitions of spatial Floquet multipliers.

Invoking the above theory then indicates that the KdV equation is the nonlinear model nearby.

New mechanism for the appearance of KdV, and it shows that KdV can also be model for water waves in infinite depth.

Mechanism for KdV?

Shallow water is not necessary for the appearance of KdV as a model equation for water waves.

Indeed, "shallow water" is neither necessary nor sufficient for the appearance of KdV as a model equation for water waves.

A new mechanism for the appearance of the KdV equation as a modulation equation.

Other mechanisms

- A dispersion relation determines whether KdV is the appropriate modulation equation.
- "Nonlinearity balances dispersion" implies the appearance of KdV as a modulation equation.