Multiscale Dynamics of 2D Rotational Compressible Euler Equations — an Analytical Approach

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Abstract. We study the 2D rotational compressible Euler equations with two singular parameters: the Rossby number for rotational forcing and the Froude/Mach number for pressure forcing. The competition of these two forces leads to a newly found parameter $\delta = \tau\sigma^{-2}$ that serves as a characteristic scale separating two dynamic regimes: $\delta \ll 1$ for the strong rotation regime [**CT08**] and $\delta \gg 1$ for the mild/weak rotation regime. The analytical novelty of this study is correspondingly two-fold. In the $\delta \ll 1$ regime, we utilize the method of iterative approximations that starts with the pressureless rotational Euler equations previously studied in [**LT04**]. The resulting approximation is a periodic-in-time flow that reflects the domination of rotation in this small regime. On the other hand, for $\delta \gg 1$, we combine fast wave analysis for nonlinear hyperbolic PDEs with Strichartz-type estimates to reveal an approximate incompressible flow. Our argument is highlighted with newly established nonlinear invariants in terms of wave interaction and is free of Fourier analysis.

1. Introduction

We investigate the 2D compressible Euler equations with two forces: the pressure gradient force and the rotational force. We focus on a prototypical case: the system of rotational shallow water (RSW) equations

$$h_t + \nabla\cdot(h\mathbf{u}) = 0 \tag{1.1}$$

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + g\nabla h = f\mathbf{u}^\perp \tag{1.2}$$

Here, the unknowns are height $h(t, x, y)$ and velocity field $\mathbf{u}(t, x, y)$. There are two physical constants in the RSW system: the gravitational constant g is associated with the pressure gradient force $g\nabla h$ and the Coriolis frequency f is associated with the rotational force $f\mathbf{u}^\perp = f\begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$. We note that in the Northern Hemisphere $f > 0$ and in the Southern Hemisphere $f < 0$. Without loss of generality, we only consider $f > 0$ in this paper.

The significance of studying RSW equations includes: 1. they are a specific case of the more general 2D barotropic Euler equations; 2. they are one of the simplest multidimensional hyperbolic PDE systems, the study of which is still largely open;

3. in geophysical sciences, the RSW equations appear as the *formal* limit of the 3D primitive equations and (incompressible) Boussinesq equations if vertical variance is asymptotically small [**Ped92**].

The dynamics of the RSW system closely hinges on the interaction of two competing forces $g\nabla h$ and $f\mathbf{u}^\perp$ coupled with nonlinear advection. The scales of the two competing forces depend on various physical regimes, leading to dramatically different effects. To this end, rewrite (1.1), (1.2) in the dimensionless form (e.g. [**Maj03**]),

$$h_t + \nabla\cdot(h\mathbf{u}) + \sigma^{-1}\nabla\cdot\mathbf{u} = 0, \tag{1.3}$$

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \sigma^{-1}\nabla h = \tau^{-1}\mathbf{u}^\perp, \tag{1.4}$$

where σ is the Froude number indicating the inverse magnitude of the pressure gradient force and τ the Rossby number for the inverse magnitude of the rotational force. The *relative* strength of these two forces, indicated by the competition of σ and τ, decides the nature of the fluid motions. For such a reason, our study is considered genuinely "multiscale" with one slow inertia scale $O(1)$ and *two* fast scales $O(\sigma^{-1})$, $O(\tau^{-1})$.

To be precise, we introduce in [**CT08**] a new parameter

$$\delta := \frac{\tau}{\sigma^2}, \tag{1.5}$$

and claim a slightly different version of the following theorem (the original theorem in [**CT08**] uses a torus domain $\mathbf{T}^2$)

Theorem 1.1. *Consider the RSW system (1.3), (1.4) endowed with initial data* $(h_0, \mathbf{u}_0) \in H^m(\mathbf{R}^2)$ *with* $m > 5$. *If the parameter* $\delta < 1$, *then the solution* $\mathbf{U}(t,\cdot) := \begin{pmatrix} h(t,\cdot) \\ \mathbf{u}(t,\cdot) \end{pmatrix}$ *stays close to an time-periodic flow* $\mathbf{U}^p(t,\cdot)$ *in the sense that*

$$\|\mathbf{U}(t,\cdot) - \mathbf{U}^p(t,\cdot)\|_{H^{m-3}} \lesssim \delta^\alpha \|(h_0,\mathbf{u}_0)\|_{H^m}, \ \textit{for } t \lesssim (1-\alpha)\ln^+\left(\frac{1}{\delta}\right). \tag{1.6}$$

Notice that the interesting case is the $\delta \ll 1$ regime where, by (1.5), the rotational force dominates the pressure force. Rotational force is commonly recognized as a stabilizing mechanism for large scale geophysical motions – see e.g. Majda[**Maj03**] , Mahalov etc.[**BMN02**], Chemin etc.[**CDGG06**]. Our result (1.6) serves as a quantification of the stabilizing effect. Also, a time-periodic approximation $\mathbf{U}^p(t,\cdot)$ is revealed, consistent with geophysical observations of the near-inertial oscillation (NIO) phenomenon which follows oceanic storms. These NIOs exhibit close-to-periodic dynamics and stay stable for relatively long time [**YJ97**].

On the other hand, in the $\delta \gg 1$ regime associated with mild/weak rotational force, the dispersive effect is predominant. We show in Section 3 a series of lemmas that lead to the following theorem.

Theorem 1.2. *Consider the RSW system (1.3), (1.4) endowed with initial data* $(h_0, \mathbf{u}_0) \in H^m(\mathbf{R}^2)$ *with* $m > 5$. *If the parameter* $\kappa := \delta^{-1} + \sigma < 1$, *then there exists an incompressible flow* $\mathbf{U}_{inc}$ *s.t. for some finite time* T *and constant* $\alpha > 0$,

$$||\mathbf{U} - \mathbf{U}_{inc}||_{L^6([0,T];W^{m-3,6}(\mathbf{R}^2))} \lesssim \kappa^\alpha \|(h_0,\mathbf{u}_0)\|_{H^m}. \tag{1.7}$$

Moreover, the incompressible flow is governed by the quasi-geostrophic equation

$$(\mathbf{U}_{inc})_t + \mathcal{P}\circ\mathcal{B}[\mathbf{U}_{inc},\mathbf{U}_{inc}] = 0, \quad (\mathbf{U}_{inc})_0 = \mathcal{P}\circ\mathbf{U}_0,$$

where the projection operator $\mathcal{P}$ is defined in Definition 3.1 below and the bilinear operator $\mathcal{B}$ in (3.6) below.

Remember that σ is the Froude number measuring the inverse scale of pressure forcing. Thus, σ majorizes the value of κ if and only if $\delta^{-1} < \sigma \leftrightarrow \sigma < \tau$, namely, when rotational force is weaker than pressure force. Our novel approach is centered around an invariant-based analysis for the particular nonlinear structure of the RSW equations — see Lemma 3.2 and identity (3.13). It does not rely on Fourier analysis and therefore avoids taunting calculation in the frequency domain. Such convenience also helps generalize our results to other equations with fluid structure and to nontrivial geometry (e.g. a nonflat surface) where Fourier analysis becomes increasingly complicated.

In the context of multiscale analysis, a major breakthrough of our results (1.6), (1.7) is the inclusion of *two* varying scales τ, σ that are free of algebraic constraints. Before, one imposes strict conditions such as $\tau \sim \sigma \ll 1$ [**EM96**] or $\sigma \ll 1 \sim \tau$ [**EM98**] for RSW and $\tau = \infty$, $\sigma \ll 1$ for non-rotational low Mach number flows [**KM81**]. Notice that each one of these constraints implies a looser constraint $\kappa \ll 1$. Therefore, these previous results are all covered by (1.7) whereas the complementing part (1.6) seems to resonate with no existing results.

2. Stabilizing effect of rotation

Our novel approach for proving (1.6), in a nutshell, is to construct a periodic approximation $\mathbf{U}^p$ using the method of (nonlinear) iterative approximation. It is of nonlinear fashion in the sense that we no longer treat the nonlinear system as a perturbed linear system. To be precise, we start our approximation by setting $h \equiv$ constant since the rotational force dominates the pressure force. As an immediate consequence, the momentum equations (1.4) are decoupled from the continuity equation (1.3). This leads to a genuinely nonlinear system – the so-called pressureless system – studied by Liu and Tadmor in [**LT04**]

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \tau^{-1}\mathbf{u}^{\perp}. \tag{2.1}$$

An ODE analysis along particle paths was employed to show that a C^1 time-periodic solution exists for all times as long as the system configuration is below certain critical threshold. We derived a corollary in [**CT08**] that C^1 solution exists for all subcritical Rossby numbers $\tau \in (0, \tau^{cr}(\nabla\mathbf{u}_0))$. It is noteworthy that, in spite of the nonlinear advection and possibly singular parameter τ, equation (2.1) admits global C^1 solutions for a large family of initial data. Putting this result in the context of hyperbolic PDE theories regarding finite time blow up (e.g. [**Lax73, Sid85**]), we see the stabilizing effect of rotational force against nonlinear advection in equation (2.1).

The next step in our new approach is to treat the pressure term $\sigma^{-1}\nabla h$ as a perturbation to (2.1). With the method of iterative approximation, one constructs a series of approximate RSW systems. In this process, a great deal of our effort is put in maintaining the global regularity and periodicity of these approximate solutions – starting with (2.1). Various techniques, such as operator splitting method, energy method, particle path method, are used to discover a globally stable, approximate RSW system with $O(\delta)$ residual terms. The final step is a classical energy norm argument.

For a detailed proof, we refer to our previous work [**CT08**].

3. Analysis of wave interaction using nonlinear invariants for fluid equations

In the previous section, we deal with the multiscale nature of RSW equations: one slow inertia scale $O(1)$, one fast pressure scale $O(\sigma^{-1})$ and one fast rotational scale $O(\tau^{-1})$. There are two *independent* fast scales in our results, which is fundamentally different from existing results with only one fast scale. The competition of these two fast scales, i.e., the associated forces, is measured by a new parameter

$$\delta = \frac{\tau}{\sigma^2}.$$

Having studied the $\delta \ll 1$ regime with dominant rotational force, we turn to the $\delta \gg 1$ regime with mild/week rotational force.

Our result (1.7) unifies and generalizes a series of previous work along the line of "fast-wave averaging" and "singular perturbation" theories [**KM81, EM96, Kre80, Sch94, Gal98**]. The main novelty of our proof is an invariant-based nonlinear analysis of wave interactions that is free of Fourier analysis. In particular, Lemma 3.2 below confirms the phenomenon of so-called "separation of fast and slow scales". We note that the interaction of fast and slow scales via nonlinear advection has been an intriguing topic for decades [**Ali95, Sid85, Sid91, Sid97, Hör87**]. Our method, however, is based on a newly found invariant (3.13) below and does not involve any taunting calculation in the frequency domain. Instead, such methodology uses the nonlinear structure inherent to fluid equations and can be easily applied to other cases – e.g. the Euler-Poisson equations. It can also play a key role whenever Fourier analysis is not convenient, e.g. on nontrivial geometry.

Once we show fast and slow scales are separated in Subsection 3.1, we utilize the classical Strichartz-type estimate for the fast, acoustic part of the solution that satisfies the linearized RSW equations

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ \mathbf{u} \end{pmatrix} = \mathcal{L}\begin{pmatrix} h \\ \mathbf{u} \end{pmatrix} := \begin{pmatrix} -\sigma^{-1}\nabla\cdot\mathbf{u} \\ -\sigma^{-1}\nabla h + \tau^{-1}\mathbf{u}^{\perp} \end{pmatrix}. \tag{3.1}$$

The slow, incompressible part of the solution, on the other hand, "survives" this dispersion-induced decay and becomes a reasonable approximation of the original solution.

To this end, we start with decomposition of the solution into slow and parts that belong to, respectively, the kernel of operator $\mathcal{L}$ as defined in (3.1) and its orthogonal complement with respect to L^2 inner product.

Definition 3.1. For any L^2 function $\mathbf{f} = \begin{pmatrix} h \\ \mathbf{u} \end{pmatrix}$, define the projection operator and its orthogonal complement,

$$\mathcal{P}[\mathbf{f}] := \mathrm{Proj}_{\ker\{\mathcal{L}\}}[\mathbf{f}], \qquad \mathcal{Q}[\mathbf{f}] := \mathbf{f} - \mathcal{P}[\mathbf{f}].$$

We then give a formula for $\mathcal{P}$ in terms of pseudodifferential operators

$$\mathcal{P}[\mathbf{f}] = \mathcal{N}^* \circ (\mathcal{N} \circ \mathcal{N}^*)^{-1} \circ \mathcal{N}[\mathbf{f}],$$

$$\text{where } \mathcal{N}[\mathbf{f}] := \frac{h}{\tau} - \frac{\partial_x u_2 - \partial_y u_1}{\sigma} = \frac{h}{\tau} - \frac{\nabla\times\mathbf{u}}{\sigma}$$

$$\text{so that } \mathcal{N}^*[g] = (\frac{g}{\tau}, -\frac{\partial_y g}{\sigma}, \frac{\partial_x g}{\sigma})^T \text{ for scalar } g$$

$$\text{and } \mathcal{N}\circ\mathcal{N}^*[g] = (\tau^{-2} - \sigma^{-2}\Delta)g.$$

Two useful facts derived from above are

$$\mathcal{P}[\mathbf{f}] = 0 \text{ iff } \mathcal{N}[\mathbf{f}] = 0 \tag{3.2}$$

$$\mathcal{L}^2 = (\sigma^{-2}\Delta - \tau^{-2}) \circ \mathcal{Q} = -\mathcal{N} \circ \mathcal{N}^* \circ \mathcal{Q}. \tag{3.3}$$

The proof of these formulas uses basic Calculus.We note that operator $\mathcal{N}$ yields the so-called "relative vorticity" $\Omega := \mathcal{N}[\mathbf{U}]$ that satisfies a conservation law similar to the continuity equation

$$\Omega_t + \nabla\cdot(\mathbf{u}\Omega) = 0. \tag{3.4}$$

Note that this is a direct consequence of applying $\mathcal{N}$ to equation (3.7) below together with the popular identity (let $\omega := \nabla \times \mathbf{u}$)

$$\nabla \times (\mathbf{u}\cdot\nabla\mathbf{u}) = \mathbf{u}\cdot\nabla\omega + \omega(\nabla\cdot\mathbf{u}) - \omega\cdot\nabla\mathbf{u}. \tag{3.5}$$

The last term (called "vorticity stretching") $\omega\cdot\nabla\mathbf{u}$ vanishes in 2D.

Correspondingly, we define the incompressible and acoustic components of the solution,

$$\mathbf{U}^P := \mathcal{P}[\mathbf{U}], \ \mathbf{U}^Q := \mathcal{Q}[\mathbf{U}].$$

Also, define the bilinear operator

$$\mathcal{B}[\mathbf{U}, \mathbf{U}] := \begin{pmatrix} \nabla\cdot(h\mathbf{u}) \\ \mathbf{u}\cdot\nabla\mathbf{u} \end{pmatrix} \tag{3.6}$$

so that the RSW equations become

$$\mathbf{U}_t + \mathcal{B}[\mathbf{U}, \mathbf{U}] = \mathcal{L}[\mathbf{U}]. \tag{3.7}$$

Then, we decompose this system by applying operators $\mathcal{P}$, $\mathcal{Q}$,

$$\mathbf{U}_t^P + \mathcal{P} \circ \mathcal{B}[\mathbf{U}, \mathbf{U}] = 0, \tag{3.8}$$

$$\mathbf{U}_t^Q + \mathcal{Q} \circ \mathcal{B}[\mathbf{U}, \mathbf{U}] = \mathcal{L}[\mathbf{U}^Q]. \tag{3.9}$$

Here we used such properties as $\mathcal{Q} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{Q}$ and $\mathcal{P} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{P} = 0$.

In the following subsections, we show a series of lemmas regarding *time integration* of nonlinear wave interaction in the bilinear $\mathcal{B}[\cdot, \cdot]$ terms in (3.8) and (3.9). The first two lemmas in Subsection 3.1, interesting on their own, help single out the "slow-slow" interaction term $\mathcal{P} \circ \mathcal{B}[\mathbf{U}^P, \mathbf{U}^P]$ as the dominant contributor of $\mathcal{P} \circ \mathcal{B}[\mathbf{U}, \mathbf{U}]$ in (3.8). In particular, we identity a new identity (3.13) below in the proof of Lemma 3.2 that eventually leads to the cancellation of "fast-fast" interaction term $\mathcal{P} \circ \mathcal{B}[\mathbf{U}^Q, \mathbf{U}^Q]$ in (3.8). Lemma 3.3, on the other hand, offers a new and simple approach to estimate the *time integral* of "slow-fast" interaction terms such as $\mathcal{B}[\mathbf{U}^P, \mathbf{U}^Q]$ and $\mathcal{B}[\mathbf{U}^Q, \mathbf{U}^P]$. Our approach utilizes the method of integrating by parts without incurring any explicit use of Fourier analysis. In Subsection 3.2, we restate Strichartz' result ([**Str77**]) in Lemma 3.5 in terms of our wave operator $\mathcal{L}$. Then, in the last Lemma 3.6, we use the Duhamel's principle to obtain decay estimate on the nonlinear system (3.9).

3.1. Estimate for (3.8). We begin with stating two lemmas.

Lemma 3.2 (Cancellation of "fast-fast" interaction). *Consider any admissible solution* $\mathbf{U}$ *to (3.7). Let* $\mathbf{U}^P$*,* $\mathbf{U}^Q$ *be defined as above and let the supscript* P *and* Q *indicate association with* $\mathbf{U}^P$ *and* $\mathbf{U}^Q$ *respectively. Then*

$$\mathcal{P}\circ\mathcal{B}[\mathbf{U},\mathbf{U}] - \mathcal{P}\circ\mathcal{B}[\mathbf{U}^P,\mathbf{U}^P] = \mathcal{P}\begin{pmatrix}0\\ \sigma\Omega^P(\mathbf{u}^Q)^\perp\end{pmatrix}. \tag{3.10}$$

Lemma 3.3 (Estimate on "slow-fast" interaction). *Consider* $\mathbf{U}$*, an admissible solution to (3.7). Then,*

$$\begin{aligned}&\left\|\int_0^T \mathbf{U}^P\otimes\mathbf{U}^Q\,dt\right\|_{H^m}\\ &\lesssim\kappa\left(\|\mathbf{U}_0\|^2_{H^{m+1}} + \|\mathbf{U}(T,\cdot)\|^2_{H^{m+1}} + \int_0^T \|\mathbf{U}\|^3_{H^{m+2}}\,dt\right)\end{aligned} \tag{3.11}$$

Here and below, we use $\otimes$ to denote the tensor product of two vectors.

Remark 3.4. An immediate consequence of these two lemmas is the following decoupling relation, obtained by integrating (3.8) w.r.t. time,

$$\begin{aligned}&\left\|\int_0^T \mathbf{U}^P_t + \mathcal{P}\circ\mathcal{B}[\mathbf{U}^P,\mathbf{U}^P]\,dt\right\|_{H^m}\\ &\lesssim\kappa\left(\|\mathbf{U}_0\|^2_{H^{m+1}} + \|\mathbf{U}(T,\cdot)\|^2_{H^{m+1}} + \int_0^T \|\mathbf{U}\|^3_{H^{m+2}}\,dt\right).\end{aligned}$$

Then, by the standard energy methods, it is easy to argue that the dynamics of $\mathbf{U}^p$ can be well approximated with an incompressible fluid equation $(\mathbf{U}_{inc})_t + \mathcal{P}\circ\mathcal{B}[\mathbf{U}_{inc},\mathbf{U}_{inc}] = 0$ such that

$$\|\mathbf{U}^P - \mathbf{U}_{inc}\|_{H^m} \le \kappa\, f_1\left(T, \|\mathbf{U}_0\|_{H^{m+2}}\right) \tag{3.12}$$

for come smooth function f_1. This approximate system turns out to be the well-known quasi-geostrophic equation up to some parametrization by σ,τ.

Proof of Lemma 3.2. We first claim the following newly found invariant

$$\mathcal{P}\circ\mathcal{B}[\mathbf{U},\mathbf{U}] = \mathcal{P}\begin{pmatrix}0\\ \sigma\Omega\mathbf{u}^\perp\end{pmatrix}. \tag{3.13}$$

With operator $\mathcal{B}$ defined in (3.6), the proof of (3.13) amounts to proving

$$\mathcal{P}\begin{pmatrix}\nabla\cdot(h\mathbf{u})\\ \mathbf{u}\cdot\nabla\mathbf{u} - \sigma\Omega\mathbf{u}^\perp\end{pmatrix} = 0$$

which, by (3.2), is equivalent to proving

$$\begin{aligned}0 &= \mathcal{N}\begin{pmatrix}\nabla\cdot(h\mathbf{u})\\ \mathbf{u}\cdot\nabla\mathbf{u} - \sigma\Omega\mathbf{u}^\perp\end{pmatrix}\\ &= \frac{1}{\tau}\nabla\cdot(h\mathbf{u}) - \frac{1}{\sigma}\nabla\times\left(\mathbf{u}\cdot\nabla\mathbf{u} - \sigma\Omega\mathbf{u}^\perp\right)\\ &= \frac{1}{\tau}\nabla\cdot(h\mathbf{u}) - \frac{1}{\sigma}\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) + \nabla\times\left[\mathbf{u}^\perp\left(\frac{h}{\tau} - \frac{\nabla\times\mathbf{u}}{\sigma}\right)\right]\\ &= \frac{1}{\sigma}\left(-\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) + \nabla\cdot(\mathbf{u}\omega)\right).\end{aligned}$$

The last line vanishes since we have identity (3.5) and the remark thereafter.

The new invariant (3.13) remains valid if we replace $\mathbf{U}$ with $\mathbf{U}^P$. Therefore, it offers a way to simplify the LHS of (3.10),

$$\begin{aligned}\mathcal{P}\circ\mathcal{B}[\mathbf{U},\mathbf{U}]-\mathcal{P}\circ\mathcal{B}[\mathbf{U}^P,\mathbf{U}^P]&=\mathcal{P}\begin{pmatrix}0\\ \sigma\Omega\mathbf{u}^\perp\end{pmatrix}-\mathcal{P}\begin{pmatrix}0\\ \sigma\Omega^P(\mathbf{u}^P)^\perp\end{pmatrix}\\ &=\mathcal{P}\begin{pmatrix}0\\ \sigma\Omega^P(\mathbf{u}^Q)^\perp\end{pmatrix}+\mathcal{P}\begin{pmatrix}0\\ \sigma\Omega^Q(\mathbf{u}^P)^\perp\end{pmatrix}+\mathcal{P}\begin{pmatrix}0\\ \sigma\Omega^Q(\mathbf{u}^Q)^\perp\end{pmatrix}.\end{aligned}$$

Using the fact that $P[\mathbf{U}^Q]=0$ and (3.2), we have $\mathcal{N}[\mathbf{U}^Q]=0$. This means $\Omega^Q=\mathcal{N}[\mathbf{U}^Q]=0$ and the above calculation implies (3.10). □

The identity (3.10) shows, both rigorously and concisely, that the bilinear interaction of "fast-fast" waves $\mathcal{B}[\mathbf{U}^Q,\mathbf{U}^Q]$ has no contribution to the dynamics of slow, incompressible part $\mathbf{U}^P$. Such "separation" of slow scale from fast scale has been confirmed in many previous work in terms of observation, computation and theories. Our Lemma 3.2 serves as a generalization of these results.

PROOF OF LEMMA 3.3. At first glance, (3.11) seems to counter our intuition: the LSH is of scale $O(1)$ while the RHS $O(\kappa)$. But *time integration* of fast scale $\mathbf{U}^Q$ actually helps reduce the scale, which is often called "cancellation of oscillation". In fact, apply operator $\mathcal{L}\circ\mathcal{Q}$ to the RSW system (3.7) and use the commutability of $\mathcal{L},\mathcal{P},\mathcal{Q}$,

$$\mathcal{L}\circ\mathcal{Q}\left(\mathbf{U}_t+\mathcal{B}[\mathbf{U},\mathbf{U}]\right)=\mathcal{L}\circ\mathcal{Q}\circ\mathcal{L}[\mathbf{U}]=\mathcal{L}^2[\mathbf{U}^Q].$$

Since by (3.3), $\mathcal{L}^2[\mathbf{U}^Q]=(\sigma^{-2}\Delta-\tau^{-2})\mathbf{U}^Q$, the above equation yields

$$\mathbf{U}^Q=(\sigma^{-2}\Delta-\tau^{-2})^{-1}\circ\mathcal{L}\circ\mathcal{Q}\left(\mathbf{U}_t+\mathcal{B}[\mathbf{U},\mathbf{U}]\right)=:\mathcal{L}_1\left(\mathbf{U}_t+\mathcal{B}[\mathbf{U},\mathbf{U}]\right)\tag{3.14}$$

Together with estimate on $\mathcal{L}_1=(\sigma^{-2}\Delta-\tau^{-2})^{-1}\circ\mathcal{L}\circ\mathcal{Q}$ (which is essentially $\mathcal{L}^{-1}$ restricted on $\ker^\perp\{\mathcal{L}\}$),

$$\|\mathcal{L}_1[\mathbf{U}^Q]\|_{H^m}\lesssim\kappa\|\mathbf{U}^Q\|_{H^{m+1}},\tag{3.15}$$

we see that (3.14) suggests $\int_0^T\mathbf{U}^Q\,dt$ be of reduced order.

Now that we have control on the size of the time integral of $\mathbf{U}^Q$, integration-by-parts is at our disposal to estimate the bilinear term on the LHS of (3.11). Or, equivalently, we use the product rule of differentiation together with (3.14),

$$\begin{aligned}\frac{\partial}{\partial t}\left(\mathbf{U}^P\otimes\mathcal{L}_1[\mathbf{U}]\right)&=\mathbf{U}^P\otimes\mathcal{L}_1[\mathbf{U}_t]+\mathbf{U}^P_t\otimes\mathcal{L}_1[\mathbf{U}]\\ &=\mathbf{U}^P\otimes\mathbf{U}^Q-\mathbf{U}^P\otimes\mathcal{L}_1\circ\mathcal{B}[\mathbf{U},\mathbf{U}]+\mathbf{U}^P_t\otimes\mathcal{L}_1[\mathbf{U}].\end{aligned}$$

Thus, upon integration w.r.t. time

$$\int_0^T\mathbf{U}^P\otimes\mathbf{U}^Q\,dt=\mathbf{U}^P\otimes\mathcal{L}_1[\mathbf{U}]\Big|_0^T+\int_0^T\mathbf{U}^P\otimes\mathcal{L}_1\circ\mathcal{B}[\mathbf{U},\mathbf{U}]-\mathbf{U}^P_t\otimes\mathcal{L}_1[\mathbf{U}]\,dt$$

so that, by (3.15), the LSH of the target inequality (3.11) is bounded

$$\left\|\int_0^T\mathbf{U}^P\otimes\mathbf{U}^Q dt\right\|_{H^m}\lesssim\kappa\left(\|\mathbf{U}\|^2_{H^{m+1}}\Big|_{t=0}+\|\mathbf{U}\|^2_{H^{m+1}}\Big|_{t=T}+\int_0^T\|\mathbf{U}\|^3_{H^{m+2}}\,dt\right).$$

□

3.2. Estimate for (3.9). In the final steps of proving (1.7) we apply Strichartz-type estimate [**Str77**] on the acoustic part of the solution $\mathbf{U}^Q = \mathbf{U} - \mathbf{U}^p$ that satisfies (3.9).

We first investigate the *linear* part of (3.9) by restating the early result of Strichartz [**Str77**, item (a), page 714].

LEMMA 3.5 (Strichartz). *Let* $\mathbf{V} = \mathcal{Q}[\mathbf{V}]$ *solves the linear system*

$$\mathbf{V}_t = \mathcal{L}[\mathbf{V}].$$

Let $\kappa = \sigma + \delta^{-1} < 1$. *Then,*

$$\|\mathbf{V}\|_{L^6(\mathbf{R};W^{m,6}(\mathbf{R}^2)))} \lesssim \kappa^\alpha \|\mathbf{V}_0\|_{H^{m+1}(\mathbf{R}^2)} \tag{3.16}$$

for some positive α

PROOF. This is a direct consequence of [**Str77**, item (a), page 714] which states that the solution to the Klein-Gordon equation

$$u_{tt} - u + \nabla u = 0$$

in 2D has time-space decay

$$\|u\|_{L^6(\mathbf{R}^{2+1})} \lesssim \|u_0, (u_0)_t\|_{H^1(\mathbf{R}^2)}.$$

Indeed, since $L^2 \circ Q = \sigma^{-2}\Delta - \tau^{-2}$ is just a rescaled Klein-Gorden operator, a simple scaling argument suffices to reveal (3.16). □

Now, by the Duhamel's principle, we show that

LEMMA 3.6. *The solution to (3.9), the acoustic part of RSW equations, satisfy estimate*

$$\left\|\mathbf{U}^Q\right\|_{L^6([0,T];W^{m,6}(\mathbf{R}^2))} \lesssim \kappa^\alpha \|\mathbf{U}_0^Q\|_{H^{m+1}(\mathbf{R}^2)} + T\kappa^\alpha \|\mathcal{B}[\mathbf{U},\mathbf{U}]\|_{L^6([0,T];H^{m+1}(\mathbf{R}^2))}$$

REMARK 3.7. An immediate consequence is that, in terms of the initial data $\mathbf{U}_0$, we have estimate

$$\left\|\mathbf{U}^Q\right\|_{L^6([0,T];W^{m,6}(\mathbf{R}^2))} \lesssim \kappa^\alpha f_2\left(T, \|\mathbf{U}_0\|_{H^{m+2}}\right) \tag{3.17}$$

for some smooth function f_2.

PROOF OF LEMMA 3.6. We use $e^{t\mathcal{L}}$ to denote the solution operator associated with the linear system $\mathbf{V}_t = \mathcal{L}[\mathbf{V}]$. Then, by the Duhamel's principle, (3.9) has solution

$$\mathbf{U}^Q(t,\cdot) = e^{t\mathcal{L}}[\mathbf{U}_0^Q] - \int_0^t e^{(t-s)\mathcal{L}}[g(s,\cdot)]\,ds$$

with $g := \mathcal{B}[\mathbf{U},\mathbf{U}]$. Thus, in terms of space norm,

$$\begin{aligned}\left\|\mathbf{U}^Q(t,\cdot)\right\|_{W^{m,6}(\mathbf{R}^2)} &\le \left\|e^{t\mathcal{L}}[\mathbf{U}_0^Q]\right\|_{W^{m,6}(\mathbf{R}^2)} + \left\|\int_0^t e^{(t-s)\mathcal{L}}[g(s,\cdot)]\,ds\right\|_{W^{m,6}(\mathbf{R}^2)} \\ &\le \left\|e^{t\mathcal{L}}[\mathbf{U}_0^Q]\right\|_{W^{m,6}(\mathbf{R}^2)} + t^{5/6}\left(\int_0^t \left\|e^{(t-s)\mathcal{L}}[g(s,\cdot)]\right\|^6_{W^{m,6}(\mathbf{R}^2)} ds\right)^{1/6}.\end{aligned}$$

Here, we used the Holder's inequality for the time integration. Take the time norm on a *finite* interval $[0, T]$,

$$\begin{aligned}\left\|\mathbf{U}^Q\right\|_{L^6([0,T];W^{m,6}(\mathbf{R}^2))} \leq &\left\|e^{t\mathcal{L}}[\mathbf{U}_0^Q]\right\|_{L^6([0,T];W^{m,6}(\mathbf{R}^2))} + \\ &T^{5/6}\left(\int_0^T\int_0^T \left\|e^{(t-s)\mathcal{L}}[g(s,\cdot)]\right\|^6_{W^{m,6}(\mathbf{R}^2)} \, ds\, dt\right)^{1/6} \\ =:&I + T^{5/6}(II)^{1/6}\end{aligned}$$

Now, apply Lemma 3.5 on I and II,

$$\begin{aligned}I \lesssim &\kappa^\alpha \|\mathbf{U}_0^Q\|_{H^{m+1}(\mathbf{R}^2)} \\ II = &\int_0^T\int_0^T \left\|e^{(t-s)\mathcal{L}}[g(s,\cdot)]\right\|^6_{W^{m,6}(\mathbf{R}^2)} \, dt\, ds \qquad \text{... switch } ds\, dt \\ \lesssim &\int_0^T \kappa^{6\alpha}\|g(s,\cdot)\|^6_{H^{m+1}(\mathbf{R}^2)}\, ds \\ = &T\cdot\kappa^{6\alpha}\cdot\|g\|^6_{L^6([0,T];H^{m+1}(\mathbf{R}^2))}\end{aligned}.$$

This shall suffice to finish the proof. □

Remark 3.8. Our result reveals the role of parameters σ, τ in controlling the decay rate of solutions. For simplicity, we did not work on optimizing this estimate. We refer to earlier work of Klainerman [**Kla85**] and Shatah [**Sha85**] for sharper estimates on nonlinear Klein-Gordon equations. Instead of Strichartz estimate, they used approaches relying on compactly supported initial data and finite speed of propagation of the linear solution operator.

Finally, construct the incompressible flow $\mathbf{U}_{inc} = \mathbf{U}^P + O(\kappa^\alpha)$ satisfying the Quasi-geostrophic equation

$$(\mathbf{U}_{inc})_t + \mathcal{P}\circ\mathcal{B}(\mathbf{U}_{inc}, \nabla\mathbf{U}_{inc}) = 0$$

so that

$$\mathbf{U} - \mathbf{U}_{inc} = \mathbf{U} - \mathbf{U}^P + (\mathbf{U}^P - \mathbf{U}_{inc}) = \mathbf{U}^Q + (\mathbf{U}^P - \mathbf{U}_{inc})$$

with the remainder terms bounded as in estimates (3.12), (3.17),

$$\begin{aligned}\|\mathbf{U}^P - \mathbf{U}_{inc}\|_{L^\infty([0,T];H^m(\mathbf{R}^2))} \lesssim &\kappa f_1\left(T, \|\mathbf{U}_0\|_{H^{m+2}}\right) \\ \left\|\mathbf{U}^Q\right\|_{L^6([0,T];W^{m,6}(\mathbf{R}^2))} \lesssim &\kappa^\alpha f_2\left(T, \|\mathbf{U}_0\|_{H^{m+2}}\right)\end{aligned}.$$

The conclusion in (1.7) follows by the Sobolev inequalities.

References

[Ali95] Serge Alinhac, *Blowup for nonlinear hyperbolic equations*, Progress in Nonlinear Differential Equations and their Applications, 17, Birkhäuser Boston Inc., Boston, MA, 1995.

[BMN02] A. Babin, A. Mahalov, and B. Nicolaenko, *Fast singular oscillating limits of stably-stratified 3D Euler and Navier-Stokes equations and ageostrophic wave fronts*, Large-scale atmosphere-ocean dynamics, Vol. I, Cambridge Univ. Press, Cambridge, 2002, pp. 126–201.

[CDGG06] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, *Mathematical geophysics*, Oxford Lecture Series in Mathematics and its Applications, vol. 32, The Clarendon Press Oxford University Press, Oxford, 2006, An introduction to rotating fluids and the Navier-Stokes equations.

[CT08] Bin Cheng and Eitan Tadmor, *Long-time existence of smooth solutions for the rapidly rotating shallow-water and Euler equations*, SIAM J. Math. Anal. **39** (2008), no. 5, 1668–1685.

[EM96] Pedro F. Embid and Andrew J. Majda, *Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity*, Comm. Partial Differential Equations **21** (1996), no. 3-4, 619–658.

[EM98] ———, *Low Froude number limiting dynamics for stably stratified flow with small or finite Rossby numbers*, Geophys. Astrophys. Fluid Dynam. **87** (1998), no. 1-2, 1–50.

[Gal98] Isabelle Gallagher, *Asymptotic of the solutions of hyperbolic equations with a skew-symmetric perturbation*, J. Differential Equations **150** (1998), no. 2, 363–384.

[Hör87] Lars Hörmander, *The lifespan of classical solutions of nonlinear hyperbolic equations*, Pseudodifferential operators (Oberwolfach, 1986), Lecture Notes in Math., vol. 1256, Springer, Berlin, 1987, pp. 214–280.

[Kla85] S. Klainerman, *Global existence of small amplitude solutions to nonlinear KleinGordon equations in four space-time dimensions*, Comm. Pure. Appl. Math. **38** (1985) 631641.

[KM81] Sergiu Klainerman and Andrew Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math. **34** (1981), no. 4, 481–524.

[KM82] ———, *Compressible and incompressible fluids*, Comm. Pure Appl. Math. **35** (1982), no. 5, 629–651.

[KM93] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), no. 9, 1221–1268.

[Kre80] Heinz-Otto Kreiss, *Problems with different time scales for partial differential equations*, Comm. Pure Appl. Math. **33** (1980), no. 3, 399–439.

[Lax73] Peter D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[LT04] Hailiang Liu and Eitan Tadmor, *Rotation prevents finite-time breakdown*, Phys. D **188** (2004), no. 3-4, 262–276.

[Maj03] Andrew Majda, *Introduction to PDEs and waves for the atmosphere and ocean*, Courant Lecture Notes in Mathematics, vol. 9, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[Mas01] Nader Masmoudi, *Incompressible, inviscid limit of the compressible Navier-Stokes system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), no. 2, 199–224.

[Ped92] J. Pedlosky, *Geophysical fluid dynamics*, Springer Verlag, Berlin, 1992.

[Sch94] Steven Schochet, *Fast singular limits of hyperbolic PDEs*, J. Differential Equations **114** (1994), no. 2, 476–512.

[Sha85] J. Shatah, *Normal forms and quadratic nonlinear KleinGordon equations*, Comm. Pure. Appl. Math. **38** (1985) 685696.

[Sid85] Thomas C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985), no. 4, 475–485.

[Sid91] ———, *The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit*, Indiana Univ. Math. J. **40** (1991), no. 2, 535–550.

[Sid97] ———, *Delayed singularity formation in* 2*D compressible flow*, Amer. J. Math. **119** (1997), no. 2, 371–422.

[Str77] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977) no. 3, 705714.

[YJ97] W. R. Young and M. Ben Jelloul, *Propogration of near-intertial oscillations through a geostrophic flow*, J. Marine Res. **55** (1997), no. 4, 735–766.

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