

A RIGOROUS TREATMENT OF MOIST CONVECTION IN A SINGLE COLUMN

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ABSTRACT. We study a single column model of moist convection in the atmosphere. We state the conditions for it to represent a stable steady state. We then evolve the column by subjecting it to an upward displacement which can release instability, leading to a time dependent sequence of stable steady states. We propose a definition of measure valued solution to describe the time dependence and prove its existence.

Key words: Rearrangement, measure-valued solutions, Lagrangian equations.

AMS subject Classifications: 35B45, 35D30, 35Q86.

1. INTRODUCTION

This paper studies a simple mathematical model of moist convection in the atmosphere set out in Bokhove *et al.* [2]. Moist convection is responsible for much of the severe weather in the extratropics, and is the main driver of the tropical circulation which is a fundamental part of the climate system. While convective storms have a very complicated structure, in which the physics of water in various phases is critical, the essential process can be captured by a one-dimensional model which only takes into account the saturation of air parcels with the associated release of latent heat. Such a model is used routinely by practising weather forecasters in interpreting the likely weather that will result from a given vertical profile of temperature and moisture, see [6], chapter 4. It also forms a key component of many theoretical studies of moist convection in the atmosphere; for instance Holt [4], Lock and Norbury [5] and Shutts [7].

The model expresses conservation of heat and moisture, together with the change of phase of moisture from vapour to and from liquid and the associated release or absorption of latent heat. This takes place at a moisture concentration which depends on temperature and pressure, and introduces a strong nonlinearity into the problem. Moist convection results from an instability of the vertical profile, which can be triggered by the upward bulk motion of the vertical profile generated by extratropical weather systems. In our model we represent the effect of this by making the saturation moisture content a monotonically decreasing function of time. This allows the model to be solved in a fixed vertical domain, which simplifies the presentation.

The conservation properties are expressed in Lagrangian form, so that a discrete version of the problem can be solved by rearranging fluid parcels as in Bokhove *et al.* [2], Holt [4] and Lock and Norbury [5]. These conservation properties have been shown to be quite accurate even in more complicated models, e.g. by Shutts and Gray [8]. The rearrangement procedure is designed to reflect the underlying physics of the problem.

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The first attempt to rigorously study this model was made by Dorian Goldman in his Master's thesis [3], where he considered a particular choice of moisture content and initial data and proved the existence of weak solutions in Lagrangian variables. However, there seems to be certain gaps in the proofs and the solution was not completely characterized. Besides, his proof does not generalize to more general choice of moisture content and initial data, which can be physically interesting.

The aim of this paper is to show that the discrete problem converges to a limit solution as the number of parcels is increased and to interpret the resulting solution as a weak Lagrangian solution of the governing equations. We take a probabilistic approach in this paper, which is completely different from [3] and allows us to deal with more general choice of moisture content function and initial data, which is physically meaningful.

The plan of the paper is as follows. In section 2 we present the problem to be solved and write it as a set of Lagrangian evolution equations. We note that we can only expect a probabilistic solution for general choices of initial data. In section 3, we describe the procedure to construct approximate(discrete) solutions given some deterministic discrete initial data, and show they satisfy the physical constraints. In section 4, we establish necessary estimates about these discrete solutions. In section 5, we come up with the notion of measure valued solutions, and show this coincides with a natural definition of the solution when the initial data and evolution is deterministic. In section 6, we take the limit of the discrete solutions as time/space step size tends to zero and obtain the existence of measure valued solutions.

2. DEFINITION OF THE PROBLEM

The problem to be studied, Bokhove *et al.* (2016), is

$$(2.1) \quad D_t(\theta + q) = 0 \text{ in } (z, t) \in [0, 1] \times (0, T).$$

$$(2.2) \quad D_t\theta = \begin{cases} 0 & \text{if } q < Q^{sat}(\theta, z, t) \\ [D_t(Q^{sat}(\theta, z, t))]^- & \text{if } q = Q^{sat}(\theta, z, t) \end{cases} \text{ in } (z, t) \in [0, 1] \times (0, T).$$

$$(2.3) \quad \frac{\partial u}{\partial z} = 0 \text{ in } (z, t) \in [0, 1] \times (0, T).$$

$$(2.4) \quad q(z, t) \leq Q^{sat}(\theta(z, t), z, t).$$

As usual, we denote $D_t = \partial_t + u \cdot \partial_z$. The equation (2.3) above should be interpreted as the divergence free condition with respect to the space variable z , namely its flow is measure preserving. The unknown functions are the potential temperature $\theta(z, t)$ and the moisture content $q(z, t)$. Equation 2.4 expresses the physical constraint that the moisture content is limited by the known saturation value Q^{sat} which is time dependent. The interesting case, which we study, is where Q^{sat} is monotonically decreasing in time. However, this is not needed in the subsequent argument. In the above, $Q^{sat} : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function in its variables, and the following strict monotonicity conditions hold:

$$(2.5) \quad \partial_\theta Q^{sat} > 0, \quad \partial_z Q^{sat} < 0, \quad \text{for any } (\theta, z, t) \in \mathbb{R}^3.$$

Physical solutions to (2.1)-(2.4) should also satisfy the following constraint:

$$(2.6) \quad z \mapsto \theta(z, t) \text{ is monotone increasing in } z \text{ for any } t \in (0, T).$$

The reason for imposing such a constraint is that physical solutions should minimize the energy functional

$$E(\bar{\theta}) = - \int_{[0,1]} z\bar{\theta}(z)dz,$$

where $\bar{\theta}(z)$ is a bounded Borel function on $[0, 1]$, among all the possible rearrangements of the particles. It is easy to see that a function $\theta(z)$ achieves the minimum of E among all the functions $\bar{\theta}$ with the same distribution as θ iff $\theta(z)$ is monotone increasing. Rearranging the parcels is a measure preserving map which does not change the distribution.

It is not hard to see that, in general, the solution does not have good regularity. Indeed, if everything is smooth, then let $F : [0, T] \times [0, 1] \rightarrow [0, 1]$ be the flow map. From (2.3), we get for each fixed $t \in [0, T]$, $F_t(\cdot)$ preserves $\mathcal{L}_{[0,1]}^1$. If F were continuous, then we can obtain that $F(z) = z$ or $F(z) = 1 - z$. But $F_0 = id$, hence by continuity in t , we would be getting $F(t, z) = z$ for all t . This is not compatible with (2.4) and (2.6) except in trivial cases. Hence F cannot be continuous. Therefore, the velocity u is defined only as a measure. It is not clear how to define weak solutions to (2.1)-(2.4) in a standard way since the set $\{q = Q^{sat}(\theta, z, t)\}$ is only a general Borel set and may not have nice regularity.

We next define the solution in Lagrangian variables. Let $F_t(z)$ be the flow map, we then get a reformulation of (2.1)-(2.4):

$$(2.7)$$

$$\partial_t(\hat{\theta} + \hat{q}) = 0 \text{ in } (z, t) \in [0, 1] \times (0, T).$$

$$(2.8)$$

$$\partial_t \hat{\theta} = \begin{cases} 0 & \text{if } \hat{q} < Q^{sat}(\hat{\theta}, F_t(z), t) \\ [\partial_t(Q^{sat}(\hat{\theta}, F_t(z), t))]^- & \text{if } \hat{q} = Q^{sat}(\hat{\theta}, F_t(z), t) \end{cases} \text{ in } (z, t) \in [0, 1] \times (0, T).$$

$$(2.9)$$

$$F_t \# \mathcal{L}_{[0,1]}^1 = \mathcal{L}_{[0,1]}^1, \text{ for any } t \in [0, T].$$

$$(2.10)$$

$$\hat{q}(z, t) \leq Q^{sat}(\hat{\theta}(z, t), F_t(z), t).$$

In the above, \hat{q} and $\hat{\theta}$ denote the corresponding variables in Lagrangian coordinates, namely $\hat{q}(t, z) = q(t, F_t(z))$, and $\hat{\theta}(t, z) = \theta(t, F_t(z))$. Here we remark that the equations can be interpreted in a natural way. Indeed, (2.7) means $\hat{\theta} + \hat{q}$ is conserved along flow lines. As for equation (2.8), notice that the right hand side of (2.8) is nonnegative, hence $\partial_t \hat{\theta}$ will be a nonnegative measure. If we can show $t \mapsto F_t(z)$ has bounded variation, then $\partial_t(Q^{sat}(\hat{\theta}, F_t(z), t))$ is a well-defined finite signed measure and its negative part can be defined. Therefore, (2.8) can be naturally defined as an equality of measures.

It will be convenient to consider the function $\Theta(w, z, t)$ as the solution θ to the equation:

$$(2.11)$$

$$\theta + Q^{sat}(\theta, z, t) = w$$

This function is well defined thanks to the assumed strict monotonicity of Q^{sat} about θ . Also we know that Θ is smooth and satisfies the strict monotonicity

$$(2.12)$$

$$\partial_w \Theta > 0, \partial_z \Theta > 0.$$

This is clear from (2.5). First we make a simple observation whose proof is elementary.

Lemma 2.1. *Define $\theta^M(t, z) = \theta(t, z) + q(t, z)$. Then $q(t, z) \leq Q^{sat}(\theta(t, z), z, t)$ is equivalent to $\theta(t, z) \geq \Theta(\theta^M(t, z), z, t)$. Equivalence holds true also if we replace the above by a strict inequality.*

We assume the initial data satisfies the physical constraint. Namely, we are given $\theta_0(z), q_0(z) \in L^\infty([0, 1])$, such that $z \mapsto \theta_0(z)$ is monotone increasing, and $q_0(z) \leq Q^{sat}(\theta_0(z), z, 0)$ for *a.e.*- z . Inspired by the previous discussions, we propose the following definition of weak Lagrangian solutions.

Definition 2.2. *Let $\theta_t(z), q_t(z) \in L^\infty([0, T] \times [0, 1]) \cap C([0, T], L^1([0, 1]))$, and $F, F^* : [0, T] \times [0, 1] \rightarrow [0, 1]$ be Borel measure preserving maps such that $F_t(\cdot), F_t^*(\cdot) \in C([0, T]; L^1([0, 1]))$, and $F_t(z) \in L^\infty([0, 1]; BV([0, T]))$. Let $\theta_0(z), q_0(z)$ be as in previous paragraph. Denote $\hat{\theta}_t(z) = \theta_t(F_t(z))$ and $\hat{q}_t(z) = q_t(F_t(z))$. Then we say (q_t, θ_t, F_t) is a weak Lagrangian solution to initial data θ_0, q_0 if the following holds:*

- (i) $\theta_t \rightarrow \theta_0, q_t \rightarrow q_0$ in $L^1([0, 1])$, $F_t \rightarrow id$ in $L^1([0, 1])$ as $t \rightarrow 0$.
- (ii) $z \mapsto \theta_t(z)$ is monotone increasing for each $t \in [0, T]$.
- (iii) For any $t \in [0, T]$, $F_t \circ F_t^*(z) = z, F_t^* \circ F_t(z) = z$ for \mathcal{L}^1 -a.e $z \in [0, 1]$.
- (iv) For \mathcal{L}^1 -a.e $z \in [0, 1]$, $\hat{\theta}_t(z) + \hat{q}_t(z) = \theta_0(z) + q_0(z)$, for \mathcal{L}^1 -a.e $t \in (0, T)$
- (v) For \mathcal{L}^1 -a.e $z \in [0, 1]$, $t \mapsto \hat{\theta}_t(z) \leq \hat{\theta}_{t'}(z)$ for \mathcal{L}^2 -a.e (t, t') with $t < t'$
- (vi) $\partial_t \hat{\theta}(\cdot, z) = [\partial_t(Q^{sat}(\hat{\theta}(\cdot, z), F_t(z), \cdot))]^- \lfloor E_z$, where $E_z = \{t \in (0, T) : \hat{q}_t^*(z) = Q^{sat}(\hat{\theta}_t^*(z), F_t(z), t)\}$, and $\hat{q}_t^*(z), \hat{\theta}_t^*(z)$ is the monotone, left continuous version of $\hat{q}_t(z), \hat{\theta}_t(z)$. chosen according to Remark 2.4.

Remark 2.3. *Let $f : (0, T) \rightarrow \mathbf{R}$ be Borel measurable, such that $f = \tilde{f}$ for \mathcal{L}^1 -a.e $t \in (0, T)$, with $\tilde{f} \in BV(0, T)$, then $\partial_t f$ is a finite signed measure, defined by*

$$\int_0^T f(t) \partial_t \zeta(t) dt = - \int_0^T \zeta(t) (\partial_t f)(dt), \forall \zeta \in C_c^1((0, T)).$$

If we choose \tilde{f} such that it is left continuous, then:

$$\partial_t f([a, b]) = \tilde{f}(b) - \tilde{f}(a), \text{ for any } [a, b] \subset (0, T).$$

Remark 2.4. *Let $f : (0, T) \rightarrow \mathbf{R}$ be Borel measurable such that for \mathcal{L}^2 -a.e $(t, t') \in (0, T)^2$ with $t < t'$, we have $f(t) \leq f(t')$, then there exists a unique function $f^* : (0, T) \rightarrow \mathbf{R}$, monotone increasing, continuous from the left, such that $f = f^*$ for \mathcal{L}^1 -a.e $t \in (0, T)$.*

It turns out that above definition of weak Lagrangian solutions is still too strong, and one cannot expect the existence of solutions in the sense defined above except for some special choice of the function Q^{sat} and the initial data.

One difficulty with the system (2.1)-(2.4) is that we do not have much regularity in space. The only regularity in space comes from the monotonicity of θ , and in general, no regularity in space for q , as well as the flow maps F_t, F_t^* . This means we lack the necessary compactness to get a function $q_t(z)$, or a measure preserving flow map F, F^* in the limit.

The evolution of θ and q is highly unstable under small perturbations of the initial data, which can be seen from the construction of the discrete problem. This suggests the use of probabilistic description of the solution. Under this description, heuristically

for each time t , we have a certain probability distribution for $\{\theta, q(z)\}_{z \in [0,1]}$, and we make a random choice of θ , which is a monotone increasing function on $[0, 1)$, also make a random choice for $q(z)$ for each z , according to this probability distribution and then evolve. This determines the probability distribution for $\{\theta, q(z)\}_{z \in [0,1]}$ at later times. In this spirit, we need to prescribe some probability distribution as initial data.

On the other hand we need the correct equation to be satisfied (point (vi) of Definition 2.2), this suggests considering some "path-spaces" which describes all the possible paths of some parcel. Inspired by the probabilistic approach of transport equation, we wish to obtain the solution as a measure in some path space, and the correct probability distribution is obtained by projecting to each t .

We will make the above heuristic discussions rigorous in section 5.

3. SOLUTION OF THE DISCRETE PROBLEM

In this section, we construct discrete solutions following the method of Bokhove *et al.* [2] and do estimates about them.

The discrete procedure is designed to reflect the underlying physics of the problem, as expressed in Met Office [6], chapter 4. It is based on a representation of the fluid as discrete parcels, so that $\hat{\theta}$ and \hat{q} are piecewise constant. The initial values satisfy the physical constraints (2.10) and (2.6) at $t = 0$. We define Q^{sat} to be a monotonically decreasing function of time and discretise the time variation. Thus after some time interval the constraint (2.10) will be violated.

The Lagrangian form of the equations (2.7)-(2.10) is solved by representing the flow map F_t as a rearrangement of the fluid parcels. The evolution of $\hat{\theta}$ and \hat{q} on each parcel is computed using (2.7) and (2.8). If (2.10) is violated for any parcel, then (2.8) is used to update $\hat{\theta}$ and set q equal to Q^{sat} . The update to $\hat{\theta}$ may result in the constraint (2.6) being violated, in which case the parcels have to be rearranged to restore the constraint. $\hat{\theta}^M = \hat{\theta} + \hat{q}$ is conserved for each parcel under the rearrangement as required by (2.7), and $\hat{\theta} \geq \Theta(\hat{\theta}^M, z, t)$ at the final positions because of Lemma 2.1.

As found by Bokhove *et al.* [2], finding this rearrangement is non-trivial because of the dependence of Q^{sat} on θ and z , and because (2.10) may be violated on several parcels simultaneously. We call these 'wet' parcels. In this case there may be many ways to satisfy the constraints. The physics of the problem requires that the final position of the wet parcel with the largest $\hat{\theta}^M$ is determined first. This is done by moving it upwards, thus increasing $\hat{\theta}$, until it encounters a larger value of $\hat{\theta}$ at some $z = z_t$. We refer to this as the parcel "beating" all other parcels with $z < z_t$. All overtaken parcels have to move down to compensate for the upward displacement. Extreme care is required in showing that this procedure has a well defined limit as the timestep tends to zero.

We now define this procedure precisely. Denote $z_i = \frac{i}{n}$, $J_i = [\frac{i-1}{n}, \frac{i}{n})$, for any integer $1 \leq i \leq n$. Let $\{\theta_j^n\}_{j=1}^n$, and $\{q_j^n\}_{j=1}^n$ be given, such that $\theta_j^n \leq \theta_{j+1}^n$, and $q_j^n \leq Q^{sat}(\theta_j^n, z_j, 0)$, for any $1 \leq j \leq n$. This means that a discrete version of (2.6) and (2.10) is satisfied. It follows from Lemma 2.1 that $\theta_j^n \geq \Theta(\theta_j^n + q_j^n, z_j, 0)$. Let $\delta t = \frac{1}{Cn}$ for some large constant $C > 0$ to be determined later on. This will be chosen so that it depends only on the function Q^{sat} , T , and the initial data. Define the wet set at time step 0 to be $W_n = \{1 \leq j \leq n : \theta_j^n < \Theta(\theta_j^n + q_j^n, z_j, \delta t)\}$. We will also denote

$\theta_j^{M,n} = \theta_j^n + q_j^n$. First we decide which parcels move to z_n . Define

$$W'_n = \{j_0 \in W_n : \text{for any } j > j_0 \text{ with } j \notin W_n, \theta_j^n < \Theta(\theta_{j_0}^{M,n}, z_j, \delta t), \text{ and if } j \in W_n, \\ \theta_{j_0}^{M,n} > \theta_j^{M,n}\}.$$

Here we make the convention that $n \in W'_n$ if and only if $n \in W_n$. The set W'_n is exactly the set of parcels which are "wet" and can beat all other parcels above up to n . We will sometimes call them "eligible". First assume $W'_n \neq \emptyset$. Let $j_0 \in W'_n$ be the parcel with the largest $\theta_j^{M,n}$ among W'_n (if there are more than one such parcels, simply choose j_0 to be largest possible), define the first rearrangement

$$\sigma_n(k) = \begin{cases} k & \text{if } 1 \leq k < j_0; \\ n & \text{if } k = j_0; \\ k - 1 & \text{if } j_0 < k \leq n. \end{cases}$$

To explain this in English, a parcel can jump to z_n only if it is wet and has the largest $\theta^{M,n}$ among all "eligible" parcels.

We also update θ_j^n after the first rearrangement in the following way:

$$\theta_j^{n,n-1} = \begin{cases} \theta_{\sigma_n^{-1}(j)}^{n-1} & \text{if } j \neq n; \\ \Theta(\theta_{j_0}^{M,n}, z_j, \delta t) & \text{if } j = n. \end{cases}$$

That is, we update the θ of parcels which jumped according to its final position, and leave the θ of other parcels unchanged. In Lagrangian coordinates, define $\hat{\theta}_j^{n,n-1} = \theta_{\sigma_n(j)}^{n,n-1}$, $\hat{q}_j^{n,n-1} = \theta_j^n + q_j^n - \hat{\theta}_j^{n,n-1}$. This is consistent with (2.7). Define the new wet set

(3.1)

$$W_{n-1} = \{1 \leq j \leq n : \theta_j^{n,n-1} < \Theta(\theta_{\sigma_n^{-1}(j)}^{M,n}, z_j, \delta t)\},$$

$$W'_{n-1} = \{j_0 \in W_{n-1} : \text{for any } j_0 < j \leq n-1 \text{ with } j \notin W_{n-1}, \theta_j^{n,n-1} < \Theta(\theta_{\sigma_n^{-1}(j_0)}^{M,n}, z_j, \delta t), \\ \text{or if } j \in W_{n-1}, \theta_{\sigma_n^{-1}(j_0)}^{M,n} > \theta_{\sigma_n^{-1}(j)}^{M,n}\}.$$

Notice that $W_{n-1} \subset \{1, 2, \dots, n-1\}$.

If $W'_n = \emptyset$, then simply take $\sigma_n = id$, and take $\theta_j^{n,n-1} = \theta_j^n$, $q_j^{n,n-1} = q_j^n$. Then we have $W_{n-1} = W_n$.

Next we repeat the above procedure to $\{\theta_j^{n,n-1}\}_{j=1}^{n-1}$, $\{q_j^{n,n-1}\}_{j=1}^{n-1}$, and the new wet set defined by (3.1). Let σ_{n-1} be the resulting rearrangement of the first $n-1$ parcels. Let $\sigma_{n-1}(n) = n$, so that it becomes the rearrangement for n parcels.

In general, let σ_k be the rearrangement when we decide which parcel moves to z_k with $\sigma_k(l) = l$ for $l > k$. Denote $\beta_k = \sigma_{k+1} \circ \dots \circ \sigma_n$, with $\beta_n = id$. Let $\{\theta_j^{n,k}\}_{j=1}^n$, $\{q_j^{n,k}\}_{j=1}^n$ be the updated θ and q after σ_{k+1} . We also denote $\theta_j^n = \theta_j^{n,n}$, and $q_j^n = q_j^{n,n}$. The wet set at this stage is given by

$$(3.2) \quad W_k = \{1 \leq j \leq n : \theta_j^{n,k} < \Theta(\theta_{\beta_k^{-1}(j)}^{M,n}, z_j, \delta t)\} \\ W'_k = \{j_0 \in W_k : \text{for any } j_0 < j \leq k \text{ with } j \notin W_k, \theta_j^{n,k} < \Theta(\theta_{\beta_k^{-1}(j_0)}^{M,n}, z_j, \delta t), \\ \text{or if } j \in W_k, \theta_{\beta_k^{-1}(j_0)}^{M,n} > \theta_{\beta_k^{-1}(j)}^{M,n}\}.$$

As before, we make the convention that if $j_0 \geq k$, then $j_0 \in W'_k$ if and only if $j_0 \in W_k$. The sets W_k, W'_k determines the evolution when we decide which parcel moves to z_{k-1} . The following inductive formula holds when $W'_{k+1} \neq \emptyset$. Let $j_* \in W'_{k+1}$ be such that $\theta_{\beta_{k+1}(j_*)}^{M,n} \geq \theta_{\beta_{k+1}(j)}^{M,n}$ for any $j \in W'_{k+1}$, then we move this parcel to z_{k+1} , namely,

$$(3.3) \quad \sigma_{k+1}(j) = \begin{cases} j & \text{if } j < j_* \text{ or } j > k+1; \\ k+1 & \text{if } j = j_*; \\ j-1 & \text{if } j_* < j \leq k+1. \end{cases}$$

We update θ_j^n accordingly:

$$(3.4) \quad \theta_j^{n,k} = \begin{cases} \theta_{\sigma_{k+1}(j)}^{n,k+1} & \text{if } j \neq k+1; \\ \Theta(\theta_{\beta_k^{-1}(k+1)}^{M,n}, z_{k+1}, \delta t) & \text{if } j = k+1. \end{cases}$$

If $W'_{k+1} = \emptyset$, then simply put $\sigma_{k+1} = id$ and $\theta_j^{n,k} = \theta_j^{n,k+1}$.

Let $\hat{\theta}_j^{n,k} = \theta_{\beta_k(j)}^{n,k}$. Define $q_j^{n,k} = q_{\sigma_{k+1}(j)}^{n,k+1} + \theta_{\sigma_{k+1}(j)}^{n,k+1} - \theta_j^{n,k}$. $\hat{q}_j^{n,k} = q_{\beta_k(j)}^{n,k}$. We observe some useful properties of above rearrangement algorithm:

Lemma 3.1. *For each index $j \in \{1, 2, \dots, n\}$, one of the following must hold:*

(i) *There exists a unique $k_1 \in \{1, 2, \dots, n\}$, such that $\beta_{k_1-1}(j) = \sigma_{k_1}(\beta_{k_1}(j)) = k_1 > \beta_{k_1}(j)$. Besides, for any $k \geq k_1$, $\beta_k(j) \leq \beta_{k+1}(j)$, and any $k \leq k_1 - 1$, $\beta_k(j) = k_1$.*

(ii) *$\beta_k(j) \leq \beta_{k+1}(j)$ for any $0 \leq k \leq n-1$.*

Morover, if for some k_2 , $\beta_{k_2}(j) \leq k_2$ and $\notin W_{k_2}$, then the second alternative must hold. On the other hand, if for some j_1, j_2 , and some k_3 , it holds $\beta_{k_3}(j_1) < \beta_{k_3}(j_2)$, but $\beta_{k_3-1}(j_1) > \beta_{k_3-1}(j_2)$, then the first alternative above holds for j_1 with $k_1 = k_3$.

This lemma says that for any given parcel, either it experiences no lifts at all among the σ_k 's, or there is a unique σ_k which lifts this parcel and it stays there in the latter rearrangements of the same time step. If a parcel becomes dry in a certain time step, then it will stay dry in the latter arrangement. The only way the order of two parcels can change is that the lower parcel experience a jump.

Proof. Fix an index j . Suppose there exists some $0 \leq k_1 \leq n-1$, for which $\beta_{k_1-1}(j) > \beta_{k_1}(j)$. From the definition of the σ_k given in (3.3), we see that if $\sigma_k(j) > j$ for some j, k , it must hold that $\sigma_k(j) = k$. Hence from $\beta_{k_1-1}(j) = \sigma_{k_1}(\beta_{k_1}(j)) > \beta_{k_1}(j)$, we see $\sigma_{k_1}(\beta_{k_1}(j)) = k_1$. If for some $k \geq k_1$, $\beta_k(j) > \beta_{k+1}(j)$, then the same argument shows $\beta_k(j) = \sigma_{k+1}(\beta_{k+1}(j)) = k+1$. Now for any $k' \leq k$, we have $\sigma_{k'}(\beta_k(j)) = \beta_k(j)$, a contradiction. This proves for $k \geq k_1$, $\beta_k(j) \leq \beta_{k+1}(j)$. The case for $k \leq k_1 - 1$ follows directly from definition of σ_k .

To see the moreover part, observe that for any $k \geq k_2$, we must have $\beta_k(j) \leq \beta_{k+1}(j)$. If not, by the argument given in the first part, we can then conclude $\beta_{k'}(j) = k+1$, for any $k' \leq k$. In particular, this means $\beta_{k_2}(j) = k+1 > k_2$, a contradiction. Since $\beta_{k_2}(j) \notin W_{k_2}$, from (3.3), we see $\beta_{k_2-1}(j) = \sigma_{k_2}(\beta_{k_2}(j)) \leq \beta_{k_2}(j)$. This means that

$$\theta_{\beta_{k_2-1}(j)}^{n,k_2-1} = \theta_{\beta_{k_2}(j)}^{n,k_2} \geq \Theta(\theta_j^{M,n}, z_{\beta_{k_2}(j)}, \delta t) \geq \Theta(\theta_j^{M,n}, z_{\beta_{k_2-1}(j)}, \delta t).$$

It follows that $\beta_{k_2-1}(j) \notin W_{k_2-1}$. Hence one conclude $\beta_{k_2-2}(j) = \sigma_{k_2-1}(\beta_{k_2-1}(j)) \leq \beta_{k_2-1}(j)$. Same argument as above applies and shows $\beta_{k_2-2}(j) \notin W_{k_2-2}$. One can apply the same argument and shows $\beta_k(j)$ is monotone decreasing in k . The "on the other hand" part follows directly from the definition of σ_k given in (3.3). \square

We want to show above define algorithm preserves a discrete version of the physical constraint:

Lemma 3.2. (i) $\theta_j^{n,k} \leq \theta_{j+1}^{n,k}$ for any $1 \leq j \leq n-1$;

$$(ii) \hat{q}_j^{n,k} + \hat{\theta}_j^{n,k} = q_j^n + \theta_j^n.$$

$$(iii) \hat{\theta}_j^{n,k} \geq \hat{\theta}_j^{n,k+1}.$$

$$(iv) q_j^{n,k} \leq Q^{\text{sat}}(\theta_j^{n,k}, z_j, 0), \text{ for } 1 \leq j \leq k, \text{ and } q_j^{n,k} \leq Q^{\text{sat}}(\theta_j^{n,k}, z_j, \delta t), \text{ for } k+1 \leq j \leq n.$$

Proof. First we prove the point (ii). From our definition, we know that $q_j^{n,k} + \theta_j^{n,k} = q_{\sigma_{k+1}^{-1}(j)}^{n,k+1} + \theta_{\sigma_{k+1}^{-1}(j)}^{n,k+1}$. From this it immediately follows $\hat{\theta}_j^{n,k} + \hat{q}_j^{n,k} = \hat{\theta}_j^{n,k+1} + \hat{q}_j^{n,k+1}$.

We prove the other three statements by induction on n . First observe that statements (i)-(iv) are true for $k = n$. (point (iii) is empty when $k = n$.) Now assume these are true for $k+1$ and above with $1 \leq k+1 \leq n$, wish to prove these for k .

Now we prove point (iii) for k , assuming $W'_{k+1} \neq \emptyset$. One can see point (iii) is equivalent to $\theta_{\sigma_{k+1}(j)}^{n,k} \geq \theta_j^{n,k+1}$, by our definition of $\hat{\theta}_j^{n,k}$. If $\sigma_{k+1}(j) \neq k+1$, then one has $\theta_{\sigma_{k+1}(j)}^{n,k} = \theta_j^{n,k+1}$ from (3.4). Now if $\sigma_{k+1}(j_*) = k+1$, then

$$\theta_{k+1}^{n,k} = \Theta(\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n}, z_{k+1}, \delta t) \geq \theta_{k+1}^{n,k+1} \geq \theta_{j_*}^{n,k+1}.$$

The first inequality above used the fact that $j_* \in W'_{k+1}$, hence it must "beat" the parcel originally at z_{k+1} . The second inequality used the induction hypothesis that point (i) holds with $k+1$. If $W'_{k+1} = \emptyset$, then we simply have $\sigma_{k+1} = id$, and $\theta_j^{n,k} = \theta_j^{n,k+1}$, so there is nothing to prove.

Then we prove point (i). We only consider the case when $W'_{k+1} \neq \emptyset$, otherwise nothing is changed by σ_{k+1} and the proof is trivial. Let $j_* \leq k+1$ be such that $\sigma_{k+1}(j_*) = k+1$.

The only nontrivial cases to check is when $j = k$ and $j = k+1$, the rest of the cases will follow from (3.3), (3.4) and the induction hypothesis that (i) holds for $k+1$. So it boils down to prove

$$(3.5) \quad \theta_{k+1}^{n,k+1} \leq \Theta(\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n}, z_{k+1}, \delta t) \leq \theta_{k+2}^{n,k+1}.$$

The first part of the inequality follows from that $j_* \in W'_{k+1}$, that is, it needs to "beat" the parcel originally at z_{k+1} in order to rise to z_{k+1} . To be precise, suppose that $k+1 \notin W_{k+1}$, then we know from $j_* \in W'_{k+1}$ and the formula for W'_k in (3.2) that $\theta_{k+1}^{k+1} < \Theta(\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n}, z_{k+1}, \delta t)$. This is exactly what we want. Now if $k+1 \in W_{k+1}$, then again

from $j_* \in W_{k+1}$ one concludes that $\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n} \geq \theta_{\beta_{k+1}^{-1}(k+1)}^{M,n}$. Hence

$$\Theta(\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n}, z_{k+1}, \delta t) \geq \Theta(\theta_{\beta_{k+1}^{-1}(k+1)}^{M,n}, z_{k+1}, \delta t) \geq \theta_{k+1}^{n,k+1}.$$

The first inequality used the monotonicity of Θ with respect to θ^M , the second inequality used that $k+1 \in W_{k+1}$. This proves the first part of (3.5).

The reason why the second part of the inequality holds is that if it were not true, then j_* would have risen to z_{k+2} instead of z_{k+1} in the rearrangement σ_{k+2} . To make this precise, let $\sigma_{k+2}(j_{**}) = j_*$, then we must have $j_{**} \geq j_*$. If not, we will have $\sigma_{k+2}(j_{**}) = k+2 = j_*$, not possible. Since $j_* \in W_{k+1}$, we have

$$\theta_{j_{**}}^{n,k+2} = \theta_{j_*}^{n,k+1} < \Theta(\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n}, z_{j_*}, \delta t) \leq \Theta(\theta_{\beta_{k+2}^{-1}(j_{**})}^{M,n}, z_{j_{**}}, \delta t).$$

This implies $j_{**} \in W_{k+2}$. Now we claim that for any j with $j_{**} < j \leq k+1$ and $j \notin W_{k+2}$, then $\theta_j^{n,k+2} < \Theta(\theta_{\beta_{k+2}^{-1}(j_{**})}^{M,n}, z_j, \delta t)$. Indeed, since $j \notin W_{k+2}$, we have $\sigma_{k+2}(j) \leq j$, $\sigma_{k+2}(j) \notin W_{k+1}$, and $\theta_j^{n,k+2} = \theta_{\sigma_{k+2}(j)}^{n,k+1}$. If the claim is not true, then

$$\Theta(\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n}, z_{\sigma_{k+2}(j)}, \delta t) \leq \Theta(\theta_{\beta_{k+1}^{-1}(j_*)}^{M,n}, z_j, \delta t) \leq \theta_j^{n,k+2} = \theta_{\sigma_{k+2}(j)}^{n,k+1}.$$

Notice $\sigma_{k+2}(j) \leq k+1$, also $\sigma_{k+2}(j) > \sigma_{k+2}(j_{**}) = j_*$, this contradicts $j_* \in W'_{k+1}$.

Let j_1 be the maximal j such that $j \geq j_{**}$, $j \in W_{k+2}$, and $\theta_{\beta_{k+2}^{-1}(j_{**})}^{M,n} \leq \theta_{\beta_{k+2}^{-1}(j)}^{M,n}$. From the induction hypothesis with $k+2$ and point (iv), we know $j_1 \leq k+2$. Consider 2 cases:

If $j_1 \notin W'_{k+2}$. First observe for any $j > j_1$ and $j \in W_{k+2}$, we must have $\theta_{\beta_{k+2}^{-1}(j)}^{M,n} < \theta_{\beta_{k+2}^{-1}(j_1)}^{M,n}$. Otherwise it will contradict the maximality of j_1 . Also for any j with $j_1 < j \leq k+1$, and $j \notin W_{k+2}$, we conclude from the claim $\theta_j^{n,k+2} < \Theta(\theta_{\beta_{k+2}^{-1}(j_{**})}^{M,n}, z_j, \delta t) \leq \Theta(\theta_{\beta_{k+2}^{-1}(j_1)}^{M,n}, z_j, \delta t)$. The only possibility remains is that $k+2 \notin W_{k+2}$, and $\theta_{k+2}^{n,k+2} \geq \Theta(\theta_{\beta_{k+2}^{-1}(j_1)}^{M,n}, z_{k+2}, \delta t)$. That is, $k+2$ is a dry parcel and cannot be beaten by j_1 . Hence

$$\theta_{k+2}^{n,k+1} \geq \theta_{k+2}^{n,k+2} \geq \Theta(\theta_{\beta_{k+2}^{-1}(j_1)}^{M,n}, z_{k+2}, \delta t) \geq \Theta(\theta_{\beta_{k+2}^{-1}(j_{**})}^{M,n}, z_{k+1}, \delta t).$$

This is what we want.

If $j_1 \in W'_{k+2}$, let $j_2 \leq k+2$ be such that $\sigma_{k+2}(j_2) = k+2$. From the definition of the procedure, we have $\theta_{\beta_{k+2}^{-1}(j_2)}^{M,n} \geq \theta_{\beta_{k+2}^{-1}(j_1)}^{M,n}$, since the parcel that actually jumps up should have the largest θ^M among all "eligible" parcels. From the inductive formula (3.4), we see

$$\theta_{k+2}^{n,k+1} = \Theta(\theta_{\beta_{k+2}^{-1}(j_2)}^{M,n}, z_{k+2}, \delta t) \geq \Theta(\theta_{\beta_{k+2}^{-1}(j_1)}^{M,n}, z_{k+2}, \delta t) \geq \Theta(\theta_{\beta_{k+2}^{-1}(j_{**})}^{M,n}, z_{k+1}, \delta t).$$

So far we finished the proof of point (i).

It only remains to show the point (iv). This is equivalent to showing $\theta_j^{n,k} \geq \Theta(\theta_j^{n,k} + q_j^{n,k}, z_j, 0)$ for $1 \leq j \leq k$, and $\theta_j^{n,k} \geq \Theta(\theta_j^{n,k} + q_j^{n,k}, z_j, \delta t)$, for $k+1 \leq j \leq n$. To see the first part, we know for $1 \leq j \leq k$, $\theta_j^{n,k} = \theta_{\sigma_{k+1}^{-1}(j)}^{n,k+1}$, and we also know from point (ii) already proved that $\theta_j^{n,k} + q_j^{n,k} = \theta_{\sigma_{k+1}^{-1}(j)}^{n,k+1} + q_{\sigma_{k+1}^{-1}(j)}^{n,k}$. Also $\sigma_{k+1}^{-1}(j) \leq k+1$, since the rearrangement σ_{k+1} never moves down a parcel by 2. Apply the induction hypothesis that (iv) holds for $k+1$ we see

$$\theta_j^{n,k} = \theta_{\sigma_{k+1}^{-1}(j)}^{n,k+1} \geq \Theta(\theta_{\sigma_{k+1}^{-1}(j)}^{n,k+1} + q_{\sigma_{k+1}^{-1}(j)}^{n,k+1}, z_{\sigma_{k+1}^{-1}(j)}, 0) \geq \Theta(\theta_j^{n,k} + q_j^{n,k}, z_j, 0).$$

To see the second part, consider first when $W'_{k+1} \neq \emptyset$, and if $j = k+1$, then we know from (3.4)

$$\theta_{k+1}^{n,k} = \Theta(\theta_{\beta_{k+1}^{-1}(k+1)}^{M,n}, z_{k+1}, \delta t) = \Theta(q_{k+1}^{n,k} + \theta_{k+1}^{n,k}, z_{k+1}, \delta t).$$

In the second equality above, we used the point (ii) already proved, and also the definition of $\theta_j^{M,n}$ given in the beginning of this section. If instead $W'_{k+1} = \emptyset$, then we know in

particular $k + 1 \notin W_{k+1}$, hence

$$\theta_{k+1}^{n,k} = \theta_{k+1}^{n,k+1} \geq \Theta(\theta_{\beta_k^{-1}(k+1)}^{M,n}, z_{k+1}, \delta t) \geq \Theta(\theta_{k+1}^{n,k} + q_{k+1}^{n,k}, z_{k+1}, \delta t).$$

If $k + 2 \leq j \leq n$, note $\sigma_{k+1}(j) = j$ we use the induction hypothesis and (3.3) to conclude

$$\theta_j^{n,k} = \theta_j^{n,k+1} \geq \Theta(\theta_j^{n,k+1} + q_j^{n,k+1}, z_j, \delta t) = \Theta(\theta_j^{n,k} + q_j^{n,k}, z_j, \delta t).$$

This finishes the proof. \square

Denote $\theta_j^n(\delta t) = \theta_j^{n,0}$, $q_j^n(\delta t) = q_j^{n,0}$. Then we have $q_j^n(\delta t) \leq Q^{sat}(\theta_j^n(\delta t), z_j, \delta t)$, for any $1 \leq j \leq n$, and $j \mapsto \theta_j^n(\delta t)$ is monotone increasing by Lemma 3.2. Define the flow map at first time step $\tilde{F}_{\delta t}^n : [0, 1) \rightarrow [0, 1)$ be such that it shifts J_i to $J_{\beta_0(i)}$ by translation, that is $\tilde{F}_{\delta t}^n(z) = z - z_i + z_{\beta_0(i)}$ for $z \in J_i$. Then $\tilde{F}_{\delta t}^n \# \mathcal{L}_{[0,1]}^1 = \mathcal{L}_{[0,1]}^1$. Apply the previous procedure to $\{\theta_j^n(\delta t)\}_{j=1}^n$, $\{q_j^n(\delta t)\}_{j=1}^n$, but with Q^{sat} evaluated at δt to get $\{\theta_j^n(2\delta t)\}_{j=1}^n$, $\{q_j^n(2\delta t)\}_{j=1}^n$, and the corresponding flow map $\tilde{F}_{2\delta t}^n : [0, 1) \rightarrow [0, 1)$. Repeating the procedure, we get a sequence of solutions at discrete times $\{\theta_j^n(k\delta t)\}_{j=1}^n$, $q_j^n(k\delta t)$, and a sequence of flow maps $\tilde{F}_{k\delta t}^n$ connecting $k\delta t$ and $(k+1)\delta t$. Denote $\theta_j^{M,n}(k\delta t) = \theta_j^n(k\delta t) + q_j^n(k\delta t)$. Here k is an integer with $0 \leq k \leq \frac{T}{\delta t} + 1$. Define $F_{k\delta t} = \tilde{F}_{k\delta t}^n \circ \dots \circ \tilde{F}_{\delta t}^n$. We will also denote $\alpha_{k\delta t}$, $\tilde{\alpha}_{k\delta t}$ be the corresponding rearrangement map on the discrete indices $\{1, 2, \dots, n\}$. Denote $\theta^n(t, z) = \theta_j^n(k\delta t)$, $q^n(t, z) = q_j^n(k\delta t)$ if $z \in J_j$ and $k\delta t \leq t < (k+1)\delta t$. Also $F^n(t, z) = F_{k\delta t}$ if $k\delta t \leq t < (k+1)\delta t$. Define $\theta_0^n(z) = \theta_j^n$, $q_0^n(z) = q_j^n$, and $\theta_0^{M,n}(z) = \theta_0^n(z) + q_0^n(z)$, for $z \in J_j$. We deduce a immediate corollary of Lemma 3.2.

Corollary 3.3. (i) $z \mapsto \theta^n(t, z)$ is monotone increasing for any $t \in [0, T)$.

(ii) Denote $\hat{\theta}^n(t, z) = \theta^n(t, F^n(t, z))$, $\hat{q}^n(t, z) = q^n(t, F^n(t, z))$, then we have $\hat{\theta}^n(t, z) + \hat{q}^n(t, z) = \theta_0^n(z) + q_0^n(z)$.

(iii) $t \mapsto \hat{\theta}^n(t, z)$ is monotone increasing for any $z \in [0, 1)$.

(iv) $q^n(t, z) \leq Q^{sat}(\theta^n(t, z), z, k\delta t)$, where k is the integer such that $k\delta t \leq t < (k+1)\delta t$.

Now we can have a more precise description of the motion of a single particle.

Lemma 3.4. Suppose that $n\delta t < \frac{\inf \partial_z \Theta}{\sup |\partial_t \Theta|}$, where both sup and inf are taken on the set

$\{(w, z, t) : |w| \leq \max_j |\theta_j^{M,n}|, z \in [0, 1], t \in [0, T]\}$, then one of the following must hold:

(i) There exists a unique $k_1 \in \{1, 2, \dots, n\}$, such that $\beta_{k_1-1}(j) = \sigma_{k_1}(\beta_{k_1}(j)) = k_1 > \beta_{k_1}(j)$. Besides, for any $k \geq k_1$, $\beta_k(j) = \beta_{k+1}(j)$, and any $k \leq k_1 - 1$, $\beta_k(j) = k_1$.

(ii) $\beta_k(j) \leq \beta_{k+1}(j)$ for any $0 \leq k \leq n - 1$.

This lemma says that if $n\delta t$ is small enough, then for any parcel experiencing a jump, the rearrangements before and after the jump will fix this parcel. In particular, if a parcel gets pushed down ($\beta_{k-1}(j) < \beta_k(j)$ for some k), then we must be in the second alternative, and by Lemma 3.1, it cannot overtake any other parcel.

Proof. The only difference between this lemma and Lemma 3.1 is that in the first alternative, we can now conclude $\beta_k(j) = \beta_{k+1}(j)$ for any $k \geq k_1$. Suppose we are in the first alternative of Lemma 3.1, and $\beta_k(j) < \beta_{k+1}(j)$ for some $k \geq k_1$, we will show that $\beta_k(j) \notin W_k$. Clearly we have $\beta_{k+1}(j) \leq k + 1$, since σ_{k+1} fix all index strictly bigger than $k + 1$. From Lemma 3.2, we know $\theta_{\beta_{k+1}(j)}^{n,k+1} \geq \Theta(\theta_j^{M,n}, z_{\beta_{k+1}(j)}, 0)$. Since

σ_{k+1} moves $\beta_{k+1}(j)$ down, it does not change the value of θ , hence $\theta_{\beta_{k+1}(j)}^{n,k+1} = \theta_{\beta_k(j)}^{n,k}$, and $\beta_{k+1}(j) \geq \beta_k(j) + 1$. It follows that:

$$\begin{aligned} \theta_{\beta_k(j)}^{n,k} &= \theta_{\beta_{k+1}(j)}^{n,k+1} \geq \Theta(\theta_j^{M,n}, z_{\beta_{k+1}(j)}, 0) \\ &\geq \Theta(\theta_j^{M,n}, z_{\beta_k(j)}, \delta t) + (\inf \partial_z \Theta) n^{-1} - \sup |\partial_t \Theta| \delta t > \Theta(\theta_j^{M,n}, z_{\beta_k(j)}, \delta t). \end{aligned}$$

The last step used the smallness of $n\delta t$. Hence $\beta_k(j) \notin W_k$. It follows that $\beta_{k-1}(j) = \sigma_k(\beta_k(j)) \leq \beta_k(j)$. Repeat the argument shown in the proof of "Moreover" part of Lemma 3.1, we see $\beta_k(j)$ will keep decreasing starting from k , and then no jump up is possible. \square

To conclude this section, we make a simple observation which will be useful in the next section.

Lemma 3.5. *Suppose for some pair of index $j_1, j_2 \in \{1, 2, \dots, n\}$, and for some k , we have $\alpha_{k\delta t}(j_1) < \alpha_{k\delta t}(j_2)$, and $\alpha_{(k+1)\delta t}(j_1) > \alpha_{(k+1)\delta t}(j_2)$, then $\theta_{j_1}^{M,n} > \theta_{j_2}^{M,n}$. In particular, $\alpha_{l\delta t}(j_1) > \alpha_{l\delta t}(j_2)$ for all $l > k$.*

In plain English, this lemma says if the index j_1 is initially below j_2 , then a necessary condition for j_1 to overtake j_2 is to have strictly larger θ^M . This means, in particular, that j_2 cannot overtake j_1 again since θ^M is invariant along parcels.

Proof. Here we write $\tilde{\alpha}_{(k+1)\delta t} = \sigma_1 \circ \dots \circ \sigma_n$ and $\beta_k = \sigma_{k+1} \circ \dots \circ \sigma_n$. Let m_0 be the maximal integer m for which $\beta_{m-1}(\alpha_{k\delta t}(j_1)) > \beta_{m-1}(\alpha_{k\delta t}(j_2))$. Then $\beta_{m_0}(\alpha_{k\delta t}(j_1)) < \beta_{m_0}(\alpha_{k\delta t}(j_2))$, and $\beta_{m_0}(\alpha_{k\delta t}(j_1)) \in W'_m$. If $\beta_{m_0}(\alpha_{k\delta t}(j_2)) \in W_m$, then we immediately have $\theta_{j_1}^{M,n} > \theta_{j_2}^{M,n}$. Otherwise,

$$\Theta(\theta_{j_1}^{M,n}, z_{\beta_{m_0}(\alpha_{k\delta t}(j_2))}, (k+1)\delta t) > \theta_{\beta_{m_0}(\alpha_{k\delta t}(j_2))}^{n,m_0}(k\delta t) \geq \Theta(\theta_{j_2}^{M,n}, z_{\beta_{m_0}(\alpha_{k\delta t}(j_2))}, (k+1)\delta t).$$

The first inequality used the definition of W'_m , the second used definition of W_m . \square

4. ESTIMATES ON THE DISCRETE SOLUTION

Next we do some estimates on the discrete solutions. Denote $M' = \|\theta_0^n\|_{L^\infty(0,1)} + \|q_0^n\|_{L^\infty(0,1)}$. In the following, we say a constant is universal, if it only depends on M', T , and Q^{sat} . We will derive the following estimates for the discrete solutions in this section. They are collected in the following theorem.

Theorem 4.1. (i) $\|\theta^n\|_{L^\infty((0,1) \times (0,T))} + \|q^n\|_{L^\infty((0,1) \times (0,T))} \leq C_1$, for some universal constant C_1 .

(ii) There exists a universal constant $C_2 > 0$, such that for any $\varepsilon > \frac{C_2}{n}$, and any $t \in [0, T)$, if $\hat{\theta}^n(t, z) > \Theta(\theta^{M,n}(t, z), F_t^n(z), t) + \varepsilon$, then we have $\hat{\theta}_{t'}(z) = \hat{\theta}_t(z)$, and $F_{t'}(z) \leq F_t(z)$ holds for any $t' - t < \frac{\varepsilon}{C_2}$.

(iii) For any $z \in [0, 1]$, $TV_{t \in [0, T)}(F_t^n(z)) \leq C_3$.

If $n\delta t < \frac{1}{C_4}$ for some universal constant $C_4' > 0$, then the following hold:

(iv) For any $\varepsilon > 0$, and any $[t - \varepsilon, t + \varepsilon] \subset [0, T)$, we have $|\hat{\theta}_{t+\varepsilon}^n(z) - \hat{\theta}_{t-\varepsilon}^n(z) - (\Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t + \varepsilon) - \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t - \varepsilon))| \leq 2C_4(\varepsilon + \delta t)$.

(v) For any $s < t \in [0, T)$, we have $\|\theta^n(t, \cdot) - \theta^n(s, \cdot)\|_{L^1([0,1])} = \|\hat{\theta}^n(t, \cdot) - \hat{\theta}^n(s, \cdot)\|_{L^1([0,1])} \leq C_5 \sqrt{t - s + \delta t}$.

(vi) For any $s < t \in [0, T]$, we have $\|F^n(t, \cdot) - F^n(s, \cdot)\|_{L^1([0,1])} \leq C_6 \sqrt{t - s + \delta t}$ for some universal constant C .

Throughout this section, we make the following conventions: when we write expressions like $\sup |\partial_t \Theta|$ and so on, they are assumed to be taken over the set $\{(w, z, t) \in \mathbb{R}^3 : |w| \leq M', z \in [0, 1], t \in [0, T]\}$ unless otherwise stated.

We start with point (i) of above theorem.

Lemma 4.1. *There exists a universal constant $C_1 > 0$, such that*

$$\|\theta^n\|_{L^\infty((0,1) \times (0,T))} + \|q^n\|_{L^\infty((0,1) \times (0,T))} \leq C_1.$$

Proof. We need to go back to the construction of the discrete solution. First we know from Corollary 3.3 point (ii) that for any $z \in [0, 1]$

$$\theta^n(t, z) + q^n(t, z) = \hat{\theta}^n(t, (F_t^n)^{-1}(z)) + \hat{q}^n(t, (F_t^n)^{-1}(z)) = \theta_0^n((F_t^n)^{-1}(z)) + q_0^n((F_t^n)^{-1}(z)).$$

Therefore $\|\theta^n + q^n\|_{L^\infty} \leq M'$.

On the other hand, from the construction of $\theta_l^n(k\delta t)$, we know either $\theta_l^n(k\delta t) = \theta_j^n((k-1)\delta t)$ for some $1 \leq j \leq n$, or $\theta_l^n(k\delta t) = \Theta(\theta_j^{M,n}, z_l, k\delta t)$. In the former case, we have $|\theta_l^n(k\delta t)| \leq \max_j |\theta_j^n((k-1)\delta t)|$. In the latter case, we have $|\theta_l^n(k\delta t)| \leq \sup |\Theta|$. Here sup is taken over the set $\{(w, z, t) : |w| \leq M', z \in [0, 1], t \in [0, T]\}$. But Θ is determined via (2.11) in terms of Q^{sat} , hence $\sup |\Theta|$ satisfies a universal bound. In any case, we have $\max_j |\theta_j^n(k\delta t)| \leq \max(\max_j |\theta_j^n((k-1)\delta t)|, \sup |\Theta|)$. It then follows easily by induction that $\|\theta^n\|_{L^\infty} \leq \max(M', \sup |\Theta|)$. The bound for q then follows automatically. \square

Next we prove the point (ii). Roughly speaking, point (ii) says if a parcel is "strictly" dry, then it will remain dry and go down for a while, the length of time this state lasts depends in a universal way how dry this parcel is.

Lemma 4.2. *There is a universal constant $\tilde{C}_2 > 0$, such that for any $\varepsilon > 0$, if for some interger k, j , it holds $\hat{\theta}_j^n(k\delta t) > \Theta(\theta_j^{M,n}, z_{\alpha_{k\delta t}(j)}, k\delta t) + \varepsilon$, we have $\hat{\theta}_j^n(l\delta t) = \hat{\theta}_j^n(k\delta t)$, and $\alpha_{l\delta t}(j) \leq \alpha_{k\delta t}(j)$ for any l with $0 \leq (l-k)\delta t \leq \frac{\varepsilon}{\tilde{C}_2}$.*

Proof. Actually we will see one can take $\tilde{C}_2 = 2 \sup |\partial_t \Theta|$. We prove this by induction on l . First observe that the statement is trivial if $l = k$. Assume this is true for some l with $(l+1-k)\delta t \leq \frac{\varepsilon}{2 \sup |\partial_t \Theta|}$. Need to show this is true also for $l+1$. Using the induction hypothesis, we can calculate

$$\begin{aligned} \hat{\theta}_j^n(l\delta t) &= \hat{\theta}_j^n(k\delta t) > \Theta(\theta_j^{M,n}, z_{\alpha_{k\delta t}(j)}, k\delta t) + \varepsilon \\ &\geq \Theta(\theta_j^{M,n}, z_{\alpha_{l\delta t}(j)}, (l+1)\delta t) + \varepsilon - \sup |\partial_t \Theta| (l+1-k)\delta t \\ &\geq \Theta(\theta_j^{M,n}, z_{\alpha_{l\delta t}(j)}, (l+1)\delta t). \end{aligned}$$

The first equality is the induction hypothesis. In the second inequality, we used the induction hypothesis that $\alpha_{l\delta t}(j) \leq \alpha_{k\delta t}(j)$.

Above calculation shows that at time $l\delta t$, the parcel $\alpha_{l\delta t}(j)$ is still "dry" by taking one more time step forward. Hence we know $\alpha_{(l+1)\delta t} = \tilde{\alpha}_{(l+1)\delta t}(\alpha_{l\delta t}(j)) \leq \alpha_{l\delta t}(j)$, and $\hat{\theta}_j^n((l+1)\delta t) = \hat{\theta}_j^n(l\delta t)$, from the procedure. \square

Now we can deduce point (ii) as a corollary of previous lemma.

Corollary 4.3. *Let $\varepsilon > 0$, $n\delta t \leq 1$. Then there exists a universal constant $C_2 > 0$, such that if $\varepsilon > \frac{C_2}{n}$ and $\hat{\theta}^n(t, z) > \Theta(\theta^{M,n}(z), F_t^n(z), t) + \varepsilon$ for some $t \in [0, T)$, we have $\hat{\theta}_t^n(z) = \hat{\theta}_t^n(z)$, and $F_{t'}^n(z) \leq F_t^n(z)$ holds for any $t' - t < \frac{\varepsilon}{C_2}$.*

Proof. First we can find integers k, j , such that $k\delta t \leq t < (k+1)\delta t$, and $z \in J_j$. It then follows from the definition of $\hat{\theta}^n$ that

$$\begin{aligned} \hat{\theta}^n(t, z) &= \hat{\theta}_j^n(k\delta t) > \Theta(\theta^{M,n}(t, z), F_t^n(z), t) + \varepsilon \\ &\geq \Theta(\theta_j^{M,n}, z_{\alpha_{k\delta t}(j)}, k\delta t) + \varepsilon - \sup |\partial_z \Theta| n^{-1} - \sup |\partial_t \Theta| \delta t. \end{aligned}$$

Since $\delta t \leq n^{-1}$, we will have $\varepsilon - \sup |\partial_z \Theta| n^{-1} - \sup |\partial_t \Theta| \delta t > \frac{\varepsilon}{2}$, if $\varepsilon > \frac{C_2'}{n}$ for some universal constant n . With such a choice of ε , we then have

$$\hat{\theta}_j^n(k\delta t) > \Theta(\theta_j^{M,n}, z_{\alpha_{k\delta t}(j)}, k\delta t) + \frac{\varepsilon}{2}.$$

Now we can conclude from Lemma 4.2 that $\hat{\theta}_j^n(l\delta t) = \hat{\theta}_j^n(k\delta t)$, and $\alpha_{l\delta t}(j) \leq \alpha_{k\delta t}(j)$ for any integer l with $0 \leq (l-k)\delta t \leq \frac{\varepsilon}{C_2'}$ for some universal constant C_2'' . This means precisely that $\hat{\theta}^n(t', z) = \hat{\theta}^n(t, z)$ and $F_{t'}^n(z) \leq F_t^n(z)$ for any $t' - t < \frac{\varepsilon}{C_2'}$. \square

Next we wish to prove point (iii). For this, we need to establish a lemma which gives control over the total variation of $t \mapsto F_t^n(z)$ in terms of the absolute bound of θ .

Lemma 4.4. *There exists a universal constant $C_3' > 0$, such that for any indices $j \in \{1, 2, \dots, n\}$,*

$$\begin{aligned} \sum_{0 \leq k \leq \frac{T}{\delta t}} \frac{1}{n} (\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+ &\leq C_3' \|\hat{\theta}^n\|_{L^\infty((0,T) \times (0,1))}, \\ \sum_{0 \leq k \leq \frac{T}{\delta t}} \frac{1}{n} |\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)| &\leq 2C_3' (\|\hat{\theta}^n\|_{L^\infty((0,T) \times (0,1))} + 2). \end{aligned}$$

Proof. First we observe that the second estimate follows from the first. Indeed, we just need to notice

$$\begin{aligned} &\sum_{0 \leq k \leq \frac{T}{\delta t}} \frac{1}{n} |\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)| \\ &= \sum_{0 \leq k \leq \frac{T}{\delta t}} \frac{2}{n} (\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+ - \sum_{0 \leq k \leq \frac{T}{\delta t}} \frac{1}{n} (\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)) \\ &\leq 2C_3' \|\hat{\theta}^n\|_{L^\infty((0,T) \times (0,1))} + 2. \end{aligned}$$

Now we only need to focus on the first estimate. Fix some k such that $\alpha_{(k+1)\delta t}(j) > \alpha_{k\delta t}(j)$. Then we know $\alpha_{(k+1)\delta t}(j) = \tilde{\alpha}_{(k+1)\delta t}(\alpha_{k\delta t}(j)) > \alpha_{k\delta t}(j)$. This means the parcel $\alpha_{k\delta t}(j)$ is "wet" at $k\delta t$, or $\hat{\theta}_j^n(k\delta t) < \Theta(\theta_j^{M,n}, z_{\alpha_{k\delta t}(j)}, (k+1)\delta t)$. After the time step, we know $\hat{\theta}_j^n((k+1)\delta t) = \Theta(\theta_j^{M,n}, z_{\alpha_{(k+1)\delta t}(j)}, (k+1)\delta t)$. Therefore

$$\begin{aligned} (4.1) \quad \hat{\theta}_j^n((k+1)\delta t) - \hat{\theta}_j^n(k\delta t) &\geq \Theta(\theta_j^{M,n}, z_{\alpha_{(k+1)\delta t}(j)}, (k+1)\delta t) - \Theta(\theta_j^{M,n}, z_{\alpha_{k\delta t}(j)}, (k+1)\delta t) \\ &\geq (\inf \partial_z \Theta)(z_{\alpha_{(k+1)\delta t}(j)} - z_{\alpha_{k\delta t}(j)})^+ = \frac{1}{n} (\inf \partial_z \Theta)(\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+. \end{aligned}$$

Now we sum (4.1) over k , the first estimate follows. \square

Then we can deduce point (iii) as a corollary

Lemma 4.5. *For any $z \in [0, 1)$*

$$PV_{t \in [0, T)}(F_t^n(z)) \leq C'_3 \|\hat{\theta}^n\|_{L^\infty((0, T) \times (0, 1))}.$$

$$TV_{t \in [0, T)}(F_t^n(z)) \leq 2C'_3 \|\hat{\theta}^n\|_{L^\infty((0, T) \times (0, 1))} + 2.$$

Here PV denotes the positive variation, and TV denotes the total variation. C'_3 is the same constant as in previous lemma.

Proof. Find indices $j \in \{1, 2, \dots, n\}$ such that $z \in J_j$, then

$$PV_{t \in [0, T)}(F_t^n(z)) = \sum_{0 \leq k \leq \frac{T}{\delta t}} (z_{\alpha_{(k+1)\delta t}(j)} - z_{\alpha_{k\delta t}(j)})^+ = \sum_{0 \leq k \leq \frac{T}{\delta t}} \frac{1}{n} (\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+$$

$$TV_{t \in [0, T)}(F_t^n(z)) = \sum_{0 \leq k \leq \frac{T}{\delta t}} |z_{\alpha_{(k+1)\delta t}(j)} - z_{\alpha_{k\delta t}(j)}| = \sum_{0 \leq k \leq \frac{T}{\delta t}} \frac{1}{n} |\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)|.$$

\square

It only remains to prove the point (iv) and (v). For this we need the following key lemma, which concludes that any given parcel can only be overtaken at a finite rate.

Lemma 4.6. *Fix $j_0 \in \{1, 2, \dots, n\}$. Let $0 \leq k < l \leq \frac{T}{\delta t}$. Define the set*

$$J_{k,l} = \{j \in \{1, 2, \dots, n\} : \alpha_{k\delta t}(j) < \alpha_{k\delta t}(j_0), \alpha_{l\delta t}(j) > \alpha_{l\delta t}(j_0)\}.$$

Then there exists a universal constant $C'_4 > 0$, such that if $n\delta t < \frac{1}{C'_4}$, we have

$$\#J \leq 2(l - k).$$

Proof. We will prove this statement with the choice of constant $C'_4 = \frac{\sup |\partial_t \Theta|}{\inf \partial_z \Theta}$. Here \sup, \inf is taken over the set $\{(w, z, t) : |w| \leq M, z \in [0, 1], t \in [0, T]\}$. With this choice, Lemma 3.4 applies. For $k \leq s \leq l - 1$, we may define

$$A_s = \{j \in \{1, 2, \dots, n\} : \alpha_{s'\delta t}(j) < \alpha_{s'\delta t}(j_0), \text{ for any } s' \text{ with } k \leq s' \leq s, \text{ and } \alpha_{(s+1)\delta t}(j) > \alpha_{(s+1)\delta t}(j_0)\}.$$

Then we have $J_{k,l} = \cup_{s=k}^{l-1} A_s$. That $J_{k,l} \subset \cup_{s=k}^{l-1} A_s$ is clear and the reverse inclusion follows from Lemma 3.5. Therefore it suffices to show $\#A_s \leq 2$ for each s , when $n\delta t$ is small. Here we use the notation of section 2 and write $\tilde{\alpha}_{(s+1)\delta t} = \sigma_1 \circ \dots \circ \sigma_n$. Here σ_k is the rearrangement map of the indices when we decide which parcel moves to z_k . Denote $\beta_k = \sigma_{k+1} \circ \dots \circ \sigma_n$. Without loss of generality, we may assume that between time step $s\delta t$ and $(s+1)\delta t$, the second alternative of Lemma 3.4 holds for $\alpha_{s\delta t}(j_0)$. Indeed, if the first alternative holds, we will have $A_s = \emptyset$, and there is nothing to prove in this case.

If $\#A_s \leq 2$ fails, then let $k_1 > k_2 > k_3$ be the 3 biggest integers for which there exists some index j such that $\alpha_{s\delta t}(j) < \alpha_{s\delta t}(j_0)$ and $\beta_{k-1}(\alpha_{s\delta t}(j)) > \beta_{k-1}(\alpha_{s\delta t}(j_0))$. Let j_1, j_2, j_3 be the index corresponding to k_1, k_2, k_3 . Namely $\alpha_{s\delta t}(j_i) < \alpha_{s\delta t}(j_0)$, but $\beta_{k_i-1}(\alpha_{s\delta t}(j_i)) > \beta_{k_i-1}(\alpha_{s\delta t}(j_0))$, $i = 1, 2, 3$. It is clear that $\beta_{k_i}(\alpha_{s\delta t}(j_i)) < \beta_{k_i}(\alpha_{s\delta t}(j_0))$, $1 \leq i \leq 3$. It is also clear that such index j_i must be unique since each σ_k lifts at most one index. From Lemma 3.4, we can see it must hold $\alpha_{s\delta t}(j_3) < \alpha_{s\delta t}(j_2) < \alpha_{s\delta t}(j_1) < \alpha_{s\delta t}(j_0)$. The last inequality is clear, since $\alpha_{s\delta t}(j_0)$

will not jump. If, say $\alpha_{s\delta t}(j_2) > \alpha_{s\delta t}(j_1)$, then $\alpha_{s\delta t}(j_1)$ overtakes $\alpha_{s\delta t}(j_2)$ since $\alpha_{s\delta t}(j_1)$ jumps first (under σ_{k_1}) and ends up above $\alpha_{s\delta t}(j_0)$. Thus $\alpha_{s\delta t}(j_2)$ will get pushed down by σ_{k_1} . But then according to Lemma 3.4, it cannot jump later on and cannot overtake any other parcel.

The plan is to show $\theta_{j_3}^{M,n} \leq \theta_{j_2}^{M,n} \leq \theta_{j_1}^{M,n}$, and also show this implies $\beta_{k_3}(\alpha_{s\delta t}(j_3)) \notin W'_{k_3}$. This will be a contradiction since we will have $\beta_{k_3}(\alpha_{s\delta t}(j_3))$ cannot jump under σ_{k_3} .

First we observe that for any k with $k_2 < k \leq k_1$, and any index j with $\beta_{k_1}(\alpha_{s\delta t}(j_1)) < \beta_{k_1}(\alpha_{s\delta t}(j)) \leq k_1$, it holds $\beta_{k-1}(\alpha_{s\delta t}(j)) = \beta_{k_1}(\alpha_{s\delta t}(j)) - 1$, while for index j with $\beta_{k_1}(\alpha_{s\delta t}(j)) < \beta_{k_1}(\alpha_{s\delta t}(j_1))$, it must hold $\beta_{k-1}(\alpha_{s\delta t}(j)) = \beta_{k_1}(\alpha_{s\delta t}(j))$. Observe it is clear with $k = k_1$. For $k < k_1$, and j satisfying $\beta_{k_1}(\alpha_{s\delta t}(j_1)) < \beta_{k_1}(\alpha_{s\delta t}(j)) \leq k_1$, they cannot jump under σ_k since they are already overtaken by $\alpha_{s\delta t}(j_1)$. For j satisfying $\beta_{k_1}(\alpha_{s\delta t}(j)) < \beta_{k_1}(\alpha_{s\delta t}(j_1))$, they cannot jump up because once they jump up under σ_k , they will jump to z_k , hence overtake $\alpha_{s\delta t}(j_0)$, contradicting our choice of k_2 . They also cannot be pushed down since if this happens, some parcel below needs to jump, again contradicting the choice of k_2 .

Now we wish to prove $\theta_{j_2}^{M,n} \leq \theta_{j_1}^{M,n}$. If not, we will show below that $\beta_{k_1}(\alpha_{s\delta t}(j_2)) \in W'_{k_1}$. This will give us a contradiction since the parcel $\beta_{k_1}(\alpha_{s\delta t}(j_1))$ does not have the largest θ^M among the parcels in the set W'_{k_1} , hence cannot jump up under σ_{k_1} . First it is clear from the "moreover" part of Lemma 3.1 that $\beta_{k_1}(\alpha_{s\delta t}(j_2)) \in W_{k_1}$. For any index j with $\beta_{k_1}(\alpha_{s\delta t}(j_2)) < \beta_{k_1}(\alpha_{s\delta t}(j)) < \beta_{k_1}(\alpha_{s\delta t}(j_1))$, and any k with $k_2 < k \leq k_1$, we conclude from last paragraph that $\sigma_k = id$, and $\theta_{\beta_{k-1}(\alpha_{s\delta t}(j))}^{n,k-1} = \theta_{\beta_{k_1}(\alpha_{s\delta t}(j))}^{n,k_1}$. Since $\beta_{k_2}(\alpha_{s\delta t}(j_2)) \in W'_{k_2}$, and no changes happen for these j 's under σ_k with $k_2 < k \leq k_1$, we see $\beta_{k_1}(\alpha_{s\delta t}(j_2))$ will beat them under rearrangement σ_{k_1} . Since we assumed $\theta_{j_2}^{M,n} > \theta_{j_1}^{M,n}$, we know $\alpha_{s\delta t}(j_2)$ beats $\alpha_{s\delta t}(j_1)$ as well. Now consider index j satisfying $\beta_{k_1}(\alpha_{s\delta t}(j_1)) < \beta_{k_1}(\alpha_{s\delta t}(j)) \leq k_1$, if $\beta_{k_1}(\alpha_{s\delta t}(j)) \in W_{k_1}$, it can be beaten by $\beta_{k_1}(\alpha_{s\delta t}(j_1))$, which means $\theta_j^{M,n} < \theta_{j_1}^{M,n}$. Since $\theta_{j_2}^{M,n} > \theta_{j_1}^{M,n}$, it can also be beaten by $\alpha_{s\delta t}(j_2)$. If $\beta_{k_1}(\alpha_{s\delta t}(j)) \notin W_{k_1}$, we know from Lemma 3.1 that $\beta_{k_2}(\alpha_{s\delta t}(j)) \notin W_{k_2}$. Hence from $\beta_{k_2}(\alpha_{s\delta t}(j_2)) \in W'_{k_2}$, we see $\theta_{\beta_{k_2}(\alpha_{s\delta t}(j))}^{n,k_2} < \Theta(\theta_{j_2}^{M,n}, z_{\beta_{k_2}(\alpha_{s\delta t}(j))}, (s+1)\delta t)$. Since they are "dry" parcels, we know their θ does not change, namely $\theta_{\beta_{k_2}(\alpha_{s\delta t}(j))}^{n,k_2} = \theta_{\beta_{k_1}(\alpha_{s\delta t}(j))}^{n,k_1}$, also we know from the observation made in previous paragraph with $k = k_2 + 1$ that $\beta_{k_2}(\alpha_{s\delta t}(j)) = \beta_{k_1}(\alpha_{s\delta t}(j)) - 1$. Hence they will be beaten by $\alpha_{s\delta t}(j_2)$ in the rearrangement σ_{k_1} . This shows $\beta_{k_2}(\alpha_{s\delta t}(j_2)) \in W'_{k_1}$.

By the same argument as above, one can also conclude $\theta_{j_3}^{M,n} \leq \theta_{j_2}^{M,n}$, following the same logic that if $\theta_{j_3}^{M,n} > \theta_{j_2}^{M,n}$, we will then conclude $\beta_{k_2}(\alpha_{s\delta t}(j_3)) \in W'_{k_2}$, thus $\alpha_{s\delta t}(j_2)$ will not jump under σ_{k_2} . So we have shown $\theta_{j_3}^{M,n} \leq \theta_{j_2}^{M,n} \leq \theta_{j_1}^{M,n}$.

Next we show $\beta_{k_3}(\alpha_{s\delta t}(j_3)) \notin W'_{k_3}$. Let j_4 be the index such that $\beta_{k_1}(\alpha_{s\delta t}(j_4)) = \beta_{k_1}(\alpha_{s\delta t}(j_1)) + 1$. Since it is overtaken by j_1 under σ_{k_1} it will remain "dry" for all later rearrangements, that is, $\beta_m(\alpha_{s\delta t}(j_4)) \notin W_m$, for any $1 \leq m < k_1$. If $\beta_{k_3}(\alpha_{s\delta t}(j_3)) \in W'_{k_3}$, in particular, one should have $\Theta(\theta_{j_3}^{M,n}, z_{\beta_{k_3}(\alpha_{s\delta t}(j_4))}, (s+1)\delta t) > \theta_{\beta_{k_3}(\alpha_{s\delta t}(j_4))}^{n,k_3} = \theta_{\beta_{k_1}(\alpha_{s\delta t}(j_4))}^{n,k_1}$. On the other hand, using Corollary 3.3, point (i), (iv), we see

$$(4.2) \quad \theta_{\beta_{k_1}(\alpha_{s\delta t}(j_4))}^{n,k_1} \geq \theta_{\beta_{k_1}(\alpha_{s\delta t}(j_1))}^{n,k_1} \geq \Theta(\theta_{j_1}^{M,n}, z_{\beta_{k_1}(\alpha_{s\delta t}(j_1))}, s\delta t).$$

Observe that $\beta_{k_3}(\alpha_{s\delta t}(j_4)) \leq \beta_{k_1}(\alpha_{s\delta t}(j_4)) - 2 = \beta_{k_1}(\alpha_{s\delta t}(j_1)) - 1$, since there are at least 2 parcels (j_1 and j_2) overtaking j_4 . Therefore

$$(4.3) \quad \begin{aligned} \theta_{\beta_{k_1}(\alpha_{s\delta t}(j_4))}^{n, k_1} &< \Theta(\theta_{j_3}^{M, n}, z_{\beta_{k_3}(\alpha_{s\delta t}(j_4))}, (s+1)\delta t) \leq \Theta(\theta_{j_3}^{M, n}, z_{\beta_{k_1}(\alpha_{s\delta t}(j_1)) - 1}, (s+1)\delta t) \\ &\leq \Theta(\theta_{j_1}^{M, n}, z_{\beta_{k_1}(\alpha_{s\delta t}(j_1))}, s\delta t) - (\inf \partial_z \Theta) n^{-1} + (\sup |\partial_t \Theta|) \delta t. \end{aligned}$$

In the second inequality above, we used above observation, and in the third inequality we used that $\theta_{j_3}^{M, n} \leq \theta_{j_2}^{M, n} \leq \theta_{j_1}^{M, n}$. Now combining (4.2) and (4.3) gives a contradiction. \square

From above discrete estimate, we can get the lemma of "finite speed of penetration".

Lemma 4.7. *Let $z_{i_0} \in [0, 1)$. Choose $n, \delta t$ such that $n\delta t = \frac{1}{2C'_4}$, where C'_4 is the constant given by previous lemma. Let $t_0 \in [0, T)$, $\varepsilon > 0$, define the set*

$$J = \{z \in [0, 1) : F_{t_0}(z) < F_{t_0}(z_{i_0}), F_{t_0+\varepsilon}(z) > F_{t_0+\varepsilon}(z_{i_0})\}.$$

Then $\mathcal{L}^1(J) \leq C(\varepsilon + \delta t)$, for some universal constant C .

Proof. Choose $j_0 \in \{1, 2, \dots, n\}$, such that $z_{i_0} \in J_{j_0}$. Choose integers k, l such that $k\delta t \leq t_0 < (k+1)\delta t$, $l\delta t \leq t_0 + \varepsilon < (l+1)\delta t$. Then we know that $J = \cup_{j \in J_{k,l}} J_j$. Here $J_{k,l}$ is defined as in previous lemma. Hence

$$\mathcal{L}^1(J) = n^{-1} \# J_{k,l} \leq \frac{2(l-k)}{n} = \frac{2(l-k)\delta t}{n\delta t} \leq 4C'_4(\varepsilon + \delta t). \quad \square$$

As an application of this lemma, we can prove point (iv) of Theorem 4.1.

Proposition 4.8. *Let $n, \delta t$ be chosen as in Lemma 4.7. There exists a universal constant $C > 0$, such that for any $\varepsilon > 0$, and $t \in [0, T)$, with $[t - \varepsilon, t + \varepsilon] \subset [0, T)$, we have*

$$(4.4) \quad |\hat{\theta}_{t+\varepsilon}^n(z) - \hat{\theta}_{t-\varepsilon}^n(z) - (\Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t-\varepsilon))|^+ \leq C(\varepsilon + \delta t).$$

Proof. Let κ be the quantity in the absolute value above. First we show $\kappa \leq C(\varepsilon + \delta t)$. Without loss of generality, we can then assume $\kappa > 0$. Then we know that $\hat{\theta}_{t+\varepsilon}^n(z) - \hat{\theta}_{t-\varepsilon}^n(z) \geq \kappa$. It follows from Corollary 4.3 that

$$(4.5) \quad \hat{\theta}_{t-\varepsilon}^n(z) \leq \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t-\varepsilon) + C\varepsilon.$$

On the other hand

$$(4.6) \quad \begin{aligned} \hat{\theta}_{t+\varepsilon}^n(z) - \hat{\theta}_{t-\varepsilon}^n(z) &\geq \Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t-\varepsilon) + \kappa \\ &\geq \Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t+\varepsilon) - \hat{\theta}_{t-\varepsilon}^n(z) - \sup |\partial_t \Theta| \delta t + \kappa. \end{aligned}$$

In the first inequality, we only used the definition of κ , and in the second inequality, we used Corollary 3.3, point (iv). If we let $t' \in [t - \varepsilon, t + \varepsilon]$ be such that $F_{t'}^n(z) = \max_{\tau \in [t-\varepsilon, t+\varepsilon]} F_{\tau}^n(z)$, we have $\hat{\theta}_{t+\varepsilon}^n(z) \leq \Theta(\theta^{M,n}(z), F_{t'}^n(z), t+\varepsilon) + \sup |\partial_t \Theta| \varepsilon$. Hence it follows from (4.6) that

$$(4.7) \quad \Theta(\theta^{M,n}(z), F_{t'}^n(z), t+\varepsilon) + \sup |\partial_t \Theta| \varepsilon \geq \Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t+\varepsilon) - \sup |\partial_t \Theta| \delta t + \kappa.$$

Noticing $\delta t \leq \varepsilon$, we obtain

$$F_{t'}^n(z) - F_{t+\varepsilon}^n(z) \geq \frac{1}{\inf \partial_z \Theta} (\kappa - 2 \sup |\partial_t \Theta| \varepsilon).$$

Now consider the set $E' = \{z' \in [0, 1] : F_{t'}^n(z') < F_{t'}^n(z), F_{t+\varepsilon}^n(z') > F_{t+\varepsilon}^n(z)\}$. Then we know $\mathcal{L}^1(E') \geq \frac{1}{\inf \partial_z \Theta} (\kappa - 2 \sup |\partial_t \Theta| \varepsilon)$. But it follows from Corollary 4.7 that $\mathcal{L}^1(E') \leq C(\varepsilon + \delta t)$. Hence $\kappa \leq C'(\varepsilon + \delta t)$, for some universal constant C' .

Next we derive a lower bound. We consider two cases. First if $\hat{\theta}_{t+\varepsilon}^n(z) > \hat{\theta}_{t-\varepsilon}^n(z)$, then as has been observed in (4.5), we know that

$$(4.8) \quad \hat{\theta}_{t+\varepsilon}^n(z) - \hat{\theta}_{t-\varepsilon}^n(z) \geq \Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t + \varepsilon) - \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t - \varepsilon) - C\varepsilon.$$

If $\Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t + \varepsilon) - \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t - \varepsilon) \geq 0$, then we can immediately conclude $\kappa \geq -C\varepsilon$. If it is negative, then we can calculate:

$$(4.9) \quad \begin{aligned} & \Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t + \varepsilon) - \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t - \varepsilon) \\ & \geq -\sup |\partial_z \Theta| (F_{t+\varepsilon}^n(z) - F_{t-\varepsilon}^n(z))^- - \sup |\partial_t \Theta| 2\varepsilon. \end{aligned}$$

Define $\tilde{E} = \{z' \in [0, 1] : F_{t-\varepsilon}^n(z') < F_{t-\varepsilon}^n(z), F_{t+\varepsilon}^n(z') > F_{t+\varepsilon}^n(z)\}$. Then Lemma 4.7 shows $\mathcal{L}^1(\tilde{E}) \leq C(\varepsilon + \delta t)$. But this means $F_{t+\varepsilon}^n(z) - F_{t-\varepsilon}^n(z) \geq -C(\varepsilon + \delta t)$. Thus from (4.8), (4.9), we know $\hat{\theta}_{t+\varepsilon}^n(z) - \hat{\theta}_{t-\varepsilon}^n(z) \geq -C(\varepsilon + \delta t)$. The conclusion follows as well.

If $\hat{\theta}_{t+\varepsilon}^n(z) = \hat{\theta}_{t-\varepsilon}^n(z)$, this means no jumps happen during $[t - \varepsilon, t + \varepsilon]$. Therefore $F_{t+\varepsilon}^n(z) \leq F_{t-\varepsilon}^n(z)$. In this case

$$\Theta(\theta^{M,n}(z), F_{t+\varepsilon}^n(z), t + \varepsilon) \leq \Theta(\theta^{M,n}(z), F_{t-\varepsilon}^n(z), t - \varepsilon) + C\varepsilon.$$

Therefore, we also have the quantity $\geq -C\varepsilon$ as well. \square

As a second application of Lemma 4.7, we finally prove point (v) of the Theorem. First we derive an obvious corollary of above lemma.

Fix some $t_0 \in [0, T)$, take $\varepsilon > \delta t$, $\kappa > 0$, and define

$$\begin{aligned} J_1 &= \{z \in [0, 1] : \sup_{t \in [t_0, t_0 + \varepsilon]} F_t^n(z) - F_{t_0}^n(z) \geq \kappa\}, \\ J_2 &= \{z \in [0, 1] : F_{t_0 + \varepsilon}^n(z) - F_{t_0}^n(z) \geq \frac{\kappa}{2}\}. \end{aligned}$$

First we observe the following lemma:

Lemma 4.9. *Let $n, \delta t$ be chosen as in Lemma 4.7. Then there exists a universal constant $C > 0$, such that if $\varepsilon + \delta t \leq \frac{\kappa}{C}$, then $J_1 \subset J_2$.*

Proof. Suppose there exists $z_0 \in J_1 - J_2$. Let $t' \in [t_0, t_0 + \varepsilon]$ be such that $F_{t'}^n(z_0) = \max_{t \in [t_0, t_0 + \varepsilon]} F_t^n(z_0)$. Then we know $F_{t'}^n(z_0) - F_{t_0}^n(z_0) \geq \kappa$. It follows that $F_{t'}^n(z_0) - F_{t_0 + \varepsilon}^n(z_0) \geq \frac{\kappa}{2}$. Consider the set

$$\hat{J}_{z_0, \varepsilon, \kappa} = \{z \in [0, 1] : F_{t'}^n(z) < F_{t'}^n(z_0); F_{t_0 + \varepsilon}^n(z) > F_{t_0 + \varepsilon}^n(z_0)\}.$$

Then we know that $\mathcal{L}^1(\hat{J}_{z_0, \varepsilon, \kappa}) \geq \frac{\kappa}{2}$. On the other hand, it follows from Lemma 4.7 that $\mathcal{L}^1(\hat{J}_{z_0, \varepsilon, \kappa}) \leq C(\varepsilon + \delta t)$ with a universal constant C . Hence we have $\frac{\kappa}{2} \leq C(\varepsilon + \delta t)$. \square

The next lemma estimates $\mathcal{L}^1(J_2)$.

Lemma 4.10. *Let $\varepsilon, n, \delta t$ be chosen as in previous lemma. Let $\kappa \geq C\varepsilon$, where C is the constant given by previous lemma, then for some universal constant C' , we have $\mathcal{L}^1(J_2) \leq \frac{C'\varepsilon}{\kappa}$.*

Proof. Let $\delta = \mathcal{L}^1(J_2)$. Since F^n measure preserving, we know

$$\int_{[0,1]} F_{t_0+\varepsilon}^n(z) - F_{t_0}^n(z) dz = 0.$$

On the other hand from the definition of J_2 ,

$$\int_{J_2} F_{t_0+\varepsilon}^n(z) - F_{t_0}^n(z) dz \geq \frac{\delta\kappa}{2}.$$

Therefore, there exists $z_1 \in [0, 1]$, such that $F_{t_0+\varepsilon}^n(z_1) - F_{t_0}^n(z_1) \leq -\frac{\delta\kappa}{2}$. Now consider

$$\hat{J}_{z_1, \varepsilon} = \{z \in [0, 1] : F_{t_0}^n(z) < F_{t_0}^n(z_1), F_{t_0+\varepsilon}^n(z) > F_{t_0+\varepsilon}^n(z_1)\}.$$

Then we have $\mathcal{L}^1(\hat{J}_{z_1, \varepsilon}) \geq \frac{\delta\kappa}{2}$. But by Lemma 4.7, we know $\mathcal{L}^1(\hat{J}_{z_1, \varepsilon}) \leq C(\varepsilon + \delta t) \leq 3C\varepsilon$. Hence $\frac{\delta\kappa}{2} \leq 3C\varepsilon$. This completes the proof. \square

With above preparation, we can obtain the following continuity estimate

Lemma 4.11. *Let $n, \delta t$ be chosen as in Lemma 4.7. Let $\varepsilon > \frac{\delta t}{2}$. Then there exists a universal constant C such that if $\varepsilon < \frac{1}{C}$, we have*

$$\|\hat{\theta}_{t_0+\varepsilon}^n - \hat{\theta}_{t_0}^n\|_{L^1(0,1)} = \|\theta_{t_0+\varepsilon}^n - \theta_{t_0}^n\|_{L^1(0,1)} \leq C'\sqrt{\varepsilon}.$$

for some universal constant C' .

Proof. From the discrete procedure, we know if $\hat{\theta}_{t_0+\varepsilon}^n(z) > \hat{\theta}_{t_0}^n(z)$,

$$\hat{\theta}_{t_0+\varepsilon}^n(z) \leq \max_{t' \in [t_0, t_0+\varepsilon]} \Theta(\theta_0^{M,n}(z), F_{t'}^n(z), t_0 + \varepsilon) + \sup |\partial_t \Theta| \varepsilon.$$

Since $\hat{\theta}_{t_0}^n(z) \geq \Theta(\theta_0^{M,n}(z), F_{t_0}^n(z), t_0) - \sup |\partial_t \Theta| \delta t$, by Corollary 3.3 point (iv), we know

$$\hat{\theta}_{t_0+\varepsilon}^n(z) - \hat{\theta}_{t_0}^n(z) \leq \sup \partial_z \Theta \cdot \max_{t \in [t_0, t_0+\varepsilon]} (F_t^n(z) - F_{t_0}^n(z)) + \sup |\partial_t \Theta| \cdot (\varepsilon + \delta t).$$

Let $\kappa \geq C\varepsilon$, where C is the universal constant given by Lemma 4.9. Combining Lemma 4.9, 4.10, we conclude $\mathcal{L}^1(J_1) \leq \mathcal{L}^1(J_2) \leq \frac{C'\varepsilon}{\kappa}$. Hence

$$\begin{aligned} \int_{[0,1]} \hat{\theta}_{t_0+\varepsilon}^n(z) - \hat{\theta}_{t_0}^n(z) dz &= \int_{J_1} \hat{\theta}_{t_0+\varepsilon}^n(z) - \hat{\theta}_{t_0}^n(z) dz + \int_{J_1^c} \hat{\theta}_{t_0+\varepsilon}^n(z) - \hat{\theta}_{t_0}^n(z) dz \\ &\leq \|\hat{\theta}\|_{L^\infty} \cdot \frac{C'\varepsilon}{\kappa} + \sup |\partial_z \Theta| \kappa + 3 \sup |\partial_t \Theta| \cdot \varepsilon. \end{aligned}$$

Now we take ε small enough such that $\sqrt{\varepsilon} \leq \frac{1}{C}$, where C is given by Lemma 4.9 and let $\kappa = \sqrt{\varepsilon}$, we obtain the continuity estimates:

$$\int_{[0,1]} |\theta_{t_0+\varepsilon}(z) - \theta_{t_0}(z)| dz = \int_{[0,1]} \hat{\theta}_{t_0+\varepsilon}^n(z) - \hat{\theta}_{t_0}^n(z) dz \leq C'\sqrt{\varepsilon}.$$

\square

Next we derive the continuity estimate for the flow maps, which follows from the continuity estimate for θ

Lemma 4.12. *Let $s, t \in [0, T]$, then for some universal constant C*

$$(4.10) \quad (F^n(t, z) - F^n(s, z))^+ \leq C(\hat{\theta}^n(t, z) - \hat{\theta}^n(s, z)).$$

Proof. As before, this follows from the discrete estimate. For any k, l integers with $0 \leq k < l \leq \frac{T}{\delta t}$, $(F_{l\delta t}^n(z) - F_{k\delta t}^n(z))^+ \leq C(\hat{\theta}_{l\delta t}^n(z) - \hat{\theta}_{k\delta t}^n(z))$. This follows from sum (4.1) from k to $l - 1$. \square

Corollary 4.13. *Let $s, t \in [0, T]$, then for the same constant as in previous lemma, we have*

$$\int_{[0,1]} |F^n(t, z) - F^n(s, z)| dz \leq 2C \int_{[0,1]} (\hat{\theta}^n(t, z) - \hat{\theta}^n(s, z)) dz.$$

Proof. Since F^n measure preserving, we know $\int_{[0,1]} (F^n(t, z) - F^n(s, z))^+ dz = \int_{[0,1]} (F^n(t, z) - F^n(s, z))^- dz$. Then the result follows by integrating (4.10) in z . \square

The point (vi) follows from previous corollary and the continuity estimate for θ .

5. DEFINITION OF MEASURE VALUED SOLUTION

In this section, we wish to define the measure valued solutions. As suggested in the discussion in the first section, we need to consider "path-spaces" which represents all the possible trajectories of an arbitrary parcel. Thanks to the point (iii) of 4.1, such paths take value in $[0, 1]$, and should have bounded variation, with uniform bound on BV .

Let $B_1 > 0$ and X_{B_1} be the set of functions $f : (0, T) \rightarrow [0, 1]$ which is left continuous and has total variation no bigger than B_1 , that is to say, for any partition of the interval $[0, T]$, denoted as $0 < t_0 < t_1 < t_2 < \dots < t_m < T$, we have

$$\sum_{i=1}^m |f(t_i) - f(t_{i-1})| \leq B_1.$$

Let d be the L^2 distance for functions in X_{B_1} , that is

$$d(f, g) = \left(\int_0^T |f(t) - g(t)|^2 dt \right)^{\frac{1}{2}}.$$

It is not hard to see that d is indeed a distance, since we required continuity from left. Also one can check (X_{B_1}, d) is a complete separable metric space, by Helly's selection principle. The physical meaning of such a space X_{B_1} is the space of all possible paths of the parcels. The reason why such paths have bounded variation is due to point (iii) of Theorem 4.1.

Let $B_2 > 0$ and Y_{B_2} be the space of monotone increasing function on $[0, 1]$, right continuous on $[0, 1)$, and with absolute bound $\leq B_2$, equipped with L^2 -distance. That is, given $h, k \in Y_{B_2}$, define their distance to be $d'(h, k) = \|h - k\|_{L^2(0,1)}$. The physical meaning of this space is all the possible profiles of potential temperature θ . We have incorporated the physical constraint that they must be monotone increasing.

Let $B_3 > 0$, put $Y = C([0, T]; Y_{B_3})$, that is, Y is the space of continuous maps from $[0, T]$ to Y_{B_3} . Here B_3 will be determined later on. We can define a metric on the space Y : given $h, k : [0, T] \rightarrow Y_{B_3}$, define $d(h, k) = \max_{t \in [0, T]} d'(h(t), k(t))$. The physical meaning of the space is the possible evolutions of the potential temperature profile.

To avoid confusion, we will denote a generic element from the space Y_B to be θ , while a generic element from the space Y will be denoted by $\tilde{\theta}$. It is easy to check both Y_B and Y are complete separable metric spaces, for any fixed $B > 0$.

First we specify the class of initial data we will be considering. Since we will be considering solutions in some "probabilistic" sense, our initial data will also be "probabilistic", namely some probability distributions. Let $B_2 > 0$ be an arbitrary positive constant.

Definition 5.1. Let $\zeta_0 \in \mathcal{P}(Y_{B_2} \times \mathbb{R} \times [0, 1])$. We say ζ_0 is an admissible data if the following holds:

- (i) $\pi_{13}\#\zeta_0 = \mu_0 \times \mathcal{L}_{[0,1]}^1$ for some $\mu_0 \in \mathcal{P}(Y_{B_2})$, and $\pi_2\#\zeta_0$ has compact support;
- (ii) $Q^{sat}(\theta(z), z, 0) \geq s - \theta(z)$ for ζ_0 - a.e. (θ, s, z) .

Remark 5.2. Heuristically, ζ_0 can be thought of prescribing the probability distribution of $\{\theta, \theta^M(z)\}_{z \in [0,1]}$, where $\theta^M = \theta + q$. Indeed, using that $\pi_{13}\#\zeta_0 = \mu_0 \times \mathcal{L}_{[0,1]}^1$ and also the disintegration theorem, we can write $\zeta_0 = \int_{Y_{B_2} \times [0,1]} d(\mu_0 \times \mathcal{L}_{[0,1]}^1)(\theta, z) \int_{\mathbb{R}} d\zeta_{\theta,z}(s)$. Here $(\theta, z) \mapsto d\zeta_{\theta,z}(s)$ is a Borel family of probability measures on \mathbb{R} . This describes the probability distribution of θ^M , hence q , given θ and z . The second point simply says the physical constraint is satisfied with probability 1.

We can define the following evaluation maps for the space X_{B_1} and Y . Fix any $t \in (0, T)$, define $e_t : X \rightarrow [0, 1]$ by $\gamma \mapsto \gamma(t)$. Similarly define $e'_t : Y \rightarrow Y_{B_3}$ by $\tilde{\theta} \mapsto \tilde{\theta}(t)$. We will frequently write $e_t(\gamma) = \gamma_t$ and $e'_t(\tilde{\theta}) = \tilde{\theta}_t$ to simplify the notation. We see from the definition of the space Y that e'_t is a continuous map. Also we can observe e_t is Borel, even if it is not continuous in general. Here we observe:

Lemma 5.3. The set $\{(\theta, s, z) \in Y_{B_2} \times \mathbb{R} \times [0, 1] : Q^{sat}(\theta(z), z, 0) \geq s - \theta(z)\}$ is a Borel subset of $Y_{B_2} \times \mathbb{R} \times [0, 1]$. Also the map e_t defined in previous paragraph is Borel map.

Proof. We just need to show the evaluation map $A : (\theta, z) \mapsto \theta(z)$ is Borel. Fix $\varepsilon > 0$, let $A_\varepsilon(\theta, z) = \varepsilon^{-1} \int_z^{z+\varepsilon} \theta(w)dw$. Here we extended $\theta(w) \equiv C$ for $w > 1$. Then for each fixed $\varepsilon > 0$, $A_\varepsilon(\theta, z)$ is continuous, and for any fixed (θ, z) , $A_\varepsilon(\theta, z) \rightarrow A(\theta, z)$, since θ is right continuous after extension. This proves A is Borel measurable.

That e_t is Borel is proved in a similar way. First we can define $f(t) = f(0)$ for $t \leq 0$. With this extension, $f(t)$ is define on $(-\infty, T)$ and is left continuous. Hence the map $f(t) \mapsto \varepsilon^{-1} \int_{t-\varepsilon}^t f(s)ds$ will converge to $f(t)$ as ε tends to 0. \square

A “deterministic” initial data takes the form $\zeta_0 = \delta_{\theta^0} \times ((\theta^0 + q^0) \times id)\#\mathcal{L}_{[0,1]}^1$. In this case, there is only one possible choice θ^0 , and for each fixed z , θ^M takes a deterministic value $\theta^0(z) + q^0(z)$.

Here we can make a definite choice of the constant B_1 and B_3 which are involved in defining the spaces X_{B_1} and Y . By point (i) in the Definition 5.1, we can assume $\pi_2\#\zeta_0 \subset [-M+1, M-1]$ for some $M > 0$. We will determine B_1 and B_3 such that they depend only on B_2, M, T and also the function Q^{sat} . Now take B_1 to be the constant C_3 given by point (iii) of Theorem 4.1 if we have the bound $\|\theta_0^n\|_{L^\infty} + \|q_0^n\|_{L^\infty} \leq M + 2B_2$. Let B_3 be the constant C_1 given by point (i) of Theorem 4.1. if we have the bound $\|\theta_0^n\|_{L^\infty} + \|q_0^n\|_{L^\infty} \leq M + 2B_2$. Without loss of generality we can assume $B_3 > B_2$ so that $Y_{B_2} \subset Y_{B_3}$.

With such a choice of constants in place, we propose the following definition of measure-valued solutions.

Definition 5.4. Let $\lambda \in \mathcal{P}(Y \times \mathbb{R} \times [0, 1] \times X_{B_1})$, and denote $\eta_t = (e'_t \times id \times id \times e_t)\#\lambda$, $\zeta_t = \pi_{124}\#\eta_t \in \mathcal{P}(Y_{B_3} \times \mathbb{R} \times [0, 1])$. Then we say λ is a measure valued solution to admissible initial data ζ_0 if the following are satisfied:

- (i) $\zeta_t \rightarrow \zeta_0$, $\pi_{34}\#\eta_t \in \Gamma(\mathcal{L}_{[0,1]}^1, \mathcal{L}_{[0,1]}^1) \rightarrow (id \times id)\#\mathcal{L}_{[0,1]}^1$ narrowly as $t \rightarrow 0$, and $t \mapsto \eta_t$ narrowly continuous.

(ii) For any $t \in (0, T)$, $\pi_{13}\#\zeta_t = \mu_t \times \mathcal{L}_{[0,1]}^1 \in \mathcal{P}(Y_{B_3} \times [0, 1])$, $\pi_2\#\zeta_t$ has compact support. Besides $Q^{sat}(\theta(z), z, t) \geq s - \theta(z)$ for ζ_t -a.e. (θ, s, z) .

(iii) For λ -a.e. $(\tilde{\theta}, s, z, \gamma)$, we have $\tilde{\theta}_t(\gamma_t) \leq \tilde{\theta}_{t'}(\gamma_{t'})$ for \mathcal{L}^2 -a.e. $(t, t') \in (0, T)^2$ and $t < t'$.

(iv) For λ -a.e. $(\tilde{\theta}, s, z, \gamma)$, we have the equality of measures:

$$\partial_t(\tilde{\theta}_t(\gamma_t)) = [\partial_t(Q^{sat}(\tilde{\theta}_t(\gamma_t), \gamma_t, t))]^- \lfloor E_{\tilde{\theta}, s, \gamma},$$

where $E_{\tilde{\theta}, s, \gamma}$ is the wet set given by $E_{\tilde{\theta}, s, \gamma} = \{t \in (0, T) : (\tilde{\theta}(\gamma))^*(t) = s - Q^{sat}((\tilde{\theta}(\gamma))^*(t), \gamma_t, t)\}$.

In the above, $(\tilde{\theta}(\gamma))^*$ is the monotone increasing, left continuous version of $\tilde{\theta}_t(\gamma_t)$ chosen according to Remark 2.4. This is possible due to point (iii) in the above definition. The notation $\Gamma(\mathcal{L}_{[0,1]}^1, \mathcal{L}_{[0,1]}^1)$ in point (i) denotes the set of Radon probability measures on $[0, 1]^2$ whose projections on both components are equal to $\mathcal{L}_{[0,1]}^1$.

Point (i) simply specifies in what sense the initial data is satisfied. ζ_t gives the probability distribution of $\{\theta, \theta^M(z)\}_{z \in [0,1]}$ at time t , and narrowly converge to ζ_0 as $t \rightarrow 0$. The second convergence simply means the ‘‘flow map’’ converges to identity as $t \rightarrow 0$. That $\pi_{34}\#\eta_t \in \Gamma(\mathcal{L}_{[0,1]}^1, \mathcal{L}_{[0,1]}^1)$ is a reformulation of the measure preserving property of the flow map, namely the incompressibility.

Point (ii) shows that ζ_t obtained satisfies the same conditions as required by the ‘‘admissibility’’ of the data. Therefore, one can take any ζ_t as initial data and evolves the solution forward.

Point (iii) shows that for all the possible choice of evolution of $\tilde{\theta}$ and the fluid path γ , $t \mapsto \tilde{\theta}_t(\gamma_t)$ is always monotone increasing in t (up to some set of Lebesgue measure 0).

Point (iv) shows that for possible choice of evolution of $\tilde{\theta}$ and the fluid path γ , the correct equation is satisfied.

Next we show if the random evolution of the solution happens to be deterministic, then Definition 2.2 and Definition 5.4 are consistent.

Lemma 5.5. *Let λ be a measure valued solution to admissible initial data ζ_0 . Assume $\zeta_0 = \delta_{\theta_0} \times ((\theta_0 + q_0) \times id) \#\mathcal{L}_{[0,1]}^1$. Assume also that $\lambda = \delta_{\theta(t)} \times ((\theta_0 + q_0) \times id \times \Phi_F) \#\mathcal{L}_{[0,1]}^1$, for some Borel map $F : [0, T] \times [0, 1] \rightarrow [0, 1]$. Let $\Phi_F(z)$ associates each z to its path $t \mapsto F_t(z)$. We also assume that there exists inverse map $F^* : [0, T] \times [0, 1] \rightarrow [0, 1]$ with $F_t^* \circ F_t = id$ and $F_t \circ F_t^* = id$ for $\mathcal{L}_{[0,1]}^1$ -a.e. z and any $t \in (0, T)$. Then $(\theta(t, z), \theta_0(F_t^*(z) + q_0(F_t^*(z))) - \theta(t, z), F)$ is a weak Lagrangian solution in the sense of Definition 2.2.*

Proof. From the definition of admissible data, we know that for \mathcal{L}^1 -a.e. $z \in [0, 1]$, it holds $Q^{sat}(\theta_0(z), z, 0) \geq q_0(z)$ for \mathcal{L}^1 -a.e. $z \in [0, 1]$. Since $\text{supp } \pi_2\#\zeta_0 = (\theta_0 + q_0)\#\mathcal{L}_{[0,1]}^1 \subset [-M, M]$, this means $\|\theta_0 + q_0\|_{L^\infty} \leq M$.

That $\theta_t \in L^\infty([0, T] \times [0, 1]) \cap C([0, T], L^1([0, 1]))$ follows from that $\theta_t \in C([0, T]; Y_{B_2})$. Let $q_t(z) = \theta_0(F_t^*(z) + q_0(F_t^*(z))) - \theta(t, z)$. From the boundedness of $\theta_0 + q_0$, we immediately get $q_t \in L^\infty([0, T] \times [0, 1])$.

That $F_t(\cdot) \in C([0, T]; L^1([0, 1]))$ follows from that $\pi_{34}\#\eta_t = (id \times F_t)\#\mathcal{L}_{[0,1]}^1$ is narrowly continuous. From $\pi_{34}\#\eta_t \in \Gamma(\mathcal{L}_{[0,1]}^1, \mathcal{L}_{[0,1]}^1)$, we see $F_t\#\mathcal{L}_{[0,1]}^1 = \mathcal{L}_{[0,1]}^1$, namely F_t is measure preserving. Combined with the assumption $F_t^* \circ F_t = id$ shows that $(id \times F_t)\#\mathcal{L}_{[0,1]}^1 = (F_t^* \times id)\#\mathcal{L}_{[0,1]}^1$, hence F_t^* is also measure preserving. As before,

the narrow continuity of $t \mapsto \pi_{34} \# \eta_t$ implies $F_t^*(\cdot) \in C([0, T]; L^1([0, 1]))$. This in turn implies $q_t \in C([0, T]; L^1([0, 1]))$.

From the definition of the space X_{B_1} , we see $F_t(z) \in L^\infty([0, 1]; BV(0, T))$, with total variation $\leq C$. Next we will check through the points (i)-(vi) in the Definition 2.2.

To see point (i), we observe that the measure $\pi_1 \# \zeta_t \rightarrow \pi_1 \# \zeta_0$ narrowly. In other words, we have $\delta_{\theta_t} \rightarrow \delta_{\theta_0}$ narrowly in $\mathcal{P}(X_0)$. This implies $\theta_t \rightarrow \theta_0$ in $L^2([0, 1])$. That $F_t \rightarrow id$ follows from that $\pi_{34} \# \eta_t = (id \times F_t) \# \mathcal{L}_{[0,1]}^1 \rightarrow (id \times id) \# \mathcal{L}_{[0,1]}^1$.

Point (ii) immediately follows from the definition of the space Y and Y_{B_2} .

Point (iii) follows from the assumption of F made in this lemma.

To see point (iv), Recall $q_t(z) = \theta_0(F_t^*(z)) + q_0(F_t^*(z)) - \theta(t, z)$. Hence for any $t \in (0, T)$ $\hat{q}_t(z) = q_t(F_t(z)) = \theta_0(F_t^* \circ F_t(z)) + q_0(F_t^* \circ F_t(z)) - \theta(t, F_t(z)) = \theta_0(z) + q_0(z) - \hat{\theta}_t(z)$, for \mathcal{L}^1 -a.e z .

Point (v) of Definition 2.2 follows from point (iii) of Definition 5.4. Indeed, from point (iii), we know that for $\pi_{14} \# \lambda - a.e (\tilde{\theta}, \gamma)$, it holds $\tilde{\theta}_t(\gamma_t) \leq \tilde{\theta}_{t'}(\gamma_{t'})$ for \mathcal{L}^2 -a.e $(t, t') \in (0, T)^2$ with $t < t'$. But $\pi_{14} \# \lambda = \delta_{\theta(t)} \times \Phi_F \# \mathcal{L}_{[0,1]}^1$. Hence for \mathcal{L}^1 -a.e z , it holds $\theta_t(F_t(z)) \leq \theta_{t'}(F_{t'}(z))$ for \mathcal{L}^2 -a.e $(t, t') \in (0, T)^2$ with $t < t'$.

To see the last point, we know from our assumption on λ that for $\lambda - a.e (\tilde{\theta}, s, z, \gamma)$, it holds $s = \theta_0(z) + q_0(z)$, and $\gamma_t = F_t(z)$. For $(\tilde{\theta}, s, z, \gamma)$ such that this holds, we know

$$\begin{aligned} E_{\theta, s, \gamma} &= E_{\theta, \theta_0(z) + q_0(z), F_t(z)} = \{t \in (0, T) : (\theta(F))_t^*(z) \\ &= \theta_0(z) + q_0(z) - Q^{sat}((\theta(F))_t^*(z), F_t(z), t)\}. \end{aligned}$$

We have seen in the above proof that for $\mathcal{L}^1 - a.e z$, we have $\hat{q}_t(z) = \theta_0(z) + q_0(z) - \theta_t(F_t(z))$. Hence if we choose the monotone and left continuous representative, we have $\hat{q}_t^*(z) = \theta_0(z) + q_0(z) - (\theta(F))_t^*(z)$. Hence for $\lambda - a.e (\tilde{\theta}, s, z, \gamma)$, $E_{\theta, s, \gamma} = E_z$, where E_z is given in point (vi) of Definition 2.2. Finally the measure theoretic equation holds $\lambda - a.e$. Since $\pi_2 \# \lambda = \mathcal{L}_{[0,1]}^1$, we see the point (vi) of Definition 2.2 holds. \square

The main existence theorem we will prove will be the following:

Theorem 5.1. *Let ζ_0 be an admissible initial data, then there exists a measure valued solution to (2.1)-(2.4) with initial data ζ_0 .*

6. EXISTENCE OF MEASURE VALUED SOLUTIONS

In this section, we will show the existence of measure valued solutions to any admissible data defined in previous section.

The plan is the following: Let ζ_0 be an admissible initial data, then we approximate ζ_0 by convex combinations of discrete and "deterministic" initial data. For each measure appearing in the convex combination, we can run the discrete procedure described in section 3. The question is to show one can take the limit of this approximation and the properties given in Definition 5.4 hold in the limit.

6.1. Discretizing the initial data. We assume $\pi_2 \# \zeta_0 \subset [-M + 1, M - 1]$ for some $M > 0$. Write $K = M - 1$. As a preliminary step, we note the following:

Lemma 6.1. *Let ζ_0 be an admissible data. Then there exists a Borel family of probability measures $\{\alpha_\theta\}_{\theta \in Y_{B_2}} \subset \mathcal{P}(\mathbb{R} \times [0, 1])$, such that for $\mu_0 - a.e \theta$, $\text{supp } \pi_1 \# \alpha_\theta \subset [-K, K]$,*

$\pi_2\#\alpha_\theta = \mathcal{L}_{[0,1]}^1$, $\theta(z) \geq \Theta(s, z, 0)$ for $\alpha_\theta - a.e (s, z)$ and for any bounded Borel function $f(\theta, s, z) : Y_{B_2} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$, it holds:

$$(6.1) \quad \int_{Y_{B_2} \times \mathbb{R} \times [0,1]} f(\theta, s, z) d\zeta_0(\theta, s, z) = \int_{Y_{B_2}} d\mu_0(\theta) \left[\int_{\mathbb{R} \times [0,1]} f(\theta, s, z) d\alpha_\theta(s, z) \right].$$

Proof. The existence of this family of probability measures satisfying the integral identity (6.1) is the standard disintegration theorem, see Theorem 5.3.1 of [1]. We just need to check that $\text{supp } \pi_1\#\alpha_\theta \subset [-K, K]$, $\pi_2\#\alpha_\theta = \mathcal{L}_{[0,1]}^1$, $\theta(z) \geq \Theta(s, z, 0)$ for $\alpha_\theta - a.e (s, z)$.

To see $\text{supp } \pi_1\#\alpha_\theta \subset [-K, K]$, take $f(\theta, s, z) = f_1(\theta)\chi_{[-K, K]^c}(s)$, and $f_1 : Y_{B_2} \rightarrow \mathbb{R}$ bounded and Borel measurable. Then we see

$$\left| \int f_1(\theta)\chi_{[-K, K]^c}(s) d\zeta_0(\theta, s, z) \right| \leq \sup |f| \int \chi_{[-K, K]^c}(s) d\zeta_0(\theta, s, z) = 0.$$

From (6.1), we know for any choice of bounded Borel function f_1 , one has

$$\int f_1(\theta) d\mu_0(\theta) \left[\int \chi_{[-K, K]^c} d\alpha_\theta(s, z) \right] = 0.$$

This implies $\int \chi_{[-K, K]^c}(s) d\alpha_\theta(s, z) = 0$ for $\mu_0 - a.e \theta$. For such θ , we have $\text{supp } \pi_1\#\alpha_\theta \subset [-K, K]$.

To see $\pi_2\#\alpha_\theta = \mathcal{L}_{[0,1]}^1$, one can similarly take $f(\theta, s, z) = f_1(\theta)f_2(z)$, using that $\pi_{13}\#\zeta_0 = \mu_0 \times \mathcal{L}_{[0,1]}^1$, one concludes for any choice of $f_2(z)$ bounded and Borel on $[0, 1]$ it holds $\int_{\mathbb{R} \times [0,1]} f_2(z) d\alpha_\theta(s, z) = \int f_2(z) dz$ for $\mu_0 - a.e \theta$. One just needs to choose a countable dense subset $\{f_2^n\}_{n \geq 1}$ of $C([0, 1])$, apply this argument with $f_2 = f_2^n$ and concludes for $\mu_0 - a.e \theta$, $\pi_2\#\alpha_\theta = \mathcal{L}_{[0,1]}^1$.

To see that $\theta(z) \geq \Theta(s, z, 0)$ for $\alpha_\theta - a.e (s, z)$, we integrate $\chi_{\{\theta(s, z): \theta(z) \geq \Theta(s, z, 0)\}}$, and use (6.1). By Lemma 5.3, such a function is bounded and Borel, hence its integral is well defined. \square

Due to Lemma 6.1, we can write $\zeta_0 = \int_{Y_{B_2}} \delta_\theta \times \alpha_\theta(s, z) d\mu_0(\theta)$, and for $\mu_0 - a.e \theta$, $\alpha_\theta(s, z)$ satisfy the "correct" condition mentioned in Lemma 6.1 Next we will construct discretization of α_θ for each fixed such θ .

Recall we have shown $\text{supp } \alpha_\theta \subset [-K, K] \times [0, 1]$, Define $K_j = [-K + \frac{(j-1)2K}{n}, -K + \frac{j \cdot 2K}{n}]$ for $1 \leq j \leq n$, call $w_j = -K + \frac{j \cdot 2K}{n}$. Suppose $\alpha_\theta(K_j \times J_i) > 0$, since we assumed $\theta(z) \geq \Theta(s, z, 0)$ for $\alpha_\theta - a.e (s, z)$, we have, for some $(\tilde{\alpha}_{ij}, z_{ij}) \in K_j \times J_i$, we have $\theta(z_{ij}) \geq \Theta(\tilde{\alpha}_{ij}, z_{ij}, 0)$. Therefore, for some universal constant $C > 2K + \frac{\sup |\partial_z \Theta|}{\inf \partial_w \Theta}$, if we define $\alpha_{ij} = w_j - \frac{C}{n}$, we have

$$\theta(z_i) \geq \theta(z_{ij}) \geq \Theta(\tilde{\alpha}_{ij}, z_{ij}, 0) \geq \Theta(\tilde{\alpha}_{ij}, z_i, 0) - \frac{\sup |\partial_z \Theta|}{n} \geq \Theta(w_j - \frac{C}{n}, z_i, 0).$$

There is no loss of generality to assume n is chosen sufficiently large so that $\frac{C}{n} < 1$. Now we can define a measure α_θ^n which is an approximation to α_θ , by putting:

$$\alpha_\theta^n = \sum_{i=1}^n \sum_{j \in H_{\theta,i}} \chi_{J_i}(z) dz \cdot n \alpha_\theta(K_j \times J_i) \delta_{\alpha_{ij}}(s).$$

Here we denote

$$(6.2) \quad H_{\theta,i} = \{j : 1 \leq j \leq n, \alpha_\theta(K_j \times J_i) > 0\}.$$

Lemma 6.2. $\alpha_\theta^n \rightarrow \alpha_\theta$ narrowly in $\mathcal{P}(\mathbb{R} \times [0, 1])$.

Proof. Let $f \in C_b(\mathbb{R} \times [0, 1])$ with Lipschitz constant 1, and we denote $H_i = H_{\theta,i}$ for simplicity.

$$\begin{aligned} \int f(s, z) d\alpha_\theta(s, z) &= \sum_{i=1}^n \sum_{J \in H_i} \int_{K_j \times J_i} f(z, s) d\alpha_\theta(s, z) \\ &= \sum_{i=1}^n \sum_{j \in H_i} \int_{K_j \times J_i} (f(z, s) - f(z_i, w_j - \frac{C}{n})) d\alpha_\theta(s, z) + \sum_{i=1}^n \sum_{j \in H_i} f(z_i, w_j - \frac{C}{n}) \alpha_\theta(K_j \times J_i) \\ &= \sum_{i=1}^n \sum_{j \in H_i} \int_{J_i} f(z, w_j - \frac{C}{n}) dz \cdot n \alpha_\theta(J_i \times K_j) + \sum_{i=1}^n \sum_{j \in H_i} \int_{J_i \times K_j} (f(z, s) - f(z_i, w_j - \frac{C}{n})) d\alpha_\theta(z, s) \\ &\quad + \sum_{i=1}^n \sum_{j \in H_i} \int_{J_i} f(z_i, w_j - \frac{C}{n}) - f(z, w_j - \frac{C}{n}) dz \cdot n \alpha_\theta(J_i \times K_j). \end{aligned}$$

The first term above is exactly the integral of f with respect to α_θ^n . The last two terms will go to zero, because in each term of the sum, the integrand is controlled by $\frac{C}{n}$. \square

For any $B > 0$ and each $\theta \in Y_B$, we can define $D^n : Y_B \rightarrow Y_B$ by putting $D^n(\theta) = \sum_{i=1}^n \theta(z_i) \chi_{J_i}$. Then for each fixed θ , it holds $D^n(\theta) \rightarrow \theta$ in Y_B . Now with choice $B = B_2$. By putting $\zeta_0^n = \int_{Y_{B_2}} \delta_{D^n(\theta)} \times \alpha_\theta^n(s, z) d\mu_0(\theta)$, we then have $\zeta_0^n \rightarrow \zeta_0$ narrowly in $\mathcal{P}(Y_{B_2} \times \mathbb{R} \times [0, 1])$ as $n \rightarrow \infty$. Besides, ζ_0^n satisfies the following properties:

- (i) $\pi_{13} \# \zeta_0^n = (D^n \# \mu_0) \times \mathcal{L}_{[0,1]}^1$, $\text{supp } \pi_{23} \# \zeta_0^n \subset [-K-1, K+1] = [-M, M]$;
- (ii) $\theta(z_i) \geq \Theta(w_j - \frac{C}{n}, z_i, 0)$, whenever $j \in H_{\theta,i}$, or $\theta(z) \geq \Theta(s, z, 0)$ for ζ_0^n - a.e. (θ, s, z) .

For simplicity of notation, we will write

(6.3)

$$\alpha_\theta(s, z) = \sum_{i=1}^n \sum_{j \in H_{\theta,i}} n \chi_{J_i}(z) \cdot \mu_{ij}^\theta \delta_{\alpha_{ij}}(s), \text{ with } \mu_{ij}^\theta = \alpha_\theta(K_j \times J_i), \text{ and } \alpha_{ij} = w_j - \frac{C}{n}.$$

6.2. Construction of approximate solutions and passage to limit. The measures ζ_0^n determines a sequence of discrete probability distributions. For each choice of θ , α_θ prescribes the probability distribution of $\theta^M(z)$ for each fixed z . More precisely, when $z \in J_i$, the possible values of θ^M are given by α_{ij} , with probability μ_{ij} .

In order to apply the discrete procedure, we will make a random choice of α_{ij} on each J_i , and this gives us a "deterministic" and discrete initial θ^M . Then we run the discrete procedure, with this θ^M as initial data and it gives us a evolution, with probability determined by the choice of α_{ij} .

First we make a random choice of the α_{ij} , allowed by the physical constraint. Denote S_θ to be the set of functions $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$, such that for each i , $\sigma(i) \in H_{\theta,i}$. Determine a discrete initial $\theta_{0,\sigma}^{M,n}$ from σ by prescribing:

$$\theta_{0,\sigma}^{M,n}(z) = \sum_{i=1}^n \alpha_{i\sigma(i)} \chi_{J_i}(z), \text{ or } \theta_{0,\sigma,i}^{M,n} = \alpha_{i\sigma(i)}.$$

The probability of such a choice is given by $n^n \mu_{1\sigma(1)} \mu_{2\sigma(2)} \cdots \mu_{n\sigma(n)}$. Notice here that $\sum_j \mu_{ij} = \frac{1}{n}$, hence $\sum_\sigma \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta = n^{-n}$. It is straightforward to check that α_θ

given by (6.3) is equal to

$$(6.4) \quad \alpha_\theta = \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta (\theta_{0,\sigma}^{M,n} \times id) \# \mathcal{L}_{[0,1]}^1.$$

After we make such a choice, we apply the discrete procedure described in section 3 to $\{\theta(z_i)\}_{i=1}^n$, and $\{\theta_{\sigma,i}^{M,n} - \theta(z_i)\}_{i=1}^n$, with time step size $\delta t = \frac{1}{nC'_4}$, here C'_4 is given by Theorem 4.1 in order for all the estimates in that theorem to hold. Notice that $\theta(z_i) \leq \theta(z_{i+1})$, and $\theta(z_i) \geq \Theta(\theta_{\sigma,i}^{M,n}, z_i, 0)$, hence it satisfies the assumptions made in the beginning of section 2. Denote $\{\theta_{\sigma,j}^n(k\delta t)\}_{1 \leq j \leq n, 0 \leq k \leq \frac{T}{\delta t}}$, $\{\theta_{\sigma,j}^{M,n}(k\delta t)\}_{1 \leq j \leq n, 0 \leq k \leq \frac{T}{\delta t}}$ to be the discrete solutions constructed according to section 2 and we adopt similar notations as there, but here with dependence on σ . Define

$$(6.5) \quad \theta_0^n(z) = \sum_i \theta(z_i) \chi_{J_i}.$$

$$(6.6) \quad \theta_\sigma^n(t, z) = \theta_{\sigma,j}^n(k\delta t),$$

$$(6.7) \quad \theta_\sigma^{M,n}(t, z) = \theta_{\sigma,j}^{M,n}(k\delta t),$$

$$(6.8) \quad \bar{\theta}_\sigma^n(t, z) = \frac{(k+1)\delta t - t}{\delta t} \theta_{\sigma,j}^n(k\delta t) + \frac{t - k\delta t}{\delta t} \theta_{\sigma,j}^n((k+1)\delta t),$$

for $k\delta t \leq t < (k+1)\delta t$ and $z \in J_j$.

Denote $F_\sigma^n(t, z)$ be the discrete flow map constructed in section 3. Since we know $\theta \in Y_{B_2}$, we see $\|\theta_0^n\|_{L^\infty} \leq B_2$, and $\|\theta_{0,\sigma}^{M,n}\|_{L^\infty} \leq M$. Hence $\|\theta_0^n\|_{L^\infty} + \|q_0^n\|_{L^\infty} \leq M + 2B_2$. We know from point (iii) of Theorem 4.1, as well as the choice of the constant B_1 made in the paragraph before Definition 5.4, that $F_\sigma(\cdot, z) \in X_{B_1}$, namely $TV_{t \in (0, T)}(F_\sigma^n(t, z)) \leq B_1$. Similarly we have $\|\theta_\sigma^n\|_{L^\infty} \leq B_3$, according to the point (i) of Theorem 4.1, as well as the choice of B_3 made before Definition 5.4. In particular, for each fixed $t \in [0, T]$, it holds $\bar{\theta}_\sigma^n(t, \cdot) \in Y_{B_3}$. Hence $\bar{\theta}_\sigma^n \in C([0, T]; Y_{B_3}) = Y$.

Let $\Phi_\sigma^n : [0, 1] \rightarrow X_{B_1}$ be defined by $\Phi_\sigma^n(\cdot, z) = F_\sigma^n(\cdot, z)$. From the discussion in the previous paragraph, we know Φ_σ^n indeed maps $[0, 1]$ into the space X_{B_1} and is easily seen to be a Borel map.

Now we form the probability measure:

$$(6.9) \quad \lambda^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta (\delta_{\bar{\theta}_\sigma^n} \times (\theta_{0,\sigma}^{M,n} \times id \times \Phi_\sigma^n) \# \mathcal{L}_{[0,1]}^1).$$

Then we see $\lambda^n \in \mathcal{P}(Y \times \mathbb{R} \times [0, 1] \times X_{B_1})$, by our previous construction, and this will be our approximate solutions. Following the notations in Definition 5.4, define $\eta_t^n = (e_t \times id \times id \times e_t') \# \lambda^n$, we have

$$(6.10) \quad \eta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta (\delta_{\bar{\theta}_\sigma^n(t)} \times (\theta_{0,\sigma}^{M,n} \times id \times F_{\sigma,t}^n) \# \mathcal{L}_{[0,1]}^1).$$

In order to take limit, we need to show that λ^n is tight. For this, we just need to show $\pi_k \# \lambda^n$ is tight, for $k = 1, 2, 3, 4$, where π_k is the projection map onto the k -th component. Indeed, we have

$$\pi_1 \# \lambda^n = \int_{Y_C} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \delta_{\theta_\sigma^n}.$$

Because of point (v) in Theorem 4.1, we see that for any $\sigma \in S_\theta$, and $s, t \in [0, T]$, with $t - s > \frac{\delta t}{2}$ we have

$$\|\theta_\sigma^n(s, \cdot) - \theta_\sigma^n(t, \cdot)\|_{L^1} \leq C_5 \sqrt{3(t-s)}.$$

The constant C_5 is given by point (v) of Theorem 4.1 when we have the bound $\|\theta_0^n\|_{L^\infty} + \|q_0^n\|_{L^\infty} \leq M + 2B_2$. Recalling (6.8), it follows that $\bar{\theta}_\sigma^n$ satisfies the same estimate, but without the restriction $t - s > \frac{\delta t}{2}$. Therefore, $\pi_1 \# \lambda^n$ is concentrated on the compact set:

$$F = \{f \in Y : \|f(t, \cdot) - f(s, \cdot)\|_{L^2} \leq C_5 \sqrt{3(t-s)} \text{ for any } s < t\}.$$

That this is a compact set follows from the Arzela-Ascoli theorem and the compactness of the space Y_{B_3} . Note that the functions in Y has bound in L^∞ , so convergence in L^p are equivalent for any $p < \infty$.

As for $\pi_2 \# \lambda^n$, since for all $\sigma \in S_\theta$, we know from the construction of discrete solutions that $\|\theta_{0,\sigma}^{M,n}\|_{L^\infty} \leq M$. Hence $\text{supp } \pi_2 \# \lambda^n \subset [-M, M]$, hence is tight.

There is nothing to prove for $\pi_3 \# \lambda^n$ and $\pi_4 \# \lambda^n$, as $[0, 1]$ and X_{B_1} are compact spaces.

So up to extracting a subsequence, we can pass to limit for the measure defined in (6.9), and we denote a limit measure to be $\lambda \in \mathcal{P}(Y \times \mathbb{R} \times [0, 1] \times X_{B_1})$. It only remains to show λ is a solution. Finally we record a lemma which will be useful in the next subsection.

Lemma 6.3. *Define $\eta_t^n = (e_t' \times id \times id \times e_t) \# \lambda^n$, then $\eta_t^n \rightarrow \eta_t$ for any $t \in (0, T)$ as $n \rightarrow \infty$.*

Proof. First we remind the reader that this is not completely obvious as the evaluation map e_t in the last component is not continuous. Instead, we will use the continuity estimates in point (vi) of Theorem 4.1.

For $\gamma \in X_{B_1}$, extend $\gamma(t) = \gamma(0+)$ for $t \leq 0$, and define the operator: $A'_\varepsilon : X_{B_1} \rightarrow X_{B_1}$, given by $\gamma \mapsto \varepsilon^{-1} \int_{t-\varepsilon}^t \gamma(s) ds$. Then it is straightforward to check A'_ε is a continuous map for each fixed ε , and $A'_\varepsilon(\gamma) \rightarrow \gamma$ for each fixed γ as $\varepsilon \rightarrow 0$.

Let $f \in C_b(Y_{B_3} \times \mathbb{R} \times [0, 1] \times [0, 1])$ and is 1-Lipschitz, then we can compute

$$\begin{aligned} (6.11) \quad & \int f(\theta, s, z, z') d\eta_t^n(\theta, s, z, z') = \int f(\tilde{\theta}_t, s, z, \gamma_t) d\lambda^n(\tilde{\theta}, s, z, \gamma) \\ & = \int (f(\tilde{\theta}_t, s, z, \gamma_t) - f(\tilde{\theta}_t, s, z, (A'_\varepsilon(\gamma))_t)) d\lambda^n(\tilde{\theta}, s, z, \gamma) + \int f(\tilde{\theta}_t, s, z, (A_\varepsilon(\gamma))_t) d\lambda^n(\tilde{\theta}, s, z, \gamma). \end{aligned}$$

Now observe that for each fixed $\varepsilon > 0$, the map $(\tilde{\theta}, s, z, \gamma) \mapsto (\tilde{\theta}_t, s, z, (A'_\varepsilon(\gamma))_t)$ is continuous. Hence for each fixed $\varepsilon > 0$, the following convergence holds as $n \rightarrow \infty$.

$$(6.12) \quad \int f(\tilde{\theta}_t, s, z, (A_\varepsilon(\gamma))_t) d\lambda^n(\tilde{\theta}, s, z, \gamma) \rightarrow \int f(\tilde{\theta}_t, s, z, (A_\varepsilon(\gamma))_t) d\lambda(\tilde{\theta}, s, z, \gamma).$$

To estimate the first term of (6.11), observe

$$\begin{aligned}
(6.13) \quad & \left| \int (f(\tilde{\theta}_t, s, z, \gamma_t) - f(\tilde{\theta}_t, s, z, (A_\varepsilon(\gamma))_t)) d\lambda^n(\tilde{\theta}, s, z, \gamma) \right| \leq \int |\gamma_t - (A_\varepsilon(\gamma))_t| d\lambda^n(\tilde{\theta}, s, z, \gamma) \\
& \leq \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{[0,1]} |F_{\sigma,t}^n(z) - (A_\varepsilon F_{\sigma,\cdot}^n(z))(t)| dz \\
& \leq \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \cdot \varepsilon^{-1} \int_{t-\varepsilon}^t \int_{[0,1]} |F_{\sigma,t}^n(z) - F_{\sigma,s}^n(z)| dz ds \leq C_6 \sqrt{\varepsilon + \delta t}.
\end{aligned}$$

In the last inequality, we used point (vi) of Theorem 4.1.

Combining (6.11)-(6.13), it follows that

$$\limsup_{n \rightarrow \infty} \left| \int f(\theta, s, z, z') d(\eta_t^n - \eta_t) \right| \leq C_6 \sqrt{\varepsilon} + \int |\gamma_t - (A_\varepsilon(\gamma))_t| d\lambda(\tilde{\theta}, s, z, \gamma).$$

Now using the bounded convergence theorem, we can conclude the integral on the right hand side tends to zero as $\varepsilon \rightarrow 0$ since $(A_\varepsilon(\gamma))_t \rightarrow \gamma_t$ as $\varepsilon \rightarrow 0$ and γ is left continuous. \square

6.3. The limit is a solution. In this section, we will show the limit λ obtained in previous subsection is a measure valued solution. Some preparations are needed before we proceed.

For any $B > 0$, we may define the "averaging" operator: $A_\varepsilon : Y_B \rightarrow Y_B$, given by $A_\varepsilon(\theta)(z) = \varepsilon^{-1} \int_z^{z+\varepsilon} \theta(w) dw$. Here we extended the definition of θ so that $\theta(z) = B$ for $z \geq 1$. It is clear that for any $\theta_1, \theta_2 \in Y_B$, one has $\|A_\varepsilon(\theta_1) - A_\varepsilon(\theta_2)\|_{L^2} \leq \|\theta_1 - \theta_2\|_{L^2}$. Hence A_ε is a continuous map for each fixed $\varepsilon > 0$. Also it is clear that for any $\theta \in Y_B$, we have $A_\varepsilon \theta \rightarrow \theta$ in Y_B as $\varepsilon \rightarrow 0$. We can define a map $A_\varepsilon : Y \rightarrow Y$ by the same formula, namely $A_\varepsilon(\tilde{\theta})(t, z) = \varepsilon^{-1} \int_z^{z+\varepsilon} \tilde{\theta}(w, t) dw$. Also one can check $A_\varepsilon : Y \rightarrow Y$ is a continuous map for each $\varepsilon \rightarrow 0$, and $A_\varepsilon(\tilde{\theta}) \rightarrow \tilde{\theta}$ in Y as $\varepsilon \rightarrow 0$.

Now choose $B = B_3$, we prove the following estimate about A_ε .

Lemma 6.4. *Let $\varepsilon > 0$. For any $n \geq 1$, and any $t \in (0, T)$, it holds*

$$\int_{Y \times X_{B_1}} |\tilde{\theta}_t(\gamma_t) - (A_\varepsilon \tilde{\theta})_t(\gamma_t)| d\pi_{14} \# \lambda^n(\theta, \gamma) \leq B_3 \varepsilon.$$

Proof. From the definition of the measure λ^n , we can calculate:

$$\begin{aligned}
& \int_{Y \times X_{B_1}} |\tilde{\theta}_t(\gamma_t) - (A_\varepsilon \tilde{\theta})_t(\gamma_t)| d\pi_{14} \# \lambda^n(\tilde{\theta}, \gamma) dt \\
& = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{[0,1]} |\bar{\theta}_{\sigma,t}^n(F_{\sigma,t}^n(z)) - (A_\varepsilon \bar{\theta}_\sigma^n)_t(F_{\sigma,t}^n(z))| dz \\
& = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{[0,1]} |\bar{\theta}_{\sigma,t}^n(z) - (A_\varepsilon \bar{\theta}_\sigma^n)_t(z)| dz.
\end{aligned}$$

In the last equality, we used the measure preserving property of $F_{\sigma,t}^n$. For each fixed θ , σ , we can estimate

$$\begin{aligned} \int_{[0,1]} |\bar{\theta}_{\sigma,t}^n(z) - (A_\varepsilon \bar{\theta}_\sigma^n)_t(z)| dz &= \varepsilon^{-1} \int_0^\varepsilon ds \int_{[0,1]} \bar{\theta}_{\sigma,t}^n(z+s) - \bar{\theta}_{\sigma,t}^n(z) dz \\ &= \varepsilon^{-1} \int_0^\varepsilon ds \left[\int_1^{1+s} \bar{\theta}_{\sigma,t}^n(z) dz - \int_0^s \bar{\theta}_{\sigma,t}^n(z) dz \right] ds \leq \varepsilon^{-1} \int_0^\varepsilon 2B_3 s ds \leq B_3 \varepsilon. \end{aligned}$$

In the first equality above, we used the monotonicity of $\bar{\theta}_\sigma^n$ in z . \square

Next we will check the properties listed in Definition 5.4 one by one.

Lemma 6.5. *Following the notation of Definition 5.4, we have $t \mapsto \eta_t \in \mathcal{P}(Y_{B_3} \times \mathbb{R} \times [0,1] \times [0,1])$ is narrowly continuous, and $\zeta_t \rightarrow \zeta_0$, $\pi_{34} \# \eta_t \in \Gamma(\mathcal{L}_{[0,1]}^1, \mathcal{L}_{[0,1]}^1) \rightarrow (id \times id) \# \mathcal{L}_{[0,1]}^1$ as $t \rightarrow 0$.*

Proof. First we check the continuity. Let $f \in C_b(Y_{B_3} \times \mathbb{R} \times [0,1] \times [0,1])$ be 1-Lipschitz. We compute

$$\int f(\theta, s, z, z') d\eta_t^n(\theta, z, s, z') = \int_{Y_{B_2}} d\mu_0(\theta) \left[\sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int f(\bar{\theta}_\sigma^n(t), z, \theta_{0,\sigma}^{M,n}(z), F_{\sigma,t}^n(z)) dz \right].$$

Now choose t, t' , with $t' > t$, one has for each fixed $\theta \in Y_{B_2}$, and each $\sigma \in S_\theta$,

$$\begin{aligned} &\int |f(\bar{\theta}_\sigma^n(t), \theta_{0,\sigma}^{M,n}(z), z, F_{\sigma,t}^n(z)) - f(\bar{\theta}_\sigma^n(t'), \theta_{0,\sigma}^{M,n}(z), z, F_{\sigma,t'}^n(z))| dz \\ &\leq \|\bar{\theta}_\sigma^n(t) - \bar{\theta}_\sigma^n(t')\|_{L^2([0,1])} + \int |F_{\sigma,t}^n(z) - F_{\sigma,t'}^n(z)| dz \leq (C_5 B_3 + C_6) \sqrt{t' - t + \delta t}. \end{aligned}$$

Here the constants C_5 and C_6 are given by the points (v) and (vi) of Theorem 4.1. Now since $\sum_{\sigma \in S} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta = 1$, we obtain for any $t < t'$:

$$(6.14) \quad \left| \int f(\theta, s, z, z') d(\eta_t^n - \eta_{t'}^n)(\theta, s, z, z') \right| \leq (C_5 B_3 + C_6) \sqrt{t' - t + \delta t}.$$

The continuity now follows from sending $n \rightarrow \infty$ in (6.14) and use Lemma 6.3. To show that $\zeta_t \rightarrow \zeta_0$, we show that for each $f \in C_b(Y_{B_3} \times \mathbb{R} \times [0,1])$, and 1-Lipschitz, one has $\sup_n \left| \int f d\zeta_t^n - \int f d\zeta_0^n \right| \leq C\sqrt{t}$, with C universal. Indeed we calculate

$$\int f(\theta, s, z') d\zeta_t^n(\theta, s, z') = \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int f(\bar{\theta}_\sigma^n(t), \theta_{0,\sigma}^{M,n}(z), F_{\sigma,t}^n(z)) dz.$$

On the other hand, from (6.4), we know that

$$\int f(\theta, s, z') d\zeta_0^n(\theta, s, z') = \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int f(\theta_0^n, \theta_{0,\sigma}^{M,n}(z), z) dz.$$

Using point (v) and (vi) in Theorem 4.1 once more, we see that for any choice of $\theta \in Y_{B_2}$, $s \in \mathbb{R}$, and $\sigma \in S_\theta$, we have

$$\begin{aligned} \int_{[0,1]} |f(\theta_0^n, s, z) - f(\bar{\theta}_\sigma^n(t), s, F_{\sigma,t}^n(z))| dz &\leq \|\theta_0^n - \bar{\theta}_{\sigma,t}^n\|_{L^2} + \int_{[0,1]} |z - F_{\sigma,t}^n(z)| dz \\ &\leq (C_5 B_3 + C_6) \sqrt{t + \delta t}. \end{aligned}$$

Then we can proceed in a similar way as before. That $\pi_{34}\#\eta_t^n \in \Gamma(\mathcal{L}_{[0,1]}^1, \mathcal{L}_{[0,1]}^1)$ follows readily from (6.10). The convergence for $\pi_{34}\#\eta_t^n$ is similar. \square

Next we check the point (ii) of Definition 5.4.

Lemma 6.6. *For any $t \in (0, T)$, $\pi_{13}\#\zeta_t = \mu_t \times \mathcal{L}_{[0,1]}^1$, $\pi_2\#\zeta_t$ has compact support. Besides, $\theta(z) \geq \Theta(s, z, t)$ for $\zeta_t - a.e$ (θ, s, z) .*

Proof. From (6.10) we conclude:

$$(6.15) \quad \zeta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \delta_{\bar{\theta}_\sigma^n(t)} \times (\theta_{0,\sigma}^{M,n} \times F_{\sigma,t}^n) \#\mathcal{L}_{[0,1]}^1.$$

Hence we conclude

$$(6.16) \quad \pi_{13}\#\zeta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \delta_{\bar{\theta}_\sigma^n(t)} \times (F_{\sigma,t}^n \#\mathcal{L}_{[0,1]}^1) = \mu_t^n \times \mathcal{L}_{[0,1]}^1.$$

In the above, we used the measure preserving property of the map $F_{\sigma,t}^n$ and here $\mu_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \delta_{\bar{\theta}_\sigma^n(t)} \in \mathcal{P}(Y_{C_1})$. Lemma 6.3 implies $\zeta_t^n \rightarrow \zeta_t$. On the other hand, we may assume $\mu_t^n \rightarrow \mu_t$. Passing to limit in (6.16), we see $\pi_{13}\#\zeta_t = \mu_t \times \mathcal{L}_{[0,1]}^1$.

Now from (6.15), one also calculate

$$\pi_2\#\zeta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \theta_{0,\sigma}^{M,n} \#\mathcal{L}_{[0,1]}^1.$$

From the construction given in the last subsection, we have $\|\theta_{0,\sigma}^{M,n}\|_{L^\infty} \leq M$, hence $\text{supp } \pi_2\#\zeta_t^n \subset [-M, M]$. Passing to the limit, the same will hold for $\pi_2\#\zeta_t$.

It remains to check that $\theta(z) \geq \Theta(s, z, t)$ for $\zeta_t - a.e$ (θ, s, z) . It suffices to show $\int (\theta(z) - \Theta(s, z, t))^- d\zeta_t(\theta, s, z) = 0$. Now we calculate

$$\begin{aligned} & \int_{Y_{B_3} \times \mathbb{R} \times [0,1]} (\theta(z) - \Theta(s, z, t)) d\zeta_t^n(\theta, s, z) \\ &= \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{[0,1]} (\bar{\theta}_\sigma^n(t, F_{\sigma,t}^n(z)) - \Theta(\theta_\sigma^{M,n}(z), F_{\sigma,t}^n(z), t))^- dz \\ &\leq \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{[0,1]} |\bar{\theta}_\sigma^n(t, F_{\sigma,t}^n(z)) - \theta_\sigma^n(t, F_{\sigma,t}^n(z))| dz + \sup |\partial_t \Theta| \delta t \\ &\leq (C_5 + \sup |\partial_t \Theta|) \sqrt{\delta t} \end{aligned}$$

In the first inequality above, we used that $\theta_\sigma^n(t, F_{\sigma,t}^n(z)) \geq \Theta(\theta_\sigma^{M,n}(z), F_{\sigma,t}^n(z), k\delta t)$, where $t \leq k\delta t < (k+1)\delta t$. This follows from point (iv) of Corollary 3.3. In the last inequality, we used the point (v) of Theorem 4.1 and the measure preserving property of the map $F_{\sigma,t}^n$. Passing to the limit as $n \rightarrow \infty$, the conclusion follows. \square

Now we check the point (iii) of Definition 5.4.

Lemma 6.7. *Fix $\varepsilon > 0$, define the function $I_\varepsilon : Y \times X_{B_1} \rightarrow \mathbb{R}$, given by*

$$(\tilde{\theta}, \gamma) \longmapsto \int_{(0,T)^2} ((A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) - (A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} dt_1 dt_2$$

is continuous.

Proof. Let $(\tilde{\theta}^k, \gamma^k) \rightarrow (\tilde{\theta}, \gamma)$ in $Y \times X_{B_1}$, we need to show $I(\tilde{\theta}^k, \gamma^k) \rightarrow I(\tilde{\theta}, \gamma)$. We prove this by showing for \mathcal{L}^1 -a.e $t \in (0, T)$, we have pointwise convergence: $(A_\varepsilon \tilde{\theta}^k)_t(\gamma^k) \rightarrow (A_\varepsilon \tilde{\theta})_t(\gamma_t)$, and then the desired convergence follows from dominated convergence theorem.

By Helly's selection principle, we know that $\gamma_t^k \rightarrow \gamma_t$ except for a countable set of t . Since $|A_\varepsilon(\tilde{\theta}^k)_t(z) - A_\varepsilon(\tilde{\theta})_t(z)| \leq \varepsilon^{-1/2} \|\tilde{\theta}_t^k - \tilde{\theta}_t\|_{L^2}$, we know that $A_\varepsilon(\tilde{\theta}^k)_t \rightarrow A_\varepsilon(\tilde{\theta})_t$ uniformly for each $t \in (0, T)$. Hence for any t with $\gamma^k(t) \rightarrow \gamma(t)$, it holds $(A_\varepsilon \tilde{\theta}^k)_t(\gamma_t^k) \rightarrow (A_\varepsilon \tilde{\theta})_t(\gamma_t)$. It follows that $((A_\varepsilon(\tilde{\theta}^k))_{t_1}(\gamma_{t_1}^k) - (A_\varepsilon \tilde{\theta}^k)_{t_2}(\gamma_{t_2}^k))^+ \rightarrow ((A_\varepsilon(\tilde{\theta}))_{t_1}(\gamma_{t_1}) - (A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2}))^+$ for \mathcal{L}^2 -a.e (t_1, t_2) . Then the result follows from bounded convergence theorem. \square

Lemma 6.8. *For any n , the following estimate holds:*

$$\int_{(0,T)^2} \int_{Y \times X_{B_1}} (\tilde{\theta}_{t_1}(\gamma_{t_1}) - \tilde{\theta}_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{14} \# \lambda^n(\theta, \gamma) \leq 2C_5 T^2 \sqrt{\delta t}.$$

Here C_5 is the universal constant given in point (v) of Theorem 4.1.

Proof. According to the definition of λ^n , we find

$$\begin{aligned} & \int_{(0,T)^2} \int_{Y \times X_{B_1}} (\tilde{\theta}_{t_1}(\gamma_{t_1}) - \tilde{\theta}_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{14} \# \lambda^n(\theta, \gamma) \\ &= \int_{Y_{B_2}} \mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{(0,T)^2} \int_{[0,1]} (\bar{\theta}_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z)) - \bar{\theta}_{\sigma,t_2}^n(F_{\sigma,t_2}^n(z)))^+ \chi_{t_1 < t_2} dt_1 dt_2 dz. \end{aligned}$$

For each fixed σ and θ , it holds:

$$\begin{aligned} & \int_{(0,T)^2} \int_{[0,1]} (\bar{\theta}_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z)) - \bar{\theta}_{\sigma,t_2}^n(F_{\sigma,t_2}^n(z)))^+ \chi_{t_1 < t_2} dt_1 dt_2 dz \\ & \leq 2T \int_0^T \int_{[0,1]} |\bar{\theta}_{\sigma,t}^n(F_{\sigma,t}^n(z)) - \theta_{\sigma,t}^n(F_{\sigma,t}^n(z))| dt dz \\ & + \int_{(0,T)^2} \int_{[0,1]} (\theta_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z)) - \theta_{\sigma,t_2}^n(F_{\sigma,t_2}^n(z)))^+ \chi_{t_1 < t_2} dt_1 dt_2 dz \\ & = 2T \int_0^T \int_{[0,1]} |\bar{\theta}_{\sigma,t}^n(z) - \theta_{\sigma,t}^n(z)| dt dz \leq 2T^2 C_5 \sqrt{\delta t}. \end{aligned}$$

In the above calculation, we used point (iii) of Corollary 3.3, hence for any $\sigma \in S_\theta$, $\theta_{\sigma,t_1}^n(F_{\sigma,t_1}^n) \leq \theta_{\sigma,t_2}^n(F_{\sigma,t_2}^n)$, for any $t_1 < t_2$. When estimating $\bar{\theta}_\sigma^n - \theta_\sigma^n$, we used (6.8) and point (v) of Theorem 4.1. \square

Lemma 6.9. *For λ -a.e $(\tilde{\theta}, s, z, \gamma) \in Y \times \mathbb{R} \times [0, 1] \times X_{B_1}$, we have*

$$\tilde{\theta}_{t_1}(\gamma_{t_1}) \leq \tilde{\theta}_{t_2}(\gamma_{t_2}), \text{ for } \mathcal{L}^2 - \text{a.e } (t_1, t_2) \in (0, T)^2 \text{ with } t_1 < t_2.$$

Proof. It suffices to show that

$$(6.17) \quad \int_{(0,T)^2} \int_{Y \times X_{B_1}} (\tilde{\theta}_{t_1}(\gamma_{t_1}) - \tilde{\theta}_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{1,4} \# \lambda(\tilde{\theta}, \gamma) = 0.$$

For any $\varepsilon > 0$, we can write

$$\begin{aligned} & \int_{(0,T)^2} \int_{Y \times X_{B_1}} (\tilde{\theta}_{t_1}(\gamma_{t_1}) - \tilde{\theta}_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{1,4} \# \lambda(\tilde{\theta}, \gamma) \\ & \leq 2T \int_0^T \int_{Y \times X_{B_1}} (\tilde{\theta}_t(\gamma_t) - (A_\varepsilon \tilde{\theta})_t(\gamma_t))^+ dt d\pi_{1,4} \# \lambda(\tilde{\theta}, \gamma) \\ & + \int_{(0,T)^2} \int_{Y \times X_{B_1}} ((A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) - (A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{1,4} \# \lambda(\tilde{\theta}, \gamma). \end{aligned}$$

For the first term, it goes to zero as $\varepsilon \rightarrow 0$, since the integrand tends to 0 for each fixed $(t, \tilde{\theta}, \gamma)$ and is clearly bounded. For the second term, we estimate:

$$\begin{aligned} & \int_{(0,T)^2} \int_{Y \times X_{B_1}} ((A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) - (A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{1,4} \# \lambda^n(\tilde{\theta}, \gamma) \\ & \leq 2T \int_0^T \int_{Y \times X_{B_1}} |\tilde{\theta}_t(\gamma_t) - (A_\varepsilon \tilde{\theta})_t(\gamma_t)| dt d\pi_{1,4} \# \lambda^n(\tilde{\theta}, \gamma) + 2C_5 T^2 \sqrt{\delta t} \leq 2B_3 T^2 \varepsilon + 2C_5 T^2 \sqrt{\delta t}. \end{aligned}$$

The last inequality follows from the Lemma 6.4, while the first inequality used Lemma 6.8. Now send $n \rightarrow \infty$, Lemma 6.7 allows us to conclude:

$$\int_{(0,T)^2} \int_{Y \times X_{B_1}} ((A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) - (A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{1,4} \# \lambda(\tilde{\theta}, \gamma) \leq 2B_3 T^2 \varepsilon.$$

The proof is completed by sending $\varepsilon \rightarrow 0$. \square

Up to now, we have checked points (i)-(iii) in Definition 5.4. It only remains to check point (iv). Our first goal will be to show the measure $\partial_t(\tilde{\theta}_t(\gamma_t))$ is concentrated on the "wet" set. As preparation, we prove the following:

Lemma 6.10. *For any $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, define the function $K_i : Y \times \mathbb{R} \times X_{B_1} \rightarrow \mathbb{R}$, $i = 1, 2$, given by*

$$K_1(\tilde{\theta}, s, \gamma) = \int_{(0,T)^2} \chi_{\{0 < t_2 - t_1 < \varepsilon_1\}} |(A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) - (A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2})| \chi_{\{(A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon_2\}} dt_1 dt_2,$$

$$K_2(\tilde{\theta}, s, \gamma) = \int_{(0,T)^2} \chi_{\{0 < t_2 - t_1 < \varepsilon_1\}} (\gamma_{t_2} - \gamma_{t_1})^+ \chi_{\{(A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon_2\}}.$$

Then K_i is lower semi-continuous, $i = 1, 2$.

Proof. Let $(\tilde{\theta}^k, s^k, \gamma^k) \rightarrow (\tilde{\theta}, s, \gamma)$, need to show $K_i(\tilde{\theta}, s, \gamma) \leq \liminf_{k \rightarrow \infty} K_i(\tilde{\theta}^k, s^k, \gamma^k)$.

As explained in the proof of Lemma 6.7, for any t such that $\gamma^k(t) \rightarrow \gamma(t)$, we have $(A_\varepsilon \tilde{\theta}^k)_t(\gamma_t^k) \rightarrow (A_\varepsilon \tilde{\theta})_t(\gamma_t)$. Hence for such t

$$\chi_{\{(A_\varepsilon \tilde{\theta})_t(\gamma_t) > \Theta(s, \gamma_t, t) + \varepsilon_2\}} \leq \liminf_{k \rightarrow \infty} \chi_{\{(A_\varepsilon \tilde{\theta}^k)_t(\gamma_t^k) > \Theta(s, \gamma_t^k, t) + \varepsilon_2\}}.$$

The integrand is lower semi-continuous with respect to $(\tilde{\theta}, s, \gamma)$ if we fix (t_1, t_2) such that $\gamma_{t_i}^k \rightarrow \gamma_{t_i}$, $i = 1, 2$. Then the lower semi-continuity of J_i follows from applying Fatou's lemma. \square

Lemma 6.11. *Let C_2 be the universal constant given by Theorem 4.1, point (ii). Then we have for any $\varepsilon > 0$*

$$(6.18) \quad \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + 2\varepsilon\}} d\lambda(\tilde{\theta}, s, z, \gamma) = 0.$$

and

$$(6.19) \quad \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int (\gamma_{t_2} - \gamma_{t_1})^+ \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + 2\varepsilon\}} d\lambda(\tilde{\theta}, s, z, \gamma) = 0.$$

Proof. We only prove (6.18). The proof of (6.19) follows similar lines and is simpler. Fix $0 < \delta < \varepsilon$. Denote $\chi_{\delta, \varepsilon} = \chi_{\{(A_\delta \tilde{\theta})_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon\}}$. From the lower semi-continuity proved in previous lemma, we conclude:

$$(6.20) \quad \begin{aligned} & \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |(A_\delta \tilde{\theta})_{t_2}(\gamma_{t_2}) - (A_\delta \tilde{\theta})_{t_1}(\gamma_{t_1})| \chi_{\delta, 2\varepsilon} d\lambda(\theta, s, z, \gamma) \\ & \leq \liminf_{n \rightarrow \infty} \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |(A_\delta \tilde{\theta})_{t_2}(\gamma_{t_2}) - (A_\delta \tilde{\theta})_{t_1}(\gamma_{t_1})| \chi_{\delta, 2\varepsilon} d\lambda^n(\tilde{\theta}, s, z, \gamma). \end{aligned}$$

We know the left hand side of (6.20) will tend to the left hand side of (6.18) as $\delta \rightarrow 0$. Next we estimate the right hand side of (6.20).

$$(6.21) \quad \begin{aligned} & \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |(A_\delta \tilde{\theta})_{t_2}(\gamma_{t_2}) - (A_\delta \tilde{\theta})_{t_1}(\gamma_{t_1})| \chi_{\delta, 2\varepsilon} d\lambda^n(\tilde{\theta}, s, z, \gamma) \\ & \leq 2T \int_0^T \int_{Y \times X_{B_1}} |(A_\delta \tilde{\theta})_t(\gamma_t) - \tilde{\theta}_t(\gamma_t)| d\pi_{14} \# \lambda^n(\tilde{\theta}, \gamma) \\ & + \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \chi_{\delta, 2\varepsilon} d\lambda^n(\tilde{\theta}, s, z, \gamma) \\ & \leq 2B_3 T^2 \delta + \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + 1.5\varepsilon\}} d\lambda^n \\ & + B_3 T \int_0^T dt_1 \int (\chi_{\delta, 2\varepsilon} - \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + 1.5\varepsilon\}})^+ d\lambda^n. \end{aligned}$$

In the second inequality, we used Lemma 6.4. For the second term of the right hand side above, we calculate:

$$(6.22) \quad \begin{aligned} & \int_{[0,T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + 1.5\varepsilon\}} d\lambda^n \\ & = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^\sigma \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{(0,T)^2} \int_0^1 \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} |\bar{\theta}_{\sigma, t_2}^n(F_{\sigma, t_2}^n(z)) - \bar{\theta}_{\sigma, t_1}^n(F_{\sigma, t_1}^n(z))| \\ & \chi_{\{\bar{\theta}_{\sigma, t_1}^n(F_{\sigma, t_1}^n(z)) > \Theta(\theta_{0, \sigma}^{M, n}(z), F_{\sigma, t_1}^n(z), t_1) + 1.5\varepsilon\}} dt_1 dt_2 dz. \end{aligned}$$

Now define:

$$\begin{aligned}\chi_1 &= \chi_{\{\bar{\theta}_{\sigma,t_1}^n(F_{\sigma,t_1}^n) > \Theta(\theta_{0,\sigma}^{M,n}(z), F_{\sigma,t_1}^n(z), t_1) + 1.5\varepsilon\}}, \\ \chi_2 &= \chi_{\{\theta_{\sigma,t_1}^n(F_{\sigma,t_1}^n) > \Theta(\theta_{0,\sigma}^{M,n}(z), F_{\sigma,t_1}^n(z), t_1) + \varepsilon\}}.\end{aligned}$$

We now estimate the right hand side of (6.22). For any $\theta \in Y_{B_2}$ and $\sigma \in S_\theta$,

$$\begin{aligned}& \int_{(0,T)^2} \int_0^1 \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} |\bar{\theta}_{\sigma,t_2}^n(F_{\sigma,t_2}^n(z)) - \bar{\theta}_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z))| \chi_1 dt_1 dt_2 dz \\ & \leq 2T \int_0^T \int_0^1 |\bar{\theta}_{\sigma,t}^n(F_{\sigma,t}^n(z)) - \theta_{\sigma,t}^n(F_{\sigma,t}^n(z))| dt dz \\ & + \int_{(0,T)^2} \int_0^1 \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} |\theta_{\sigma,t_2}^n(F_{\sigma,t_2}^n(z)) - \theta_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z))| \chi_1 dt_1 dt_2 dz \\ & \leq 2T^2 C_5 \sqrt{\delta t} + \int_{(0,T)^2} \int_0^1 \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{C_2}\}} |\theta_{\sigma,t_2}^n(F_{\sigma,t_2}^n(z)) - \theta_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z))| \chi_2 dt_1 dt_2 dz \\ & + 2B_1 \int_{(0,T)^2} \int_0^1 (\chi_1 - \chi_2)^+ dt_1 dt_2 dz.\end{aligned}$$

In the last inequality, we used the point (v) of Theorem 4.1. The second term of right hand side above is 0, due to point (iv) of Theorem 4.1. To estimate the last term, we notice

$$\begin{aligned}& \int_{(0,T)^2} \int_0^1 (\chi_1 - \chi_2)^+ dt_1 dt_2 dz \leq \int_{(0,T)^2} \int_0^1 \chi_{\{\bar{\theta}_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z)) - \theta_{\sigma,t_1}^n(F_{\sigma,t_1}^n(z)) > 0.5\varepsilon\}} \\ & \leq \frac{2T}{\varepsilon} \int_0^T \int_0^1 |\bar{\theta}_{\sigma,t}^n(z) - \theta_{\sigma,t}^n(z)| dz dt \leq \frac{2TC_5 \sqrt{\delta t}}{\varepsilon}.\end{aligned}$$

For the last term of (6.21), we have

$$\begin{aligned}& \int_0^T dt_1 \int (\chi_{\delta, 2\varepsilon} - \chi_{\{\bar{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon\}})^+ d\lambda^n \\ & \leq \int_0^T dt_1 \int \chi_{\{|(A_\delta \bar{\theta})_{t_1}(\gamma_{t_1}) - \bar{\theta}_{t_1}(\gamma_{t_1})| \geq \varepsilon\}} d\lambda^n \leq \varepsilon^{-1} \int_0^T dt_1 \int |(A_\delta \bar{\theta})_{t_1}(\gamma_{t_1}) - \bar{\theta}_{t_1}(\gamma_{t_1})| d\lambda^n \\ & \leq \varepsilon^{-1} B_3 T \delta.\end{aligned}$$

In the last inequality, we use Lemma 6.4 again. Combining the calculations above, we obtain the left hand side of (6.20) $\leq \varepsilon^{-1} B_3 T \delta + 2B_3 T^2 \delta$. The proof follows from sending $\delta \rightarrow 0$. \square

By Remark 2.4, and Lemma 6.9, we know for $\pi_{14} \# \lambda - a.e$ (θ, γ) , one can determine a unique monotone increasing, and left continuous function $(\theta(\gamma))^*$, which equals $\theta_t(\gamma_t)$ for $\mathcal{L}^1 - a.e$ $t \in (0, T)$. We will simply denote this function by $\alpha(t)$ in the following lemma.

Lemma 6.12. *Let $\varepsilon > 0$, then for $\lambda - a.e$ $(\tilde{\theta}, s, z, \gamma) \in Y \times X_{B_1}$, the following property holds:*

For any $t \in (0, T)$ such that $\alpha(t) > \Theta(s, \gamma_t, t) + \varepsilon$, it holds $\alpha(t') = \alpha(t)$, and $\gamma_{t'} \leq \gamma_t$, for any $0 < t' - t < \frac{\varepsilon}{2C_2}$. Here C_2 is the constant given by point (ii), Theorem 4.1.

Proof. Let (θ, γ) be chosen so that the statement of Lemma 6.9 holds true, so that we can define $\alpha(t)$. Let $(\tilde{\theta}, s, z, \gamma)$ also satisfy that

$$(6.23) \quad \int_{[0, T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{2C}\}} |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon\}} dt_1 dt_2 = 0,$$

and

$$(6.24) \quad \int_{[0, T]^2} \chi_{\{0 < t_2 - t_1 < \frac{\varepsilon}{2C}\}} dt_1 dt_2 \int (\gamma_{t_2} - \gamma_{t_1})^+ \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon\}} = 0.$$

We know from Lemma 6.11 that (6.23) and (6.24) holds for λ -a.e $(\tilde{\theta}, s, z, \gamma)$, for λ -a.e $(\tilde{\theta}, s, z, \gamma)$. Choose some $(\tilde{\theta}, s, z, \gamma)$ so that (6.23) and (6.24) hold, then we have $\tilde{\theta}_{t_2}(\gamma_{t_2}) = \tilde{\theta}_{t_1}(\gamma_{t_1})$ and $\gamma_{t_2} \leq \gamma_{t_1}$, for \mathcal{L}^2 -a.e (t_1, t_2) with $\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon$ and $0 < t_2 - t_1 < \frac{\varepsilon}{2C}$.

Fix any (t_1, t_2) with $0 < t_2 - t_1 < \frac{\varepsilon}{2C_2}$ and $\alpha(t_1) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon$. By left continuity, there exists $\delta > 0$, such that $\alpha(t'_1) > \Theta(s, \gamma_{t'_1}, t'_1) + \varepsilon$ for any $t'_1 \in (t_1 - \delta, t_1)$. By Fubini's theorem, we know for \mathcal{L}^1 -a.e $t'_1 \in (t_1 - \delta, t_1)$, we have $\alpha(t'_2) = \tilde{\theta}_{t'_2}(\gamma_{t'_2}) = \tilde{\theta}_{t'_1}(\gamma_{t'_1})$ and $\gamma_{t'_2} \leq \gamma_{t'_1}$ for \mathcal{L}^1 -a.e $t'_2 \in (t'_1, t'_1 + \frac{\varepsilon}{2C_2})$. By left continuity of $\alpha(t)$ and $\gamma(t)$, we conclude $\alpha(t'_2) = \tilde{\theta}_{t'_1}(\gamma_{t'_1})$ and $\gamma_{t'_2} \leq \gamma_{t'_1}$ for any $t'_2 \in (t'_1, t'_1 + \frac{\varepsilon}{2C_2})$. This is true for \mathcal{L}^1 -a.e $t'_1 \in (t_1 - \delta, t_1)$. Hence we can find a sequence $\{t_1^n\}_{n=1}^\infty \subset (t_1 - \delta, t_1)$ such that this is true for t_1^n and $t_1^n \rightarrow t_1$ as $n \rightarrow \infty$. We can assume $\tilde{\theta}_{t_1^n}(\gamma_{t_1^n}) = \alpha(t_1^n)$ holds. Define $t_2^n = t_1^n + t_2 - t_1$, then $t_2^n \rightarrow t_2$, and we have $\alpha(t_2^n) = \alpha(t_1^n)$ and $\gamma_{t_2^n} \leq \gamma_{t_1^n}$. Let $n \rightarrow \infty$, and use left continuity of α and γ one more time, we can conclude $\alpha(t_2) = \alpha(t_1)$ and $\gamma_{t_2} \leq \gamma_{t_1}$. \square

Lemma 6.13. *For any $C > 0$, $\varepsilon > 0$, define the function $H_\varepsilon : Y \times \mathbb{R} \times X_{B_1} \rightarrow \mathbb{R}$, given by*

$$H_\varepsilon(\tilde{\theta}, s, \gamma) = \int_{(0, T)^2} \left(|(A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2}) - (A_\varepsilon \tilde{\theta})_{t_1}(\gamma_{t_1}) - (\Theta(s, \gamma_{t_2}, t_2) - \Theta(s, \gamma_{t_1}, t_1))^+| - C(t_2 - t_1) \right)^+ \chi_{t_1 < t_2} dt_1 dt_2.$$

Then H_ε is continuous.

Proof. The proof of this lemma is quite similar to Lemma 6.9. We already noted that $(\tilde{\theta}^k, \gamma^k) \rightarrow (\tilde{\theta}, \gamma)$ implies $(A_\varepsilon \tilde{\theta}^k)_t(\gamma_t^k) \rightarrow (A_\varepsilon \tilde{\theta})_t(\gamma_t)$ for any t such that pointwise convergence of γ_t^k happens. The proof follows then from dominated convergence since everything is bounded. \square

Lemma 6.14. *Let C_4 be the constant given in point (iv), Theorem 4.1, then we have:*

$$(6.25) \quad \int_{(0, T)^2} \int \left(|\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1}) - (\Theta(s, \gamma_{t_2}, t_2) - \Theta(s, \gamma_{t_1}, t_1))^+| - C_4(t_2 - t_1) \right)^+ \chi_{t_1 < t_2} dt_1 dt_2 d\lambda = 0.$$

Proof. Write the left hand side of (6.25) to be $\int H(\tilde{\theta}, s, \gamma) d\lambda$, with the definition of H similar to H_ε in the last lemma (without A_ε). Then we can estimate

$$\int_{Y \times \mathbb{R} \times [0, 1] \times X_{B_1}} H(\tilde{\theta}, s, \gamma) d\lambda(\tilde{\theta}, s, z, \gamma) = \int (H - H_\varepsilon)(\tilde{\theta}, s, \gamma) d\lambda + \lim_{n \rightarrow \infty} \int H_\varepsilon(\tilde{\theta}, s, \gamma) d\lambda^n.$$

The first term will tend to zero as $\varepsilon \rightarrow 0$. For the second term, we have

$$\begin{aligned} \int H_\varepsilon(\tilde{\theta}, s, \gamma) d\lambda^n &\leq \int |H_\varepsilon - H| d\lambda^n + \int H(\tilde{\theta}, s, \gamma) d\lambda^n \\ &\leq 2T \int_0^T \int |\tilde{\theta}_t(\gamma_t) - (A_\varepsilon \tilde{\theta})_t(\gamma_t)| d\lambda^n + \int H(\tilde{\theta}, s, \gamma) d\lambda^n \\ &\leq 2T^2 B_3 \varepsilon + \int H(\tilde{\theta}, s, \gamma) d\lambda^n. \end{aligned}$$

In the last inequality above, we used Lemma 6.4. To deal with the remaining term, first we can write:

$$\int H(\tilde{\theta}, s, \gamma) d\lambda^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \int_{[0,1]} H(\bar{\theta}_\sigma^n, \theta_{0,\sigma}^{M,n}(z), F_\sigma^n(z)) dz.$$

Fix some $\theta \in Y_{B_2}$, and $\sigma \in S_\theta$, we can calculate:

$$\begin{aligned} \int H(\bar{\theta}_\sigma^n, \theta_{0,\sigma}^{M,n}(z), F_\sigma^n(z)) dz &\leq 2T \int_0^T \int_{[0,1]} |\bar{\theta}_{\sigma,t}^n(F_{\sigma,t}^n(z)) - \theta_{\sigma,t}^n(F_{\sigma,t}^n(z))| dt dz \\ &+ \int H(\theta_\sigma^n, \theta_\sigma^{M,n}(z), F_\sigma^n(z)) dz \leq 2T^2 C_5 \sqrt{\delta t} + T^2 C_4 \delta t. \end{aligned}$$

In the second inequality above, we used the point (iv) and (v) of Theorem 4.1, and that $F_{\sigma,t}^n$ is measure preserving.

Now the proof is finished by first letting $n \rightarrow \infty$ and then let $\varepsilon \rightarrow 0$. \square

As a corollary, we deduce that

Corollary 6.15. *For $\lambda - a.e$ $(\tilde{\theta}, s, z, \gamma)$, it holds:*

$$|\alpha(t_2) - \alpha(t_1) - (\Theta(s, \gamma_{t_2}, t_2) - \Theta(s, \gamma_{t_1}, t_1))^+| \leq C_4(t_2 - t_1),$$

for any $0 < t_1 < t_2 < T$. Here C_4 is the constant given in point (iv) of Theorem 4.1.

With above preparation, we can check the point (iv) of Theorem 5.4.

Proposition 6.16. *For $\lambda - a.e$ $(\tilde{\theta}, s, z, \gamma)$, we have the equality of measures:*

$$(6.26) \quad \partial_t(\theta_t(\gamma_t)) = [\partial_t(Q^{sat}(\theta_t(\gamma_t), \gamma_t, t))]^- \lfloor E_{\theta,s,\gamma}.$$

where $E_{\theta,s,\gamma}$ is the wet set given by

$$E_{\theta,s,\gamma} = \{t \in (0, T) : (\theta(\gamma))^*(t) = s - Q^{sat}((\theta(\gamma))^*(t), \gamma_t, t)\}.$$

Proof. We choose $(\tilde{\theta}, s, z, \gamma)$ such that the statement of Lemma 6.9, 6.12, and Corollary 6.15 hold. The plan is to apply Lemma 7.1 to the functions: $f(t) = \alpha(t)$, and $g(t) = s - Q^{sat}(\alpha(t), \gamma(t), t)$. Here $\alpha(t)$ is the monotone increasing and left continuous version of $\theta_t(\gamma_t)$ chosen according to Remark 2.4. This is possible since Lemma 6.9 holds.

First we verify that for $\lambda - a.e$ $(\tilde{\theta}, s, z, \gamma)$, it holds $\alpha(t) \geq s - Q^{sat}(\alpha(t), \gamma(t), t)$, for any $t \in (0, T)$. This is the same as $\alpha(t) \geq \Theta(s, \gamma(t), t)$. This follows from Lemma 6.6. Indeed, we have shown there that for any $t \in (0, T)$, $\theta(z') \geq \Theta(s, z', t)$ for $\zeta_t - a.e$ (θ, s, z') . Recall the definition of ζ_t , this is the same as saying for any fixed $t \in (0, T)$, $\theta_t(\gamma_t) \geq \Theta(s, \gamma_t, t)$ for $\lambda - a.e$ $(\tilde{\theta}, s, z, \gamma)$. Now choose a countable dense subset D of the set $\{t \in (0, T) : \alpha(t) = \theta_t(\gamma_t)\}$. Then $D \subset (0, T)$ is also dense. Then for $\lambda - a.e$

$(\tilde{\theta}, s, z, \gamma)$, it holds $\alpha(t_i) \geq \Theta(s, \gamma_{t_i}, t_i)$, for any $t_i \in D$. Since D is dense and both $\alpha(t)$, $\gamma(t)$ continuous from left, we see it is true for all $t \in (0, T)$.

Next we verify points (i)-(iii) in Lemma 7.1.

The point (i) follows from Lemma 6.12. Indeed, if $\alpha(t) > s - Q^{sat}(\alpha(t), \gamma(t), t) + \varepsilon$, then we know $\alpha(t) > \Theta(s, \gamma(t), t) + \frac{\varepsilon}{C_7}$ for some universal constant C_7 . Now Lemma 6.12 implies that $\alpha(t') = \alpha(t)$, $\gamma(t') \leq \gamma(t)$, for any $0 < t' - t < \frac{\varepsilon}{C_8}$. Hence $\alpha(t') = \alpha(t) > s - Q^{sat}(\alpha(t'), \gamma(t'), t') - \sup |\partial_t Q^{sat}|(t' - t) + \varepsilon$. If we still have $\varepsilon > 2 \sup |\partial_t Q^{sat}|(t' - t)$, then we have $f(t') > g(t')$. Hence $f(t') = f(t)$ and $f(t') > g(t')$ as long as $0 < t' - t < \frac{\varepsilon}{C_8 + 2 \sup |\partial_t Q^{sat}|}$. This verifies point (i).

Next we verify point (ii). This follows from Corollary 6.15. Indeed, from that corollary, we can deduce $\alpha(t^+) - \alpha(t) = (\Theta(s, \gamma(t^+), t) - \Theta(s, \gamma(t), t))^+$, by fixing any $t_1 = t$, and $t_2 \searrow t$. If $\alpha(t^+) = \alpha(t)$, we conclude $\Theta(s, \gamma(t^+), t) \leq \Theta(s, \gamma(t), t)$. Therefore $\gamma(t^+) \leq \gamma(t)$ by strict monotonicity of Θ in z variable. Then $Q^{sat}(\alpha(t^+), \gamma(t^+), t) \geq Q^{sat}(\alpha(t), \gamma(t), t)$. If $\alpha(t^+) > \alpha(t)$, we must have $\alpha(t) = \Theta(s, \gamma(t), t)$, or $\alpha(t) = s - Q^{sat}(\alpha(t), \gamma(t), t)$, otherwise it contradicts Lemma 6.12. But then $\alpha(t^+) = \Theta(s, \gamma(t^+), t)$. Hence $\alpha(t^+) = s - Q^{sat}(\alpha(t^+), \gamma(t^+), t)$. In any case, we have $f(t^+) - f(t) = (g(t^+) - g(t))^+$. This verifies point (ii).

Point (iii) again follows from Lemma 6.12. Indeed, from the monotonicity of Q^{sat} , we just need to show for any $[a, b] \subset \{f > g\}$, it holds $\alpha(t_2) = \alpha(t_1)$, and $\gamma(t_2) \leq \gamma(t_1)$, for any $t_1 < t_2$, $t_1, t_2 \in [a, b]$. Let $t^* = \sup\{t \in [t_1, t_2] : \alpha(t') = \alpha(t_1), \gamma(t') \leq \gamma(t_1), \text{ for any } t' \leq t\}$. Then we must have $t^* = t_2$. Otherwise, since $f(t^*) > g(t^*)$, Lemma 6.12 allows us to push beyond t^* , giving a contradiction. \square

7. APPENDIX

Lemma 7.1. *Let $f : (0, T) \rightarrow \mathbb{R}$ be monotone increasing, and $g : (0, T) \rightarrow \mathbb{R} \in BV((0, T))$, both continuous from left and bounded, with $f(t) \geq g(t)$, $\forall t \in (0, T)$. Suppose that for some constant $C > 0$*

(i) $f(t') \equiv f(t)$, $f(t') > g(t')$ for any $t \in (0, T)$ with $f(t) > g(t) + \varepsilon$ and any t' with $t' - t < \frac{\varepsilon}{C}$.

(ii) $f(t^+) - f(t) = [g(t^+) - g(t)]^+$, for any $t \in (0, T)$.

(iii) $g(t_2) - g(t_1) \leq C(t_2 - t_1)$ for any $t_1 < t_2 \in [a, b]$ with $[a, b] \subset \{f > g\}$.

Then $\partial_t f$ is concentrated on the set $\{f = g\}$ and for any Borel set $E \subset \{f = g\}$, one has $\partial_t f(E) = (\partial_t g)^+(E)$.

Before we prove this result, we prove the following lemma as a preparation.

Lemma 7.2. *Under the assumptions of previous lemma, we have for any $[a, b] \subset (0, T)$*

$$(7.1) \quad (\partial_t g)^+([a, b]) \leq f(b) - f(a) + C \mathcal{L}^1([a, b] \cap \{f > g\}).$$

Proof. First recall

$$(\partial_t g)^+([a, b]) = \sup \left\{ \sum_{i=1}^n (g(t_i) - g(t_{i-1}))^+ : a = t_0 < t_1 < \dots < t_n = b \right\}.$$

We fix a partition appearing in the right hand side above. For each i , define

$$\begin{aligned} t'_{i-1} &= \sup\{t \in [t_{i-1}, t_i] : [t_{i-1}, t] \cap \{f = g\} = \emptyset\}, \\ t'_i &= \inf\{t \in [t_{i-1}, t_i] : [t, t_i] \cap \{f = g\} = \emptyset\}. \end{aligned}$$

Let D be the set of i for which $[t_{i-1}, t_i] \cap \{f = g\} \neq \emptyset$. Then for $i \in D$ one can calculate:

If $t_i > t'_i$, choose $\varepsilon < t_i - t'_i$, since $[t'_i + \varepsilon, t_i] \subset \{f > g\}$, we see from point (iii):

$$\begin{aligned} g(t_i) - g(t'_{i-1}) &= g(t_i) - g(t'_i + \varepsilon) + g(t'_i + \varepsilon) - g(t'_{i-1}) \\ &\leq C(t_i - t'_i - \varepsilon) + g(t'_i + \varepsilon) - g(t'_{i-1}). \end{aligned}$$

Let $\varepsilon \rightarrow 0$, we see

$$g(t_i) - g(t'_{i-1}) \leq C(t_i - t'_i) + g((t'_i)^+) - g(t'_{i-1}).$$

On the other hand, since $[t_{i-1}, t'_{i-1}] \subset \{f > g\}$, we can conclude

$$g(t'_{i-1}) - g(t_{i-1}) \leq C(t_{i-1} - t'_{i-1}).$$

Combining above calculations, we get:

$$\begin{aligned} &\sum_{i=1}^n (g(t_i) - g(t_{i-1}))^+ = \sum_{i \in D} (g(t_i) - g(t_{i-1}))^+ + \sum_{i \notin D} (g(t_i) - g(t_{i-1}))^+ \\ \leq &\sum_{i \in D, t_i > t'_i} C(t_i - t'_i) + \sum_{i \in D} C(t'_{i-1} - t_{i-1}) + \sum_{i \notin D} C(t_i - t_{i-1}) \\ &+ \sum_{i \in D, t_i > t'_i} (g((t'_i)^+) - g(t'_{i-1}))^+ + \sum_{i \in D, t_i = t'_i} (g(t_i) - g(t'_{i-1}))^+ \\ \leq &C\mathcal{L}^1([a, b] \cap \{f > g\}) + \sum_{i \in D, t_i > t'_i} (g((t'_i)^+) - g(t'_{i-1}))^+ + \sum_{i \in D, t_i = t'_i} (g(t_i) - g(t'_{i-1}))^+. \end{aligned}$$

In the first inequality above, we used condition (iii). If $i \in D$ and $t_i > t'_i$, we will have

$$g((t'_i)^+) - g(t'_{i-1}) \leq f(t_i) - f(t'_{i-1}).$$

Similarly for $i \in C$ and $t_i = t'_i$, we see

$$g(t_i) - g(t'_{i-1}) \leq f(t_i) - f(t'_{i-1}).$$

Hence

$$\begin{aligned} \sum_{i=1}^n (g(t_i) - g(t_{i-1}))^+ &\leq C\mathcal{L}^1([a, b] \cap \{f > g\}) + \sum_{i \in D} (f(t_i) - f(t'_{i-1})) \\ &\leq C\mathcal{L}^1([a, b] \cap \{f > g\}) + f(b) - f(a) \end{aligned}$$

So the desired result follows. □

Corollary 7.3. *For any $(a, b) \subset (0, T)$,*

$$(7.2) \quad (\partial_t g)^+((a, b)) \leq f(b) - f(a^+) + C\mathcal{L}^1((a, b) \cap \{f > g\}).$$

Proof. Apply previous lemma to $[a + \varepsilon, b)$, and send $\varepsilon \rightarrow 0$. □

Now we can prove Lemma 7.1 with the help of previous lemma.

Proof. Here $\partial_t f$ is the Radon measure defined on $[0, T]$ such that $\partial_t f([a, b]) = f(b) - f(a)$ for any $[a, b] \subset [0, T]$.

First we show that $\partial_t f$ is concentrated on the set $\{f = g\}$. We show that $\partial_t(\{f > g\}) = 0$. Indeed, if $t_0 \in \{f > g\}$, let $\beta_{t_0} = \sup\{t' \geq t_0 : f(t') = f(t_0)\}$. From the assumption, we know $\beta_{t_0} > t_0$. Also we know $\{f > g\} = \cup_{f(t) > g(t)} [t, \beta_t)$. Each component of the set $\{f > g\}$ contains a non-degenerate interval, hence there are only countably many components. Therefore, $\{f > g\} = \cup_i C_i$, where C_i are connected components, and has form $[a_i, b_i)$ or (a_i, b_i) . It is not hard to see that f remains a constant on C_i . Otherwise, by continuity from the left, and also the point(i), one can conclude $f(t) = g(t)$ for some $t \in C_i$, contradiction. So $\partial_t f(C_i) = f(b_i) - f(a_i) = 0$, or $\partial_t f(C_i) = f(b_i) - f(a_i^+) = 0$.

Next let $E \subset \{f = g\}$ be a Borel set, we want to show $\partial_t f(E) = (\partial_t g)^+(E)$. First we show $(\partial_t g)^+(E) \leq \partial_t f(E)$. Fix $\varepsilon > 0$, then from the outer regularity of the Radon measure $\partial_t f$ and \mathcal{L}^1 , we can find an open set U , with $E \subset U$, $\partial_t f(U - E) < \varepsilon$, and $\mathcal{L}^1(U - E) < \varepsilon$. Write $U = \cup_i (a_i, b_i)$, with (a_i, b_i) pairwise disjoint. Then we know $\sum_i \partial_t f((a_i, b_i)) \leq \partial_t f(U) < \varepsilon$, and $\sum_i \mathcal{L}^1((a_i, b_i) - E) < \varepsilon$.

Then from previous corollary, we know

$$\begin{aligned} (\partial_t g)^+(E) &\leq \sum_i (\partial_t g)^+((a_i, b_i)) \leq \sum_i (f(b_i) - f(a_i^+)) + \sum_i C \mathcal{L}^1((a_i, b_i) \cap \{f > g\}) \\ &\leq \partial_t f(E) + \varepsilon + C \sum_i \mathcal{L}^1((a_i, b_i) - E) \leq \partial_t f(E) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that $(\partial_t g)^+(E) \leq \partial_t f(E)$.

Now to prove the reverse inequality, Again we choose a cover $E \subset U$, with $U = \cup_i (a_i, b_i)$, (a_i, b_i) pairwise disjoint, such that $(\partial_t g)^+(E) \geq \sum_i (\partial_t g)^+((a_i, b_i)) - \varepsilon$.

We can assume $(a_i, b_i) \cap E \neq \emptyset$ for each i . Denote

$$\begin{aligned} a'_i &= \sup\{t \in (a_i, b_i) : (a_i, t) \cap \{f = g\} = \emptyset\}, \\ b'_i &= \inf\{t \in (a_i, b_i) : [t, b_i) \cap \{f = g\} = \emptyset\}. \end{aligned}$$

For those i with $b'_i < b_i$, we can decrease b_i , so that $f(b_i) - f((b'_i)^+) + |g(b_i) - g((b'_i)^+)| < \varepsilon 2^{-i}$. From the left continuity of f and g , one has $f(b'_i) = g(b'_i)$. Now if $b_i = b'_i$, then

$$(\partial_t g)^+((a_i, b_i)) \geq g(b_i) - g(a_i^+) \geq f(b_i) - f(a_i^+).$$

If $b_i > b'_i$, then

$$\begin{aligned} (\partial_t g)^+((a_i, b_i)) &\geq (g((b'_i)^+) - g(b'_i))^+ + (g(b'_i) - g(a_i^+))^+ \\ &\geq f((b'_i)^+) - f(b'_i) + f(b'_i) - f(a_i^+) \geq f(b_i) - f(a_i^+) - \varepsilon 2^{-i}. \end{aligned}$$

So sum up, we get

$$\sum_i (\partial_t g)^+((a_i, b_i)) \geq \sum_i (f(b_i) - f(a_i^+)) - \varepsilon \geq \partial_t f(E) - \varepsilon.$$

So the proof is complete. \square

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