# The 3-dimensional wave equation derived from Maxwell's equations

Maxwell's equations are

$$\nabla\cdot\underline{E} = \rho/\epsilon_0, \quad \nabla\cdot\underline{H} = 0, \quad \nabla\times\underline{E} = -\mu_0\frac{\partial\underline{H}}{\partial t}, \quad \nabla\times\underline{H} = \underline{J}+\epsilon_0\frac{\partial\underline{E}}{\partial t}.$$

In free space (no charge, no current) these become

$$\nabla \cdot \underline{E} = 0 \quad \text{(a)}, \qquad \nabla \cdot \underline{H} = 0 \quad \text{(b)},$$

$$\nabla \times \underline{E} = -\mu_0\frac{\partial\underline{H}}{\partial t} \quad \text{(c)}, \qquad \nabla \times \underline{H} = \epsilon_0\frac{\partial\underline{E}}{\partial t} \quad \text{(d)}.$$

Now, consider problem (5) in chapter 14, which asks you to show, for any vector field $\underline{B}$, that $\nabla \times (\nabla \times \underline{B}) = -\nabla^2\underline{B} + \nabla(\nabla \cdot \underline{B})$. Here is how this is done.

First, note that a general vector field, that depends on the three space co-ordinates and on time, $t$, can be written as

$$\underline{B} = \underline{i}B_x(x,y,z,t) + \underline{j}B_y(x,y,z,t) + \underline{k}B_z(x,y,z,t),$$

where $B_x$, $B_y$ and $B_z$ are any differentiable functions of $x, y, z$ and $t$.

Now consider the question: what is $\nabla^2\underline{B}$? Writing it out in full, we have

$$\nabla^2\underline{B} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) \cdot \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right)\underline{B}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\underline{B}$$

and expanding this, we get

$$\nabla^2\underline{B} = \underline{i}\left(\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_x}{\partial y^2} + \frac{\partial^2 B_x}{\partial z^2}\right) + \underline{j}\left(\frac{\partial^2 B_y}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_y}{\partial z^2}\right)$$

$$+\underline{k}\left(\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2}\right).$$

Now to the identity to be proved in question (5). First of all,

$$\nabla \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - \underline{j}\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + \underline{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right),$$

and so

$$\nabla \times (\nabla \times \underline{B}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) & -\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) & \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \end{vmatrix}.$$

The $x$-component of this is

$$\underline{i}\left[\frac{\partial^2 B_y}{\partial y \partial x} - \frac{\partial^2 B_x}{\partial y^2} - \frac{\partial^2 B_x}{\partial z^2} + \frac{\partial^2 B_z}{\partial x \partial z}\right]$$

$$= \underline{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) - \frac{\partial^2 B_x}{\partial y^2} - \frac{\partial^2 B_x}{\partial z^2}\right]$$

$$= \underline{i}\left[\frac{\partial}{\partial x}\left(\nabla \cdot \underline{B} - \frac{\partial B_x}{\partial x}\right) - \left(\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_x}{\partial y^2} + \frac{\partial^2 B_x}{\partial z^2}\right) + \frac{\partial^2 B_x}{\partial x^2}\right]$$

$$= \underline{i}\left[\frac{\partial}{\partial x}(\nabla \cdot \underline{B}) - \left(\frac{\partial^2 B_x}{\partial y^2} + \frac{\partial^2 B_x}{\partial y^2} + \frac{\partial^2 B_x}{\partial z^2}\right)\right].$$

You should recognise the first term in the above as the $x$-component of $\nabla(\nabla \cdot \underline{B})$ and the last three terms as the $x$-component of $\nabla^2 \underline{B}$. Applying the same argument to the $y$- and $z$-componets, we finally obtain

$$\nabla \times (\nabla \times \underline{B}) = -\nabla^2 \underline{B} + \nabla(\nabla \cdot \underline{B}). \qquad (1)$$

Now we come back to Maxwell's equations in free space. Taking the curl of (c), we have

$$\nabla \times (\nabla \times \underline{E}) = \mu_0 \frac{\partial}{\partial t}(\nabla \times \underline{H})$$

and using (d) to replace $\nabla \times \underline{H}$, we obtain

$$\nabla \times (\nabla \times \underline{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}.$$

Our identity, equation (1), then tells us that

$$\nabla \times (\nabla \times \underline{E}) = -\nabla^2 \underline{E} + \nabla(\nabla \cdot \underline{E}) = -\nabla^2 \underline{E},$$

since, by (a), $\nabla \cdot \underline{E} = 0$ in free space. Hence, finally,

$$\boxed{\nabla^2 \underline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}}$$

which is the 3-dimensional wave equation.