A Model of Political Voting Behaviours across Different Countries

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Abstract

This paper analyses, models mathematically, and compares national voting behaviours across seven democratic countries that have a long-term election history, focusing on re-election rates, leaders’ reputation with voters and the importance of friends’ and family influence. A three-state Markov model is fitted to election data to describe the relative transition rates of individual political leaders and political parties between the states: elected for the first time, re-elected once and re-elected more than once. It is found that this model fits the data well. The model supports the hypothesis that the re-election rates of leaders are determined by the number of times a leader has previously been elected, irrespective of the number of years that the leader has been in power. The seven countries can be divided into those in which there is a high probability that leaders will be re-elected and those in which incumbents have relatively less success.

A simple phenomenological dynamical model of electoral districts in which voters may be influenced by social neighbours, political parties and political leaders is then created to explore differences in voter behaviours in the countries. It is found that the model can fit the election data and supports the thesis that an unsuccessful leader has a greater negative influence on individual voters than a successful leader. Furthermore, the model shows that increasing the influence on voters of social neighbours leads to a decrease in the average re-election rate of leaders, but has no impact on the average amount of time the dominant party is in charge.

Keywords: Sociophysics, voting behaviour, hierarchical system, statistical model, complex science

1 Introduction

Understanding national voting behaviour and analysing various influences on voters has a long history of research in terms of developing fair election systems \cite{41, 44, 48, 53}, but also due to it being a prototypical example of a complex system \cite{47, 51, 52}. In order to explore these various influences, several models (both social science and mathematical) have been proposed looking at basic forms and constitutions of Government \cite{6, 8, 15, 26, 37}, voter systems \cite{12, 14, 16, 35, 42} and election behaviours \cite{20–22, 50}.

Research on voting behaviour \cite{10, 27, 28, 39, 49, 56} has shown that the reputation of a political party’s leader has an influence on voters. Studies frequently show that a leader’s unsuccessful reputation reduces the chances of re-election more than a successful one enhances them \cite{10, 11, 28, 49}. Voters’ perceptions of leadership also shape their evaluation of party policy \cite{27, 28, 39, 49, 56}.

Differences between voting behaviour in various countries have also been observed due to a variety of cultural and constitutional differences. In the past, in the United Kingdom and Australia, voters were more influenced by factors including class, age, gender, religion, and ethnicity \cite{1, 24, 40}, whereas a more general observation is that western European voters elect parties that present clear political alternatives \cite{35}. It was also observed
that throughout history, American voters were strongly influenced by their family’s party preferences [5]. Hence, there is a complex interaction of factors and influences determining the outcome of individual elections.

In order to untangle the intricate reasons for voter behaviour, several simple conceptual voting models have been proposed. One of the first was proposed by Campbell et al. [9], who used the layout of the funnel of causality model to describe voting behaviour in a two-party system at a point in time, such that current understanding of voting influences can be understood. In Campbell et al.’s model, voters inherit their political party preference from their parents. This party identification then influences the individual’s future affiliation’s to a party which is based on how the individual regards both parties, the issues of domestic and foreign policy and also compares how the two parties deal with governmental affairs. All these characteristics influence the individual’s voting decision. This simple model was able to predict 87% of voting decisions, suggesting that American voters are significantly more influenced by their families than by party policy. However, Campbell et al.’s conclusions have been criticised on the grounds that they used unrepresentative election results from the 1950s and that voters are modelled as uninformed and lacking interest in politics, with the result that every voter’s party preference was represented as being entirely dependent on loyalty towards a party to the exclusion of other individual influences, such as class, age and gender [46]. Furthermore, national influences such as the perception of a leader or media are not included but are known to have major impacts on voter behaviours; see [10]. Despite all these issues, Campbell et al.’s model is widely used due to it explaining a large part of American voter behaviour and supporting the hypothesis that American voters are “mostly” influenced by their immediate social network.

Galam [19–22] proposed several simple mathematical voter models in order to investigate the effect of various individual influences on group decision and voting. Instead of analysing party preferences, their hierarchical model analyses the election process where voters choose one of the two proposed policies. Once the policy preference has been chosen, a random voter is selected to represent the policy in the above hierarchical level. Then the voters gather together to form various groups. Each voter selected keeps their policy preference, but again, one of the elected is randomly chosen and put forward into the following level. This process continues until only one member is elected to represent their policy. Galam proposed using the classical Physics Ising model to describe how voters make policy choices [19]. In the simplest form of this model, the voters can choose between one or two policies and a voter’s preference is influenced by an initial choice and the votes of their closest neighbours, e.g. family/friends. It is found that depending on the strength of the influence of a voter’s immediate social network one either observes the group of voters selecting one policy with a 50% preference, or (for strong influence) regions where voters all vote for one policy i.e., segregation. While these voting models have allowed researchers to investigate some of the fundamental underlying mechanisms for voter behaviour, the models generally only capture local individual influences on voter behaviour and ignore the national influence of media/leaders. Furthermore, few of these models have been compared to election data and so there is a need to develop models that explore conceptual mechanisms for voter behaviour while also being able to reproduce gross averages of the election data.

The aim of this paper is to first look at election rates of seven different countries with a long democratic election history. We will concentrate on election averages rather than individual elections throughout this paper hence the effect of precise election policies/manifestos are significantly reduced. One naturally expects that subsequent elections depend on those that went before and hence to investigate this time dependence, we carry out a detailed data analysis by fitting a simple three-state Markov model to the election data describing the transition from first-time elected, re-election once and re-elected multiple times of individual leaders. We find that such a model fits the election averages extremely well, suggesting that a fundamental mechanism for voter behaviour is based on how many times a leader has been elected and that a suitable time-scale is per election. Furthermore, due to the excellent model fit, we are able to carry out a detailed analysis and comparison of voting behaviour in the seven different countries. Since we concentrate on election averages and have largely ignored policies/constitutions, one possible explanation for the similarities and differences seen in the voting behaviour is due to the influence of social network and a leader’s reputation on individual voters. In order to investigate this hypothesis, we construct a voting model that incorporates the fundamental voter
mechanism found from the Markov model analysis with the influence of family/friends/etc. and a leader’s reputation on voters using Galam’s model as a basis. The voting model carries out time steps every election and is inline with the time-scale found from the Markov model analysis. We then compare our voting model to the election averages to investigate different national voting behaviours.

Neither the Markov model or voting model takes the choice of constitution into account, even though it is known that it does impact the voting turn out and indirectly the national voting behaviour. While our models will be very simple, we find that we can draw several important conclusions about various national voting behaviour and some basic voting mechanisms. Crucially, we note that we are not attempting to predict a single election but rather explain some common mechanisms for voter behaviour that can explain the similarities and differences seen across different countries.

The outline of the paper is as follows. With the aid of a Markov model, §2 analyses the data on the seven national election histories. In §3 a voting model is constructed and simulated to investigate the combined effects of a leader’s reputation and a voter’s social network influence on overall voting behaviour. Finally, in §4 we discuss the results and draw conclusions.

2 Election Data and Analysis

2.1 Data

In this section, we analyse the election data of a democratic voting system. A democratic system is divided into three powers: legislative, executive, and judiciary. Part of the legislative power consists of the legislators representing their political parties, who are elected by the voters in electoral districts. The democratic leader of the country is a member of the executive council and is the party leader whose political party has the most representatives in the legislative. The final power is the judiciary, who interprets the laws of the country. This power is not taken into consideration in this research. The focus of the data collected was on the election results for the executive council from seven democratic countries, in particular on the status of each country’s leader, whether newly elected, re-elected once, or re-elected multiple times. The countries chosen have had at least 20 democratic elections and have a near universal suffrage. Two of the oldest democratic voting systems are France and the USA allowing us to analyse a large number of election results. The five original Commonwealth countries Australia, Canada, Republic of Ireland, New Zealand, and the UK all have similar government structures but different to France and the USA, and also have long election histories\(^1\). The election data for other democratic countries were not considered as they have had few general elections, yielding high, volatile election averages. Hence, we decide to concentrate on France, USA, Australia, Canada, Republic of Ireland, New Zealand, and the UK countries; for the sources for the data see [2–4, 7, 13, 15, 17, 23, 25, 26, 29–33, 38, 45, 54, 55]. The election results are collected up until the election year 2011. As the leader’s term in office in the year 2011 had not come to an end, the leader’s status of being elected, re-elected once or re-elected multiple times is not included in the data.

We will simplify each country’s government system to a two-party system, where the two parties are the dominant party and the opposition. For some countries like the USA, throughout history only two parties have been in power, therefore making their political system a good fit for our research. Other countries for example France, have had numerous political parties in charge throughout history and to fit our classification of a two-party system to the French voting history, we subjectively classified the parties to have either left or right wing political views. Doing so allows us to categorise them into a two party system. The party in charge for more than 50% of the time is defined as the dominant party.

In Table 1 and Table 2, we list the percentages that the dominant party is elected to power and the chance of a party leader is re-elected for the original Commonwealth countries and France and the USA, respectively. Percentages are calculated in terms of number of elections.

\(^1\)Although South Africa was part of the original Commonwealth, we have decided to exclude South Africa due to the apartheid system during the 20th century.
<table>
<thead>
<tr>
<th></th>
<th>Australia</th>
<th>Canada</th>
<th>Rep.of Ireland</th>
<th>New Zealand</th>
<th>UK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dominant party in charge (%)</td>
<td>61.5</td>
<td>57</td>
<td>58</td>
<td>62.5</td>
<td>60</td>
</tr>
<tr>
<td>Chance of being re-elected (%)</td>
<td>51</td>
<td>63</td>
<td>53</td>
<td>66</td>
<td>47</td>
</tr>
</tbody>
</table>

Table 1: List of the percentages in terms of elections for the five original Commonwealth countries.

<table>
<thead>
<tr>
<th></th>
<th>France</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dominant party in charge (%)</td>
<td>51.8</td>
<td>65.9</td>
</tr>
<tr>
<td>Chance of being re-elected (%)</td>
<td>21.4</td>
<td>30.2</td>
</tr>
</tbody>
</table>

Table 2: List of the percentages gathered by the election results from France and the USA.

In both tables we see that the dominant party was elected between 52-66% of the time for all countries. However, the biggest differences in the election data can be seen from the re-election percentages of the democratic leadership varying from around 47-66% for the five Commonwealth countries but only between 21-30% for France and the USA. Even though the leaders of the USA can only be re-elected once since the 1950s, prior this only President Roosevelt was re-elected multiple times.

It is perhaps not unexpected that the election percentages for the five original Commonwealth countries are almost identical given that they have similar election systems. However, the fact that France and the USA have a dominant party in charge with a percentage similar to the five original Commonwealth countries, but have a significantly lower re-election percentages raises several interesting questions. In order to analyse and explore the differences and the effect of previous elections in the re-election percentages we will first try and fit a simple Markov model to the data.

### 2.2 Markov Model

To investigate the data and identify further possible underlying mechanisms for the voting behaviour observed in §2.1, we construct a simple three-state Markov model describing the time evolution from one election to the next keeping track of the probabilities of election/re-election and re-election more than once, of a democratic leadership and also of the dominant and opposition party. We then fit the Markov model to the election data and investigate the goodness-of-fit of the model’s results to the data. The Markov model then allows us to investigate the voter behaviour in each country and what this means in terms of the underlying voter behaviour for each electorate.

The Markov model is set-up as follows; see Figure 1 for a diagram of the model depicting the transition probabilities from each of the three states (newly elected, re-elected and re-elected more than once). Let $x_n \in \mathbb{R}^3$ be the three probabilities of a democratic leadership or party being newly elected, re-elected once and re-elected multiple times at each election $n = 1, 2, 3, \ldots$ i.e.,

$$x_n = [\text{newly elected}, \text{re-elected once}, \text{re-elected multiple times}]^T,$$

where $T$ denotes the transpose of a vector. The probabilities $x_n$ are updated at each election via the Markov model based on the previous election i.e.,

$$x_n = P x_{n-1},$$

where $P$ is a $3 \times 3$ probability matrix describing the transition probabilities between the three states (newly elected, re-elected and re-elected more than once), and is given by

$$P = \begin{bmatrix}
P_{11} & P_{21} & (1 - P_{31} - P_{32}) \\
(1 - P_{11}) & P_{22} & P_{32} \\
0 & (1 - P_{21} - P_{22}) & P_{31}
\end{bmatrix}.$$
Let us briefly describe the choice of the probability matrix and the diagram of the model in Figure 1 in terms of leaders being elected. At the start of each election history there will always be a newly-elected democratic leader. At the next election, with a probability $P_{11}$ another newly-elected democratic leader is possible or the original leader is re-elected with a probability $1 - P_{11}$ since these are the only two possibilities. Once a leader has been re-elected once, there are three possible outcomes at the next election, namely a new leader is elected (with probability $P_{22}$), the leader is re-elected once with probability $P_{21}$, or the leader is re-elected more than once with probability $1 - P_{21} - P_{22}$. The transition of $P_{21}$ describes when a democratic leader has already been re-elected once, but for some reason cannot finish their term, then an unelected leader takes their place. If this leader is elected in the following election, it is considered they are re-elected for the first time, not just elected, as the unelected leader was already in power before the election. Once a democratic leader has been re-elected more than once, the following are possible: the leader is re-elected again with probability $P_{31}$, a leader goes to being just re-elected once (which occurs when an unelected leader comes to power during a term), or a newly elected leader is elected. This model is also suitable to analyse the election/re-election of the dominant and opposition party.

We note that one could argue that the transition probabilities $P_{21}$ and $P_{32}$ should not be there if one is considering just a single individual or party. However, we are interested in uncovering fundamental mechanisms for elector behaviour and one possible influence is that being in power is significantly more advantageous. Hence, if an unelected leader comes to power before the next election, then the benefit/hinderance of being in power during the election should be taken account of.

In order to fit and calculate the probability transition matrix $P$, for each country we calculate the entries of $P$ via the equation \[ P_{ij} = \frac{m_{ij}}{m_i}, \] where $i$ and $j$ denotes one of the three states, $m_{ij}$ is the total number of leaders up to a certain election $n$ who made the transition from state $i$ to state $j$ from the data, and $m_i$ is the total number of leaders in state $i$ also from the data. This calculation is done for each election for all data up to that election. This leads to a maximum likelihood fit of the Markov model to the data and is equivalent to carrying out a standard

Figure 1: An overview of the Markov model showing the three states of a leader/party and the corresponding transition probabilities.
Least Squares fit of the model with the data [36]. One highly useful statistic a Markov model yields is the steady state equilibrium distribution of the Markov model, which is evaluated as the number of elections tend to infinity i.e. $n \to \infty$. By comparing the steady state equilibrium distributions to the data allows us to find out if the voters behaviour is stable over time or still changing. In order to calculate the steady state equilibrium distribution from a Markov model one has to solve the steady state relation

$$x = Px,$$

where $x$ is the steady state probability vector. Solving (2.3) is equivalent to finding the unit eigenvector of $P$ and re-normalising the eigenvector so the probabilities add up to one. When presenting our findings, we plot the effect of fitting the elections up to a certain election with (2.2) on the steady state equilibrium distribution of the Markov model. We then test the goodness-of-fit by carrying out a standard Chi-square test.

2.3 Markov Model Results

We present the analysis of the fitted Markov model to each country’s election data of a leader and party. In Figure 2, we plot the expected steady state equilibrium distribution from the Markov model fit for a democratic leader to be newly-elected, re-elected once, and re-elected more than once. In each of the figures, the solid line indicates the steady state equilibrium distribution from the Markov model fit of a new leader being elected, the asterisk line the steady state equilibrium distribution of the current leader being re-elected once, and the thin dotted line the probability of the current leader being re-elected multiple times. The highlighted areas indicate the elections that took place during a World War, and the solid black line in the France figure denotes the interruption to French government due to the German invasion of France during World War II.

For all the countries we see in Figures 2, an initial transience of the computed steady state equilibrium distribution due to the small number of elections initially used in the fit but as the number of elections increases beyond ten then the computed probabilities settle down suggesting stable voting behaviour in each country. The computed election probabilities for all the countries apart from the USA remain largely unaffected by the World Wars. In the USA’s case, the spike in the re-election probability was due to the election of Franklin D. Roosevelt to a fourth term in office.

Since the voting behaviour for each of the countries appears to have stabilised after ten elections, it strongly suggests that the voting behaviour for each country is robust as addition of more data points would make little change to the computed probabilities. In order to see if the voting behaviour of each country has changed over time or if the change in the probabilities is a “small numbers” issue we used the same fitting method but reversing the data points by using the most recent election as the initial election. We found that in this case the steady state equilibrium distribution converged after ten elections, indicating that the initial instabilities before the first ten elections are due to the small numbers of elections used rather than any change in voting behaviours.

The computed probabilities highlight some countries with similar and different voting behaviours. In particular, both Canada and New Zealand have steady state equilibrium distribution of being newly elected, re-elected once and re-elected more than once roughly equal at a third. For all the countries except Canada and New Zealand the steady state equilibrium distribution of a newly elected leader is higher than the other two probabilities. In particular, for France and the USA the probability of a newly elected leader is very high > 60% compared to all the other countries.

For both the Republic of Ireland and New Zealand we see that for a long time (up to 20 elections) the probability of being re-elected multiple times was significantly higher than the other two probabilities. France, the UK and the USA all have a significant probability of a leader being re-elected once but a negligible probability of a leader being re-elected more than once suggesting that out of all the original Commonwealth countries, the UK has a voting behaviour closest to France and the USA. However, if one
Figure 2: The steady state equilibrium distribution for voter behaviour for leaders in the Commonwealth countries, France and the USA in the three-state Markov model.
sums up the probabilities of re-elected once and many-times then we find that for all the Commonwealth countries the chance of a leader remaining in power is significantly greater than a new leader being elected, as observed in §2.1.

In order to test the goodness-of-fit of the Markov model and the choice of a three-state model over a two- or many-state Markov model, a Chi-square test was performed on all of the calculated steady state equilibrium distribution results for a two-state up to eight-state Markov model. The maximum number of states considered is eight, as no leader was elected/re-elected more than eight times. The Chi-square test looks at the expected frequencies of a leader being in one of the states calculated by the model and compares them with the frequencies observed in the data. From the Chi-square test, one is then able to compute the goodness-of-fit for all the countries for a two- up to eight-state Markov models. It is found that a two-state Markov model poorly captures the voter behaviour with the goodness-of-fit for New Zealand evaluated at \( p \sim 0.01 \). On the other hand, the results for a three-state Markov model the goodness-of-fit is \( 0.88 < p < 1 \) for all countries. As expected, the goodness-of-fit increases as the number of states increased. However, the three-state Markov model describes all the countries’ voting behaviours with excellent accuracy while remaining simple enough to yield some fundamental mechanisms and similarities of various country’s voting behaviours.

Finally, we focus on the election/re-election of the political parties. In Figure 3, we plot the expected steady state equilibrium distribution from the Markov model fit for a party to be newly-elected, re-elected once, and re-elected more than once. As in Figure 2, the fit and corresponding steady state equilibrium distribution are re-computed after every election using the data up to that election to form the probability matrix and calculated the resulting probabilities for each state via solving (2.3). Again, in each of the figures, the solid line indicates the steady state equilibrium distribution from the Markov model fit of a new party being elected, the asterisk line the steady state equilibrium distribution of the current party being re-elected once, and the thin dotted line the probability of the current party being re-elected multiple times. The highlighted areas indicate the elections that took place during a World War.

For France and New Zealand the steady state equilibrium distribution does not seem to have stabilised, which could be due to our arbitrary classification of the parties from the data. The party election rates for all the other countries is stable. It can be seen that the parties are more likely to be re-elected multiple times, except for the UK and the USA. For the UK there is a higher chance of the opposition party being elected, whereas for the USA the election of the opposition party and a party being re-elected multiple times occurs at a similar rate. The results for a three-state Markov model analysing the party election, the goodness-of-fit for the steady state equilibrium distributions for all countries are between 0.87 < \( p < 0.98 \). Comparing Figure 3 to Figure 2 shows the party election differs to the leaders steady state equilibrium distribution. Therefore, when analysing voting behaviour a distinction between party and leader election needs to be made.

Overall, we have found that the voter behaviour for each country’s leader is stable and that a simple three-state Markov model dependent only on the previous election and if a leader had been re-elected or not, provides a good fit of the election data and suggests that the “memory” in the system beyond the most recent past elected does not matter. This suggests that an underlying fundamental mechanism for voter behaviour is the basis of looking at who was in power in the previous term. Given the similarities of the French and the USA voting behaviour compared to that of the Commonwealth countries it appears that being in power is a major benefit to leaders (which is not influenced by the leader’s party) in the Commonwealth countries and is a hinderance for leaders in France and the USA. This analysis raises several interesting questions. What are the fundamental influences on voting behaviour? Since we have ignored differences in the constitution of each country, we look at possible mechanisms that would be generic. As mentioned by other researchers [5, 46], one hypothesis for the differences in the voter behaviour is that the country’s voters are more strongly influenced by their family/friends/social network. Another possible explanation for the different re-election rates of leaders is the impact of the reputation voters. As shown in research [10] voters perceive the leader to be either successful or unsuccessful, which could have various degrees of impact on the voters. In order to construct a model to examine these two hypotheses, we will need to construct a voting
Figure 3: The steady state equilibrium distribution for voter behaviour for political parties in the Commonwealth countries, France and the USA in the three-state Markov model.
model that incorporates the fundamental voter behaviour mechanism described by the three-state Markov model and the influence of family/friends/etc. and reputation of a political leader on individual voters.

3 Voting Model

In the previous section, the voting behaviour of each county was analysed, showing the Commonwealth countries to have similar voting behaviour, as do France and the USA. As noted in the previous section, a possible explanation of these similarities and differences of voter behaviour is due to the influence of family/friends/etc. and leader’s reputation on individual voters. Hence, in order to explain the election data we need a model that incorporates these influences on individual voters but also has the underlying mechanism that voters decide who to elect based on who was previously in power. Since these mechanisms operate at different stages in an election one needs to construct a descriptive model of the election process.

The voting model is broken down into three stages or levels, namely the micro-, meso- and macro-levels; see Figure 4 for an overview of the model. At the micro-level, individual voters are grouped into \( N_e \) equal-sized electoral districts with \( N_v \) voters in each district. The voters in each electoral district are modelled using a modified version of Galam’s model that incorporates influence from nearest social neighbours, their previous vote and the success score \( \beta \) from the previously elected leader. In each of these electoral districts, each voter elects one of two parties (denoted by \(-1\) and \(+1\)) and the mean of the vote in each district determines which legislator is elected into the meso-level, where two parties are formed.

The meso-level represents the legislature. The political party in charge before the election keeps the democratic leader as the party leader only if their term was successful; the success of a leader’s term in office is randomly scored to be either successful or unsuccessful. If it was unsuccessful, the party votes for a new party leader. Each party elects a leader based on a randomly assigned leadership score from a standard Uniform distribution assigned to each legislator (this is only re-assigned if a new legislator is elected i.e., there has been a change of party elected in an electoral district), and on each legislator’s political complexion, which
is based on the mean vote in their electoral district. The leadership battle is then decided by the following system. The two legislators in a party with the highest leadership score become the two candidates that the rest of the legislators in the party will vote for. The legislators then vote for the leader who has the closest political complexion to there own. The motivation for this type of leadership election is as follows. If most of the legislators elected are very secure in their seats then they are likely to have strong views in order to reflect their electorate and hence they will look for a party leader who also reflects those strong views and visa-versa.

The leader of the party with the most legislators is selected as democratic leader for the executive council in the macro-level. Based on the leadership score of the democratic leader, it is determined if the leader has had a successful tenure (based on whether a random number is lower/higher than the leader’s leadership score) and this forms the basis for the influence on the voters in each of the electoral districts. The leader’s reputation has an either successful or unsuccessful term denoted as $\beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$, respectively, influencing the voters in the following election. In both cases, if the values of $\beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$ are positive then the voter’s will experience an increase in preference to vote for the opposition. Only two types of reputation are considered, as research has shown that voters perceive the leader’s term in office to be either successful or unsuccessful, see [10]. We note that the implementation of a negative reputation on individual voters and that a voter’s next vote is partially based on their previous vote, reflects the fundamental mechanism uncovered by the Markov model since if a leader is re-elected and has a negative impact on voters then in the next re-election their support will decrease.

During one execution of the model one democratic leader is elected and to obtain an election history one runs the model multiple times. Since we use random variables, the election history needs to be re-run multiple times in order to yield election averages that we can compare with the election statistics in §2.1.

We will now describe in more detail the modification of Galam’s model [18, 34, 43] that we use in each electoral district at the micro-level of the model. A tension $T_i$ is created between an individual voter’s party preference and their neighbour’s voting decision. If the individual voter and their social neighbours are likeminded then there is no tension between them, whereas if the individual voter has a different party preference to their social neighbours then there is a larger tension between them. The coupling constant $J$ determines the strength of the nearest social neighbours on their own vote, thus impacting the tension. An external influence given by the election district’s previously elected legislator’s party and the democratic leader’s reputation also influence the individual voter’s party preference. The tension for each voter is given by

$$T_i = -\frac{J}{2} s_i \sum_{<iz>} s_z - \mu s_i,$$

(3.1)

where $s_i = \pm 1$ is the vote of each voter $i$, $J$ is the strength of the nearest social neighbours’ influence, $<iz>$ restricts the individual voter $i$ to be influenced by its $z$ nearest neighbours and $\mu$ is the strength of the external influence from the leader. A small influence of a voter’s nearest neighbours corresponds to a small value of $J$. The external influence is chosen to be how the electoral district previously voted and if the previous overall party in power is $\pm 1$ multiplied by the reputation score of the leader. In particular, if an electoral district voted the same way as the overall outcome of the election, then it decreases the tension in the system. On the other hand, if an electoral district voted counter to the overall outcome of the election then there is an increased tension for the voters to choose the opposition. Hence, the external influence is defined as

$$\mu = \text{sign}(\bar{s}_p) - \beta \sum_{j=1}^{N_c} \text{sign}(\bar{s}_p),$$

(3.2)

where $N_c$ is the total number of electoral districts, $\bar{s}_p$ the mean vote in the previous election of the voter’s electoral district, $\bar{s}_p$ is the mean vote in the previous election of all electoral districts, and $\beta$ is a number given by

$$\beta = \begin{cases} 
\beta_{\text{suc}} & \text{leader’s term was successful} \\
\beta_{\text{unsuc}} & \text{leader’s term was unsuccessful}
\end{cases}$$
As previously noted, both $\beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$ have a negative effect on voters if chosen to be positive values. In order to determine if a leader was successful or not, we draw a random number from the standard Uniform distribution and determine that the leader was successful if this random number is less than the leadership score of the leader and unsuccessful otherwise.

We use the Boltzmann distribution to determine the probability that an electoral district is in a certain configuration (i.e., voted in a certain manner)

$$P(s) = \frac{e^{-\sum_{i=1}^{N_v} T_i}}{\sum_{s=\pm1} e^{-T_i}}$$

where $N_v$ is the number of voters in an electoral district. We have set the “temperature” and “Boltzmann constant” in Boltzmann distribution to unity. The Boltzmann distribution is chosen, as the most likely voter configuration is found by maximising the probability (3.3) that is equivalent to minimising the total tension ($\sum_{i=1}^{N_v} T_i$) for each electoral district. This minimisation is carried out using the Metropolis-Hastings algorithm.

For the first two elections, the external influence $\mu$ is set to zero so that the voters are only influenced by their neighbours, as in the first election their is no leader. For the third and later elections, the external influence $\mu$ has an effect on the voters.

### 3.1 Initial analysis and calibration

Before we run simulations of the voting model, we first need to narrow down the parameter space we wish to explore. In particular, it is not clear what values one should take for the nearest social neighbour influence parameter $J$ or the values determining the strength of the leader’s influence $\beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$. In order to ascertain a range for the parameters $J, \beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$, we first start to explore a single electoral district looking in particular at the mean vote in an electoral district with infinitely-many voters to investigate the effect of an external influence from a leader.

Let us first describe the two general types of behaviour we expect of an individual electoral district model. For a small coupling constant $J$, we expect the tension of each voter to be governed mostly by the leader’s influence. Starting from a uniform random distribution of voters, the mean vote $\bar{\pi}$ in an electoral district will be approximately zero. Hence, the external influence from the leader is also approximately zero and the mean vote in an electoral district will remain close to zero. For a large coupling constant $J$, the influence of the leader can be neglected leading to clusters of similar minded voters. In this case, we expect the dominant party to be in charge all the time. The point at which a bifurcation occurs, where the mean vote of an electoral district turns form 0 to $\pm 1$, is at the critical coupling constant $J_c = 0.44$ for a system with no external impacts. So that our model can have changes of government, we require the coupling constant to be close to the critical coupling constant $J_c$ where clustering occurs. The influence of the leader will change the value of the $J_c$ and we find an approximation for how this critical value changes.

In order to find the approximate effect of the external influence on how the electoral district will vote, we carry out a mean field approximation of the tension for each voter, $T_i$ by

$$T(s_i) \approx -\left(\frac{Jz}{2}\bar{\pi} + \mu\right) s_i =: T_{\text{approx}} s_i$$

where the nearest neighbour interaction is approximated by the mean vote and $\mu$ is given by (3.2). Now since the tension of each voter is decoupled, the probability for a single voter $s_i$ is given by

$$p(s_i) = \frac{e^{-T_{\text{approx}} s_i}}{e^{-T_{\text{approx}}} + e^{T_{\text{approx}}}}$$

and in order to have consistency with the mean-field approximation, the expected mean value vote calculated...
from the probability distribution by (3.5) should be equal to the mean vote i.e.,

$$\bar{s} = \sum_{s_i = \pm 1} p(s_i) s_i = \tanh(T_{\text{approx}}). \quad (3.6)$$

Hence, we have the mean field equation

$$\bar{s} = \tanh \left( \frac{Jz}{2} \bar{s} + \mu \right), \quad (3.7)$$

to solve. The effect of the external influence $\mu$, will lead to an $\bar{s}$ that is the same sign as $\mu$; see Figure 5 for a graphical sketch of the solutions of (3.7) for $\mu > 0$ and $\mu < 0$ and $Jz$ small/large. Hence, for small $\beta$, there is an increased preference to vote for the same overall outcome ($\pm 1$) as before and a constantly elected dominant party is expected to occur.

Figure 5: Blue line is the lefthand side (3.7) and red line is the righthand side of (3.7). A solution of (3.7) corresponds to intersections of the two graphs denoted by circles for (a) $Jz < 1$ and $\mu > 0$ (b) $Jz < 1$ and $\mu < 0$ (c) $Jz > 1$, $\mu > 0$ and (d) $Jz > 1$, $\mu < 0$. Linear stability is denoted full circles and linearly instability denoted by empty circles.

One may also consider the case where all the electoral districts vote with the same mean $\bar{s}$ where $\bar{s}$ is very close to $\pm 1$. In this case, we are able to consider the effect of multiple runs of the model. For $\bar{s}$ close to $\pm 1$, we may adapt $\mu$ by the following smooth function

$$\mu = \text{sign}(\bar{s})(1 - \beta) \approx \tanh(\bar{s})(1 - \beta). \quad (3.8)$$

A necessary condition for a non-trivial solution of (3.7) is that the derivative of the righthand side of (3.7) at the origin $\bar{s} = 0$ is greater than unity. This leads to the condition

$$\frac{Jz}{2} + (1 - \beta) > 1, \quad \Rightarrow \quad J > \frac{2\beta}{z}. \quad (3.9)$$

In this case, we expect only one leader to get elected and be re-elected for a coupling constant $J$ satisfying (3.9). For one to satisfy the condition $|\bar{s}| \approx 1$, we require $J$ to be significantly larger than this bound but it gives us a starting point for model investigations.
Although the critical coupling constant $J$ cannot be determined, equation (3.9) highlights the gross effect that the reputation $\beta$ and the coupling constant $J$ on the system. In particular, it can be seen for a reputation $\beta > 0$ has a negative impact of voters, thus reducing the chance of a leader to be re-elected, whereas for $\beta < 0$ increase their chance. Therefore, for the model to recreate the different voting behaviours observed in §2.1 both parameters need to be varied.

### 3.2 Political Voting Model Results

For all our simulations, we choose the number of electoral districts $N_c = 11$ with the number of voters $N_v = 100 \times 100$. The voters in each electoral district are set on a torus, where each voter is influenced by the four closest neighbours to the north, east, south and west. The model produces stochastic results, so multiple runs are performed and the results averaged. As shown in §2.3 when analysing voting behaviour a distinction between leader and political party election needs to be made. The results of the voting model present averages of the number of re-elected leaders and how often the parties are in charge over 50 elections.

We start by first looking at the effect of the leadership influence, $\beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$, and the nearest neighbour influence, $J$, on the election averages from the model. In Figure 6 we show a two parameter plot of the re-election rate of a leader varying the parameters $J$ and $\beta_{\text{suc}}$ and the unsuccessful reputation parameter set to one i.e., $\beta_{\text{unsuc}} = 1$ or in words, an unsuccessful leader has a strong impact on the voters. To counteract the influence of the unsuccessful leader, the successful reputation is set to have a positive impact on voters ranging from 0, such that the successful reputation has no impact, to $-0.9$ leading to the successful reputation having a stronger impact on the re-election of the leader. Note that the successful reputation is not as strong as the unsuccessful one, as from research it is known that the unsuccessful reputation has a stronger influence on voters than a successful one [10, 11, 28, 49]. From Figure 6 it can be seen that there are between 58-63% of leaders constantly being re-elected. From the election data it is known that only 58% of Commonwealth leaders and 26% of French or American leaders are re-elected. Hence, in order for our model to produce re-election rates similar to the French or American countries, the successful reputation cannot have a positive influence on voters.

![Figure 6](image_url)

**Figure 6:** A plot of the average re-election rates over 50 realisations of the voter model for leaders, where the unsuccessful reputation is set at $\beta_{\text{unsuc}} = 1$, such that voters are strongly influenced by the unsuccessful term of a leader. The successful reputation has a positive influence on voters, such that they are more likely to re-elect the leader. The colours indicate the re-election rate of a leader, where constantly between 58-63% of leaders are re-elected.

In Figure 7 a two parameter plot of the re-election rate of a leader varying the parameters $J$ and $\beta_{\text{unsuc}}$ is shown where the successful reputation parameter is set to zero, $\beta_{\text{suc}} = 0$, i.e. a successful leader has...
no impact on the voters. We see that, no matter how great the unsuccessful reputation parameter and the influence of friends/family on a voter, around 60% of leaders are re-elected consistently except for a very small, non-zero unsuccessful reputation parameter. As these results still do not produce the re-election rates similar to the French or American countries, we require that a successful reputation must have a slight negative impact on voters. Another interesting observation from Figure 7, is that as the social neighbour influence on voters $J$ is increased this leads to a general decrease in the re-election rate of the leaders.

In Figure 8, we set $\beta_{\text{unsuc}} = 0$, i.e. the unsuccessful reputation is set to have no impact on voters while we vary the coupling constant $J$ and the successful reputation $\beta_{\text{suc}}$ and plot the re-election rate of the leaders. Note the successful reputation has a negative impact on voters. In this case we see that there is a very low re-election rate for almost all values. In particular, only if a successful reputation is set to have no impact are there any re-elected leaders, which in turn means that a successful reputation can not have a stronger negative impact on voters than an unsuccessful reputation.

In Figure 9 the results on the left have the coupling constant, $J$ set to zero i.e., voters are not influenced by their friends’/family’s party preference and vary the parameters $\beta_{\text{suc}}, \beta_{\text{unsuc}}$ whereas in the results on the right in Figure 9 the same parameters are varied but the coupling constant is set to one i.e. voters are likely to adopt their neighbours’ party preference. Comparing these two figures and Figure 7, we see that the greater the influence of friends’/family’s party preference, the less likely it is for a leader to be re-elected. This is a surprising result since one might expect lock-in of votes to occur and an increase in the leaders being re-elected. However, due to the effect of a leader’s reputation, lock-in does not occur.

Next we carry out a trial-and-error calibration of the model to the election data of the Commonwealth countries and then vary one of the parameters $J, \beta_{\text{suc}}, \beta_{\text{unsuc}}$ while keeping the others fixed to see if the model can reproduce similar election averages to France and the USA.

We fix the coupling constant set for both model fits at $J = 0.48$ (just above the critical coupling constant for a system with no external influence), such that voters are slightly influenced by their friends’/family’s party preference. We match the model’s results for the re-election of a leader and how often a party was in power to the data in §2.1 for the Commonwealth countries. Setting $\beta_{\text{suc}} = 0.03$ and $\beta_{\text{unsuc}} = 0.4$, we find that the dominant party is in power for approximately 54% of the elections and approximately 56% of leaders being re-elected; see Figure 10 for one realisation of an election history. This matches the election averages for all the Commonwealth countries where one party is found to be in charge for approximately 57% of the elections.
Successful reputation

Figure 8: A plot of the average re-election rates over 50 realisations of the voter model for leaders, where the unsuccessful reputation is set at $\beta_{\text{unsuc}} = 0$, where the x-axis is the negative impact of a successful leader on voters. Only for $\beta_{\text{suc}} = 0$ are there any re-elected leaders.

with around 59% of leaders being re-elected once. Since a new leader is only elected if their party is not re-elected, we find there is a strong correlation between the number of parties re-elected and the number of leaders re-elected (it is possible the leader is not re-elected in their electoral district hence there is a change of leader). We also see in Figure 10, that for these parameters it is possible and fairly common for a leader to be re-elected twice but we do not observe more re-elections.

The re-election rates of the leaders for France and the USA are half those of the Commonwealth countries. For the model to yield re-election rates similar to France and the USA, one possibility is that we fix $J = 0.48$ and change the reputation scores to $\beta_{\text{suc}} = 0.08$ and $\beta_{\text{unsuc}} = 0.3$. On average, approximately 26% of French and American leaders were re-elected, and the dominant party was on average 59% of the time in charge. In Figure 11, we see one election history realisation of the model and that on average one party is in power for approximately 62% of the time and that 27% of leaders are re-elected. We note that while we do see a leader being re-elected twice, this is rare and that most of the time a new leader and party is elected. The biggest change in the parameters from the Commonwealth fit, is in the value of the negative reputation score $\beta_{\text{suc}}$ where we have increased the value to 63% from the Commonwealth value suggesting that a possible explanation for the differences in the countries is that successful leaders in countries similar to France and the USA have a bigger negative influence than in the Commonwealth countries and hence maybe why the re-election rates of the leaders in France and the USA are significantly lower than for the Commonwealth countries.

We next fix, $\beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$ to the values obtained for France and the USA and vary $J$ to see if we can reproduce the election averages of France and the USA. In Figure 12, the thick black line, fitted to the results of the model, denotes the average of how often the dominant party was in charge, which stays at around 51%, showing that the two parties were in charge for around the same amount of time. The thin red line is fitted to the results of the re-election rates, representing the average number of times leaders were re-elected. From Figure 12 it can be seen, the greater influence of the coupling constant $J$ on voters, the fewer leaders are re-elected. The decrease rate of the thin red line, describing the average number of times leaders were re-elected, is around 1.5%.

Finally we fix the $\beta_{\text{suc}}$ and $\beta_{\text{unsuc}}$ for the values obtained for France and the USA and vary $J$, which as in the previous case, the re-election rate of a leader decreases with the increase of the coupling constant $J$. The party election is unaffected by the increase of the coupling constant, as shown in Figure 13. In this case
Figure 9: A plot of the average re-election rates over 50 realisations of the voter model for leaders, where in the figure on the left, the coupling constant is set at $J = 0$, so that voters are not influenced by their neighbours. In the figure on the right, the coupling constant is set at $J = 1$, such that the voters are likely to take on their neighbour’s party preference. Comparing these two results shows the unsuccessful reputation cannot have a more negative influence on voters than a successful one.

Figure 10: Overview of one electoral history for Commonwealth countries. The light blue and dark red fields indicate the leader’s political party. A solid line dividing the elections indicates a new leader has been elected, whereas a dotted line indicates a re-elected leader.

the decrease of the re-election rate in this figure is around 3%.

Therefore, these results show that a stronger impact of social neighbours $J$ on voters has consequently a negative influence on the leader’s re-election rate. This indirect influence is dependent on the impact of the successful reputation of the leader. Furthermore, these results show that the coupling constant $J$ does not impact how often a party is in charge. This implies that, for a larger coupling constant $J$, there is a greater change of a political party changeover, than for a smaller coupling constant. This would support the hypothesis that the influence of friends/social neighbours might be important, but other influences such as the democratic leader’s reputation can also explain the differences observed in the election data from various countries. The model does support the theory that just the two reputation scores for a successful and unsuccessful leader are enough to understand the data. In all cases for the model to reproduce the election averages of the data, we require that an unsuccessful reputation has a far greater negative impact on voters than a successful (negative) impact on voters.
Figure 11: Overview of one electoral history for France and the USA. The light blue and dark red fields indicate the leader’s political party. A solid line dividing the elections indicates a new leader has been elected, whereas a dotted line indicates a re-elected leader.

Figure 12: A plot of the average re-election rates over 50 realisations of the voter model for leaders and the dominant party, where the success scores for the Commonwealth countries are kept the same, while the coupling constant $J$ increases steadily. The thick black line shows the average of how often the dominant party is in charge, whereas the thin red line shows on average how often a leader was re-elected. The average of how often a party remains in charge stays constant throughout the increase of the coupling constant, whereas the re-election rate decreases by 1.5%.

Figure 13: A plot of the average re-election rates over 50 realisations of the voter model for leaders and the dominant party, where the success scores for the France and the USA are kept the same, while the coupling constant $J$ increases steadily. The thick black line shows the average of how often the dominant party is in charge, whereas the thin red line shows on average how often a leader was re-elected. The average of how often a party remains in charge stays constant throughout the increase of the coupling constant, whereas the re-election rate decreases by 3%.
4 Conclusion

In this paper we have looked at the election averages for the leaders and political parties of seven different countries. It is clear that looking at just the re-election averages of parties and leaders while ignoring vastly different constitutions/cultures etc, one sees that the Commonwealth countries are all roughly similar but significantly different from France and the USA. In particular, it is found that the original Commonwealth countries re-elect around 58% of their leaders while France and the USA re-elect only around half as many.

Rather than just simply treating the election data as static, we fitted a time dependent three-state Markov model to investigate the effect of previous elections. It is found that the Markov model fits the data well suggesting that on average voters only consider the most recent past election in making their next vote. In order to further investigate the underlying voting mechanism, a descriptive dynamical model is created to test two possible theories namely, the influence of social nearest neighbours and a leader’s reputation on voters. From this model, we find that while an increase in the influence of social nearest neighbours on individual voters has a negative impact on the re-election rates of leaders, the most important factor for the model to reproduce the election averages from the data is the two reputation scores for a successful and unsuccessful leader. Both these reputation scores have a negative impact on voters but crucially we require that the impact of a successful leader needs to be significantly less than an unsuccessful leader.

From the literature, we find several authors support the hypothesis that voters in the USA are more influenced by family party preference, whereas other countries such as the UK are more influenced by parties that present clear political alternatives [5, 35]. Furthermore, the possibility of an unsuccessful leader having a greater negative influence on voters than a successful one is also supported by several authors [11, 28, 49]. However, the major novelty of this work is to show that even if one disregards constitutional and cultural differences, the election averages can be explained by some simple rules implemented in a conceptual mathematical model. This does not mean that the precise context of a specific election is irrelevant rather that if we make the assumptions that all voters follow a simple set of general rules then we can reproduce the election data averages.

We highlight that this work tries to draw out a few fundamental mechanisms for voter behaviour from specific elections and that there may be other types of general behaviour of voters not investigated here. However, this work has shown that simple conceptural models are able to reproduce the gross averages from election data and support several hypotheses for the underlying rules governing electoral systems. We believe that the type of modelling carried out in this paper will help support and investigate several hypotheses that various researchers have found from analysing particular elections.

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