Comment on "Josephson vortices at tricrystal boundaries"

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We consider a recent work done by Kogan, Clem and Kirtley [Phys. Rev. B **61**, 9122 (2000)]. We discuss a specific case of a tricrystal boundary with a π-junction as one of the three arms. It has been discussed that a vortex with total flux 3/2 of a one-flux-quantum ϕ_0 is metastable. In this Comment we complete the analysis by pointing out that the flux is unstable, except at some unphysical combinations of the Josephson lengths of the junctions. We discuss systems that can have a stable $(n+1/2)\phi_0$ state.

In a recent paper Kogan, Clem and Kirtley[1] consider Josephson vortices at tricrystal boundaries. When one of the three Josephson junctions is a π-junction, there is a half-integer flux spontenously generated and attached to the joint. They consider a general case when the Josephson lengths of the junctions λ_J's are not the same. Besides this $\phi_0/2$ state, they notice also the existence of $3\phi_0/2$ flux. They conclude that the latter state is in general metastable. This result is used to analyze the stability of $3\phi_0/2$ flux in film geometry where they wrote 'In thicker films, ... the $3\phi_0/2$ state may become stable as is the case in the bulk.' In this Comment we will show analytically that even in bulk crystals the $3\phi_0/2$ flux is actually unstable, unless the junctions have the same Josephson length λ_J. Taking one λ_J different from the others can break the stability. Combining the results presented in Ref. 1 with ours gives a clearer explanation why $3\phi_0/2$ state has never been recorded in experiments.

The time-dependent governing equation of the phase difference along the junctions is described by the following perturbed sine-Gordon equation

$$\lambda_i^2\phi^i_{xx} - \phi^i_{tt} = \theta^i \sin\phi^i + \alpha\phi^i_t, \tag{1}$$

with $i = 1, 2, 3,\ x > 0,\ t > 0$ and α is a positive damping coefficient. The subscript J of Josephson length is omitted for brevity. The index i numbers the junction. The parameter θ^i represents the type of the ith junction. The boundary conditions at the intersection are

$$\begin{aligned} &\phi^1 + \phi^2 + \phi^3 = 0,\\ &\phi^1_x = \phi^2_x = \phi^3_x, \end{aligned} \tag{2}$$

all evaluated at $x = 0$.

This system of three Josephson junctions coupled via the boundary conditions has been derived first by Nakajima, Onodera and Ogawa.[2] The derivation of boundary conditions (2) using an electrical analogue is given by Nakajima and Onodera.[3] The dynamic behavior of integer fluxes in this system when the three junctions are of the same type and have the same Josephson length has been discussed in Refs. 4,5.

In this Comment we consider $\theta^1 = -\theta^2 = -\theta^3 = -1$. A time-independent solution of Eq. (1) representing a $3\phi_0/2$ flux is given by

$$\begin{aligned} \phi^1_0 &= 4\tan^{-1}(e^{(x-x_1)/\lambda_1}) - \pi,\\ \phi^2_0 &= 4\tan^{-1}(e^{(x-x_2)/\lambda_2}),\\ \phi^3_0 &= 4\tan^{-1}(e^{(x-x_3)/\lambda_3}) - 2\pi. \end{aligned} \tag{3}$$

For simplicity we scale λ_1 to 1 such that in the calculation we need to consider only λ_2 and λ_3.

In this Comment we discuss the linear stability of the solution (3). The first case we consider is $\lambda_i = 1$ for all i for which it corresponds to $x_1 = x_2 = x_3 = 0$.[1] First we linearize about the solution ϕ^i_0. We write $\phi^i(x,t) = \phi^i_0 + u^i(x,t)$ and substitute the spectral ansatz $u^i = e^{\omega t}v^i(x)$. Retaining the terms linear in u^i gives the following eigenvalue problem

$$v^i_{xx} - (\omega^2 + \alpha\omega + \theta^i\cos\phi^i_0)v^i = 0, \tag{4}$$

with boundary conditions at $x = 0$ given by

$$\begin{aligned} &v^1 + v^2 + v^3 = 0,\\ &v^1_x = v^2_x = v^3_x. \end{aligned} \tag{5}$$

The spectrum ω consists of the essential spectrum and the point spectrum (isolated eigenvalues). The essential spectrum is given by those ω for which there exist a solution to

$$v^i_{xx} - (\omega^2 + \alpha\omega + (\lim_{x\to\infty}\theta^i\cos\phi^i_0))v^i = 0,$$

or

$$v^i_{xx} - (\omega^2 + \alpha\omega + 1)v^i = 0 \tag{6}$$

of the form $v^i = e^{i\kappa x}$, with κ real.

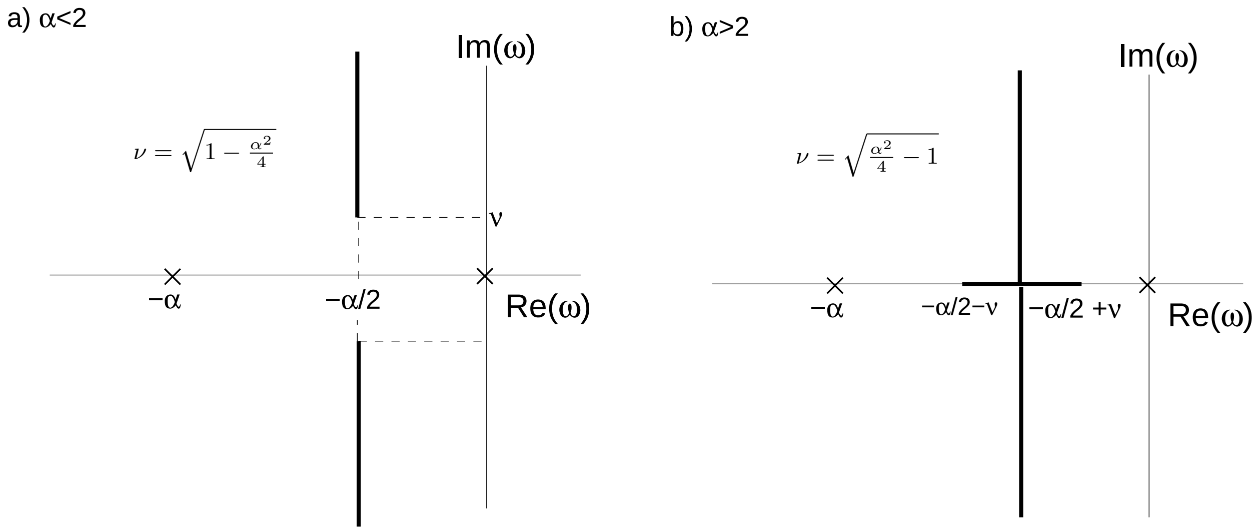

FIG. 1: A sketch of the point spectra (crosses) and the essential spectrum Eq. (7) (thick lines) for two cases of α: (a) $\alpha < 2$ and (b) $\alpha > 2$. When $\alpha \geq 2$, there is a part of the boundary lines that is at the negative real-line from point $(-\alpha/2 - \sqrt{\alpha^2/4-1}, 0)$ to point $(-\alpha/2 + \sqrt{\alpha^2/4-1}, 0)$. There is no spectrum with positive real part implying the linear stability of solution (3).

It follows that

$$\omega = \frac{-\alpha \pm \sqrt{\alpha^2 - 4(1+\kappa^2)}}{2}. \tag{7}$$

It is easy to see that there is no ω with positive real part as a solution to the above equation. Equation (7) is plotted in Fig. 1, with κ as parameter.

The above stability analysis shows that solution (3) can be stable. We cannot conclude whether the solution is linearly stable or not before analysing the point spectrum.

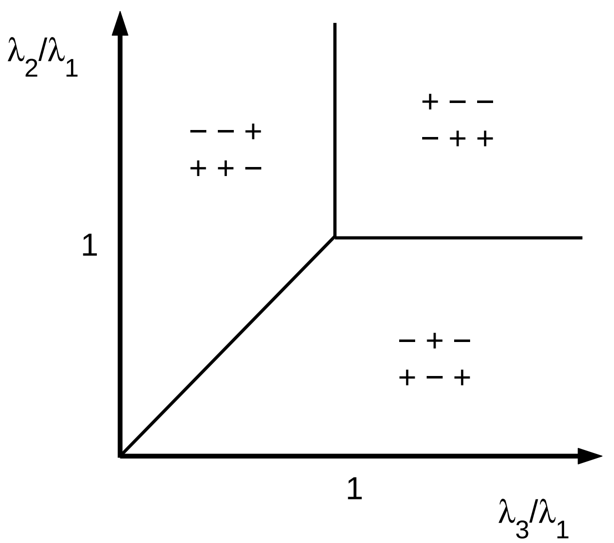

FIG. 2: The correct plot of the bifurcation diagram presented in Fig. 4 of Ref. 1.

To complete the analysis, our next task is to find the point spectrum ω and the corresponding eigenfunction v^i. The point spectrum corresponds to the solution v^i that tends to 0 as $x \to \infty$.

The eigenfunction v^i that corresponds to the largest eigenvalue is given by[6,7]

$$\begin{aligned} &v^i(x) = c_i e^{\mu x}(\tanh x - \mu), \\ &\mu^2 = \omega^2 + \alpha\omega + 1, \quad \mathrm{Re}(\mu) < 0, \end{aligned} \tag{8}$$

where c_i is to be determined. The coefficient c_i is determined by requiring that v^i has to satisfy Eq. 5 at $x = 0$ for all i. Hence, we obtain

$$\begin{aligned} &\mu(c_1 + c_2 + c_3) = 0, \\ &c_1(1-\mu^2) = c_2(1-\mu^2) = c_3(1-\mu^2). \end{aligned}$$

The fact that v^i cannot be zero for all i implies that $\mu = 0$ or $\mu = \pm 1$. From the condition that $\mathrm{Re}(\mu) < 0$, we obtain $\mu = -1$ or $\omega = 0, -\alpha$ with the corresponding eigenfunctions given by

$$\begin{aligned} &[v^1, v^2, v^3] = [1, 0, -1]e^x(\tanh x - 1) = [-1, 0, 1] \text{ sech } x, \\ &[v^1, v^2, v^3] = [1, -1, 0]e^x(\tanh x - 1) = [-1, 1, 0] \text{ sech } x. \end{aligned}$$

This result shows that there are quadruple eigenvalues at zero when the damping term is absent. Double zero eigenvalue bifurcates to the left half-plane when α is nonzero. The value of $\mu = 0$ gives bounded but not decaying eigenfunctions from which we obtain the edge of the essential spectrum ($\kappa = 0$ in Eq. (7)). A sketch of the locations and the bifurcation of the point spectra is presented in Fig. 1.

Hence, we conclude that Eq. (3) is linearly stable.

We have used numerical simulations of Eq. (1) to assist the analysis. In the scheme we take $\phi_t^i(x, 0) = 0$ and

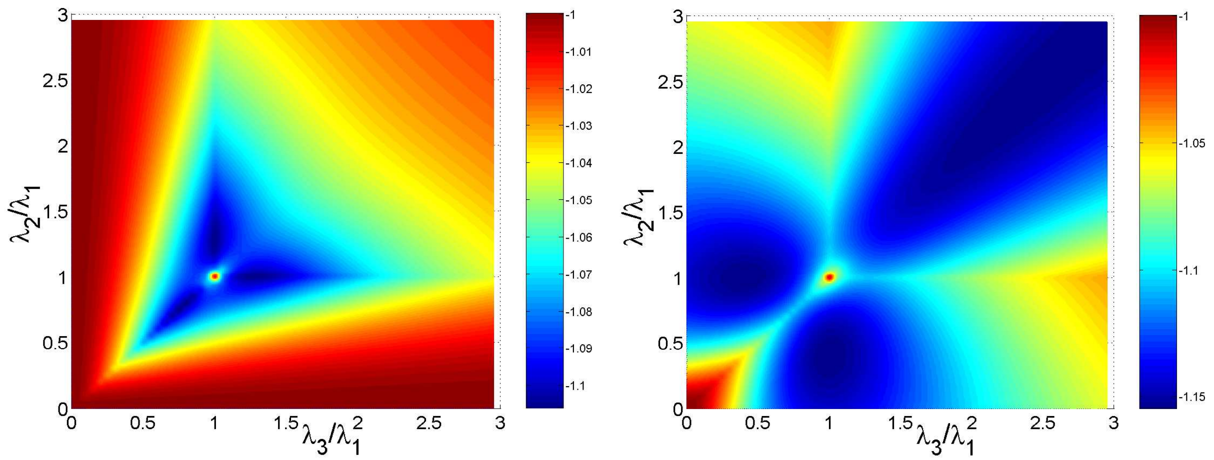

FIG. 3: (Color online) The contourplot of the smallest μ as a function λ_2/λ_1 and λ_3/λ_1 for the $3\phi_0/2$ solution with: a) one +; b) two +'s in the (sign(x_1),sign(x_2),sign(x_3)).

$\phi^i(x,0) = \phi_0^i$ as the initial conditions. Indeed we find the stability.

To search for a stable $3\phi_0/2$ state, the authors of Ref. 1 suggest to look at tricrystals with the Josephson length of the π-arm being larger than those of the 0-arms. We will show below that there is a positive eigenvalue bifurcate from the zero eigenvalue implying the instability if there is a Josephson length different from the others.

To do the stability analysis for this general case, we need to use the bifurcation diagram presented in Fig. 4 of Ref. 1. Unfortunately the picture is incorrect. The correction is shown in Fig. 2. With this bifurcation diagram, we know the combination of (sign(x_1),sign(x_2),sign(x_3)) that satisfies the governing equations. A solution with two +'s has a higher energy than the corresponding solution with one +.[1]

To obtain the eigenvalues of the $3\phi_0/2$ state in this general case, we take the following procedures. First, we set $c_2 = 1$. Then, we solve

$$v^1 + v^2 + v^3\big|_{x=0} = 0 \quad \text{and} \quad v_x^1(0) = v_x^2(0)$$

for c_1 and c_3. The only equation we have to solve now is $v_x^1(0) = v_x^3(0)$ or $v_x^2(0) = v_x^3(0)$ from which we obtain a polynomial of order five for μ. The eigenfunction v^i for nonzero x_i can be obtained by replacing variable x in the right hand side of Eq. (8) with $(x - x_i)/\lambda_i$. For the sake of simplicity, we do not write down the cumbersome expression of $\mu = \mu(\lambda_2, \lambda_3)$. In Fig. 3, we present the plot of the largest $|\mu|$ with $\mathrm{Re}(\mu) < 0$ for both combinations.

Remembering that $\mu^2 = \omega^2 + \alpha\omega + 1$, the two plots inform us that when the Josephson lengths do not have the same length, a pair of eigenvalues at the real-line bifurcates from the quadruple zero. If one searches for other negative μ's then he will find that all the solutions representing $3\phi_0/2$ flux have at least two negative μ's. It can be checked as well that if μ solves the polynomial equation for the stability of the solution with one + then $-\mu$ solves the polynomial equation of the corresponding solution with two +'s.

The calculation we have done has proven that there is no stable $3\phi_0/2$ state in tricrystal junctions with one π-arm, except at some unphysical combinations of the Josephson lengths.

With the above analysis, it can be easily shown that $\phi_0/2$ state is stable and $5\phi_0/2$ state is unconditionally unstable.

One can easily show using the same analysis that $3\phi_0/2$ state will be unconditionally stable in tetracrystals with one π-arm. One can also calculate that $5\phi_0/2$ state will be linearly stable in pentacrystals with one π-arm. Using this pattern, we can make a generalization that to obtain a stable $(1/2 + n)\phi_0$ state, we need at least $(1 + 2n)$ junctions connected to a joint with one of the arms is a π-junction. All the stable states require the maximum field to be at the joint (see Fig. 1 in Ref. 1).

To summarize, we have shown the instability of a $3\phi_0/2$ state. Combining the result presented in Ref. 1 with ours gives a clearer explanation why $3\phi_0/2$ state is and will never be observed in experiments, especially in film geometry. The stability analysis we present here can be easily applied to discuss the stability of solutions of other Josephson junction systems. We also have described systems that can have a stable $(1/2 + n)\phi_0$ state. An interesting question on the dynamic behavior of these fluxes is still to be addressed.

H. S. thanks T. P. P. Visser for many illuminating discussions, B. J. Geurts for explanations on numerical schemes and J. R. Kirtley for fruitful discussions on experiments and manipulations of half-fluxons in tricrystal junctions. This work is supported by the Royal Netherlands Academy of Arts and Sciences (KNAW) and partially by the PiShift Programme.

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