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PERTURBATIONS OF EMBEDDED EIGENVALUES FOR THE BILAPLACIAN ON A CYLINDER

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Abstract. Perturbation problems for operators with embedded eigenvalues are generally challenging since the embedded eigenvalues cannot be separated from the rest of the spectrum. In this paper we study a perturbation problem for embedded eigenvalues for the bilaplacian with an added potential, when the underlying domain is a cylinder. We show that the set of nearby potentials, for which a simple embedded eigenvalue persists, forms a smooth manifold of finite codimension.

1. **Introduction.** Embedded eigenvalues occur in many applications arising in physics. In quantum mechanics, for instance, eigenvalues of the energy operator correspond to energy bound states that can be attained by the underlying physical system. If such an eigenvalue is embedded in the continuous spectrum, it is of fundamental importance to determine whether it, and therefore the corresponding bound state, persists upon perturbing the potential (see [7, 15] for examples). Alternatively, embedded eigenvalues in inverse scattering problems correspond to soliton-type structures for the original integrable problems whose robustness under perturbations is therefore again determined by the fate of the embedded eigenvalue [12, 13].

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A different motivation for the same question arises in systems that support nonlinear waves (e.g. water waves) or vortex solutions (e.g. in photonic lattices or other nonlinear optical systems). The linear and nonlinear stability of such waves is then determined by the spectrum of the linearisation of the underlying nonlinear PDE or the energy operator about the wave. For both water waves and certain vortex solutions, the key difficulty are again embedded eigenvalues which will always be present due to the specific nature of the nonlinear systems describing them. For instance, it is often of interest to construct other types of waves from a given nonlinear wave: starting with a water wave, for example, with a localised profile in one spatial direction, one may wish to glue several well-separated copies of the original water wave together to yield a wave with several elevated humps [5]. The potential appearing in the energy operator corresponding to the newly constructed wave consists then of several copies of the potential of the original wave and, to determine the stability properties of the new water wave, one needs to investigate the spectrum of the new energy operator using information from the original operator. In this situation, the fate of embedded eigenvalues under large perturbations (gluing widely separated potentials together is not a small regular perturbation) is the crucial issue that determines the stability of the new waves.

For the sake of clarity, we focus in this paper on the perturbation problem for the bilaplacian on a cylinder: this is the simplest possible PDE example of the perturbation problem for embedded eigenvalues, and it serves as a prototype for other self-adjoint problems, including systems of partial differential equations. We emphasize that our method is applicable to a much larger class of self-adjoint, and non-selfadjoint problems, and an outline of these extensions is given in §5.

Our goal is to show that the set of potentials for which a given embedded eigenvalue persists forms an infinite-dimensional submanifold of finite codimension in an appropriate space of perturbations. Furthermore, we will prove that the codimension is, in fact, given by the multiplicity of the continuous spectrum, computed at the embedded eigenvalue.

To set the scene, we consider as the underlying domain the cylinder $\{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 = 1\}$. It is convenient to describe the cylinder using cylindrical coordinates (z, φ) where $(x, y) = (\cos\varphi, \sin\varphi)$ for $\varphi \in S^1 := \mathbb{R}/2\pi\mathbb{Z}$. Using these coordinates, we identify the cylinder with $\mathbb{R} \times S^1$. By $L^2(\mathbb{R} \times S^1)$ we mean the space of all square integrable functions with respect to the Haar measure $2\pi d\varphi dz$. The Laplacian on the cylinder is described in cylindrical coordinates by

$$\Delta := \frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}.$$

The bilaplacian Δ^2, i.e. the composition of Δ with itself, is a closed densely defined operator on $L^2(\mathbb{R} \times S^1)$ with spectrum $[0, \infty)$.

The initial potential, and the perturbations we add to it later on, have to decay algebraically as $|z| \to \infty$ with a sufficiently large algebraic rate. To make this precise, we fix $\alpha \in (0, 1)$ and define

$$X_\beta := \Big\{ \rho \in C^{0,\alpha}(\mathbb{R} \times S^1);\ \|\rho\|_{X_\beta} : \sup_{(z_j,\varphi_j)\in\mathbb{R}\times S^1} \frac{|\rho(z_1,\varphi_1) - \rho(z_2,\varphi_2)|}{|(z_1,\varphi_1) - (z_2,\varphi_2)|^\alpha} + \sup_{(z,\varphi)\in\mathbb{R}\times S^1} |\rho(z,\varphi)|(1 + |z|)^\beta < \infty \Big\}$$

for $\beta > 1$. We are then interested in the operator

$$\mathcal{L} := \Delta^2 + \theta$$

where $\theta \in X_\beta$ for some $\beta > 1$. The operator $\mathcal{L}$ is densely defined with domain $H^4(\mathbb{R} \times S^1)$.

Since $\mathcal{L}$ is self-adjoint, its spectrum is a subset of the real line. The continuous spectrum of $\mathcal{L}$ is the half line $[0, \infty)$ and is generated by the continuum eigenfunctions

$$u(z, \varphi) = e^{ikz} e^{in\varphi}, \qquad k \in \mathbb{R},\ n \in \mathbb{Z} \tag{1}$$

of Δ^2, which satisfy $\Delta^2 u = (k^2 + n^2)^2 u$ and can be turned into approximate eigenfunctions of $\mathcal{L}$ by shifting them in space towards ∞ and truncating them using cutoff functions as in [6, Lemma 2 in appendix to §5]. We record that if $(m-1)^4 < \lambda < m^4$ for some $m \in \mathbb{Z}$, then Δ^2 admits $4m-2$ linearly independent continuum eigenfunctions obtained by choosing $|n| = 0, \ldots, m-1$ in (1) and afterwards $k \in \mathbb{R}$ so that $\lambda = (k^2 + n^2)^2$. Thus, it is reasonable to refer to $4m - 2$ as the multiplicity of the continuous spectrum at $\lambda \in ((m-1)^4, m^4)$. A rigorous definition of the multiplicity of elements in the continuous spectrum can be found in [2, Definition 2 in §85]: using the spectral resolution of the Fourier transform of Δ^2 on L^2, it is not difficult to see that this definition also gives $4m - 2$, thus justifying the preceding formal calculation.

Recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{L}$ if there exists an element $u \in (\operatorname{Dom} \mathcal{L}) \setminus \{0\} = H^4(S^1 \times \mathbb{R}) \setminus \{0\}$ such that

$$\mathcal{L}u = \lambda u, \tag{2}$$

and we refer to any such function u as an eigenfunction of $\mathcal{L}$ belonging to the eigenvalue λ. The multiplicity of an eigenvalue λ is the maximal number of linearly independent eigenfunctions corresponding to λ. An embedded eigenvalue is an eigenvalue which also belongs to the continuous spectrum.

We are interested in the case when λ_0 is an embedded eigenvalue, i.e. when $\lambda_0 > 0$, since when λ_0 is a simple eigenvalue which is isolated from the rest of the spectrum, then the persistence of eigenvalues is well known [9, pp. 213–215]. We also need to assume that λ_0 avoid the branch points $\lambda = m^4$ with $m \in \mathbb{Z}$ where the multiplicity of the continuous spectrum changes:

Hypothesis 1. $\mathcal{L}$ has a simple embedded eigenvalue $\lambda_0 > 0$ with $\lambda_0^{1/4} \notin \mathbb{Z}$.

The following example shows that for each $\lambda_0 > 0$ there is a potential in X_β, in fact even in the class of potentials with compact support, so that Hypothesis 1 is met[1]:

Example 1. Let $\lambda_0 > 0$, and choose $u > 0$ so that

$$(-\Delta + \sqrt{\lambda_0} I)u = f,$$

for some $f \in C_0^\infty(\mathbb{R} \times S^1)$: Take for instance u to be a smooth approximation of a fundamental solution of $-\Delta + \sqrt{\lambda_0} I$ on $\mathbb{R} \times S^1$. Let

$$\theta = \frac{(\Delta + \sqrt{\lambda_0} I) f}{u}$$

[1]This is in contrast to the laplacian for which the potential has to be slowly decaying and oscillating in order for embedded eigenvalues to exist [15, §XIII.13].

and note that $\theta \in C_0^\infty(\mathbb{R} \times S^1)$. A calculation shows that

$$(\Delta^2 + \theta)u = \lambda_0\, u,$$

and λ_0 is an embedded eigenvalue for $\mathcal{L}$.

Our goal is to analyse the persistence of the eigenvalue λ_0 when a small perturbation $\rho \in X_\beta$ is added to the potential θ: Thus, we consider the operator $\mathcal{L} + \rho$ and wish to characterize the set

$$\mathcal{M}_\delta := \{\rho \in X_\beta;\ \text{there exists } \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \\ \text{such that } \lambda \text{ is an eigenvalue of } \mathcal{L} + \rho\}$$

in a neighborhood of $\rho = 0$ for $\delta > 0$ sufficiently small. Our main result is:

Theorem 1.1. *Suppose that Hypothesis 1 holds, fix $\beta > 1$, and let m be the smallest nonnegative integer such that $\lambda_0 < m^4$. Then there exist a $\delta > 0$ and a neighborhood $\mathcal{O}$ of 0 in X_β such that $\mathcal{M}_\delta \cap \mathcal{O}$ is a smooth manifold of codimension $4m - 2$.*

This result agrees with the recent result in [1] where a class of operators was studied using methods from spectral theory. The advantage of our approach compared to the abstract theory developed in [1] are the extensions to systems and non-selfadjoint problems on cylindrical domains that we mentioned earlier. In addition, the technique used here can be used for operators that are posed on $\mathbb{R}^n$ (see our forthcoming paper [3]): In fact, our motivation came from the persistence problem of embedded eigenvalues for PDE operators on the plane. In this case, the continuous spectrum has infinite multiplicity which complicates the analysis significantly.

To solve the perturbation problem we use dynamical systems techniques. The eigenvalue problem can be written as a system of ODEs on a function space consisting of functions defined on the circle S^1. The dynamical variable is the spatial coordinate along the axis of the cylinder, i.e, the variable $z \in \mathbb{R}$. It will be shown that this dynamical system has an exponential dichotomy, i.e. for each $z \in \mathbb{R}$ it has a stable subspace consisting of initial points corresponding to the exponentially decaying solutions at ∞ and an unstable subspace of initial points corresponding to the exponentially decaying solutions at $-\infty$. The eigenvalue problem can be rephrased as the problem of determining whether these stable and unstable subspaces intersect nontrivially. By using Lyapunov-Schmidt reduction, we show that for small perturbations, there are finitely many conditions that need to be satisfied in order for these subspaces to have a nontrivial intersection. The number of conditions needed is the same as the multiplicity of the continuous spectrum of $\mathcal{L}$ at λ_0. Finally, we use the implicit function theorem to show that the perturbations for which the embedded eigenvalue persist form a manifold of the appropriate codimension.

2. **The spatial dynamics setting.**

Throughout the rest of the paper, we shall assume that Hypothesis 1 holds. In intrinsic coordinates (z, φ), the perturbed eigenvalue problem reads

$$\frac{\partial^4 u}{\partial z^4} + 2\frac{\partial^4 u}{\partial z^2 \partial \varphi^2} + \frac{\partial^4 u}{\partial \varphi^4} + (\theta + \rho)u = \lambda u. \tag{3}$$

As we are looking for eigenvalues λ near λ_0, we can restrict to eigenvalues λ with $(m-1)^4 < \lambda < m^4$ or $\lambda < 0$. In the spatial dynamics setting, the variable z plays the role of the dynamic variable. Therefore, the function u is seen as a function of one variable z, taking values in a space of functions on the circle S^1. Thus, defining $\partial : H^k(S^1) \to H^{k-1}(S^1)$ to denote differentiation with respect to φ and denoting by $'$ differentiation with respect to z, the eigenvalue problem (3) can be rewritten as the system (by putting $u = u_1$, $u' = u_2$, etc)

$$U' = A(z; \lambda, \rho)U, \tag{4}$$

with

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad \text{and} \quad A(z; \lambda, \rho) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda - \partial^4 - \theta - \rho & 0 & -2\partial^2 & 0 \end{pmatrix}. \tag{5}$$

The potential θ acts on functions in $C(\mathbb{R}, H^k(S^1))$ as follows:

$$\theta u(z) = \theta(z, \cdot)\, u(z),$$

and the perturbation ρ acts in the same way. The matrix $A(z; \lambda, \rho)$ is an unbounded operator on $X := H^3(S^1) \times H^2(S^1) \times H^1(S^1) \times L^2(S^1)$ with domain $Y := H^4(S^1) \times H^3(S^1) \times H^2(S^1) \times H^1(S^1)$. The norm on X is defined by

$$\|U\|_X^2 = \|u_1\|_{H^3(S^1)}^2 + \|u_2\|_{H^2(S^1)}^2 + \|u_3\|_{H^1(S^1)}^2 + \|u_4\|_{L^2(S^1)}^2.$$

With this set-up, the following lemma can be proved.

Lemma 2.1. *$u \in H^4_{loc}(\mathbb{R} \times S^1)$ satisfies* (3) *if and only if the dynamical system* (4) *has a solution $U \in C^{0,\alpha}(\mathbb{R}; Y)$.*

Proof. If $u \in H^4_{loc}(\mathbb{R} \times S^1)$ is a solution of (3), then since θ and ρ are Hölder continuous with exponent α, it follows from the eigenvalue equation that $\Delta^2 u \in C^{0,\alpha}(\mathbb{R} \times S^1)$. Hence by elliptic regularity theory, $u \in C^{4,\alpha}(\mathbb{R} \times S^1)$. In particular $u \in C^{0,\alpha}(\mathbb{R}; C^4(S^1)) \cap C^{1,\alpha}(\mathbb{R}; C^3(S^1)) \cap C^{2,\alpha}(\mathbb{R}; C^2(S^1)) \cap C^{3,\alpha}(\mathbb{R}; C^1(S^1))$. It follows that the corresponding $U = (u, u', u'', u''')^T$ belongs to $C^{0,\alpha}(\mathbb{R}; C^4(S^1) \times C^3(S^1) \times C^2(S^1) \times C^1(S^1)) \subset C^{0,\alpha}(\mathbb{R}; Y)$.

Conversely, if $U \in C^{0,\alpha}(\mathbb{R}; Y)$ is a solution of (4), we let $u = U_1$. By (4) and (5), $u^{(4)} \in C^{0,\alpha}(\mathbb{R}; L^2(S^1))$, $u'' \in C^{0,\alpha}(\mathbb{R}; H^2(S^1))$, and $u \in C^{0,\alpha}(\mathbb{R}; H^4(S^1))$. Hence

$$\Delta^2 u \in C^{0,\alpha}(\mathbb{R}; L^2(S^1)) \subset L^2_{loc}(\mathbb{R} \times S^1),$$

and so $u \in H^4_{loc}(\mathbb{R} \times S^1)$. It also follows from (4) and (5) that u satisfies (3). □

We will see in Lemma 3.2 that any eigenfunction in fact decays exponentially. Initial values of asymptotically decaying solutions of non-autonomous linear systems such as (4) can be found as intersections of stable and unstable subspaces. To describe those stable and unstable subspaces, the system at infinity and the notion of exponential dichotomies are introduced first.

2.1. **The system at infinity.** The system at infinity is the equation

$$U' = A_\infty(\lambda)U, \tag{6}$$

where

$$A_\infty(\lambda) = \lim_{|z| \to \infty} A(z; \lambda, \rho) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda - \partial^4 & 0 & -2\partial^2 & 0 \end{pmatrix}.$$

To analyse the solutions of the system at infinity (6), we expand the function $U = (u_1, u_2, u_3, u_4)^T$ as a Fourier series in the φ variable and denote its kth Fourier coefficient by $\widehat{U}_k(z)$. Then for $k \in \mathbb{Z}$

$$\widehat{U}_k'(z) = \widehat{A}_\infty(k, \lambda)\widehat{U}_k(z),$$

where

$$\widehat{A}_\infty(k, \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda - k^4 & 0 & 2k^2 & 0 \end{pmatrix}.$$

For any $k \in \mathbb{Z}$, the characteristic equation of $\widehat{A}_\infty(k, \lambda)$ is $\mu^4 - 2k^2\mu^2 - \lambda + k^4 = 0$ or $\mu^2 = k^2 \pm \sqrt{\lambda}$.

It is straightforward to verify that the eigenvector corresponding to an eigenvalue μ is $(1/\mu^3, 1/\mu^2, 1/\mu, 1)^T$. Define

$$M_k := \begin{pmatrix} -(k^2+\sqrt{\lambda})^{-3/2} & -(k^2-\sqrt{\lambda})^{-3/2} & (k^2-\sqrt{\lambda})^{-3/2} & (k^2+\sqrt{\lambda})^{-3/2} \\ (k^2+\sqrt{\lambda})^{-1} & (k^2-\sqrt{\lambda})^{-1} & (k^2-\sqrt{\lambda})^{-1} & (k^2+\sqrt{\lambda})^{-1} \\ -(k^2+\sqrt{\lambda})^{-1/2} & -(k^2-\sqrt{\lambda})^{-1/2} & (k^2-\sqrt{\lambda})^{-1/2} & (k^2+\sqrt{\lambda})^{-1/2} \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{7}$$

and let

$$D_k := \begin{pmatrix} -\sqrt{k^2+\sqrt{\lambda}} & 0 & 0 & 0 \\ 0 & -\sqrt{k^2-\sqrt{\lambda}} & 0 & 0 \\ 0 & 0 & \sqrt{k^2-\sqrt{\lambda}} & 0 \\ 0 & 0 & 0 & \sqrt{k^2+\sqrt{\lambda}} \end{pmatrix}. \tag{8}$$

Then $\widehat{A}_\infty(k, \lambda) = M_k D_k M_k^{-1}$.

Going back to the full asymptotic operator $A_\infty(\lambda)$, we can show the following:

Lemma 2.2. *Let m be the smallest nonnegative integer such that $\lambda < m^4$. Then $A_\infty(\lambda) : X \to X$ is a closed, densely defined operator with spectrum*

$$\sigma(A_\infty(\lambda)) \left\{ \pm\sqrt{k^2+\sqrt{\lambda}}, \pm\sqrt{k^2-\sqrt{\lambda}};\ k \in \mathbb{Z} \right\}.$$

Thus $A_\infty(\lambda)$ has m pairs of purely imaginary eigenvalues $\mu = \pm i\sqrt{\sqrt{\lambda} - k^2}$, $k = 0, \dots, m-1$. All other eigenvalues are real, being $\mu = \pm\sqrt{k^2+\sqrt{\lambda}}$, $k = 0, \dots, m-1$, and $\mu = \pm\sqrt{k^2 \pm \sqrt{\lambda}}$, $k^2 \geq m^2$. All eigenvalues with $k \neq 0$ have a two-dimensional eigenspace. The smallest positive eigenvalue of $A_\infty(\lambda)$ is $\mu_{\min}(\lambda) := \min(\lambda^{1/4}, \sqrt{m^2 - \sqrt{\lambda}})$.

Proof. It is clear that $Y = \mathrm{Dom}(A_\infty(\lambda))$ is dense in X. In order to prove that $A_\infty(\lambda)$ is closed on X, let $U_i \in Y$ and $U, f \in X$ be such that $U_i \xrightarrow{X} U$ and $A_\infty(\lambda)U_i \xrightarrow{X} f$. By the definition of $A_\infty(\lambda)$, it follows that $f_1 = u_2$, $f_2 = u_3$, $f_3 = u_4$ and that $f_{4,i} := (\lambda I - \partial^4)u_{1,i} - 2\partial^2 u_{3,i} \to f_4$ in $L^2(S^1)$. By the estimate

$$\begin{aligned} \|u_{1,i} - u_{1,j}\|_{H^4} &\leq C\|\partial^4 u_{1,i} - \partial^4 u_{1,j}\|_{L^2} + C\|u_{1,i} - u_{1,j}\|_{L^2} \\ &= C\|\lambda(u_{1,i} - u_{1,j}) - 2\partial^2(u_{3,i} - u_{3,j}) - f_{4,i} + f_{4,j}\|_{L^2} + C\|u_{1,i} - u_{1,j}\|_{L^2}, \end{aligned}$$

it follows that $u_{1,i}$ is a Cauchy sequence in $H^4(S^1)$ (note that $u_{3,i} \to f_2$ in $H^2(S^1)$). Then $u_{1,i}$ converges also in $H^4(S^1)$, and the limit has to be $u_1 \in H^4(S^1)$. Hence

$$(\lambda I - \partial^4)u_{1,i} - 2\partial^2 u_{3,i} \to (\lambda I - \partial^4)u_1 - 2\partial^2 u_3$$

in $L^2(S^1)$ and proves that $f_4 = (\lambda - \partial^4)u_1 - 2\partial^2 u_3$. Altogether we can conclude that $U \in Y$ and that $f = A_\infty(\lambda)U$. This completes the proof that $A_\infty(\lambda)$ is closed on X.

Finally we consider the spectrum. Let

$$l_s^2 = \{f : \mathbb{Z} \to \mathbb{R};\ \sum_{k=-\infty}^{\infty} (1+k^2)^s |f(k)|^2 < \infty\},$$

and let

$$\widehat{Y} = l_4^2 \times l_3^2 \times l_2^2 \times l_1^2,$$
$$\widehat{X} = l_3^2 \times l_2^2 \times l_1^2 \times l^2.$$

Note that $U \in X$ if and only if its Fourier expansion $\widehat{U} \in \widehat{X}$, etc.

Define $\widehat{A}_\infty(\lambda) : \widehat{X} \to \widehat{X}$ to be

$$(\widehat{A}_\infty(\lambda)\widehat{U})_k : \widehat{A}_\infty(k,\lambda)\widehat{U}_k,$$

and note that $\widehat{A}_\infty(\lambda)$ is closed and densely defined with domain $\widehat{Y}$. It is clear that $(A_\infty(\lambda) - \mu I) : X \to X$ has a bounded inverse if and only if $(\widehat{A}_\infty(\lambda) - \mu I) : \widehat{X} \to \widehat{X}$ has a bounded inverse. It is also clear that the eigenvalues of $\widehat{A}_\infty(k,\lambda)$ belong to the spectrum of $\widehat{A}_\infty(\lambda)$. For $k \in \mathbb{N}$, the operators $\widehat{A}_\infty(\pm k, \lambda)$ have the same eigenvalues, hence leading to two-dimensional eigenspaces for the original operator $A_\infty(\lambda)$.

We need to show that there are no other points in the spectrum. Define $\widehat{M} : l^2 \times l^2 \times l^2 \times l^2 \to \widehat{X}$ by

$$(\widehat{M}\widehat{U})_k = M_k U_k,$$

and note that $\widehat{M}$ is a linear homeomorphism between these spaces. Define also the unbounded operator $\widehat{D}$ on $l^2 \times l^2 \times l^2 \times l^2$ by

$$(\widehat{D}\widehat{U})_k = D_k \widehat{U}_k.$$

Note that $\widehat{D}$ is a densely defined closed operator with domain $l_1^2 \times l_1^2 \times l_1^2 \times l_1^2$, and that $\sigma(\widehat{D}) = \cup_{k\in\mathbb{Z}} \sigma(A_\infty(k,\lambda))$. If $\mu \notin \sigma(\widehat{D})$, then

$$(\widehat{A}_\infty(k,\lambda) - \mu I)^{-1} = \widehat{M}(\widehat{D} - \mu I)^{-1}\widehat{M}^{-1}.$$

It can now be seen that $(\widehat{A}_\infty(k,\lambda) - \mu I)^{-1}$ is bounded, which is what we needed to prove. □

Let $X^u(\lambda)$ be the closure in X of the span of the eigenvectors of $A_\infty(\lambda)$ corresponding to the positive eigenvalues of $A_\infty(\lambda)$, and let $X^s(\lambda)$ be the closure of the span of the eigenvectors corresponding to the negative eigenvalues. We denote by $X^c(\lambda)$ the span of the eigenvectors corresponding to the purely imaginary eigenvalues. By Lemma 2.2, $X = X^s(\lambda) \oplus X^c(\lambda) \oplus X^u(\lambda)$. Note that both $X^u(\lambda)$ and $X^s(\lambda)$ are infinite dimensional while $X^c(\lambda)$ is finite dimensional. Let $P^u(\lambda)$, $P^s(\lambda)$ and $P^c(\lambda)$ be the spectral projections onto $X^u(\lambda)$, $X^s(\lambda)$ and $X^c(\lambda)$, respectively.

Lemma 2.3. *Let m be the smallest nonnegative integer such that $\lambda < m^4$. Then the operators $A_\infty(\lambda)P^s(\lambda)$, $-A_\infty(\lambda)P^u(\lambda)$, $A_\infty(\lambda)(I - P^u(\lambda))$, and $-A_\infty(\lambda)(I - P^s(\lambda))$ generate analytic semigroups on $X^s(\lambda)$, $X^u(\lambda)$, $X^s(\lambda) \oplus X^c(\lambda)$ and $X^u(\lambda) \oplus X^c(\lambda)$, respectively.*

Proof. Let $\eta \in (0, \mu_{\min}(\lambda))$ (recall that $\mu_{\min}(\lambda)$ is the smallest positive real eigenvalue of $A_\infty(\lambda)$). We define $V(z) = e^{\pm\eta z}U(z)$ and the ODE for V is $V'(A_\infty(\lambda) \pm \eta I)V$. It suffices to show that there exists a constant $C > 0$ such that

$$\left\|(A_\infty(\lambda) \pm \eta I - i\mu I)^{-1}\right\|_{\mathcal{L}(X,X)} \leq \frac{C}{1+|\mu|}$$

for all $\mu \in \mathbb{R}$. As in the proof of Lemma 2.2, we note that this estimate follows if there exists a constant $\widetilde{C} > 0$ such that for every $\widehat{U} \in l^2 \times l^2 \times l^2 \times l^2$

$$\left\|(\widehat{D} \pm \eta I - i\mu I)^{-1}\widehat{U}\right\|_{l^2\times l^2\times l^2\times l^2} \leq \frac{\widetilde{C}}{1+|\mu|}\|\widehat{U}\|_{l^2\times l^2\times l^2\times l^2}.$$

It is easy to see that there is a constant $\widetilde{C} > 0$ such that for $k \in \mathbb{Z}$ we have

$$\left|(D_k \pm \eta I - i\mu I)^{-1}\widehat{U}_k\right|^2 \leq \frac{\widetilde{C}\,|\widehat{U}_k|^2}{(1+|\mu|)^2}.$$

Hence

$$\begin{aligned}\left\|(\widehat{D} \pm \eta I - i\mu I)^{-1}\widehat{U}\right\|^2_{l^2\times l^2\times l^2\times l^2} &= \sum_{k\in\mathbb{Z}}\left|(D_k \mp \eta I - i\mu I)^{-1}\widehat{U}_k\right|^2 \\ &\leq \frac{\widetilde{C}^2}{(1+|\mu|)^2}\sum_{k\in\mathbb{Z}}|\widehat{U}_k|^2.\end{aligned}$$

□

2.2. **Exponential dichotomies.** We define exponential dichotomies following [14].

Definition 2.4. Let $J = \mathbb{R}_-$, $\mathbb{R}_+$, or $\mathbb{R}$. The system (4) is said to possess an exponential dichotomy in X on the interval J if there exists a family of projections $P(z) \in \mathcal{L}(X,X)$, $s \in J$, such that the projections satisfy $P(\cdot)U \in C(J;X)$ for any $U \in X$, and there exist constants K and $\kappa^s < 0 < \kappa^u$ with the following properties:

(i) For any $z_0 \in J$ and $U \in X$ there exists a unique solution $\Phi^s(z,z_0)U$ of (4) defined for $z \geq z_0$, $z, z_0 \in J$ such that $\Phi^s(z_0,z_0)U = P(z_0)U$ and

$$\|\Phi^s(z,z_0)U\|_X \leq Ke^{\kappa^s(z-z_0)}\|U\|_X$$

for all $z \geq z_0$, $z, z_0 \in J$.

(ii) For any $z \in J$ and $U \in X$ there exists a unique solution $\Phi^u(z,z_0)U$ of (4) defined for $z \leq z_0$, $z, z_0 \in J$ such that $\Phi^u(z_0,z_0)U(I - P(z_0))U$ and

$$\|\Phi^u(z,z_0)U\|_X \leq Ke^{\kappa^u(z-z_0)}\|U\|_X$$

for all $z \leq z_0$, $z, z_0 \in J$.

(iii) The solutions $\Phi^s(z,z_0)U$ and $\Phi^u(z,z_0)U$ satisfy

$$\begin{aligned}&\Phi^s(z,z_0)U \in \operatorname{Ran} P(z) &&\text{for all } z \geq z_0 \in J,\\ &\Phi^u(z,z_0)U \in \ker P(z) &&\text{for all } z \leq z_0 \in J.\end{aligned}$$

With this definition, Lemma 2.3 immediately implies the following corollary:

Corollary 1. *Let m be the smallest nonnegative integer such that $\lambda < m^4$. Suppose that $\eta \in (0, \mu_{\min}(\lambda))$. Then the operator $A_\infty(\lambda) + \eta I$ possesses exponential dichotomies on $\mathbb{R}$, with $\kappa^s = -\mu_{\min}(\lambda) + \eta$ and $\kappa^u = \eta$, while the operator $A_\infty(\lambda) - \eta I$ possesses exponential dichotomies on $\mathbb{R}$, with $\kappa^s = -\eta$ and $\kappa^u = \mu_{\min}(\lambda) - \eta$.*

Proof. For $A_\infty(\lambda) + \eta I$, put $P(z) = P^s(\lambda)$ and

$$\Phi^s(z, z_0) = e^{P^s(A_\infty(\lambda)-\eta I)(z-z_0)} P^s(\lambda),$$
$$\Phi^u(z, z_0) = e^{(I-P^s)(A_\infty(\lambda)-\eta I)(z-z_0)} (I - P^s(\lambda)).$$

and for $A_\infty(\lambda) - \eta I$, put $P(s) = I - P^u(\lambda)$ and

$$\Phi^s(s, t) = e^{(I-P^u(\lambda))(A_\infty(\lambda)-\eta I)(s-t)} (I - P^u(\lambda)),$$
$$\Phi^u(s, t) = e^{P^u(\lambda)(A_\infty(\lambda)-\eta I)(s-t)} P^u(\lambda).$$

It is clear that these two pairs of operators satisfy the conditions of exponential dichotomies. □

The full system is

$$U' = \big(A_\infty(\lambda) + B(z; \rho)\big)U, \tag{9}$$

where

$$B(z; \rho) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\theta(z) - \rho(z) & 0 & 0 & 0 \end{pmatrix}.$$

To be able to define exponential dichotomies, we will consider an exponentially weighted variable $V_\pm(z) = e^{\pm\eta z} U(z)$. The equation for $V_\pm$ is

$$V_\pm' = \big(A_\infty(\lambda) \pm \eta I + B(z; \rho)\big)V_\pm. \tag{10}$$

These systems have exponential dichotomies on $\mathbb{R}_\pm$.

Lemma 2.5. *Let m be the smallest nonnegative integer such that $\lambda < m^4$. Suppose that $\rho \in X_\beta$. Then for any $\eta \in (0, \mu_{\min}(\lambda))$, the systems in* (10) *have exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$:*

- (i) *the system for V_+ has an exponential dichotomy with rates $\kappa^s = -\mu_{\min}(\lambda) + \eta$ and $\kappa^u = \eta$;*
- (ii) *the system for V_- has an exponential dichotomy with $\kappa^s = -\eta$ and $\kappa^u = \mu_{\min}(\lambda) - \eta$*

For V_+, the projection on $\mathbb{R}_+$ is denoted by $P_+^s(\cdot; \lambda, \rho)$ and the projection on $\mathbb{R}_-$ is denoted by $I - P_-^{cu}(\cdot; \lambda, \rho)$. For V_-, the projection on $\mathbb{R}_-$ is denoted by $I - P_-^u(\cdot; \lambda, \rho)$ and the projection on $\mathbb{R}_+$ is denoted by $P_+^{cs}(\cdot; \lambda, \rho)$. The corresponding evolutions operators are denoted by $\Psi_+^s(\cdot, \cdot; \lambda, \rho)$ and $\Psi_+^{cu}(\cdot, \cdot; \lambda, \rho)$ for V_+ on $\mathbb{R}_+$, etc.

Moreover, the above dichotomy constants, projections and evolution operators depend smoothly on λ and ρ for (λ, ρ) in a neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times X_\beta$.

Proof. We note that the result follows from [14, Theorem 1] if we can verify the conditions (H1), (H2), (H3) and (H5) of the same paper. We have already verified in Lemma 2.3 that condition (H1) holds. We will verify (H2) with $\alpha = 0$ and $\epsilon > 0$ arbitrary. Since θ and ρ are Hölder continuous in the z variable and decay as $z \to \pm\infty$, it is not difficult to see that (H2) holds for any $\epsilon > 0$. To verify (H3) (compactness of $(A_\infty \pm \eta I)^{-1}$), we need to show that $(A_\infty(\lambda) \pm \eta I)^{-1}$ is compact in $\mathcal{L}(X, X)$. This follows since $\mathrm{Dom}(A_\infty(\lambda)) \subset X$ is compactly embedded in in X. Condition (H5) follows from [10, Theorem 2.5], after rewriting the first order system (4) as a second order system using the variables $(u_1, u_3) \in H^3(S^1) \times H^1(S^1)$.

Smoothness of the exponential dichotomies with respect to the parameters (λ, ρ) is a consequence of [14], where it is shown that the exponential dichotomies can be found as solutions Φ of a certain linear equation of the form $\mathcal{T}\Phi = h$ on appropriate

Banach spaces where $\mathcal{T}$ is invertible and h is given. Since the operator $\mathcal{T}$ is invertible for $(\lambda, \rho) = (\lambda_0, 0)$ (see [14]) and depends smoothly on (λ, ρ), so do the solutions to the linear equation $\mathcal{T}\Phi = h$. $\square$

Using Lemma 2.5, we define corresponding evolution operators for the system in (9) on $\mathbb{R}_+$ and $\mathbb{R}_-$:

$$\begin{aligned} \Phi^s_\pm(z, z_0; \lambda, \rho) &= e^{-\eta(z-z_0)}\, \Psi^s_\pm(z, z_0; \lambda, \rho), \\ \Phi^{cu}_\pm(z, z_0; \lambda, \rho) &= e^{-\eta(z-z_0)}\, \Psi^{cu}_\pm(z, z_0; \lambda, \rho), \\ \Phi^u_\pm(z, z_0; \lambda, \rho) &= e^{\eta(z-z_0)}\, \Psi^u_\pm(z, z_0; \lambda, \rho), \\ \Phi^{cs}_\pm(z, z_0; \lambda, \rho) & e^{\eta(z-z_0)}\, \Psi^{cs}_\pm(z, z_0; \lambda, \rho). \end{aligned} \tag{11}$$

It is straight forward to check that these operators satisfy the conditions of Definition 2.4.

3. **Decay of eigenfunctions.** The first step in the proof of Theorem 1.1 is to show that any eigenfunction of (2) decays exponentially. We start proving this by deriving an expression for bounded solutions of the dynamical system (4). This expression will be used to show that all eigenfunctions are exponentially decaying.

Lemma 3.1. *Let $U(z)$ be a solution of* (4). *There exists $L_* > 0$ such that*

(i) *if U is bounded on $\mathbb{R}_+$, then for every $L \geq L_*$, there exist a $U^s_0 \in X^s$ and a $U^c_0 \in X^c$ such that for $z \geq L$*

$$\begin{aligned} U(z) = {} & e^{A_\infty P^s(\lambda)(z-L)} U^s_0 + e^{A_\infty P^c(\lambda)(z-L)} U^c_0 \\ & + \int_L^z e^{A_\infty P^s(\lambda)(z-\xi)} P^s(\lambda) B(\xi; \rho) U(\xi)\, d\xi \\ & - \int_z^\infty e^{A_\infty P^{cu}(\lambda)(z-\xi)} P^{cu}(\lambda) B(\xi; \rho) U(\xi)\, d\xi. \end{aligned} \tag{12}$$

(ii) *if U is bounded on $\mathbb{R}_-$, then for every $L \geq L_*$, there exist a $V^u_0 \in X^u$ and a $V^c_0 \in X^c$ such that for $z \leq -L$*

$$\begin{aligned} U(z) = {} & e^{A_\infty P^u(\lambda)(z+L)} V^u_0 + e^{A_\infty P^c(\lambda)(z+L)} V^c_0 \\ & - \int_z^{-L} e^{A_\infty P^u(\lambda)(z-\xi)} P^u(\lambda) B(\xi; \rho) U(\xi)\, d\xi \\ & + \int_{-\infty}^z e^{A_\infty P^{cs}(\lambda)(z-\xi)} P^{cs}(\lambda) B(\xi; \rho) U(\xi)\, d\xi. \end{aligned} \tag{13}$$

Proof. Define $M : \tilde{X} \to X$ with $\tilde{X} = L^2(S^1) \times L^2(S^1) \times L^2(S^1) \times L^2(S^1)$ by

$$MU(\varphi) = \sum_{k \in \mathbb{Z}} M_k \widehat{U}_k e^{ik\varphi},$$

where M_k is defined in (7) and $\widehat{U}_k$ are the Fourier coefficients of U as before. Note that M is a linear homeomorphism. Define also the unbounded operator $D(\lambda)$ on $\tilde{X}$ by

$$DU(\varphi) = \sum_{k \in \mathbb{Z}} D_k \widehat{U}_k e^{ik\varphi},$$

where D_k is defined in (8). Note that D is closed and densely defined with domain $H^1(S^1) \times H^1(S^1) \times H^1(S^1) \times H^1(S^1)$.

By changing variables $U = MV$, the system (4) can be written as

$$V' = DV + \tilde{B}(z;\rho)V, \tag{14}$$

where $\tilde{B}(z,\rho) := M^{-1}B(z,\rho)M$. Then $U \in X$ if and only if $V \in \tilde{X}$. Note that $\tilde{B}$ has the same decay properties as B. In particular,

$$(1+|z|)^{\beta}\|\tilde{B}(z;\rho)\|_{\mathcal{L}(\tilde{X},\tilde{X})} < \infty.$$

and we introduce the space $\tilde{X}_\beta$ with norm defined by

$$\|\tilde{B}\|_{\tilde{X}_\beta} : \sup_{z\in\mathbb{R}}(1+|z|)^{\beta}\|\tilde{B}(z;\rho)\|_{\mathcal{L}(\tilde{X},\tilde{X})}.$$

In the rest of this proof we also use the notation $\tilde{P}^s := M^{-1}P^s$, $\tilde{P}^u := M^{-1}P^u$ $\tilde{P}^c := M^{-1}P^c$, and $\tilde{X}^s := \operatorname{Ran}\tilde{P}^s$, etc.

We first consider the case that U is bounded on $\mathbb{R}_+$. Introduce $V^j(z) := \tilde{P}^jV(z)$, for $j \in \{s,c,u\}$. By the variation of constants formula (see e.g. [6]), it follows after projecting onto $\tilde{X}^u$, $\tilde{X}^c$ and $\tilde{X}^u$, respectively, that any solution of (14) satisfies

$$\begin{aligned}
V^s(z) &= e^{D\tilde{P}^s(z-\zeta)}V^s(\zeta) + \int_\zeta^z e^{D\tilde{P}^s(z-\xi)}\tilde{P}^s\tilde{B}(\xi;\rho)V(\xi)\,d\xi, && z\ge\zeta\ge 0,\\
V^c(z) &= e^{D\tilde{P}^c(z-\zeta)}V^c(\zeta) + \int_\zeta^z e^{D\tilde{P}^c(z-\xi)}\tilde{P}^c\tilde{B}(\xi;\rho)V(\xi)\,d\xi, && z,\zeta\ge 0,\\
V^u(z) &= e^{D\tilde{P}^u(z-\zeta)}V^u(\zeta) + \int_\zeta^z e^{D\tilde{P}^u(z-\xi)}\tilde{P}^u\tilde{B}(\xi;\rho)V(\xi)\,d\xi, && \zeta\ge z\ge 0.
\end{aligned} \tag{15}$$

As $U(z)$ is bounded, also $V(z)$ is bounded as $z\to\infty$, and therefore V^s, V^c and V^u are all bounded.

We first look at V^u and let $\zeta\to\infty$ in the last equation of (15). Since V^u is bounded, it follows that

$$V^u(z) = -\int_z^\infty e^{D\tilde{P}^u(z-\xi)}\tilde{P}^u\tilde{B}(\xi;\rho)V(\xi)\,d\xi.$$

Next, we study the equation for V^c. The integral in the right hand side of the second equation of (15) converges as $\zeta\to\infty$ since $\|e^{DP^c\xi}\|$ is uniformly bounded for $\xi\in\mathbb{R}$, and since

$$\begin{aligned}
\int_z^\infty \|\tilde{B}(\xi;\rho)V(\xi)\|\,d\xi &\le \|\tilde{B}(\cdot,\rho)\|_{\tilde{X}_\beta}\|V\|_{L^\infty}\int_z^\infty \frac{1}{(1+\xi)^\beta}\,d\xi\\
&= \frac{1}{\beta-1}\|\tilde{B}(\cdot,\rho)\|_{\tilde{X}_\beta}\|V\|_{L^\infty}\frac{1}{(1+z)^{\beta-1}},
\end{aligned} \tag{16}$$

where $\beta > 1$. Since the left hand side of the second equation of (15) does not depend on ζ, $\lim_{\zeta\to\infty} e^{-D\tilde{P}^c\zeta}V^c(\zeta) =: V^c(\infty)$ exists. Hence there exists $V_0^c \in \tilde{X}^c$ such that

$$V^c(z) = e^{D\tilde{P}^c z}V_0^c - \int_z^\infty e^{D\tilde{P}^c(z-\xi)}\tilde{P}^c\tilde{B}(\xi;\rho)V(\xi)\,d\xi.$$

For V^s, we choose $\zeta = L \ge 0$ arbitrarily, so that by (15) for $z \ge L$

$$\begin{aligned}
V(z) = e^{D\tilde{P}^s(z-L)}V^s(L) + e^{D\tilde{P}^c z}V_0^c + \int_L^z e^{D\tilde{P}^s(z-\xi)}\tilde{P}^s\tilde{B}(\xi;\rho)V(\xi)\,d\xi\\
- \int_z^\infty e^{D\tilde{P}^{cu}(z-\xi)}\tilde{P}^{cu}\tilde{B}(\xi;\rho)V(\xi)\,d\xi.
\end{aligned}$$

Now write $V_0^s = V^s(L)$ and transform back to the U variable. By putting $U_0^s := MV_0^s$, $U_0^c := MV_0^c$ and $U_0^u := MV_0^u$ we obtain the expression in the lemma.

The proof for $\mathbb{R}_-$ is similar to the proof above. □

With the expression in Lemma 3.1, we use a contraction mapping argument to show that any eigenfunction is exponentially decaying.

Lemma 3.2. *Fix $\rho \in X_\beta$, and assume that λ is an eigenvalue of the perturbed operator $\mathcal{L} + \rho$ with eigenfunction $u \in H^4(\mathbb{R} \times S^1)$, where $\lambda^{1/4} \notin \mathbb{Z}$. We denote by $U \in C^{0,\alpha}(\mathbb{R}; Y)$ the corresponding solution of the dynamical system (4). Let $\tilde{\kappa} := \frac{1}{2}\mu_{\min}(\lambda) > 0$, then there exists a constant $K > 0$ such that*

$$\|U(z)\|_X \leq K e^{-\tilde{\kappa}|z|}$$

for all $z \in \mathbb{R}$.

Proof. Since $u \in H^4(\mathbb{R}\times S^1)$ and $U = (u, u', u'', u''')^T$, it follows that $U \in H^1(\mathbb{R}; X)$. By standard arguments, $\|U(z)\|_X \to 0$ as $|z| \to \infty$. We will concentrate on the case where $z \to \infty$, since the proof for $z \to -\infty$ is similar. We estimate the integrals in (12) and use the notation $B(\xi)$ in place of $B(\xi; \rho)$:

$$I_1 = \int_L^z e^{A_\infty P^s(z-\xi)} P^s B(\xi) U(\xi)\, d\xi \quad \text{and} \quad I_2 = \int_z^\infty e^{A_\infty P^{cu}(z-\xi)} P^{cu} B(\xi) U(\xi)\, d\xi$$

By (16) since $\|e^{A_\infty P^{cu}\tau} P^{cu}\| \leq 1$ for any $\tau > 0$, it follows for I_2 that

$$|I_2| \leq \frac{1}{\beta - 1} \|B\|_{X_\beta} \|U\|_{L^\infty} \frac{1}{(1+z)^{\beta-1}}. \tag{17}$$

To estimate I_1, we use that for any $\alpha > 1$

$$\lim_{r\to\infty} \int_1^r \left(\frac{r}{s}\right)^\alpha e^{-(r-s)}\, ds = 1,$$

which implies that there exists a constant $C > 0$ such that for every $r \geq 1$,

$$\int_1^r \left(\frac{r}{s}\right)^\alpha e^{-(r-s)}\, ds \leq C. \tag{18}$$

Write $\kappa = \mu_{\min}(\lambda)$. By the substitution $\kappa(1+\xi) = \tau$ and equation (18) we have for $z \geq L$ (see Corollary 1 for the estimate on $e^{P^s D(z-\xi)} P^s$)

$$\begin{aligned} |I_1| &\leq \frac{\|B\|_{X_\beta} \|U\|_{L^\infty}}{(1+z)^\beta} \int_L^z e^{-\kappa(z-\xi)} \left(\frac{1+z}{1+\xi}\right)^\beta d\xi \\ &= \frac{\|B\|_{X_\beta} \|U\|_{L^\infty}}{\kappa(1+z)^\beta} \int_{\kappa(1+L)}^{\kappa(1+z)} \left(\frac{\kappa(1+z)}{\tau}\right)^\beta e^{-(\kappa(1+z)-\tau)}\, d\tau \\ &\leq C \frac{\|B\|_{X_\beta} \|U\|_{L^\infty}}{\kappa(1+z)^\beta} \end{aligned} \tag{19}$$

provided that $\kappa(1+L) \geq 1$ which in particular holds if $L \geq 1/\kappa$. This proves that I_1 and I_2 converge to 0 as $z \to \infty$. Then by (12) it follows that $U(z) \to U^c(\infty)$ as $z \to \infty$. As $U(z) \to 0$ as $z \to \infty$ (this follows from the fact that $U(z) \to 0$ for $z \to \infty$) this implies that $U^c(\infty) = 0$.

Hence for any $z \geq L \geq 1/\kappa$, $U(z)$ satisfies

$$\begin{aligned} U(z) &= e^{A_\infty P^s(z-L)}U^s(L) + \int_L^z e^{A_\infty P^s(z-\xi)}P^s B(\xi)U(\xi)\,d\xi \\ &\quad - \int_z^\infty e^{A_\infty P^{cu}(z-\xi)}P^{cu}B(\xi)U(\xi)\,d\xi. \end{aligned} \tag{20}$$

For $L \geq 1/\kappa$ fixed and $\eta \geq 0$ we define the spaces

$$Y_\eta = \{W \in C([L,\infty);\mathcal{L}(X,X));\ \|W\|_{Y_\eta} : \sup_{z\geq L} e^{\eta z}\|W(z)\|_X < \infty\}.$$

Then for $\eta \in [0,\kappa)$ we let $F : Y_\eta \to Y_\eta$ be given by

$$\begin{aligned} F(W)(z) &= e^{A_\infty P^s(z-L)}U^s(L) + \int_L^z e^{A_\infty P^s(z-\xi)}P^s B(\xi)W(\xi)\,d\xi \\ &\quad - \int_z^\infty e^{A_\infty P^{cu}(z-\xi)}P^{cu}B(\xi)W(\xi)\,d\xi. \end{aligned}$$

To see that the right hand side belongs to Y_η, note that $P^sU^s = U^s$ and and that $e^{A_\infty P^s\tau}P^s$ is exponentially decaying like $e^{-\kappa\tau}$, for $\tau \to \infty$, hence $e^{A_\infty P^s(z-L)}U^s(L) \in Y_{\eta,L}$. To see that the integrals belong to Y_η, similar arguments as those below can be used.

Let $W_1, W_2 \in Y_{\eta,L}$. It follows that

$$\begin{aligned} &\|F(W_1) - F(W_2)\|_{Y_\eta} \\ &\leq \|W_1 - W_2\|_{Y_\eta}\Big(\sup_{z\geq L}\int_L^z \|e^{A_\infty P^s(z-\xi)}P^s\|_{\mathcal{L}(X,X)}\|B(\xi)\|_{\mathcal{L}(X,X)}e^{\eta(z-\xi)}\,d\xi \\ &\qquad + \sup_{z\geq L}\int_z^\infty \|e^{A_\infty P^{cu}(z-\xi)}P^{cu}\|_{\mathcal{L}(X,X)}\|B(\xi)\|_{\mathcal{L}(X,X)}e^{\eta(z-\xi)}\,d\xi\Big) \\ &\leq K\|B\|_{X_\beta}\|W_1 - W_2\|_{Y_\eta}\left(\sup_{z\geq L}\frac{1}{(1+z)^\beta}\int_L^z e^{-(\kappa-\eta)(z-\xi)}\left(\frac{1+z}{1+\xi}\right)^\beta d\xi\right. \\ &\qquad \left. + \sup_{z\geq L}\int_z^\infty \frac{1}{(1+\xi)^\beta}\,d\xi\right) \\ &\leq K\|B\|_{X_\beta}\|W_1 - W_2\|_{Y_\eta}\left(\frac{C}{(\kappa-\eta)(1+L)^\beta} + \frac{1}{(\beta-1)(1+L)^{\beta-1}}\right). \end{aligned}$$

By choosing L large enough, we see that F is a contraction on Y_η. Hence F has a unique fixed point in Y_η for any $\eta \in [0,\kappa)$.

Since $U \in Y_{0,L}$ and solves (20), it is a fixed point for F when $\eta = 0$. But $Y_\eta \subset Y_0$ for $\eta > 0$, so by uniqueness of the fixed point in Y_0 these fixed points must be the same. Hence $U \in Y_\eta$ for any $\eta \in [0,\kappa)$. This shows that U decays exponentially. □

4. **Lyapunov–Schmidt reduction.** After the preparatory work on the decay of the eigenfunctions, we can prove the main theorem using Lyapunov–Schmidt reduction.

Let u_* be the eigenfunction associated with the embedded eigenvalue of the unperturbed problem, i.e, $\mathcal{L}u_* = \lambda_0 u_*$. We may without loss of generality assume that u_* is normalized so that $\int_{-\infty}^\infty \|u_*(z)\|^2_{L^2(S^1)}\,dz = 1$. Write

$$U_* := \left(u_*, u_*', u_*'', u_*'''\right)^T,$$

i.e. U_* is the solution of the unperturbed system (4) with $\rho = 0$ and $\lambda = \lambda_0$.

Using the evolution operators and related projections, defined in (11), we define the stable and unstable subspaces as

$$E^s_+ = \{U \in Y;\ P^s_+(0;\lambda_0,0)U = U\},$$
$$E^u_- = \{U \in Y;\ P^u_-(0;\lambda_0,0)U = U\}$$

Roughly speaking, E^s_+ consists of the initial values of solutions of the unperturbed system which decay exponentially as $z \to \infty$ and E^u_- consists of the initial values of solutions which decay exponentially as $z \to -\infty$. We have $E^s_+ \cap E^u_-$ span$\{U_*(0)\}$ since λ_0 is an eigenvalue with multiplicity 1.

Corollary 2. *Let $U(z)$ be a solution of* (4)*. If $U(0) \in E^s_+ \cap E^u_-$, then $U^c_0 = 0$ in equation* (12) *and* (13) *respectively.*

To find embedded eigenvalues, we define the mapping $\iota : E^s_+ \times E^u_- \times \mathbb{R} \times X_\beta \to X$ by

$$\iota(U^s_0, U^u_0; \lambda, \rho) P^s_+(0;\lambda,\rho)U^s_0 - P^u_-(0;\lambda,\rho)U^u_0.$$

Lemma 4.1. *λ is an embedded eigenvalue of $\mathcal{L}+\rho$ if and only if there exist $U^s_0 \in E^s_+$ and $U^u_0 \in E^u_-$ with $(U^s_0, U^u_0) \neq 0$ such that*

$$\iota(U^s_0, U^u_0; \lambda, \rho) = 0. \tag{21}$$

Proof. If (21) holds, then $P^s_+(0;\lambda,\rho)U^s_0 = P^u_-(0;\lambda,\rho)U^u_0$ which implies that the solution of (4) with initial value $P^s_+(0;\lambda,\rho)U^s_0 = P^u_-(0;\lambda,\rho)U^u_0$ decays exponentially as $|z| \to \infty$ as it is both in the stable and unstable subspaces of the perturbed problem. By Lemmas 2.1 and 3.2, it follows that λ is an eigenvalue of the perturbed operator for $\mathcal{L}+\rho$. Conversely, if λ is an eigenvalue near λ_0, then Lemma 3.2 guarantees that (4) has a solution $U(z)$ which decays exponentially as $|z| \to \infty$ with rate at least $\hat{\kappa}$. Thus, $U(0) = P^s_+(0;\lambda,\rho)U(0)$ and $U(0) = P^u_-(0;\lambda,\rho)U(0)$ which mean that (21) holds. □

To solve (21), note that for any $(U^s_0, U^u_0) \in E^s_+ \times E^u_-$ we have $\iota(U^s_0, U^u_0; \lambda_0, 0) = U^s_0 - U^u_0$, hence $\operatorname{Ran}\iota(\cdot,\cdot;\lambda_0,0) = E^s_+ + E^u_-$. Next we will show that the codimension of $E^s_+ + E^u_-$ is one higher than the dimension of E^c_∞.

Lemma 4.2. *We have* $\operatorname{codim}(E^s_+ + E^u_-) = \dim(E^c_\infty) + 1 = 4m - 1$.

Proof. Using [16] (see also [17] for similar arguments), we show that $\iota(\cdot,\cdot;\lambda_0,0)$ is Fredholm with Fredholm index

$$\operatorname{ind}(\iota) = -\dim(E^\infty_c) = -(4m-2).$$

Since $\operatorname{Ran}\iota(\cdot,\cdot;\lambda_0,0) = E^s_+ + E^u_-$, the definition of the Fredholm index implies that

$$\begin{aligned}\operatorname{codim}(E^s_+ + E^u_-) &= \dim(\ker(\iota(\cdot,\cdot;\lambda_0,0)) - \operatorname{ind}(\iota(\cdot,\cdot;\lambda_0,0)) \\ &= 1 - (-(4m-2)) = 4m - 1.\end{aligned}$$

As in [16, §2], we define a reference frame based on the asymptotic system. Let $\eta \in (0, \mu_{\min}(\lambda_0))$. Then the reference system

$$U' = [A_\infty(\lambda_0) - \eta I]\, U$$

has an exponential dichotomy on $\mathbb{R}$ with projection P^u on the unstable subspace (see Corollary 1).

As shown in Lemma 2.5, the dynamical system

$$U' = [A_\infty + B(z;0) + \eta\ \operatorname{sign}(z)\, I]\, U$$

has exponential dichotomies on $\mathbb{R}_-$ with projection P_-^u and on $\mathbb{R}_+$ with projection $I - P_+^s$. Define the projections $\widehat{P}^s(s) : X^s \oplus X^c \to \mathrm{Ran}(P_+^s(s))$ and $\widehat{P}^u(s) : \mathrm{Ran}(P_-^u(s)) \to X^u$ as

$$\widehat{P}^s = P_+^s\big|_{X^s \oplus X^c} \quad \text{and} \quad \widehat{P}^u = P_-^u\big|_{\mathrm{Ran}(P_-^u(s))}$$

In [16, Theorem 2.2], it is shown that $\widehat{P}^s$ and $\widehat{P}^u$ are Fredholm and that the Fredholm index is independent of s. So we can define the relative Morse indices ([16, Definition 2.3])

$$i_- = \mathrm{ind}(\widehat{P}^u) \quad \text{and} \quad i_+ = \mathrm{ind}(\widehat{P}^s).$$

As the Morse indices are independent of s, we will determine them by using the limits for $s \to \pm\infty$. By definition, $\lim_{s\to\infty} P_+^s(s) = P^s$, and so it follows that $\mathrm{Ker}(\widehat{P}^s(\infty)) = X^c$. Thus $i_+ = \dim(E_\infty^c) = 4m - 2$. Similarly, $\lim_{s\to\infty} P_-^u(s)P^u$. Thus $\widehat{P}^u(\infty)) = I$ and $i_- = 0$.

Define the operator $\mathcal{T} : H^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, Y) \to L^2(\mathbb{R}, X)$ as

$$\mathcal{T} = \frac{d}{dz} - (A_\infty + B(z;\theta_0)) - \eta \,\mathrm{sign}(z)\, I$$

In [16, Theorem 2.6], it is shown that $\mathcal{T}$ is Fredholm with $\mathrm{ind}(\mathcal{T}) = i_- - i_+$. From the proof of [16, Theorem 2.6], it follows that ι is Fredholm with $\mathrm{ind}(\iota) = \mathrm{ind}(T)$ (via the operator $\mathcal{K}$ in the proof of [16, Theorem 2.6]). Hence we can conclude that $\mathrm{ind}(\iota) = 0 - \dim(E_\infty^c)$. □

Now let Q be a projection in X onto $\mathrm{Ran}\,\iota(\cdot,\cdot;\lambda_0,0) = E_+^s + E_-^u$. Lemma 4.2 implies that $\dim(\ker(Q)) = 4m - 1$. Equation (21) can then be rewritten as

$$\begin{aligned} Q\iota(U_0^s, U_0^u; \lambda, \rho) &= 0, \\ (I - Q)\iota(U_0^s, U_0^u; \lambda, \rho) &= 0. \end{aligned} \tag{22}$$

Lemma 4.3. *Let $D \subset E_+^s \times E_-^u$ be an affine hyperplane which satisfies $D \cap \mathrm{span}\{(U_*(0), U_*(0))\} = \{(U_*(0), U_*(0))\}$. For (λ, ρ) close to $(\lambda_0, 0)$ the equation $Q\iota(\cdot,\cdot;\lambda,\rho) = 0$ has a unique solution $(U_0^s, U_0^u) = (U_0^s(\lambda,\rho), U_0^u(\lambda,\rho)) \in D$ in a neighbourhood of $(U_*(0), U_*(0))$. Moreover, U_0^s and U_0^u are smooth in their arguments.*

Proof. For (λ, ρ) fixed, $Q\iota$ is a linear mapping from $E_+^s \times E_-^u$ to $\mathrm{Ran}\,Q = E_+^s + E_-^u$, and $\ker Q\iota(\cdot,\cdot;\lambda,\rho) = \mathrm{span}\{(U_*(0), U_*(0))\}$. By the definition of Q and since P_+^s and P_-^u are continuous in λ and ρ, it follows that $Q\iota(\cdot,\cdot;\lambda,\rho)$ is surjective for (λ,ρ) in a neighbourhood of $(\lambda_0, 0)$.

Now the implicit function theorem gives that for (λ, ρ) close to $(\lambda_0, 0)$ the equation $0 = Q\iota(\cdot,\cdot;\lambda,\rho)$ has a unique solution $(U_0^s, U_0^u) = (U_0^s(\lambda,\rho), U_0^u(\lambda,\rho)) \in D$ in a neighbourhood of $(U_*(0), U_*(0))$, which is smooth in its arguments. □

Using the variation of constants formula (see [6]), it is easy to see that

$$\begin{aligned} \iota(U_0^s, U_0^u; \lambda, \rho) = U_0^s - U_0^u &+ \int_{-\infty}^0 \Phi_-^{cs}(0, z; \lambda, 0)\rho(z) B \Phi_-^u(z, 0; \lambda, \rho) U_0^u \, dz \\ &+ \int_0^\infty \Phi_+^{cu}(0, z; \lambda, 0)\rho(z) B \Phi_+^s(z, 0; \lambda, \rho) U_0^s \, dz, \end{aligned}$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

To solve the second equation of (22), define

$$F : \mathbb{R} \times X_\beta \to \ker Q \tag{23}$$

as

$$\begin{aligned} F(\lambda,\rho) :&- \iota(U_0^s(\lambda,\rho), U_0^u(\lambda,\rho); \lambda,\rho) = -(I-Q)\,\iota(U_0^s(\lambda,\rho), U_0^u(\lambda,\rho);\lambda,\rho) \\ &= \int_{-\infty}^0 (I-Q)\,\Phi_-^{cs}(0,z;\lambda_0,0)(\lambda-\lambda_0-\rho(z))B\Phi_-^u(z,0;\lambda,\rho)U_0^u(\lambda,\rho)\,dz \\ &+ \int_0^\infty (I-Q)\,\Phi_+^{cu}(0,z;\lambda_0,0)(\lambda-\lambda_0-\rho(z))B\Phi_+^s(z,0;\lambda,\rho)U_0^s(\lambda,\rho)\,dz. \end{aligned}$$

Note that F is a smooth function and that solving the (22) is equivalent to solving $F(\lambda,\rho) = 0$.

To solve this, we use the dual space of X and the adjoint equation. The dual space of X is $X' = H^{-3}(S^1) \times H^{-2}(S^1) \times H^{-1}(S^1) \times L^2(S^1)$ (using the L^2 pairing). The adjoint equation of the unperturbed equation (4) with $\rho = 0$ and $\lambda = \lambda_0$ is

$$W' = -(A_\infty + B(z;\lambda_0,0))^* W. \tag{24}$$

If the flow map of the adjoint system is denoted by $\Psi(s,t)$, then $\Psi(z,z_0) = \Phi(z_0,z)^*$, where Φ is the flow map of the unperturbed system (4). This implies that for any $U \in Y$ and $W \in X'$ and for any $z, z_0 \in \mathbb{R}$ we have

$$\langle \Psi(z,z_0)W, \Phi(z,z_0)U\rangle \langle W, \Psi(z,z_0)^*\Phi(z,z_0)U\rangle \langle W, \Phi(z_0,z_0)U\rangle = \langle W, U\rangle.$$

In other words, the pairing of any two solutions of the linear system and the adjoint system is constant. Furthermore, the adjoint system has an exponential dichotomy, similar to the one for the unperturbed system (4), but with the roles of $\mathbb{R}_+$ and $\mathbb{R}_-$ swapped, see [16, Lemma 5.1].

It is straight forward to check that $U_*^\perp : (-u_*''' + 2\partial^2 u_*', u_*'' - 2\partial^2 u_*, -u_*', u_*)^T$ solves the adjoint equation (24) and is exponentially decaying for $|z| \to \infty$. Furthermore, as the pairing between $U_*^\perp$ and any solution of the adjoint system is constant, we get that for any $U^s + U^u \in E_+^s + E_-^u$

$$\langle U_*^\perp(0), U^s + U^u\rangle \langle U_*^\perp(z), \Phi_+^s(z)U^s\rangle + \langle U_*^\perp(-z), \Phi_-^u(-z)U^u\rangle = 0,$$

as all of $U_*^\perp(z)$, $U_*^\perp(-z)$, $\Phi_+^s(z)U^s$ and $\Phi_-^u(-z)U^u$ decay to zero for $z \to \infty$. This implies that for any $U \in Y$, we have $\langle U_*^\perp(0), QU\rangle = 0$, thus also $\langle Q^* U_*^\perp(0), U\rangle = 0$, for any $U \in Y$. As Y is dense in X, this gives immediately that $U_*^\perp(0) \in \ker Q^*$ or $(I-Q)^* U_*^\perp = U_*^\perp$.

By taking the pairing of $F(\lambda,\rho)$ with $U_*^\perp(0)$, the following lemma can be proved.

Lemma 4.4. *The equation $\langle U_*^\perp(0), F(\lambda,\rho)\rangle = 0$ defines a smooth function $\lambda(\rho)$ such that $\lambda(0) = \lambda_0$ in a neighbourhood of 0, i.e. $\langle U_*^\perp(0), F(\lambda(\rho),\rho)\rangle = 0$ and $\lambda(0) = \lambda_0$. Furthermore, for any $\rho \in X_\beta$*

$$\lambda'(0)\rho \int_{-\infty}^\infty (u_*(z), \rho(z)u_*(z))_{L^2(S^1)}\,dz. \tag{25}$$

Proof. The pairing of $F(\lambda,\rho)=0$ with $U_*^{\perp}(0)$ gives the equation

$$\begin{aligned}0 = F_*(\lambda,\rho) &:= \langle U_*^{\perp}(0), F(\lambda,\rho)\rangle \\ &= \int_{-\infty}^{0} \langle U_*^{\perp}(0), \Phi_-^{cs}(0,z;\lambda_0,0)(\lambda-\lambda_0-\rho(z))B\Phi_-^{u}(z,0;\lambda,\rho)U_0^u(\lambda,\rho)\rangle\, dz \\ &+ \int_{0}^{\infty} \langle U_*^{\perp}(0), \Phi_+^{s}(z,0;\lambda_0,\rho)(\lambda-\lambda_0-\rho(z))B\Phi_+^{s}(z,0;\lambda,\rho)U_0^u(\lambda,\rho)\rangle\, dz \\ &= \int_{-\infty}^{0} \langle U_*^{\perp}(z), (\lambda-\lambda_0-\rho(z))B\Phi_-^{u}(z,0;\lambda,\rho)U_0^u(\lambda,\rho)\rangle\, dz \\ &+ \int_{0}^{\infty} \langle U_*^{\perp}(z), (\lambda-\lambda_0-\rho(z))B\Phi_+^{s}(z,0;\lambda,\rho)U_0^u(\lambda,\rho)\rangle\, dz.\end{aligned}$$

By Lemma 2.5, F_* is a smooth function of λ and ρ in a neighbourhood of $(\lambda_0,0)$. Moreover, $F_*(\lambda_0,0)=0$ and

$$\frac{\partial F_*}{\partial\lambda}(\lambda_0,0)\int_{-\infty}^{\infty}\langle U_*^{\perp}(z), BU_*(z)\rangle\, dz = \int_{-\infty}^{\infty}\|u_*(z)\|^2_{L^2(S^1)}\, dz = 1,$$

as u_* is normalized so that $\int_{-\infty}^{\infty}\|u_*(z)\|^2_{L^2(S^1)}\, dz = 1$.

This allows us to use the implicit function theorem to solve for λ and obtain $\lambda(\rho)$ in a neighbourhood of $\rho=0$ with $\lambda(0)=\lambda_0$. Differentiating the equation $F_*(\lambda(\rho),\rho)=0$ at $\rho=0$ gives

$$\lambda'(0)\rho = -\int_{-\infty}^{\infty}\langle U_*^{\perp}(z), R(z)U_*(z)\rangle\, dz = \int_{-\infty}^{\infty}(u_*(z),\rho(z)u_*(z))_{L^2(S^1)}\, dz.$$

□

Now the proof of Theorem 1.1 can be finished by showing that there is a smooth codimension $4m-2$ manifold in X_β with $F(\lambda(\rho),\rho)=0$ for all ρ in this manifold.

Proof of Theorem 1.1. We note that $\ker Q$ is $(4m-1)$-dimensional, and hence there are $4m-2$ conditions left to verify. Again we will use the adjoint equation. The adjoint projection Q^* has a $(4m-1)$-dimensional kernel, just as Q itself. We have seen that $U_*^{\perp}(0)\in\ker Q$. Define $W_k(0)\in X'$, $k=1,\dots 4m-2$, to be such that $\{W_k(0);\ k=1,\dots 4m-2\}\cup\{U_*^{\perp}(0)\}$ is a basis for $\ker Q^*$. For $k=1,\dots,4m-2$, let $W_k(z)$ be solutions of the adjoint unperturbed system with initial condition $W_k(0)$.

For $k=1,\dots,4m-2$, we let

$$F_k(\rho) := \langle W_k(0), F(\lambda(\rho),\rho)\rangle,$$

and note that $F_k : X_\beta\to\mathbb{R}$ are smooth functions. If for some ρ, $F_k(\rho)=0$ for all $k=1,\dots,4m-2$, then $F(\lambda(\rho),\rho)=0$ as $\{W_k(0);\ k1,\dots,4m-2\}\cup\{U_*^{\perp}(0)\}$ is a basis for $\ker Q^*$. To prove Theorem 1.1, we show that there is a codimension $4m-2$ manifold of perturbations ρ, such that $F_k(\rho)=0$ for all $k=1,\dots,4m-2$.

With the notation

$$U(z;\rho) : \begin{cases}\Phi_+^s(z,0;\lambda(\rho),\rho)U_0^s(\lambda(\rho),\rho), & \text{for } z\ge 0,\\ \Phi_-^u(z,0;\lambda(\rho),\rho)U_0^u(\lambda(\rho),\rho), & \text{for } z\le 0,\end{cases}$$

we have

$$F_k(\rho) = \int_{-\infty}^{\infty} \langle W_k(z), (\lambda(\rho) - \lambda_0 - \rho(z))BU(z;\rho)\rangle \, dz$$
$$= \int_{-\infty}^{\infty} (w_k(z), (\lambda(\rho) - \lambda_0 - \rho(z))u(z;\rho))_{L^2(S^1)} \, dz,$$

where

$$W_k = (-w_k''' + 2\partial^2 w_k', w_k'' - 2\partial^2 w_k, -w_k', w_k)^T$$

and $u(z;\rho)$ is the first component of $U(z;\rho)$. Obviously $F_k(0) = 0$ for $k = 1, \ldots, 4m-2$ as $\lambda(\rho) - \lambda_0 - \rho(z) = 0$ for $\rho = 0$.

We claim that $F_k'(0)$ for $k = 1, \ldots, 4m-2$ are linearly independent. To see this, first observe that for all $\rho \in X_\beta$

$$F_k'(0)\rho \int_{-\infty}^{\infty} (w_k(z), ([\lambda'(0)\rho] + \rho(z))u_*(z))_{L^2(S^1)} \, dz,$$

with $\lambda'(0)\rho$ as in (25). Now assume that there exist $\alpha_1, \ldots, \alpha_{4m-2} \in \mathbb{R}$ such that for every $\rho \in X_\beta$

$$0 = \sum_{k=1}^{4m-2} \alpha_k F_k'(0)\rho \int_{-\infty}^{\infty} (w(z), ([\lambda'(0)\rho] + \rho(z))u_*(z))_{L^2(S^1)} \, dz, \qquad (26)$$

where we use the notation

$$w := \sum_{k=1}^{4m-2} \alpha_k w_k.$$

We will show that $\alpha_i = 0$ for all $i = 1, \ldots, 4m-2$. After a rearrangement and by (25), we see that (26) is equivalent to

$$\int_{-\infty}^{\infty} (w(z) + \alpha_* u_*(z), \rho(z)u_*(z))_{L^2(S^1)} \, dz = 0$$

for every $\rho \in X_\beta$, where

$$\alpha_* := \int_{-\infty}^{\infty} (w(z), u_*(z))_{L^2(S^1)} \, dz.$$

It follows that

$$w(z) + \alpha_* u_*(z) = 0 \qquad (27)$$

for every $z \in \mathbb{R}$ such that $u_*(z) \neq 0$. Since u_* and w are continuous functions and since by unique continuation u_* cannot vanish on an interval, we see that (27) holds for every $z \in \mathbb{R}$.

At $z = 0$, the functions $W_k(0)$, $k = 1, \ldots, 4m-2$, and $U_*^\perp(0)$ form a basis for $\ker(Q^*)$, hence are linearly independent. Therefore also the functions $w_k(0)$, $k = 1, \ldots, 4m-2$, and $u_*(0)$ are linearly independent. This gives that $\alpha_* = 0$ and $\alpha_i = 0$ for all $i = 1, \ldots, 4m-2$ and hence $F_k'(0)$ are linearly independent for $k = 1, \ldots, 4m-2$.

Define $\mathcal{F} : X_\beta \to \mathbb{R}^{4m-2}$ by

$$\mathcal{F}(\rho) := (F_1(\rho), \ldots, F_{4m-2}(\rho))^T.$$

The linear independence of the components of $\mathcal{F}'(0)$ implies that we have a decomposition

$$X_\beta = (\ker \mathcal{F}'(0)) \oplus M,$$

where M is $(4m-2)$-dimensional. Moreover, the map $\mathcal{F}'(0)$ is surjective. For $\rho \in X_\beta$, we write $\rho = \xi + \eta$, where $\xi \in \ker \mathcal{F}(0)$ and $\eta \in M$.

Now define $\mathcal{G} : (\ker \mathcal{F}'(0)) \times M \to \mathbb{R}^{4m-2}$ by $\mathcal{G}(\xi, \eta) = \mathcal{F}(\xi + \eta)$. The Fréchet derivative with respect to η which we denote by $\mathcal{G}'_\eta(0,0)$ is invertible, and so by the implicit function theorem the equation $\mathcal{G}(\xi, \eta) = 0$ is solvable for η in terms of ξ, and this equation defines a smooth manifold of codimension $4m-2$ in a neighbourhood of 0. $\square$

5. **Discussion.** Our result shows that a simple eigenvalue λ_0 which is embedded in the continuous spectrum of the bilaplacian persists upon adding a small algebraically decaying potential if and only if the potential lies in a certain submanifold of finite codimension in an appropriate space of admissible potentials. We also proved that this codimension is equal to the multiplicity of the continuous spectrum at λ_0.

The main technical obstacle for proving these results is the fact the implicit function theorem is not directly applicable as the relevant operator does not have closed range (since the eigenvalue λ_0 is embedded in the continuous spectrum). To overcome this difficulty, we employed a spatial-dynamics formulation of the eigenvalue problem for which we can use exponential dichotomies. This allowed us to reduce the eigenvalue problem to a reduced matching equation to which we can eventually apply the implicit function theorem. This approach is very general and applies not only to the bilaplacian but to general selfadjoint PDE operators, including systems, posed on cylindrical domains of the form $\mathbb{R} \times \Omega$ where $\Omega \subset \mathbb{R}^n$ is a smooth, bounded, open set (see, for instance, [14, 16]).

We now discuss possible extensions of our results.

5.1. **Multiple eigenvalues.** The Lyapunov–Schmidt reduction we carried out in §4 works completely analogous near embedded eigenvalues $\lambda_0 \notin \mathbb{Z}^{1/4}$ with geometric multiplicity $k > 1$: We now have $\dim(E^s_+ \cap E^u_-) = k$ and $\operatorname{codim}(E^s_+ + E^u_-) = \dim(E^c_\infty) + k = 4m - 2 + k$ and, consequently, the reduced system obtained in this fashion is of the form

$$\tilde{F}(\lambda, \rho) d = 0 \tag{28}$$

where $d \in \mathbb{R}^k$ and $\tilde{F}(\lambda, \rho) \in \mathbb{R}^{(4m-2+k)\times k}$ for $(\lambda, \rho) \in \mathbb{R} \times X_\beta$. Each of the k columns of $\tilde{F}$ has the same form as the reduced map $F(\lambda, \rho)$ in (23). To solve the reduced system, we need to find all (λ, ρ) near $(\lambda_0, 0)$ so that there is a nonzero $d \in \mathbb{R}^k$ for which (28) is met. The set of solutions (λ, ρ) will no longer be a smooth manifold: Instead, we except that it will be the union of k manifolds which correspond to the set of potentials ρ for which $\mathcal{L} + \rho$ has at least one embedded eigenvalue. Their intersections will consist of those potentials for which there are more than two embedded eigenvalues. Proving this may, however, be difficult as multiple eigenvalues of symmetric operators may not be differentiable when considered as functions of more than two parameters (see [9, Example 5.12 in chapter II]). Thus, we leave this an an open problem.

5.2. **Branch points.** So far, we have excluded the case when the initial eigenvalue λ_0 is a branch point, that is, when $\lambda_0 \in \mathbb{Z}^{1/4}$. In this case, the system (6) at infinity has a Jordan block of length at most 4. The characterization of eigenfunctions given in §3 remains valid with identical proof, at least when $\beta > 4$ to compensate for the growth in the center directions due to this Jordan block. Working in exponentially weighted spaces, our methods give the following result:

Theorem 5.1. *Assume that $\lambda_0 = (m-1)^4$ is a branch point and fix any η with $0 < \eta < \frac{1}{2}$. Then the set of potentials for which $\mathcal{L} + \rho$ has an eigenvalue near λ_0*

with an eigenfunction that decays exponentially with rate at least η is a manifold in X_β of codimension $4m-2$.

Using a combination of the results in [8] and this paper, it should also be possible to characterize the sets

$$\mathcal{M}(\lambda) := \{\rho \in X_\beta;\ \lambda \text{ is an eigenvalue of } \mathcal{L}+\rho\}$$

as manifolds of finite codimension. More precisely, if $\lambda_0 = (m-1)^4$, then we believe that

$$\operatorname{codim} \mathcal{M}(\lambda) = \begin{cases} 4(m-1)-3 = 4m-7 & \text{for } \lambda < \lambda_0 \\ 4m-3 & \text{for } \lambda \geq \lambda_0 \end{cases}$$

provided $|\lambda - \lambda_0|$ is sufficiently small. The union of the manifolds $\mathcal{M}(\lambda)$ for $\lambda > \lambda_0$, or separately for $\lambda < \lambda_0$, gives the manifold $\mathcal{M}_\delta$ described in Theorem 1.1.

5.3. **Non-selfadjoint operators.** The same ideas work for certain non-selfadjoint eigenvalue problems. Assume that $\mathcal{L}$ is a reaction-diffusion or fourth-order PDE operator posed on a cylindrical domain $\mathbb{R}\times\Omega$ that has a simple eigenvalue λ_0 which is embedded in a curve of continuous spectrum of multiplicity one. It follows from this assumption as in §3 that any eigenfunction of $\mathcal{L}$ belonging to the embedded eigenvalue λ_0 decays exponentially with rate larger than κ where $\kappa > 0$ is chosen so that any eigenvalue of the asymptotic system $A_\infty(\lambda_0)$ that does not lie on the imaginary axis has distance at least 2κ to the imaginary axis.

In this situation, we would like to determine the set of perturbations for which the eigenvalue persists *and* the associated eigenfunction decays exponentially to zero as $|z| \to \infty$ with rate larger than κ. This problem is exactly of the form as the one discussed in §3 and §4. Our analysis shows that the eigenvalue persists with a uniformly decaying eigenfunction if and only if the potential lies in a submanifold of codimension one. If we require in addition that the eigenvalue is still embedded, then the corresponding submanifold of admissible potentials has codimension two.

Alternatively, we may ask for which perturbations the eigenvalue persists without any restrictions on the decay of the eigenfunction other than that it lies in L^2. Typically, the persisting eigenvalue will leave the continuous spectrum, and the associated eigenfunction will decay only slowly to zero with a rate that comes from the spatial eigenvalue of $A_\infty(\lambda)$ that lies on the imaginary axis for the unperturbed problem and generates the continuous spectrum. This question can be analysed in a fashion similar to [11, 4, 8] where operators on $\mathbb{R}$ were considered, and we refer to these papers for details.

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