Dimension breaking of nonlinear elliptic PDEs: satisfying the spectral condition geometrically

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Abstract

Dimension breaking occurs when the solution of a nonlinear partial differential equation (PDE) depending on n independent variables bifurcates to one depending on $n+1$. A central hypothesis in the theory of dimension breaking is that a certain operator should have a non-zero purely imaginary eigenvalue. This hypothesis is difficult to verify in general. We present a geometric theory for verifying this hypothesis. Moreover, for a large class of partial differential equations, namely multi-symplectic Hamiltonian PDEs, we show that the verification of this hypothesis is encoded in the basic state. The theory is demonstrated by obtaining new results on dimension breaking of localized states for three examples: the (2+1)-Boussinesq equation, the Zakharov-Kuznetsov equation and the Kadomtsev-Petviashvili equation.

Table of Contents

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1 Introduction

Given a nonlinear partial differential equation in $n+1$ independent variables, and a basic solution that depends only on n independent variables, *dimension breaking* arises when this n-dimensional solution bifurcates to one depending on $n+1$ dimensions. As far as we are aware the first rigorous formulation of the dimension-breaking problem in unbounded domains is due to Kirchgässner & Schaaf [16] and Hărăguş & Kirchgässner [11, 12] and has been developed further in recent work by Dias & Hărăguş-Courcelle [8].

A prototypical example of dimension breaking is given by the semilinear elliptic PDE with $n=1$,

$$u_{xx} + u_{yy} - u + u^2 = 0\,, \quad (x,y) \in \mathbb{R}^2\,, \tag{1}$$

introduced in [12]. A y-independent solution of (1), denoted by $\hat{u}(x)$, satisfies,

$$\hat{u}_{xx} - \hat{u} + \hat{u}^2 = 0\,. \tag{2}$$

This ODE has an explicit localized solution,

$$\hat{u}(x) = \tfrac{3}{2}\,\mathrm{sech}^2\tfrac{1}{2}(x+x_0)\,, \quad \text{for any } x_0 \in \mathbb{R}\,. \tag{3}$$

To determine whether there exists a solution of (1) nearby the state (3), let $u(x,y) = \hat{u}(x) + v(x,y)$ and substitute in (1),

$$v_{xx} + v_{yy} + (2\hat{u}(x) - 1)v + v^2 = 0\,. \tag{4}$$

The idea of [16] and [11] is to formulate the system (4) as a dynamical system in the y-direction,

$$\frac{d}{dy}\mathbf{v} = \mathcal{L}\mathbf{v} + \mathcal{N}(\mathbf{v})\,, \quad \mathbf{v} = \begin{pmatrix} v \\ v_y \end{pmatrix} \in \mathbb{H}\,, \tag{5}$$

where $\mathbb{H}$ is some chosen function space and

$$\mathcal{L} = \begin{bmatrix} 0 & 1 \\ -\frac{\partial^2}{\partial x^2} + 1 - 2\hat{u}(x) & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{N}(\mathbf{v}) = \begin{pmatrix} 0 \\ -v^2 \end{pmatrix}\,. \tag{6}$$

Dimension breaking occurs when there exists a solution of (5) which is bounded for all $y \in \mathbb{R}$ and localized in the x direction. Since it is solutions nearby $\hat{u}$ that are of interest, the starting point for answering this question is the linearized equation, and the spectral problem associated with the linearization,

$$\mathcal{L}\mathbf{v} = \mu\,\mathbf{v}\,, \quad \mathbf{v} \in \mathbb{H}^{\mathbb{C}}\,, \tag{7}$$

where $\mathbb{H}^{\mathbb{C}}$ is the complexification of $\mathbb{H}$.

Equation (7) is a central equation in the theory of dimension breaking. Bounded solutions of the linearized problem are associated with purely imaginary eigenvalues in the spectrum of $\mathcal{L}$. The necessary spectral condition for dimension breaking is that there exist at least one eigenvalue of $\mathcal{L}$ on the imaginary axis. Establishing the existence of *nonzero* purely imaginary eigenvalues is difficult in general. (Existence of zero eigenvalues of $\mathcal{L}$ is relatively straightforward, and will not be considered here.)

Before proceeding with a further discussion of this necessary spectral condition, we will verify that it is satisfied for the example (4). For this example, the spectral problem (7) reduces to

$$v_{xx} + (-1 + \mu^2 + 2\hat{u}(x))\, v = 0\,. \tag{8}$$

The discrete spectrum of this problem can be computed explicitly by transforming (8) to an associated Legendre equation (cf. [12]). There are exactly 5 discrete eigenvalues and two are nonzero and purely imaginary,

$$\sigma_p = \{\, \pm\tfrac{\sqrt{3}}{2}, 0, \pm \mathrm{i}\tfrac{\sqrt{5}}{2} \,\}\,. \tag{9}$$

Given the existence of purely imaginary eigenvalues, the strategy of [11, 12] is to apply a centre-manifold reduction to the nonlinear problem to prove the existence of nonlinear states that are bounded for all $y \in \mathbb{R}$. For the example (4), the nonlinear bifurcating states which break dimension are localized in the x-direction with periodic modulation in the y-direction.

Computing the discrete spectrum analytically of spectral problems of the form (6) will be possible only in very exceptional cases. Indeed, according to [12] the most difficult hypothesis to verify in the dimension-breaking theory is the spectral hypothesis, particularly that there exist nonzero purely imaginary eigenvalues in the spectrum of $\mathcal{L}$.

The purpose of this paper is twofold: we propose a secondary dynamical systems argument that can be used to show the existence of at least one nonzero purely imaginary eigenvalue. Secondly, we introduce a geometric framework for a large class of nonlinear PDEs, namely Hamiltonian systems with a multi-symplectic structure, where an abstract general condition can be given for the existence of at least one nonzero purely imaginary eigenvalue in the spectrum of $\mathcal{L}$.

The secondary dynamical systems argument arises by reformulating the spectral problem (7)-(8) as a dynamical system in the x variable,

$$\frac{d}{dx}\mathbf{u} = \mathbf{A}(x,\kappa)\mathbf{u}\,, \quad \mu = i\kappa\,, \quad \mathbf{u} = \begin{pmatrix} v \\ v_x \end{pmatrix} \tag{10}$$

and

$$\mathbf{A}(x,\kappa) = \begin{bmatrix} 0 & 1 \\ 1+\kappa^2 - 2\hat{u}(x) & 0 \end{bmatrix}\,. \tag{11}$$

A dynamical systems view of the spectral problem (7) is: for some real positive value of κ, does there exist a localized solution – in $L_2(\mathbb{R})$ for example – of (10)? This setup is very similar to that for the *Evans function* associated with the linearized stability of solitary waves (cf. Evans [9], Alexander, Gardner & Jones [1]).

Since the basic state $\hat{u}$ is localized in x, the matrix $\mathbf{A}(x,\kappa)$ satisfies

$$\lim_{x\to\pm\infty} \mathbf{A}(x,\kappa) = \mathbf{A}^\infty(\kappa)\,. \tag{12}$$

When κ is real and positive, $\mathbf{A}^\infty(\kappa)$ has one strictly postive and one strictly negative real eigenvalue for each fixed κ. Therefore, by standard theory for ODEs (e.g. Coddington &

Levinson [7], page 91) there exists a pair of solutions of (10), one of which decays exponentially as $x \to +\infty$ and one which decays exponentially as $x \to -\infty$. Denote these solutions by $\mathbf{u}^+(x,\kappa)$ and $\mathbf{u}^-(x,\kappa)$, respectively. If for some real positive value of κ, the solutions $\mathbf{u}^+(x,\kappa)$ and $\mathbf{u}^-(x,\kappa)$ are linearly dependent, i.e.

$$\mathbf{u}^+(x,\kappa) \wedge \mathbf{u}^-(x,\kappa) = 0\,, \tag{13}$$

where $\wedge$ is the wedge product, then there exists a solution of (10) in $L_2(\mathbb{R})$. This real value of κ is then a nonzero purely imaginary value for μ, and the spectral necessary condition for dimension breaking would be satisfied.

Introduce a volume form for $\mathbb{C}^2$, for example let $\mathbf{e}_1$ and $\mathbf{e}_2$ be an orthonormal basis and take $\mathcal{V} = \mathbf{e}_1 \wedge \mathbf{e}_2$ to be the volume form. Then the wedge product in (13) can be written,

$$\mathbf{u}^+(x,\kappa) \wedge \mathbf{u}^-(x,\kappa) = D(\kappa)\,\mathcal{V}\,, \quad \text{with} \quad D(\kappa) = \det\left[\mathbf{u}^+(x,\kappa)\,|\,\mathbf{u}^-(x,\kappa)\right]\,. \tag{14}$$

If $D(\kappa) = 0$ for some real finite value of κ, then the spectral hypothesis for dimension breaking is satisfied. Therefore, we call $D(\kappa)$ the dimension-breaking function. Its construction is similar to the Evans function. In the dimension-breaking setting, it is the real values of κ which are of interest, although $D(\kappa)$ can be extended into the complex κ-plane.

For the example (4), the function $D(\kappa)$ can be explicitly computed,

$$D(\kappa) = D_0(\kappa)\,\kappa^2\,\left(\kappa^2 - \tfrac{5}{4}\right)\left(\kappa^2 + \tfrac{3}{4}\right)\,, \tag{15}$$

where

$$D_0(\kappa) = \left(\frac{8}{15 p_\infty}\right)^2\,, \quad p_\infty = 1 + \frac{1}{5}(5 + 8\kappa^2) + \left(2 + \frac{8}{15}\kappa^2\right)\sqrt{1+\kappa^2}\,.$$

It is evident from the expression (15) that $D(\kappa)$ has exactly one real positive root.

In general, an explicit expression for the dimension-breaking function $D(\kappa)$ will not be available, and so we have replaced one intractable problem – finding elements in the spectrum of the operator $\mathcal{L}$ – with another intractable problem: finding roots of an abstractly defined function. However, the existence of a real positive root of $D(\kappa)$ in (15) follows easily from abstract properties of this function. Note that

$$D(0) = 0\,, \quad D'(0) = 0\,, \quad D''(0) < 0\,, \quad \text{and} \quad \lim_{\kappa \to +\infty} D(\kappa) = +1\,,$$

and therefore the existence of a positive root of $D(\kappa)$ follows from the intermediate value theorem. These properties of $D(\kappa)$ are reminiscent of the geometric study of the Evans function in Pego & Weinstein [18] and Bridges & Derks [5].

The proposed strategy is to formulate the spectral problem as the study of the real positive zeros of $D(\kappa)$. This function is abstractly defined and in general it is not obvious how to determine these zeros. However, in the multi-symplectic setting it is possible to deduce general results about the derivatives of $D(\kappa)$ at the origin, and its sign for large κ. For a large class of systems, we will prove that

$$D(0) = 0\,, \quad D'(0) = 0 \quad \text{and} \quad D''(0) = \Pi\,\mathcal{C}'(\alpha)\,, \tag{16}$$

where $\mathcal{C}(\alpha)$ is a function associated with the basic unperturbed solitary wave, and Π is a sign associated with a geometric property of the basic solitary wave. In fact it is possible to prove a more general expression for $D''(0)$ for cases where the basic solitary wave is more exotic, for example, is biasymptotic to a nontrivial state at infinity. This geometric characterization of $D''(0)$ is similar to the analysis of the symplectic Evans function in Bridges & Derks [5, 6].

In the analysis of the example, it was clear that a dynamical systems setting was useful in both the x and y directions. In fact, for the example (1), the dynamical system (5) can be formulated as a Hamiltonian system, and the secondary linear dynamical system (10) is Hamiltonian also, but with respect to a different symplectic structure. Therefore we propose a *multi-symplectic structures* framework for the analysis of the dimension-breaking problem.

A multi-symplectic Hamiltonian PDE in two space dimensions x and y takes the following canonical form

$$\mathbf{K}Z_x + \mathbf{L}Z_y = \nabla S(Z), \qquad Z \in \mathbb{R}^{2n}, \tag{17}$$

where $\mathbf{K}$ and $\mathbf{L}$ are $2n \times 2n$ constant skew-symmetric matrices and $S(Z)$ is a function, and the gradient is taken with respect to a standard inner product on $\mathbb{R}^{2n}$ (cf. [3, 5, 6]).

Before proceeding with general aspects of the multi-symplectic formulation and its implications for dimension breaking, it may be useful to reconsider the example (1) from a multi-symplectic point of view.

Introduce new variables

$$v = u_x + p_y\,, \quad w = u_y - p_x \quad \text{with} \quad v_y - w_x = 0\,,$$

then (1) is recovered by $v_x + w_y = u - u^2$. Combining these four equations leads to a system of the form (17) with

$$\mathbf{K} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad Z = \begin{pmatrix} u \\ v \\ w \\ p \end{pmatrix}, \tag{18}$$

and

$$S(Z) = \frac{1}{2}(v^2 + w^2 - u^2) + \frac{1}{3}u^3\,. \tag{19}$$

Setting $\partial_y = 0$ results in (17) becoming a standard Hamiltonian system on $\mathbb{R}^4$ in the x-direction, and setting $\partial_x = 0$ results in (17) becoming a standard Hamiltonian system on $\mathbb{R}^4$ in the y-direction. A basic state $\widehat{Z}(x)$ is a homoclinic orbit of the Hamiltonian system

$$\mathbf{K}\widehat{Z}_x = \nabla S(\widehat{Z})\,, \quad \widehat{Z} \in \mathbb{R}^4\,. \tag{20}$$

In these coordinates, the basic state $\hat{u}(x)$ in (3) takes the form

$$\widehat{Z}(x) = \hat{u}(x) \begin{pmatrix} 1 \\ -\tanh\frac{1}{2}(x+x_0) \\ 0 \\ 0 \end{pmatrix}. \tag{21}$$

The linearization about this state in the full system (17) is then

$$\mathbf{K}\widetilde{Z}_x + \mathbf{L}\widetilde{Z}_y = D^2S(\widehat{Z})\widetilde{Z}\,. \tag{22}$$

Therefore, the spectral problem, obtained by setting $\widetilde{Z}(x,y) = \mathrm{Re}(Z(x)e^{i\kappa y})$, takes the geometrical form

$$\mathbf{K}Z_x = [D^2S(\widehat{Z}) - i\kappa\mathbf{L}]Z \,. \tag{23}$$

This system is a linear complex symplectic system when κ is real. The state transition matrix $\Phi(x,\kappa)$ for the ODE (23) satisfies $\Phi^*\mathbf{K}\Phi = \mathbf{K}$, as is easily verified, where the superscript $*$ indicates complex conjugate transpose. We will study (23) and its generalizations using an analogue of the Evans function theory, but taking full account of the complex symplectic geometry encoded in the system (23).

All this geometry goes a long way in decoding properties of the spectral problem. However, there is a subtlety which we have not addressed yet: where do $\mathcal{C}$ and α in (16) come from? The basic state (3) or its extension (21) does not depend on any parameters other than the phase x_0. On the other hand, it can be embedded in a one-parameter family, by letting $\widehat{Z}(x)$ depend on $x+\alpha y$. Then $\widehat{Z}(x;\alpha)$ satisfies

$$[\mathbf{K}+\alpha\mathbf{L}]\,\widehat{Z}_x = \nabla S(\widehat{Z})\,, \tag{24}$$

An important feature of this equation is that α multiplies the gradient of a functional $\mathcal{C}$, defined by

$$\mathcal{C}(\widehat{Z}) = \int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{L}\widehat{Z}_x, \widehat{Z}\rangle \,\mathrm{dx}\,, \tag{25}$$

where $\langle\cdot,\cdot\rangle$ is a standard Euclidean inner product. This functional is associated with the *transverse symplectic structure*, with symplectic form $\omega^{(3)}(U,V) = \langle \mathbf{L}U, V\rangle$. It is precisely the properties of this functional which encode information about the second derivative of the dimension-breaking function $D(\kappa)$. For the example (1) the function $\mathcal{C}(\alpha)$ can be explicitly computed,

$$\mathcal{C}(\alpha) = \frac{6}{5}\frac{\alpha}{\sqrt{1+\alpha^2}}\,, \tag{26}$$

and so

$$\mathcal{C}'(\alpha) = \frac{6}{5}\frac{1}{(1+\alpha^2)^{3/2}} > 0\,.$$

For the example it is straightforward to show that $\Pi = -1$ and so $D''(0) < 0$. Note that $\mathcal{C}'(\alpha)$ can be evaluated at $\alpha = 0$. In other words, information about the sign of $D''(0)$ for a basic state independent of y can be deduced by embedding it in the one parameter family, taking derivatives and then evaluating everything at $\alpha = 0$.

The starting point for the geometric theory is a $(2+1)$-dimensional wave equation with a multi-symplectic structure,

$$\mathbf{M}Z_t + \mathbf{K}Z_x + \mathbf{L}Z_y = \nabla S(Z), \qquad Z \in \mathbb{R}^{2n}\,, \tag{27}$$

where $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ are skew-symmetric matrices and $S(Z)$ is a function. The two-forms related to the matrices are

$$\omega^{(1)}(Z_1,Z_2) = \langle \mathbf{M}Z_1\,,\,Z_2\rangle,\ \omega^{(2)}(Z_1,Z_2) = \langle \mathbf{K}Z_1\,,\,Z_2\rangle,\ \text{and } \omega^{(3)}(Z_1,Z_2) = \langle \mathbf{L}Z_1\,,\,Z_2\rangle\,.$$

Later we will often use the two-form associated with translated or rotated coordinates,

$$\Omega(Z_1, Z_2) = \langle \mathbf{J}_{c,\alpha} Z_1\,,\, Z_2 \rangle\,, \quad \text{with} \quad \mathbf{J}_{c,\alpha} = \mathbf{K} - c\mathbf{M} + \alpha \mathbf{L}\,. \tag{28}$$

Examples of such wave equation are numerous; many examples of Hamiltonian PDEs that can be reformulated as multi-symplectic systems are in [3, 5, 6]. In this paper we will concentrate on three examples to illustrate the multi-symplectic framework for dimension breaking. The first example is,

$$2u_t = (u_{xx} + u_{yy} + f(u))_x\,. \tag{29}$$

where $f(u)$ is some given nonlinear function. This PDE is a model equation related to many physical systems as described in Hărăguş & Kirchgässner [12]. If $f(u) = u^2 - u$, the equation is the Zakharov-Kuznetsov equation [19]. If we define

$$Z = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{L}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then with $u = q_1$, this equation has a multi-symplectic structure on $\mathbb{R}^6$ with

$$S(Z) = -F(q_1) + q_1 p_2 - \frac{1}{2}p_1^2 - \frac{1}{2}p_3^2\,,$$

where $F(q_1)$ is defined by $F'(q_1) = f(q_1)$.

The second example is a (2+1)-dimensional Boussinesq equation as derived by Johnson [13]. A generalised version of this equation is

$$u_{tt} = (f(u) + \varepsilon u_{xx})_{xx} + \sigma u_{yy}. \tag{30}$$

For $f(u) = u - u^2$ and $\sigma = \varepsilon = 1$, this is the equation as derived by Johnson. For $\varepsilon = -1$, this equation is a two-dimensional extension of the "good" Boussinesq equation (cf. Bona & Sachs [2] and references therein). Define

$$\mathbf{L}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S(Z) = -F(q_1) - \frac{1}{2\varepsilon}p_1^2 + \frac{1}{2}p_2^2 - \frac{\sigma}{2}p_3^2,$$

then this wave equation has a multi-symplectic structure on $\mathbb{R}^6$ of the form (27).

For the third example, we consider the Kadomtsev-Petviashvili equation (cf. KADOMTSEV & PETVIASHVILI [14]):

$$(2u_t + u_{xxx} + f(u)_x)_x + \sigma u_{yy} = 0. \tag{31}$$

Define

$$\mathbf{L}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S(Z) = -F(q_1) - q_1 p_2 - \frac{1}{2}p_1^2 - \frac{1}{2}(2C+\sigma)p_3^2,$$

then with $q_1 = u$, this PDE has a multi-symplectic structure of the form (27) with C an arbitrary constant. (Note: $C = 0$ gives a structure very similar to the Boussinesq system and $C = -\frac{\sigma}{2}$ results in additional symmetry.)

The first step in the dimension-breaking analysis is to characterize the basic localized state. The details of this characterization are given in §2. The examples we consider in this paper, and many other PDEs where dimension breaking is of interest, have additional symmetry (acting on $\mathbb{R}^{2n}$), and this results in multi-parameter families of localized states. In the multi-symplectic setting, these states have a natural geometry associated with *multi-symplectic relative equilibria*.

If we denote the action of the symmetry group by G_θ, with $\theta = (\theta_1, \ldots, \theta_q)$, then a multi-parameter localized basic state of the system (27), when it exists, will be taken to be of the form

$$Z(t,x,y) = G_{\theta(t,x,y)} \mathcal{T}_{\tau(t,y)} \widetilde{Z}(x), \tag{32}$$

with

$$\theta(t,x,y) = (a_1 t + b_1 x + d_1 y, \ldots, a_q t + b_q x + d_q y) \quad \text{and} \quad \tau(t,y) = ct - \alpha y.$$

Although time is included, the basic states are assumed to be steady relative to a moving frame of reference. We will call this basic state a solitary wave if $\lim\limits_{|x|\to\infty} \widetilde{Z}(x)$ exists and there is some $\beta > 0$ such that

$$\lim_{|x|\to\infty} e^{\beta|x|} \widetilde{Z}_x(x) \text{ exists and } \lim_{|x|\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}(e^{\beta|x|}\widetilde{Z}_x(x)) = 0\,. \tag{33}$$

If the q generators of the Lie group are denoted by $\xi_1(Z), \ldots, \xi_q(Z)$, it follows from multi-symplectic Noether theory [3] that there exist vector functions

$$P_1(Z), \ldots, P_q(Z)\,, \quad Q_1(Z), \ldots, Q_q(Z) \quad \text{and} \quad R_1(Z), \ldots, R_q(Z)\,,$$

such that

$$\mathbf{M}\xi_i(Z) = \nabla P_i(Z), \quad \mathbf{K}\xi_i(Z) = \nabla Q_i(Z) \quad \text{and} \quad \mathbf{L}\xi_i(Z) = \nabla R_i(Z), \quad \text{for} \quad i = 1, \ldots, q.$$

The properties (33) and the relations above imply that $\widetilde{Z}(x)$ is bounded, decays exponentially to the state at infinity, and satisfies the ODE

$$\mathbf{J}_{c,\alpha}\,\widetilde{Z}_x = \nabla\left[S(\widetilde{Z}) - \sum_{i=1}^{q}(a_i P_i(\widetilde{Z}) + b_i Q_i(\widetilde{Z}) + d_i R_i(\widetilde{Z}))\right]. \tag{34}$$

The gradient structure of this equation leads naturally to a constrained variational principle: find

$$\text{crit}\ \{\mathcal{H}_{a,b,d}(Z) - \mathcal{K}(Z) \mid \mathcal{I}(Z) = \gamma, \quad \mathcal{C}(Z) = C\} \tag{35}$$

where

$$\begin{aligned}
\mathcal{H}_{a,b,d}(Z) &= \int\left[S(Z) - \sum_{i=1}^{q}(a_i P_i(Z) + b_i Q_i(Z) + d_i R_i(Z))\right] dx,\\
\mathcal{K}(Z) &= \tfrac{1}{2}\int \langle \mathbf{K}Z\,,\,Z_x\rangle\,dx,\\
\mathcal{I}(Z) &= \tfrac{1}{2}\int \langle \mathbf{M}Z\,,\,Z_x\rangle\,dx, \quad \text{and} \quad \mathcal{C}(Z) = \tfrac{1}{2}\int \langle \mathbf{L}Z\,,\,Z_x\rangle\,dx.
\end{aligned}$$

For specific choices of the functions $f(u)$ in the three examples, one can indeed find such solutions explicitly (cf. §§6-8). It is the parameter structure of this variational principle which encodes information about the sign of $D''(0)$.

The basic solitary wave is said to break dimension if there exist nearby solutions with non-trivial transverse variation. In other words, dimension breaking occurs if the system (27) has solutions of the form

$$Z(t,x,y) = G_{\theta(t,x,y)}\mathcal{T}_{\tau(t,y)}(\widetilde{Z}(x) + \widehat{U}(x,y)),$$

with $\widehat{U}$ small.

Substitution of this form of the solution, and linearizing about the basic state, leads to the following linear PDE,

$$\mathbf{J}_{c,\alpha}\widehat{U}_x + \mathbf{L}\widehat{U}_y = \mathbf{B}(x;c,\alpha,a,b,d)\widehat{U},$$

where

$$\mathbf{B}(x;c,\alpha,a,b,d) = D^2S(\widetilde{Z}) - \sum_{i=1}^{q}(a_i D^2P_i(\widetilde{Z}) + b_i D^2Q_i(\widetilde{Z}) + d_i D^2R_i(\widetilde{Z})).$$

Solutions of this system can be written as $\widehat{U}(x,y) = \Re(e^{i\kappa y}U(x;\kappa))$, where

$$U_x = \mathbf{A}(x,\kappa;c,\alpha,a,b,d)\,U\,, \quad U \in \mathbb{C}^{2n}\,, \tag{36}$$

and

$$\mathbf{J}_{c,\alpha}\mathbf{A}(x,\kappa;c,\alpha,a,b,d) = [\mathbf{B}(x;c,\alpha,a,b,d) - i\kappa\mathbf{L}]\,. \tag{37}$$

The equation (36) is the central equation associated with verifying the spectral necessary condition for dimension breaking. By deriving this equation within the multi-symplectic setting, it contains significant geometric information. For example, when $\kappa \in \mathbb{R}$ the linear system (36) is complex symplectic, with respect to the symplectic operator $\mathbf{J}_{c,\alpha}$. In general, complex symplectic spaces can be much more exotic than real symplectic spaces (cf. Everitt & Markus [10]), but this complex symplectic space is a straightforward extension of real symplectic structure, because the symplectic operator $\mathbf{J}_{c,\alpha}$ is real. The necessary condition for dimension breaking requires the existence for some $\kappa \in \mathbb{R}$ a bounded function $U(x)$ which satisfies (36). The analysis of this problem is very similar to the analysis of the symplectic Evans function which arises in the linear instability problem for solitary waves of multi-symplectic PDEs, and therefore we will be able to appeal to recent results on this in [5, 6].

The paper is organized as follows. In §2 we consider properties of the basic state. In §3 we introduce *the symplectic dimension-breaking function*, which is based on a geometric analysis of (36). In §4 we compute derivatives of the symplectic dimension-breaking function at the origin, and then deduce a geometric condition for the existence of nonzero purely imaginary eigenvalues of the dimension-breaking spectral problem in §5. Here we rely of aspects of the theory developed in [6]. We can also appeal to existing theory for large κ asymptotics. Then in §§6-8 the theory is applied in detail to deduce new results on dimension breaking for three examples. In the final section, we discuss the stability of the bifurcating states.

2 Multi-parameter families of localized basic states

The basic state will be some localized state, such as a solitary wave. When this state does not depend on any parameters, such as the example in the introduction, the idea will be to embed it in a one-parameter family of states in order to deduce information about the sign of $D''(0)$. In this section we consider general aspects of this embedding.

In addition, many PDEs of interest have a nontrivial symmetry group, as well as being multi-symplectic. For these cases we will consider a generalization of the localized state where the state at infinity may be nontrivial. The symmetry results in multi-parameter families of solitary waves. The extra parameter structure associated with the symmetry provides additional information about the sign of $D''(0)$.

There is also symmetry inherent in the multi-symplectic system (27) due to the fact that $S(Z)$ does not depend *explicitly* on x, y and t, and is therefore equivariant with respect to translations in the t, x and y direction. The $x-$direction will be identified as the preferred direction, and the action of translation in the x-variable will be denoted by $\mathcal{T}_\tau$, i.e., $\mathcal{T}_\tau Z(t,x,y) = Z(t,x-\tau,y)$. Although the direction of the basic state may not be aligned perfectly with the x-axis, it will always be transverse to the y-direction. The y-direction is the dimension-breaking direction and is called the *transverse* direction.

To be precise about the role of symmetry acting on the phase space $\mathbb{R}^{2n}$, we assume there is a q-dimensional compact or affine Lie group $\mathcal{G}$ acting on vectors in $\mathbb{R}^{2n}$, and the system (27) is equivariant with respect to the action of this Lie group. If the q generators of the Lie group are denoted by $\xi_1(Z),\ldots,\xi_q(Z)$, it follows from multi-symplectic Noether theory [3] that there exist vector functions $P_1(Z),\ldots,P_q(Z)$, $Q_1(Z),\ldots,Q_q(Z)$ and $R_1(Z),\ldots,R_q(Z)$ such that

$$\mathbf{M}\xi_i(Z) = \nabla P_i(Z), \quad \mathbf{K}\xi_i(Z) = \nabla Q_i(Z) \quad \text{and} \quad \mathbf{L}\xi_i(Z) = \nabla R_i(Z), \quad \text{for} \quad i = 1,\ldots,q.$$

For the three examples considered in this paper, both q_2 and q_3 are cyclic coordinates, and so the multi-symplectified systems are equivariant with respect to the action of a 2-dimensional affine Lie group, with actions

$$G_\theta Z = G_{(\theta_1,\theta_2)} Z = Z + \theta_1 E_2 + \theta_2 E_3,$$

where $E_2 = (0,1,0,0,0,0)^T$ and $E_3 = (0,0,1,0,0,0)^T$. Its generators are $\xi_1 = E_2$ and $\xi_2 = E_3$. So the associated vector functions are

$$P_1(Z) = q_1, \quad P_2(Z) = 0, \quad Q_1(Z) = -p_2, \quad \text{and} \quad Q_2(Z) = -p_3.$$

The vector functions $R_i(Z)$ differ for each of the three examples,

$$R_1^1(Z) = 0 = R_1^2(Z), \quad R_1^3(Z) = -C\,p_3, \quad R_2^1(Z) = p_1, \quad \text{and} \quad R_2^2(Z) = q_1 = R_2^3(Z).$$

The assumption that $\widetilde{Z}(x)$ is bounded for $|x| \to \infty$ implies that we can define $Z_0^\pm = \lim\limits_{x\to\pm\infty} \widetilde{Z}(x)$. Then Z_0 satisfies

$$\nabla \left[S(Z_0) - \sum_{i=1}^{q} (a_i P_i(Z_0) + b_i Q_i(Z_0) + d_i R_i(Z_0)) \right] = 0. \tag{38}$$

If Z_0 satisfies this equation, then $G_\theta Z_0$ satisfies it also, because of the equivariance properties of the system. Therefore we restrict to the case where all solutions of (38) are related by a group action in $\mathcal{G}$. This set of solutions of (38) is called the manifold at infinity, denoted by $\mathcal{M}_\infty(a,b,d)$. Since (38) does not depend on α nor c, we can choose Z_0^- to be independent of c and α. Then there exists some $\theta(c,\alpha;a,b,d)$ such that $Z_0^+ = G_{\theta(c,\alpha;a,b,d)} Z_0^-$. A complete expression for the basic multi-parameter family of localized states is then given by (32), with the shape of the wave satisfying the constrained variational principle (35). Geometrically this state is a *multi-symplectic relative equilibrium.*

When the basic state does not depend on any parameters, the basic ODE (34) reduces to

$$\mathbf{K}\widetilde{Z}_x = \nabla S(\widetilde{Z}) \quad \text{with embedding} \quad \mathbf{J}_{0,\alpha}\widetilde{Z}_x = \nabla S(\widetilde{Z})\,.$$

The embedded equation then has a characterization as a constrained variational principle,

$$\text{crit}\ \{\mathcal{H}(Z) - \mathcal{K}(Z) \mid \quad \mathcal{C}(Z) = C\} \tag{39}$$

This variational principle is non-degenerate precisely when $\mathcal{C}'(\alpha) \neq 0$.

3 Symplectic dimension breaking and the Evans function

The linearization about the basic state results in (36) which is the basic equation to be analyzed to determine if the spectral necessary condition is satisfied. The coefficient matrix $\mathbf{A}(x,\kappa;c,\alpha,a,b,d)$ reduces to a parameter-dependent matrix when $x \to \pm\infty$. As in the Evans function theory, it is these matrices at infinity which dictate the properties of the solutions of (36).

Define

$$\mathbf{B}^\pm(c,\alpha;a,b,d) = \mathbf{B}(Z_0^\pm) \quad \text{and} \quad \mathbf{A}^\pm(\kappa,c,\alpha;a,b,d) = J_{c,\alpha}^{-1}(\mathbf{B}^\pm - i\kappa\mathbf{L}),$$

then the system at infinity is

$$J_{c,\alpha}U_x = (\mathbf{B}^\pm - i\kappa\mathbf{L})U \quad \text{or} \quad U_x = \mathbf{A}^\pm U.$$

The spectral problem associated with the matrices $\mathbf{A}^\pm$ is

$$\Delta(\mu,\kappa) = \det(\mathbf{B}^\pm - i\kappa\mathbf{L} - \mu\mathbf{J}_{c,\alpha}) = 0 \quad \Leftrightarrow \quad \det(\mathbf{A}^\pm - \mu\mathbf{I}) = 0\,.$$

The function Δ satisfies $\Delta(\kappa,\mu) = \overline{\Delta(\kappa,-\overline{\mu})}$, since $\mathbf{B}$ is symmetric, and $\mathbf{L}$, $\mathbf{J}_{c,\alpha}$ are skew symmetric. Hence if μ solves the spectral problem, then also $-\overline{\mu}$ solves the spectral problem.

At $\kappa = 0$, we will assume that there are only two eigenvalues with non-zero real part: $\pm\beta$. (This is what occurs in the three examples, and can be modified for other examples.) Recall that β is the decay rate of the basic state.

When $\kappa \neq 0$ there may be other eigenvalues with strictly negative real part. Let p be the number of eigenvalues with strictly negative real part when κ is real and positive. Assume that these p eigenvalues are simple. Furthermore, it is assumed that all $p-1$ eigenvalues which converge to 0 for $\kappa \to 0$ have eigenvectors which converge to p independent generators of the Lie group $\mathcal{G}$. The p eigenvalues with negative real part are denoted by $\mu_1(\kappa), \ldots \mu_p(\kappa)$, with

$$\Re\mu_1(\kappa) \leq \ldots \leq \Re\mu_p(\kappa).$$

Next we define the eigenvectors $\zeta_i^\pm(\kappa)$ and their symplectic adjoints $\eta_i^\pm$ to be such that

$$A^\pm(\kappa)\zeta_i^\pm = \mu_i\zeta_i^\pm \quad \text{and} \quad (A^\pm(\kappa))^* J_{c,\alpha}\overline{\eta_i^\pm} = \overline{\mu_i}J_{c,\alpha}\overline{\eta_i^\pm} \Rightarrow A^\pm(\kappa)\overline{\eta_i^\pm} = -\overline{\mu_i}\overline{\eta_i^\pm}$$

The relation between Z_0^- and Z_0^+ implies that

$$\zeta_i^+ = DG_\theta\zeta_i^- \quad \text{and} \quad \eta_i^+ = DG_\theta\eta_i^-.$$

Furthermore, ζ_i and η_i are normalised by $\Omega(\eta_i^\pm,\zeta_j^\pm) = \delta_{ij}$, where the symplectic form Ω is defined in (28).

As $\kappa \to 0$, the eigenvalues $\mu_2,\ldots,\mu_p$ accumulate on the imaginary axis. In a similar way as in [6, §4] it can be shown that if

$$\begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_a^i & \mathcal{Q}_a^i & \mathcal{R}_a^i \\ \mathcal{P}_b^i & \mathcal{Q}_b^i & \mathcal{R}_b^i \\ \mathcal{P}_d^i & \mathcal{Q}_d^i & \mathcal{R}_d^i \end{pmatrix} \begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix} \neq 0, \quad \text{with} \quad \mathcal{P}_a^i = \frac{\partial}{\partial a}P_i(\widetilde{Z}), \quad \text{etc.,} \tag{40}$$

then the vectors η_i have a single pole at $\kappa = 0$. The regularized eigenvectors will be denoted by

$$\widetilde{\eta}_i(\kappa,\cdot) = \kappa\eta_i(\kappa,\cdot), \quad i = 2,\ldots,p.$$

These definitions of the eigenvectors imply that

$$\zeta_1^+(0) = \lim_{x\to\infty} e^{\beta x}\widetilde{Z}_x(x) \quad \text{and} \quad \eta_1^-(0) = \chi_{00}^- \lim_{x\to-\infty} e^{-\beta x}\widetilde{Z}_x(x),$$

where

$$\chi_{00}^- = \lim_{x\to\infty} e^{2\beta x}\Omega(\widetilde{Z}_x(-x)\,,\, DG_\theta^T\widetilde{Z}_x(x)).$$

And $\text{span}\{\widetilde{\eta}_2^{\pm}(0),\dots,\widetilde{\eta}_p^{\pm}(0)\}$ and $\text{span}\{\zeta_2^{\pm}(0),\dots,\zeta_p^{\pm}(0)\}$ form p-dimensional subspaces in the Lie algebra related to $\mathcal{G}$. For more details about this, see [6].

Next we return to the non-autonomous system (36). Define $U_i^+(x;\kappa,\cdot)$ and $W_i^-(x;\kappa,\cdot)$ to be such that U_i^+ satisfies the linearised system

$$U_x = \mathbf{A}(x;\kappa,\cdot)U \quad \text{and} \quad \lim_{x\to\infty} e^{-\mu_i x}U_i^+(x) = \zeta_i^+ \tag{41}$$

and W_i^- satisfies the symplectic adjoint equation

$$\overline{W}_x = \mathbf{A}(x;\kappa,\cdot)\overline{W} \quad \text{and} \quad \lim_{x\to-\infty} e^{\mu_i x}W_i^-(x) = \eta_i^- . \tag{42}$$

Also we define the regularised solution vectors

$$\widetilde{W}_i(x;\kappa,\cdot) = \kappa W_i(x;\kappa,\cdot), \quad i = 2,\dots,p.$$

The dimension-breaking function is then modelled on the definition of the Evans function in Alexander, Gardner & Jones [1]. Let $\text{Tr}(\kappa)$ denote the trace of $\mathbf{A}(x;\kappa,\cdot)$, then the following function is independent of x,

$$\widehat{D}(\kappa) = e^{-\text{Tr}(\kappa)x}Y_+(x,\kappa)\wedge Y_-(x,\kappa)\,,$$

where Y_+ and Y_- are forms representing the subspaces which decay as x goes to $\pm$ infinity respectively. This function is proportional to a volume form on $\mathbb{C}^{2n}$. Choosing a volume form and denoting it by $\mathcal{V}$, the above expression can be written,

$$\widehat{D}(\kappa) = D(\kappa)\,\mathcal{V}\,. \tag{43}$$

It then follows from results in [6], that the function $D(\kappa)$ can be represented in a form analogous to the symplectic Evans matrix,

$$D(\kappa) = \det\begin{pmatrix} \Omega(W_1^-\,,\,U_1^+) & \cdots & \Omega(W_1^-\,,\,U_p^+) \\ \vdots & \ddots & \vdots \\ \Omega(W_p^-\,,\,U_1^+) & \cdots & \Omega(W_p^-\,,\,U_p^+)\,. \end{pmatrix} \tag{44}$$

The linearised system (36) has a bounded (at most algebraicly growing) solution if and only if $D(\kappa)=0$. If κ_0 is real and positive, then $D(\kappa_0)=0$ satisfies the necessary spectral condition for dimension breaking in a multi-symplectic system.

Writing (44) as $D(\kappa) = \det[\mathbf{\Omega}(\kappa)]$, the $p\times p$ matrix $\mathbf{\Omega}(\kappa)$ will be called the symplectic dimension breaking (DB) matrix, and its entries will be denoted by $\Omega_{ij} = \Omega(W_i^-\,,\,U_j^+)$. This matrix is singular at $\kappa=0$, so we define a regularised symplectic DB matrix as $\widetilde{\mathbf{\Omega}}$ and denote its entries by

$$\widetilde{\Omega}_{ij} = \Omega(\widetilde{W}_i^-\,,\,U_j^+) = \kappa\,\Omega(W_i^-\,,\,U_j^+), \quad 2\le i,j\le p.$$

4 Properties of $\mathbf{\Omega}(\kappa)$ for κ near zero

For the entries in the symplectic DB matrix, there is nice relation between the derivative with respect to κ and the transverse symplectic structure associated with $\mathbf{L}$.

Lemma 1 *For all $i,j=1,\dots,p$, we have*

$$\frac{\partial}{\partial\kappa}\Omega(W_j^-\,,\,U_i^+) = i\int_S^R \omega^{(3)}(W_j^-\,,\,U_i^+)\,dx + \Omega(W_j^-\,,\,\partial_\kappa U_i^+)\Big|_{x=-S} + \Omega(\partial_\kappa W_j^-\,,\,U_i^+)\Big|_{x=R},$$

for any $S,R>0$.

Proof Let $i,j=1,\dots,p$ and $S,R>0$. Differentiation of (41) and (42) with respect to κ gives

$$\begin{aligned}\mathbf{J}_{c,\alpha}(\partial_\kappa U_i^+)_x &= [\mathbf{B}(x)-i\kappa\mathbf{L}]\partial_\kappa U_i^+ - i\mathbf{L}U_i^+;\\ \mathbf{J}_{c,\alpha}(\partial_\kappa W_j^-)_x &= [\mathbf{B}(x)+i\kappa\mathbf{L}]\partial_\kappa W_j^- + i\mathbf{L}W_j^-.\end{aligned} \tag{45}$$

Using this and (41) and (42), we get

$$\begin{aligned} i\,\omega^{(3)}(W_j^-\,,\,U_i^+) &= \langle \mathbf{J}_{c,\alpha}(\partial_\kappa W_j^-)_x - [\mathbf{B}(x)+i\kappa\mathbf{L}]\partial_\kappa W_j^-\,,\,U_i^+\rangle\\ &= -\langle\partial_\kappa W_j^-\,,\,[\mathbf{B}(x)-i\kappa\mathbf{L}]U_i^+\rangle + \langle(\mathbf{J}_{c,\alpha}\partial_\kappa W_j^-)_x\,,\,U_i^+\rangle\\ &= -\langle\partial_\kappa W_j^-\,,\,\mathbf{J}_{c,\alpha}(U_i^+)_x\rangle + \langle(\mathbf{J}_{c,\alpha}\partial_\kappa W_j^-)_x\,,\,U_i^+\rangle\\ &= \tfrac{\mathrm{d}}{\mathrm{d}x}\Omega(\partial_\kappa W_j^-\,,\,U_i^+)\end{aligned}$$

and in a similar way

$$i\,\omega^{(3)}(W_j^-\,,\,U_i^+) = -\frac{\mathrm{d}}{\mathrm{d}x}\Omega(W_j^-\,,\,\partial_\kappa U_i^+).$$

Hence

$$\begin{aligned}-i\int_{-S}^R \omega^{(3)}(W_j^-\,,\,U_i^+)\,dx &= \Omega(W_j^-\,,\,\partial_\kappa U_i^+)\Big|_{x=-S} - \Omega(W_j^-\,,\,\partial_\kappa U_i^+)\Big|_{x=0}\\ &\quad - \Omega(\partial_\kappa W_j^-\,,\,U_i^+)\Big|_{x=0} + \Omega(\partial_\kappa W_j^-\,,\,U_i^+)\Big|_{x=R}\\ &= -\partial_\kappa\Omega(W_j^-\,,\,U_i^+) + \Omega(W_j^-\,,\,\partial_\kappa U_i^+)\Big|_{x=-S} + \Omega(\partial_\kappa W_j^-\,,\,U_i^+)\Big|_{x=R},\end{aligned}$$

since $\Omega(W_j^-\,,\,U_i^+)$ is x-independent. □

We will study the behaviour of the entries in the symplectic DB matrix near $\kappa=0$. At $\kappa=0$ the solutions of the linearised system (36) and its adjoint satisfy (see [6])

$$U_1^+(x,0,\cdot) = \widetilde{Z}_x(x,\cdot) \quad\text{and}\quad W_1^-(x;0,\cdot) = \chi_{00}^-\widetilde{Z}_x(x,\cdot).$$

Also, for $i=2,\dots,p$:

$$U_i^+(x;0,\cdot)\in\mathfrak{g} \quad\text{and}\quad W_i^-(x;0,\cdot),\ \widetilde{W}_i^-(x;0,\cdot)\in\mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra related to the Lie group $\mathcal{G}$. To be explicit, we can define generators $\xi_i\in\mathfrak{g}$ and $\chi_{ij}^-\in\mathbb{R}$ such that

$$U_i^+(x;0,\cdot) = \xi_i(Z(x,\cdot)) \quad\text{and}\quad \widetilde{W}_i^-(x;0,\cdot) = \sum_{j=2}^p \chi_{ji}^-\xi_j(Z(x,\cdot)).$$

From [6, Lemma 18] it follows that

Lemma 2 *For any fixed* a, b, d, c, α *and* $1 \le i, j \le p$ *we have* $\widetilde{\Omega}_{ij}(0, \cdot) = 0$.

We can also find explicit expressions for the derivative with respect to κ of the solutions of linearised system (36) and its adjoint.

Lemma 3 *There exist constants* $c_1^{\pm}(c, \alpha, a, b, d)$ *such that*

$$\begin{aligned}\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} U_1^+(x; \kappa, c, \alpha, \cdot) &= i\tfrac{\partial}{\partial\alpha}\widetilde{Z}(x; c, \alpha, \cdot) + c_1^+(c, \alpha, \cdot)\widetilde{Z}_x(x; c, \alpha, \cdot)\\ \tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} W_1^-(x; \kappa, c, \alpha, \cdot) &= -i\chi_{00}^-\tfrac{\partial}{\partial\alpha}\widetilde{Z}(x; c, \alpha, \cdot) + c_1^-(c, \alpha, \cdot)\widetilde{Z}_x(x; c, \alpha, \cdot)\end{aligned}$$

Proof By substituting $\kappa = 0$ into (45) we get for $1 \le i \le p$

$$\begin{aligned}[\, J_{c,\alpha} D_x - \mathbf{B}(x; c, \alpha, \cdot)\,]\ \left(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} U_i^+(x; \kappa, c, \alpha, \cdot)\right) &= -i\mathbf{L}U_i^+(x; 0, c, \alpha, \cdot)\,;\\ [\, J_{c,\alpha} D_x - \mathbf{B}(x; c, \alpha, \cdot)\,]\ \left(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} W_i^-(x; \kappa, c, \alpha, \cdot)\right) &= i\mathbf{L}W_i^-(x; 0, c, \alpha, \cdot)\,.\end{aligned}$$

For $i = 1$ this gives

$$\begin{aligned}[\, J_{c,\alpha} D_x - \mathbf{B}(x; c, \alpha, \cdot)\,]\ \left(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} U_1^+(x; \kappa, c, \alpha, \cdot)\right) &= -i\mathbf{L}\widetilde{Z}_x(x; c, \alpha, \cdot);\\ [\, J_{c,\alpha} D_x - \mathbf{B}(x; c, \alpha, \cdot)\,]\ \left(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} \widetilde{W}_1^-(x; \kappa, c, \alpha, \cdot)\right) &= i\chi_{00}^-\mathbf{L}\widetilde{Z}_x(x; c, \alpha, \cdot)\,.\end{aligned} \tag{46}$$

Furthermore, the existence of a multi-parameter family of travelling waves implies that we can differentiate the Euler-Lagrange equation (34) with respect to α. This gives

$$[\, J_{c,\alpha} D_x - \mathbf{B}(x; c, \alpha, \cdot)\,]\tfrac{\partial}{\partial\alpha}\widetilde{Z}(x; c, \alpha, \cdot) = -\mathbf{L}\widetilde{Z}_x(x; c, \alpha, \cdot). \tag{47}$$

Combining this with the expressions (46) for $\frac{\partial}{\partial\kappa}\big|_{\kappa=0} U_1^+(x; \kappa, c, \alpha, \cdot)$ and $\frac{\partial}{\partial\kappa}\big|_{\kappa=0} \widetilde{W}_1^-(x; \kappa, c, \alpha, \cdot)$, we get

$$\begin{aligned}[\, J_{c,\alpha} D_x - \mathbf{B}(x; \cdot)\,]\left(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} U_1^+(x; \kappa, c, \alpha, \cdot) - i\tfrac{\partial}{\partial\alpha}\widetilde{Z}(x; c, \alpha, \cdot)\right) &= 0\,;\\ [\, J_{c,\alpha} D_x - \mathbf{B}(x; \cdot)\,]\left(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} W_1^-(x; \kappa, c, \alpha, \cdot) + i\chi_{00}^-\tfrac{\partial}{\partial\alpha}\widetilde{Z}(x; c, \alpha, \cdot)\right) &= 0\,.\end{aligned}$$

Hence $\frac{\partial}{\partial\kappa}\big|_{\kappa=0} U_1^+(x; \kappa, c, \alpha, \cdot) = i\frac{\partial}{\partial\alpha}\widetilde{Z}(x; c, \alpha, \cdot) + y_1^+(x; c, \alpha, \cdot)$, where $y_1^+(x; c, \alpha, \cdot)$ is some element from the kernel of $[J_{c,\alpha}D_x - \mathbf{B}(x; c, \alpha, \cdot)]$. Both $\frac{\partial}{\partial\kappa}\big|_{\kappa=0} U_1^+(x; \kappa, c, \alpha, \cdot)$ and $\frac{\partial}{\partial c}\widetilde{Z}(x; c, \alpha, \cdot)$ are exponentially decaying for $x \to \infty$. By hypothesis, the only exponentially decaying solution in the kernel of $[J_{c,\alpha}D_x - \mathbf{B}(x; c, \alpha, \cdot)]$ is $\widetilde{Z}_x$, hence there exists a constant $c_1^+(c, \alpha, \cdot)$ such that $y_1^+(x; c, \alpha, \cdot) = c_1^+(c, \alpha, \cdot)\widetilde{Z}_x(x; c, \alpha, \cdot)$. In a similar way we can derive the other expression in the lemma. □

From [6, Lemma 19 and Lemma 20] it follows that

Lemma 4 *For any fixed* c, α, a, b, d

$$\tfrac{\partial}{\partial\kappa}\,|_{\kappa=0}\ \widetilde{\Omega}_{11}(\kappa; \cdot) = \tfrac{\partial}{\partial\kappa}\,|_{\kappa=0}\ \widetilde{\Omega}_{1j}(\kappa; \cdot) = \tfrac{\partial}{\partial\kappa}\,|_{\kappa=0}\ \widetilde{\Omega}_{i1}(\kappa; \cdot) = 0, \quad \text{for } i, j \ge 2\,.$$

For $2 \le i, j \le p$, *define*

$$(\partial\mathcal{D})_{i-1,j-1} = \tfrac{\partial}{\partial\kappa}\big|_{\kappa=0}\widetilde{\Omega}_{i,j}(\kappa; \cdot) = \tfrac{\partial}{\partial\kappa}\big|_{\kappa=0}\Omega(\widetilde{W}_{i+1}^-, U_{j+1}^+)\,;$$

then,

$$(\partial\mathcal{D})_{i-1,j-1} = \sum_{k=1}^{q}\left[\Omega(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0}\widetilde{\eta}_i^-, \xi_k(Z_0^-)) + \chi_{k,i-1}^-\Omega(\xi_k(Z_0^+), \tfrac{\partial}{\partial\kappa}\big|_{\kappa=0}\zeta_j^+)\right].$$

In [6, Lemma 16] it is shown that

Lemma 5 *If the manifold at infinity satisfies* (40)*, then the matrix ∂D is equal to the identity matrix.*

The next step involves analyzing the second derivative of Ω_{11}.

Lemma 6 *For any c, α, a, b, d,*

$$\frac{\partial^2}{\partial\kappa^2}\Big|_{\kappa=0} \widetilde{\Omega}_{11}(\kappa; c, \alpha, \cdot) = \frac{\partial^2}{\partial\kappa^2}\Big|_{\kappa=0} \Omega(\widetilde{W}_1^-, U_1^+) = 2\chi_{00}^- \frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \chi_{00}^- \omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+)$$

Proof ¿From Lemma 3 it follows

$$\begin{aligned} i\int_{-\infty}^{\infty} \omega^{(3)}\left(\widetilde{W}_1^-(x;0,\cdot),\, \tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} U_1^+(x;\kappa,\cdot)\right)\, dx &= i\int_{-\infty}^{\infty} \omega^{(3)}\left(\chi_{00}^- \widetilde{Z}_x(x;\cdot), i\tfrac{\partial}{\partial\alpha}\widetilde{Z}(x;\cdot) + c_1^+\widetilde{Z}_x(x;\cdot)\right) \\ &= \chi_{00}^- \int_{-\infty}^{\infty} \omega^{(3)}(\widetilde{Z}_\alpha(x;\cdot)\,,\, \widetilde{Z}_x(x;\cdot)) \\ &= \chi_{00}^- \tfrac{d}{d\alpha}\mathcal{C}(\widetilde{Z}) - \tfrac{1}{2}\chi_{00}^- \omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+). \end{aligned}$$

In a similar way we can show that

$$i\int_{-\infty}^{\infty} \omega^{(3)}\left(\tfrac{\partial}{\partial\kappa}\big|_{\kappa=0} \widetilde{W}_1^-(x,\kappa;\cdot)\,,\, U_1^+(x,0;\cdot)\right)\, dx = \chi_{00}^- \tfrac{d}{d\alpha}\mathcal{C}(\widetilde{Z}) - \tfrac{1}{2}\chi_{00}^- \omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+).$$

Finally we analyze the boundary terms. Using Lemma 3, we get at $\kappa = 0$

$$\begin{aligned} \tfrac{\partial}{\partial\kappa}\Omega\left(\tfrac{\partial}{\partial\kappa}\widetilde{W}_1^-(x;\kappa,\cdot), U_1^+(x;\kappa,\cdot)\right) &= \Omega(-i\chi_{00}^- \tfrac{\partial}{\partial\alpha}\widetilde{Z}(x;\cdot) + c_1^-\widetilde{Z}_x(x;\cdot), i\tfrac{\partial}{\partial\alpha}\widetilde{Z}(x;\cdot) + c_1^+\widetilde{Z}_x(x;\cdot)) \\ &\quad + \Omega\left(\tfrac{\partial^2}{\partial\kappa^2}\widetilde{W}_1^-(x,\kappa;\cdot)\,,\, \widetilde{Z}_x(x;\cdot)\right) \end{aligned}$$

So for $x \to -\infty$ this expression vanishes at $\kappa = 0$. In a similar way we can show that the other boundary terms disappear. □

5 Satisfying the spectral necessary condition geometrically

We now have all the information needed to give a geometric condition for the existence of a positive real root of $D(\kappa)$.

Theorem 7 *For some fixed $(a, b, d; c, \alpha)$, let $\kappa_\infty \in \mathbb{R}$ be some positive value and let $d_\infty = D(\kappa_\infty, \cdot)$. Define*

$$(\chi_{00}^-)^{-1} = \lim_{x\to\infty} e^{2\beta x}\Omega(\widetilde{Z}_x(-x)\,,\, DG_\theta^T \widetilde{Z}_x(x)).$$

If the manifold at infinity associated with the basic state satisfies (40) *and if*

$$d_\infty \chi_{00}^- \left[\tfrac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \tfrac{1}{2}\omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+)\right] < 0, \tag{48}$$

then there is some real positive value of κ such that the symplectic dimension-breaking function $D(\kappa)$ vanishes.

Proof Lemmas 2 and 4 imply that

$$\widetilde{\Omega}_{ij}(0;\cdot) = \frac{\partial}{\partial\kappa}\widetilde{\Omega}_{ij}\Big|_{\kappa=0}(\kappa;\cdot) = 0, \quad \text{for } i=1 \text{ or } j=1 \quad .$$

Hence we can conclude that for $\kappa\to 0$

$$\widetilde{\Omega}_{ij}(\kappa;\cdot) = \mathcal{O}(\kappa^2), \quad \text{for } i=1 \text{ or } j=1.$$

Moreover, with Lemma 1 and Lemma 6 we have

$$\widetilde{\Omega}_{11}(\kappa;\cdot) = \kappa^2\chi_{00}^- \left[\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\omega^{(3)}(Z_0^+,\partial_\alpha Z_0^+)\right] + \mathcal{O}(\kappa^3), \quad \text{as } \kappa\to 0\,.$$

Furthermore, Lemmas 2 and 4 show that $\widetilde{\Omega}_{ij}(0;\cdot)=0$ and $\frac{\partial}{\partial\kappa}\big|_{\kappa=0}\widetilde{\Omega}_{ij}(\kappa;\cdot) = (\partial\mathcal{D})_{(i-1)(j-1)}$, so altogether

$$\widetilde{\Omega}_{ij}(\kappa;\cdot) = \kappa(\partial\mathcal{D})_{(i-1)(j-1)} + \mathcal{O}(\kappa^2), \quad \text{for } \kappa\to 0, \text{ if } i,j\geq 2.$$

So we are in the following situation:

$$\widetilde{\Omega}(\kappa;\cdot) = \left(\begin{array}{c|c} \kappa^2\chi_{00}^-\left[\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\,\omega^{(3)}(Z_0^+,\partial_\alpha Z_0^+)\right] + \mathcal{O}(\kappa^3) & \mathcal{O}(\kappa^2) \\ \hline \mathcal{O}(\kappa^2) & \kappa(\partial\mathcal{D}) + \mathcal{O}(\kappa^2) \end{array}\right)$$

$$= \kappa^p \left(\begin{array}{c|c} \kappa\chi_{00}^-\left[\frac{d}{d\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\chi_{00}^-\,\omega^{(3)}(Z_0^+,\partial_\alpha Z_0^+)\right] + \mathcal{O}(\kappa^2) & \mathcal{O}(\kappa) \\ \hline \mathcal{O}(\kappa) & \partial\mathcal{D} + \mathcal{O}(\kappa) \end{array}\right).$$

From [6, Appendix 7.1], it follows that the determinant of such a matrix is

$$\det(\widetilde{\Omega}(\kappa;\cdot)) = \kappa^{1+p}\,\chi_{00}^- \left[\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\chi_{00}^-\,\omega^{(3)}(Z_0^+,\partial_\alpha Z_0^+)\right]\det(\partial\mathcal{D}) + \mathcal{O}(\kappa^{2+p}).$$

Using that by definition $\det(\widetilde{\Omega}(\kappa;\cdot)) = \kappa^{p-1}\det(\Omega(\kappa;\cdot))$, we get that the dimension breaking function near $\kappa=0$ is given by

$$D(\kappa) = \kappa^2\,\chi_{00}^- \left[\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\,\omega^{(3)}(Z_0^+,\partial_\alpha Z_0^+)\right]\det(\partial\mathcal{D}) + \mathcal{O}(\kappa^3).$$

By using that the dimension breaking function is continuous for $\kappa>0$, we get that D has to vanish at some positive value of κ if d_∞ and $\chi_{00}^-\,(\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\chi_{00}^-\,\omega^{(3)}(Z_0^+,\partial_\alpha Z_0^+))\det(\partial\mathcal{D})$ have opposite signs. Finally, the fact that the manifold at infinity satisfies (40) (hence ∂D is the identity matrix) gives the expression of the Theorem. □

There are essentially three properties of the basic state that go into the goemetric condition (48). The parameter χ_{00}^- is a geometric property of the shape of the wave, $\omega^{(3)}(Z_0^+,\partial_\alpha Z_0^+)$ is a property of the "tail", the state at infinity. For example if the basic state decays exponentially to zero as $x\to\pm\infty$ then this term vanishes. The third property is $\mathcal{C}'(\alpha)$ which arises from the embedding, and it is the most important term. The only other required information is d_∞ which is a property of the linearization. This parameter can be obtained a number of ways, for example from the linearization in the limit as $\kappa\to\infty$. For the examples in this paper, we can appeal to the large-κ asymptotics, which are analogous to similar asymptotics used in analyzing the Evans function, to determine d_∞.

6 The Zakharov-Kuznetsov equation

The Zakharov-Kuznetsov (ZK) equation is a model equation for a range of physical problems, for example, long waves on a thin liquid film, Rossby waves in a rotating atmoshere, and nonlinear ion-acoustic waves, see Zakharov & Kuznetsov [19]. The ZK equation is given by (29), with $f(u) = u^2 - u$. In a one-dimensional setting, the ZK equation reduces to the Korteweg-de Vries equation, and it is well-known that this equation has solitary wave solutions. In the multi-symplectic setting, this basic solitary wave lifts to a multi-parameter family for and b_1, c such that $1 + 2(b_1 - c) > 0$, and is given by

$$\widetilde{Z}(x) = \begin{pmatrix} u(x) \\ -\int_{-\infty}^{x}(u(s)+b_1)\,ds + q_2^\infty \\ q_3^\infty \\ -d_2 + u_x(x) \\ -c(u(x)+b_1) + a_1 + f(-b_1) \\ b_2 + \alpha u_x(x) \end{pmatrix} \quad \text{where} \quad u(x) = A\,\text{sech}^2(Bx) - b_1,$$

with

$$B = \frac{1}{2\sqrt{1+\alpha^2}}\sqrt{1+2(b_1-c)} \quad \text{and} \quad A = \frac{3}{2}(1+2(b_1-c)) = 6(1+\alpha^2)B^2\,,$$

and q_2^∞, q_3^∞ arbitrary real numbers. Hence for $\beta = 2B$, we have

$$\Psi^\pm = \lim_{x\to\pm\infty} e^{\pm\beta x}\widetilde{Z}_x(x) = 4A\,(\mp 2B, -1, 0, 4B^2, \pm 2Bc, 4B^2\alpha)$$

and $\frac{\mathrm{d}}{\mathrm{d}x}(\lim_{|x|\to\infty} e^{\beta|x|}\widetilde{Z}_x(x)) = 0$. Furthermore,

$$\mathcal{C}(\widetilde{Z}) = \frac{172}{5}\,\alpha B^5(1+\alpha^2)^2 = \frac{6}{5\sqrt{1+\alpha^2}}\alpha\sqrt{1+2(b_1-c)}(2(b_1-c)+1)^2,$$

thus

$$\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) = \frac{192}{5}B^5(1+\alpha^2).$$

Also, the manifold at infinity is

$$M_\infty(a_1,b_1,b_2,d_2) = \{G_{\theta_1,\theta_2}(-b_1,0,0,-d_2,a_1+f(-b_1),b_2) \mid \theta_1,\,\theta_2 \in \mathbb{R}\}.$$

Since $P_1(Z_0) = -b1$, $Q_1(Z_0) = -a_1 - f(-b_1)$, $Q_2(Z_0) = -b_2$ and $R_2(Z_0) = -d_2$, we have

$$\begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_a^1 & \mathcal{Q}_a^1 & \mathcal{R}_a^1 \\ \mathcal{P}_b^1 & \mathcal{Q}_b^1 & \mathcal{R}_b^1 \\ \mathcal{P}_d^1 & \mathcal{Q}_d^1 & \mathcal{R}_d^1 \end{pmatrix} \begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix} = f'(b_1) \neq 0$$

and

$$\begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_a^2 & \mathcal{Q}_a^2 & \mathcal{R}_a^2 \\ \mathcal{P}_b^2 & \mathcal{Q}_b^2 & \mathcal{R}_b^2 \\ \mathcal{P}_d^2 & \mathcal{Q}_d^2 & \mathcal{R}_d^2 \end{pmatrix} \begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix} = -(1+\alpha^2) \neq 0.$$

Also, if $Z_0^- = (-b_1, 0, 0, -d_2, a_1 + f(-b_1), b_2)$, then

$$Z_0^+ = (-b_1, -\frac{2A}{B}, 0, -d_2, a_1 + f(-b_1), b_2)$$

since $-\int_{-\infty}^{\infty}(u(s) + b_1)\,ds = -\frac{2A}{B}$. Only B depends on α, hence

$$\omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+) = 0.$$

For the system at infinity, the matrix $\mathbf{A}^+ = \mathbf{A}^- =: \mathbf{A}$. The characteristic polynomial for the matrix $\mathbf{A}$ is

$$p(\mu,\kappa) = \frac{1}{(1+\alpha^2)^2}\,\mu^2(\mu^2(1+\alpha^2) + 2i\mu\kappa\alpha - \kappa^2)(\mu^2(1+\alpha^2) + 2i\mu\kappa\alpha - \kappa^2 + f'(-b_1)).$$

So the eigenvalues are

$$\mu = 0, \quad \mu = \frac{\pm\kappa - i\kappa\alpha}{1+\alpha^2} \quad \text{or} \quad \mu = \frac{\pm\sqrt{\kappa^2 + 4B^2(1+\alpha^2)^2} - i\kappa\alpha}{1+\alpha^2}.$$

The eigenvectors are

$$v_1(\kappa,\mu) = (-\mu, 1, 0, -\mu^2, c\mu, -\mu(i\kappa + \mu\alpha))$$

for $\mu = 0$ or $\mu = \frac{\pm\sqrt{\kappa^2+4B^2(1+\alpha^2)^2} - i\kappa\alpha}{1+\alpha^2}$ and

$$v_2(\kappa,\mu) = (0, 0, i\kappa(i\kappa\alpha - \mu(1+\alpha^2)), \mu\kappa^2, 0, \kappa^2(\mu\alpha - i\kappa))$$

for $\mu = \frac{\pm\kappa - i\kappa\alpha}{1+\alpha^2}$.

Hence there are two eigenvalues with negative real part for $\kappa \neq 0$:

$$\mu_1(\kappa) = \frac{-\sqrt{\kappa^2 + 4(1+\alpha^2)^2 B^2} - i\kappa\alpha}{1+\alpha^2} \quad \text{and} \quad \mu_2(\kappa) = \frac{-|\kappa| - i\kappa\alpha}{1+\alpha^2}$$

Furthermore,

$$\chi_{00}^- = (\Omega(\Psi^-, \Psi^+))^{-1} = -\frac{1}{9216\,B^7(1+\alpha^2)^3}$$

Also, $\zeta_2(0), \widetilde{\eta}_2(0) \in \text{span}\{E_3\}$.

By using Theorem 7, we get that for any value of the parameters, the solitary wave will be capable of dimension breaking.

Theorem 8 *For any α, $c \in \mathbb{R}$, a_1, b_2, $d_2 \in \mathbb{R}$, and $1 + 2(b_1 - c) > 0$, the multi-symplectic ZK-equation has a solitary wave solution. If $d_\infty > 0$, then there is some real positive value of κ such that the symplectic dimension breaking function $D(\kappa)$ vanishes.*

Proof From the observations above,

$$\chi_{00}^- \left[\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+) \right] = -\frac{1}{240\,B^2(1+\alpha^2)^2} < 0.$$

□

Since the structure of the linearized system for large κ is similar to that for the Evans function analysis of the Boussinesq model with large λ analyzed in [6], we expect that similar arguments will show that $d_\infty = +1$. Moreover, we expect that many of the existing arguments in the literature on the large-λ behaviour of the Evans function carry over to large-κ behaviour of the dimension-breaking function. Other examples of large-λ analyses of the Evans function can be found in [18, 15] and references therein.

7 A (2+1)-dimensional Boussinesq equation

The (1+1)-Boussinesq equation is a model for a wide range of physical phenomena such as water waves, nonlinear strings and condensed matter physics. The (2+1)-Boussinesq equation is given by (30), with $f(u) = D(u^2 - u)$, where $D = \pm 1$. In JOHNSON [13], the equation with $D = -1$, $\varepsilon = 1$ and $\sigma = 1$ is derived as a model for water waves in shallow water.

In one space dimension, the (2+1)-Boussinesq equation reduces to the standard Boussinesq equation. If $\varepsilon = -1$ and $D = -1$, this is the socalled good Boussinesq equation (cf. [2]). This equation has a number of well-known solitary wave solutions. By including symmetry, this equation has a multi-parameter family of solitary waves. Under the parametric conditions,

$$D^2 - 4D(d_2 + a_1) > 0 \quad \text{and} \quad (\alpha^2\sigma - c^2 \pm \sqrt{D^2 - 4D(d_2 + a_1)})\varepsilon < 0.$$

this family takes the following form in the multi-symplectic coordinates,

$$\widetilde{Z}(x) = \begin{pmatrix} u(x) \\ c\int_{-\infty}^{x}(u(s) - u_0)\,ds + q_2^\infty \\ \sigma\alpha\int_{-\infty}^{x}(u(s) - u_0)\,ds + q_3^\infty \\ \varepsilon u_x(x) \\ -c(u(x) - u_0) - b1 \\ \frac{b_2}{\sigma} + \alpha(u(x) - u_0) \end{pmatrix} \quad \text{where} \quad u(x) = A\,\text{sech}^2(Bx) + u_0,$$

with u_0 such that u_0 solves $f(u_0) = -(a_1 + d_2)$, i.e., $u_0 = \frac{1}{2} \pm \frac{1}{2D}\sqrt{D^2 - 4D(d_2 + a_1)}$,

$$B = \frac{1}{2|\varepsilon|}\sqrt{-\varepsilon(\alpha^2\sigma - c^2 \pm \sqrt{D^2 - 4D(d_2 + a_1)})}, \quad \text{and} \quad A = \frac{6\varepsilon B^2}{D}.$$

Furthermore, q_2^∞, q_3^∞ are arbitrary real numbers. Note that if $\varepsilon < 0$ and $\sigma > 0$, then for each pair of wave speeds (c, α), there exist at most one solitary wave (the "+"-sign in the expression for B). For other signs of ε and σ, there can exist pairs of wave speeds (c, α) with two different solitary wave solutions. Note that not all these cases correspond to well-posed evolution equations. For example JOHNSON'S [13] equation is not well-posed – although it is still a good model for shallow water waves!

Hence for $\beta = 2B$, we have

$$\Psi^\pm = \lim_{x\to\pm\infty} e^{\pm\beta x}\widetilde{Z}_x(x) = \frac{6\varepsilon B^2}{D}(\mp 2B, c, \alpha\sigma, 4\varepsilon B^2, \pm 2cB, \mp 2B\alpha)$$

and $\frac{\mathrm{d}}{\mathrm{d}x}(\lim_{|x|\to\infty} e^{\beta|x|}\widetilde{Z}_x(x)) = 0$. Thus

$$\mathcal{C}(\widetilde{Z}) = -\frac{A\alpha\sigma}{3B}(4A + 3u_0) = -\frac{6\sigma\varepsilon B\alpha}{D}\left(\frac{8\varepsilon B^2}{D} + u_0\right)$$

and

$$\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) = -\frac{6\sigma\varepsilon u_0}{2D}\frac{\partial}{\partial\alpha}(\alpha B) - \frac{12\sigma\varepsilon B}{D^2}(4\varepsilon B^2 - 3\alpha^2\sigma)$$

The manifold at infinity is

$$M_\infty(a_1, b_1, b_2, d_2) = \{G_{\theta_1,\theta_2}(u_0, 0, 0, 0, -b_1, \frac{b_2}{\sigma}) \mid \theta_1,\, \theta_2 \in \mathbb{R}\}.$$

Since $P_1(Z_0) = u_0 = \frac{1}{2} \pm \frac{1}{2D}\sqrt{D^2 - 4D(d_2 + a_1)}$, $Q_1(Z_0) = b_1$, $Q_2(Z_0) = -\frac{b_2}{\sigma}$ and $R_2(Z_0) = u_0 = \frac{1}{2} \pm \frac{1}{2D}\sqrt{D^2 - 4D(d_2 + a_1)}$,

$$\begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_a^1 & \mathcal{Q}_a^1 & \mathcal{R}_a^1 \\ \mathcal{P}_b^1 & \mathcal{Q}_b^1 & \mathcal{R}_b^1 \\ \mathcal{P}_d^1 & \mathcal{Q}_d^1 & \mathcal{R}_d^1 \end{pmatrix} \begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix} = 1 \mp \frac{c^2}{\sqrt{D^2 - 4D(d_2 + a_1)}}$$

and

$$\begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_a^2 & \mathcal{Q}_a^2 & \mathcal{R}_a^2 \\ \mathcal{P}_b^2 & \mathcal{Q}_b^2 & \mathcal{R}_b^2 \\ \mathcal{P}_d^2 & \mathcal{Q}_d^2 & \mathcal{R}_d^2 \end{pmatrix} \begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix} = -\frac{1}{\sigma} \mp \frac{\alpha^2}{\sqrt{D^2 - 4D(d_2 + a_1)}}.$$

Both expressions don't vanish for generic values of c and α.

Also, if $Z_0^- = (u_0, 0, 0, 0, -b_1, \frac{b_2}{\sigma})$, then

$$Z_0^+ = (u_0, \frac{12c\varepsilon B}{D}, \frac{12\alpha\sigma\varepsilon B}{D}, 0, -b_1, \frac{b_2}{\sigma})$$

since $\int_{-\infty}^{\infty} (u(s) - u_0)\, ds = \frac{12\varepsilon B}{D}$. Only B depends on α, hence

$$\omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+) = -\frac{12\varepsilon\sigma u_0}{D} \frac{\partial}{\partial\alpha}(\alpha B).$$

For the system at infinity, the matrix $\mathbf{A}^+ = \mathbf{A}^- =: \mathbf{A}$. The characteristic polynomial for the matrix $\mathbf{A}$ is

$$p(\mu, \kappa) = \mu^2 \left(\mu^4 - \mu^2 B^2 + \frac{2i\kappa\alpha\sigma}{\varepsilon}\mu - \frac{\kappa^2\sigma}{\varepsilon} \right).$$

For κ large, the non-zero eigenvalues are

$$\mu = \mu_0 \sqrt{\kappa}(1 + \mathcal{O}(\frac{1}{\kappa})), \quad \kappa \to \infty$$

where μ_0 solves $\mu^4 = \frac{\sigma}{\varepsilon}$. For κ small, there is a pair of eigenvalues $\mu = \pm B + \mathcal{O}(\kappa)$ and a pair of eigenvalues $\mu = (i\alpha \pm \frac{1}{\sigma}\sqrt{-\alpha^2\sigma^2 - \varepsilon\sigma B^2})\kappa + \mathcal{O}(\kappa^2)$.

For $\sigma\varepsilon > 0$, it can be verified that there are always four eigenvalues on the imaginary axis plus one eigenvalue with positive real part and one eigenvalue with negative real part, hence $p = 1$. For $\sigma\varepsilon < 0$ and $B^2 > -\frac{9\sigma}{8\varepsilon}\alpha^2$, it can be shown that there are always two eigenvalues with positive real part, two eigenvalues with negative real part and two eigenvalues at zero, hence $p = 2$.

The eigenvectors are

$$v(\kappa, \mu) = (\mu^2, c\mu, (\mu\alpha + i\kappa)\sigma, \varepsilon\mu^3, -c\mu^2, \mu(\mu\alpha + i\kappa)).$$

Furthermore,

$$\chi_{00}^- = (\Omega(\Psi^-, \Psi^+))^{-1} = -\frac{D^2}{576\,\varepsilon^3 B^7}.$$

Also, if $p = 2$, then $\zeta_2(0), \widetilde{\eta}_2(0) \in \operatorname{span}\{(0, c, \mathrm{i}\,\mathrm{sgn}(\sigma)\sqrt{-\sigma(\varepsilon \mathrm{B}^2 + \alpha^2\sigma)}, 0, 0, 0)\}$.

By using Theorem 7, we get that for part of the parameter space the solitary wave will be capable of dimension breaking.

Theorem 9 *For any $c \in \mathbb{R}$, $a_1, b_2 \in \mathbb{R}$,*

$$D^2 - 4D(d_2 + a_1) > 0 \quad \textit{and} \quad (\alpha^2\sigma - c^2 \pm \sqrt{D^2 - 4D(d_2 + a_1)})\varepsilon < 0,$$

the multi-symplectic 2D-Boussinesq-equation has a solitary wave solution and there is some real positive value of κ such that the symplectic dimension breaking function $D(\kappa)$ vanishes if $d_\infty > 0$ and

- $\sigma\varepsilon > 0$ *and* $\varepsilon(4\alpha^2\sigma - c^2 \pm \sqrt{D^2 - 4D(d_2 + a_1)}) > 0$*;*
- $\sigma\varepsilon < 0$ *and* $\varepsilon(7\sigma\alpha^2 + c^2 \mp \sqrt{D^2 - 4D(d_2 + a_1)}) > 0$*.*

Proof ¿From the observations above,

$$\chi_{00}^{-}\left[\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+)\right] = -\frac{D^2}{48B^6}\,\frac{\sigma}{\varepsilon}\,(3\alpha^2\frac{\sigma}{\varepsilon} - 4B^2) < 0$$

if $\frac{\sigma}{\varepsilon}(3\alpha^2\frac{\sigma}{\varepsilon} - 4B^2) > 0$. Hence if $\sigma\varepsilon < 0$, this condition is trivially satisfied, but the number of eigenvalues with negative real part is only constant for all real values of κ, if $B^2 > -\frac{9\sigma}{8\varepsilon}\alpha^2$. □

The structure of the linearized system for large κ is similar to that for the Zakharov-Kuznetsov system analyzed in the previous section, and therefore we expect that $d_\infty = +1$.

8 The Kadomtsev-Petviashvili equation

The Kadomtsev-Petviashvili equation – or KP equation – is a cornerstone of 20th-century mathematical physics. It was derived initially as a model for studying the transverse instability of the KdV solitary wave [14]. It was subsequently found to be a model equation in a wide range of applications such as water waves and plasma physics, it was found to be integrable, and to have important implications in algebraic geometry. The dimension-breaking problem can be studied explicitly using the integrability of the equation. The purpose of this section is to illustrate how the basic KdV solitary wave can be lifted into a multi-parameter family, and how the geometric theory leads to dimension breaking without the need for explicit calculations. Moreover the theory also applies to the case where $f(u)$ is not quadratic, which is no longer integrable.

The (generalised) KP equation is given by (31) with $f(u) = D(u^2 - u)$. In a one-dimensional setting, the KP equation reduces to the KdV equation, which has a well-known solitary wave solution. Lifting this state into a multi-parameter family as in §1 (using $C = 0$ in $\mathbf{L}_3$), and placing the following constraints on the parameters,

$$-\alpha^2\sigma + 2c - 2Db_1 + D > 0$$

the family of solitary waves has the following form in multi-symplectic coordinates,

$$\widetilde{Z}(x) = \begin{pmatrix} u(x) \\ \int_{-\infty}^{x}(u(s) - b_1)\,ds + q_2^\infty \\ \sigma\alpha\int_{-\infty}^{x}(u(s) - b_1)\,ds + q_3^\infty \\ -u_x(x) \\ -c(u(x) - b_1) - f(b_1) - (a_1 + d_2) \\ \frac{b_2}{\sigma} + \alpha(u(x) - u_0) \end{pmatrix} \quad \text{where} \quad u(x) = A\,\mathrm{sech}^2(Bx) + b_1,$$

with

$$B = \frac{1}{2}\sqrt{-\alpha^2\sigma + 2c - 2Db_1 + D} \quad \text{and} \quad A = \frac{6B^2}{D} = \frac{6}{D}(-\alpha^2\sigma + 2c - 2Db_1 + D)\,.$$

Furthermore, q_2^∞, q_3^∞ are arbitrary real numbers.

Hence for $\beta = 2B$, we have

$$\Psi^\pm = \lim_{x\to\pm\infty} e^{\pm\beta x}\widetilde{Z}_x(x) = \frac{6B^2}{D}(\mp 2B, 1, \alpha\sigma, 4B^2, 0, \mp 2B\alpha)$$

and $\frac{\mathrm{d}}{\mathrm{d}x}(\lim_{|x|\to\infty} e^{\beta|x|}\widetilde{Z}_x(x)) = 0$. Thus

$$\mathcal{C}(\widetilde{Z}) = -\frac{6\sigma B\alpha}{D}\left(8B^2 + Db_1\right)$$

and

$$\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) = -\frac{12\sigma B}{D^2}\left(4B^2 - 3\sigma\alpha^2\right) - \frac{6\sigma b_1}{D}\frac{\partial}{\partial\alpha}(B\alpha)$$

The manifold at infinity is

$$M_\infty(a_1, b_1, b_2, d_2) = \left\{ G_{\theta_1,\theta_2}\left(b_1, 0, 0, 0, -f(b_1) - a_1 + d_2, \frac{b_2}{\sigma}\right) \mid \theta_1,\, \theta_2 \in \mathbb{R} \right\}.$$

Since $P_1(Z_0) = b_1$, $Q_1(Z_0) = f(b_1) + a_1 + d_2$, $Q_2(Z_0) = -\frac{b_2}{\sigma}$, $R_1(Z_0) = 0$ and $R_2(Z_0) = b_1$, we get

$$\begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_a^1 & \mathcal{Q}_a^1 & \mathcal{R}_a^1 \\ \mathcal{P}_b^1 & \mathcal{Q}_b^1 & \mathcal{R}_b^1 \\ \mathcal{P}_d^1 & \mathcal{Q}_d^1 & \mathcal{R}_d^1 \end{pmatrix} \begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix} = -2c + f'(b_1) \neq 0 \quad \text{for generic values of } c$$

and

$$\begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_a^2 & \mathcal{Q}_a^2 & \mathcal{R}_a^2 \\ \mathcal{P}_b^2 & \mathcal{Q}_b^2 & \mathcal{R}_b^2 \\ \mathcal{P}_d^2 & \mathcal{Q}_d^2 & \mathcal{R}_d^2 \end{pmatrix} \begin{pmatrix} -c \\ 1 \\ \alpha \end{pmatrix} = -\frac{1}{\sigma} \neq 0.$$

Also, if $Z_0^- = \left(b_1, 0, 0, 0, -f(b_1) - (a_1 + d_2), \frac{b_2}{\sigma}\right)$, then

$$Z_0^+ = \left(b_1, \frac{12B}{D}, \frac{12\alpha\sigma B}{D}, 0, -f(b_1) - (a_1 + d_2), \frac{b_2}{\sigma}\right)$$

since $\int_{-\infty}^{\infty}(u(s) - u_0)\,ds = \frac{12B}{D}$. Only B depends on α, hence

$$\omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+) = -\frac{12\sigma b_1}{D}\frac{\partial}{\partial\alpha}(B\alpha).$$

For the system at infinity, the matrix $\mathbf{A}^+ = \mathbf{A}^- =: \mathbf{A}$. The characteristic polynomial for the matrix $\mathbf{A}$ is

$$p(\mu, \kappa) = \mu^2(\mu^4 - B^2\mu^2 + 2i\kappa\alpha\sigma\mu - \kappa^2\sigma).$$

This polynomial is the same as the one in the previous section on the Boussinesq equation if we take $\varepsilon = 1$. Hence if $\sigma > 0$, then $p = 1$ and if $\sigma < 0$ and $B^2 > -\frac{9}{8}\sigma\alpha^2$, then $p = 2$.

The eigenvectors are

$$v(\kappa, \mu) = (\mu^2, \mu, (\alpha\mu + i\kappa)\sigma, \mu^3, -\mu^2 c, \mu(\alpha\mu + i\kappa)).$$

Furthermore,

$$\chi_{00}^- = (\Omega(\Psi^-, \Psi^+))^{-1} = -\frac{D^2}{576\, B^5}.$$

Also, $\zeta_2(0), \widetilde{\eta}_2(0) \in \text{span}\{(0, 1, \sigma\sqrt{4\sigma B^2 + \alpha^2\sigma^2}), 0, 0, 0)\}$.

By using Theorem 7, we get that for part of the parameter space the solitary wave will be capable of dimension breaking.

Theorem 10 *For any $c \in \mathbb{R}$, a_1, $b_2 \in \mathbb{R}$, and $-\alpha^2\sigma + 2c - 2Db_1 + D > 0$ the multi-symplectic Kadomtsev-Petviashvili equation has a solitary wave solution and there is some real positive value of κ such that the symplectic dimension breaking function $D(\kappa)$ vanishes if $d_\infty > 0$ and*

- *$\sigma > 0$ and $4\alpha^2\sigma - 2c + 2Db_1 - D > 0$;*
- *$\sigma < 0$ and $7\alpha^2\sigma + 4c - 4Db_1 + 2D > 0$.*

Proof ¿From the observations above,

$$\chi_{00}^- \left[\frac{\partial}{\partial\alpha}\mathcal{C}(\widetilde{Z}) - \frac{1}{2}\omega^{(3)}(Z_0^+, \partial_\alpha Z_0^+) \right] = \frac{\sigma}{48B^4}\,(4B^2 - 3\alpha^2\sigma) < 0$$

if $\sigma(4B^2 - 3\sigma\alpha^2) < 0$. Hence if $\sigma < 0$, this condition is trivially satisfied, but the number of eigenvalues with negative real part is only constant for all real values of κ, if $B^2 > \frac{9}{8}\sigma\alpha^2$. □

Again, we expect that it can be shown that $d_\infty = +1$ in a similar way as before.

9 Concluding remarks

Although time appears in the general form of the equations we studied, it played a passive role. The basic state could be time dependent, but is steady in a moving frame.

On the other hand, in the paper, we showed that there is a close connection between the Evans function, which is used to study the linearized stability of solitary waves, and the dimension-breaking function associated the the spectral problem in dimension breaking. Therefore it is natural to ask whether the framework could combine both features: dimension breaking and stability. For example, how can one study the stability of the bifurcating states?

The bifurcating states depend essentially on two coordinates, the coordinate of the basic state and the transverse coordinate. Therefore the linearization about this state will be a PDE, and is not reducible to an ODE, and therefore the Evans function, which is based essentially on the structure of ODEs no longer applies. There are a number methods in the literature for approaching the transverse instability problem, see Kivshar & Pelinovsky [17] for a review. However, in the present case, the multi-symplectic geometry will be present and it may therefore be possible to derive a geometric contion using other means. For example, in Bridges [4], a geometric condition for transverse instability of solitary waves is introduced using a Lyapunov-Schmidt type argment, and some generalization of this condition may apply to bifurcating states that arise from dimension breaking.

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