Hodge duality and the Evans function

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Abstract

Two generalisations of the Evans function, for the analysis of the linearisation about solitary waves, are shown to be equivalent. The generalisation introduced by Alexander, Gardner and Jones (1990) is based on exterior algebra and the generalisation introduced by Swinton (1992) is based on a matrix formulation and adjoint systems. In regions of the complex plane where both formulations are defined, the equivalence is geometric: we show that the formulations are dual and the duality can be made explicit using Hodge duality and the Hodge star operator. Swinton's formulation excludes potential branch points at which the Alexander, Gardner and Jones formulation is well-defined. Therefore we consider the implications of equivalence on the analytic continuation of the two formulations.

1. Introduction

The linearisation about a travelling solitary wave state, of a partial differential equation on the real line, leads to linear systems of the form

$$U_x = \mathbf{A}(x,\lambda)U\,, \quad x\in\mathbb{R}\,, \quad \lambda\in\Lambda\,, \quad U\in\mathbb{C}^n\,. \tag{1}$$

The matrix $\mathbf{A}(x,\lambda)$ depends continuously on x and is an analytic function of λ for $\lambda\in\Lambda$, where Λ is some open simply-connected subset of the complex λ-plane (Λ is precisely defined below after equation (3)). The system (1) is the basis for a dynamical systems analysis of the spectral problem. The complex parameter λ is the spectral parameter and corresponds to a linear stability exponent: if (1) has a solution which is bounded for all $x\in\mathbb{R}$, with $\mathrm{Re}(\lambda)>0$ then this solution corresponds to an unstable eigenfunction of the linearised stability problem for the solitary wave. The dynamical systems formulation introduced by Evans [1] is based on the properties of the system (1) at $x=\pm\infty$. Let

$$\mathbf{A}_\infty(\lambda) = \lim_{x\to\pm\infty}\mathbf{A}(x,\lambda)\,, \tag{2}$$

assuming the limits exist (which is to be expected when (1) corresponds to the linearisation about a solitary wave). It is straightforward to allow for the limits in (2) to differ at $x=\pm\infty$, but it would add unnessary detail that would detract from the main point of this paper.

For any $\lambda\in\Lambda$, define

$$\begin{aligned} E^s(\lambda) &= \{\,\xi\in\mathbb{C}^n \;:\; \lim_{x\to+\infty} e^{\mathbf{A}_\infty(\lambda)x}\xi = 0\,\}\\ E^u(\lambda) &= \{\,\xi\in\mathbb{C}^n \;:\; \lim_{x\to-\infty} e^{\mathbf{A}_\infty(\lambda)x}\xi = 0\,\}\,. \end{aligned} \tag{3}$$

The primary hypothesis on the set Λ is that it be an open simply-connected subset of $\mathbb{C}$ such that the dimension of both $E^s(\lambda)$ and $E^u(\lambda)$ is the same for each $\lambda\in\Lambda$. A secondary hypothesis on Λ will be introduced after equation (7). Therefore we can define

$$m_u(\Lambda) = \dim E^u(\lambda)\,, \quad m_s(\Lambda) = \dim E^s(\lambda)\,. \tag{4}$$

In the original work of Evans it is assumed that

$$\min(m_u(\Lambda), m_s(\Lambda)) = 1 \quad \text{and} \quad m_u(\Lambda)+m_s(\Lambda) = n\,. \tag{5}$$

The first hypothesis in (5) is generalised in [2] and [3] to

$$\min(m_u(\Lambda), m_s(\Lambda)) = k \quad \text{with} \quad 1 \le k \le \tfrac{1}{2}n \tag{6}$$

with the second hypothesis, $m_u(\Lambda) + m_s(\Lambda) = n$, retained.

For definiteness, assume that $m_s(\Lambda) = k$ and $m_u(\Lambda) = n - k$, and order the eigenvalues of $\mathbf{A}_\infty(\lambda)$, $\mu_1(\lambda), \ldots, \mu_n(\lambda)$, by

$$\mathrm{Re}(\mu_1(\lambda)) \le \cdots \le \mathrm{Re}(\mu_k(\lambda)) < 0 < \mathrm{Re}(\mu_{k+1}(\lambda)) \le \cdots \le \mathrm{Re}(\mu_n(\lambda)), \quad \forall\, \lambda \in \Lambda\,. \tag{7}$$

The secondary hypothesis on Λ is that the eigenvalues $\mu_1(\lambda), \ldots, \mu_k(\lambda)$ are simple for each $\lambda \in \Lambda$. This hypothesis is not needed for the formulation of the Evans function in [2] but it is needed for the formulation in [3]. Note that it is not necessary that the eigenvalues $\mu_{k+1}(\lambda), \ldots, \mu_n(\lambda)$ be simple.

With the above hypotheses on the constant coefficient systems at $x = \pm\infty$, standard theory on asymptotic properties of linear systems can be invoked (cf. [4], [5]). Suppose

$$\int_{-\infty}^{+\infty} \|\mathbf{A}(x,\lambda) - \mathbf{A}_\infty(\lambda)\|\, dx < +\infty \quad \forall\, \lambda \in \Lambda\,, \tag{8}$$

then, for any $\lambda \in \Lambda$, there exists k linearly independent functions

$$\{\, U_1(x,\lambda), \cdots, U_k(x,\lambda) \mid x \in \mathbb{R},\ \lambda \in \Lambda \,\} \tag{9}$$

depending analytically on λ, satisfying (1) and, since each eigenvalue of $\mathbf{A}_\infty(\lambda)$ is simple, these functions satisfy the asymptotic estimates

$$\lim_{x \to +\infty} e^{-\mu_j(\lambda)x} U_j(x,\lambda) = \xi_j(\lambda), \quad \forall\, \lambda \in \Lambda\,, \quad j = 1, \ldots, k\,, \tag{10}$$

where $\xi_j(\lambda)$ is an eigenvector of $\mathbf{A}_\infty(\lambda)$ associated with the eigenvalue $\mu_j(\lambda)$, for all $j = 1, \ldots, n$.

Similarly there exist $n - k$ linearaly independent functions

$$\{\, U_{k+1}(x,\lambda), \cdots, U_n(x,\lambda) \mid x \in \mathbb{R},\ \lambda \in \Lambda \,\} \tag{11}$$

depending analytically on λ and satisfying (1) and

$$\lim_{x \to -\infty} \|U_j(x,\lambda)\| = 0 \quad \forall\, \lambda \in \Lambda\,, \quad j = k+1, \ldots, n\,. \tag{12}$$

In this context, the formulation of the Evans function in Alexander, Gardner and Jones [2] can be described as follows. Let

$$Y_+(x,\lambda) = U_1(x,\lambda) \wedge \cdots \wedge U_k(x,\lambda) \in \textstyle\bigwedge^k(\mathbb{C}^n)\,, \tag{13}$$

where $\wedge$ is the wedge product and $\bigwedge^k(\mathbb{C}^n)$ is the k^{th} exterior power of the vector space $\mathbb{C}^n$ (cf. [6], [7], [8]). Similarly, define

$$Y_-(x,\lambda) = U_{k+1}(x,\lambda) \wedge \cdots \wedge U_n(x,\lambda) \in \textstyle\bigwedge^{n-k}(\mathbb{C}^n)\,. \tag{14}$$

The forms $Y_+(x,\lambda)$ and $Y_-(x,\lambda)$ satisfy the asymptotic estimates

$$\lim_{x \to +\infty} e^{-\sigma_+(\lambda)x} Y_+(x,\lambda) = Y_+^\infty(\lambda) \quad \text{and} \quad \lim_{x \to -\infty} e^{-\sigma_-(\lambda)x} Y_-(x,\lambda) = Y_-^\infty(\lambda)\,,$$

where $Y_+^\infty(\lambda) \in \bigwedge^k(\mathbb{C}^n)$ and $Y_-^\infty(\lambda) \in \bigwedge^{n-k}(\mathbb{C}^n)$ and

$$\sigma_+(\lambda) = \mu_1(\lambda) + \cdots + \mu_k(\lambda) \quad \text{and} \quad \sigma_-(\lambda) = \mu_{k+1}(\lambda) + \cdots + \mu_n(\lambda)\,. \tag{15}$$

Then the Evans function is defined in [2] to be

$$\widetilde{D}(\lambda) = e^{-\int_0^x \mathrm{Tr}(\mathbf{A}(s,\lambda))\, ds}\, Y_-(x,\lambda) \wedge Y_+(x,\lambda) \quad \text{for any } \lambda \in \Lambda\,. \tag{16}$$

It is proved in [2] that $\widetilde{D}(\lambda)$ is independent of x, depends analytically on λ and if $\widetilde{D}(\lambda)=0$ for some $\lambda\in\Lambda$, then there exists a solution of (1) which decays exponentially as $x\to\pm\infty$ and is square integrable over the real line.

Since $\widetilde{D}(\lambda)\in\bigwedge^n(\mathbb{C}^n)$, it can be represented as a complex function times a volume form on $\mathbb{C}^n$. Let $\mathbf{e}_1,\ldots,\mathbf{e}_n$ be any unitary basis for $\mathbb{C}^n$ and fix the volume form to be

$$\mathcal{V}=\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n\in\bigwedge^n(\mathbb{C}^n)\,. \tag{17}$$

Then

$$\widetilde{D}(\lambda)=D_{AGJ}(\lambda)\,\mathcal{V}\,. \tag{18}$$

Clearly $\widetilde{D}(\lambda)=0$ means $D_{AGJ}(\lambda)=0$. The function $D_{AGJ}(\lambda)$ can be analytically continued to a larger set Λ^+ which includes values of λ where eigenvalues in (7) coalesce. Analytic continuation is possible because $\sigma_+(\lambda)$ and $\sigma_-(\lambda)$ are *simple* eigenvalues of $\mathbf{A}_\infty(\lambda)$ restricted to $\bigwedge^k(\mathbb{C}^n)$, and are therefore analytic functions of λ, even at values of λ where the eigenvalues in (7) coalesce. Analyticity of $D_{AGJ}(\lambda)$ is discussed further in §4.

The formulation introduced by Swinton [3] is based on adjoint systems and is constructed as follows. The adjoint system associated with (1) is

$$W_x=-\mathbf{A}(x,\lambda)^*W\quad\ W\in\mathbb{C}^n\,, \tag{19}$$

where $\mathbf{A}(x,\lambda)^*$ is the adjoint of $\mathbf{A}(x,\lambda)$ based on an inner product on $\mathbb{C}^n$. To be precise, if U and W are any vectors in $\mathbb{C}^n$ with components $u_1,\cdots,u_n$ and $w_1,\ldots,w_n$ respectively, then the inner product $\langle U,W\rangle=\sum_{j=1}^n\overline{u}_jw_j$, where $\overline{u}_i$ denotes the complex conjugate of u_i.

The systems at $x=\pm\infty$ associated with (19) are

$$W_x=-\mathbf{A}_\infty(\lambda)^*W\,,\quad W\in\mathbb{C}^n\,,\quad \lambda\in\Lambda\,, \tag{20}$$

and clearly the spectrum of the matrix $-\mathbf{A}_\infty(\lambda)^*$ is minus the complex conjugate of the spectrum of $\mathbf{A}_\infty(\lambda)$. Therefore there exists k functions

$$\{\,W_1(x,\lambda),\cdots,W_k(x,\lambda)\mid x\in\mathbb{R},\quad \lambda\in\Lambda\,\} \tag{21}$$

with each $\overline{W_j(x,\lambda)}$ depending analytically on λ. The functions (21) satisfy (19) and, since the eigenvalues $\{\mu_1(\lambda),\ldots,\mu_k(\lambda)\}$ are simple, they satisfy the asymptotic estimates

$$\lim_{x\to-\infty}e^{+\overline{\mu_j(\lambda)}x}W_j(x,\lambda)=\eta_j(\lambda)\,,\quad \forall\,\lambda\in\Lambda\,,\quad j=1,\ldots,k\,, \tag{22}$$

where $\eta_j(\lambda)$ is the eigenvector of $\mathbf{A}_\infty(\lambda)^*$ associated with the eigenvalue $\overline{\mu_j(\lambda)}$. Similarly one can define $n-k$ functions $W_j(x,\lambda)$, $j=k+1,\ldots,n$, which decay to zero exponentially as $x\to+\infty$. Define the *Evans matrix*,

$$\mathbf{E}(\lambda)=\begin{bmatrix}\langle W_1(x,\lambda),U_1(x,\lambda)\rangle & \cdots & \langle W_1(x,\lambda),U_k(x,\lambda)\rangle\\ \vdots & \ddots & \vdots\\ \langle W_k(x,\lambda),U_1(x,\lambda)\rangle & \cdots & \langle W_k(x,\lambda),U_k(x,\lambda)\rangle\end{bmatrix}\,,\quad\text{for any }\lambda\in\Lambda\,. \tag{23}$$

Swinton [3] proves that each entry of the the Evans matrix is independent of x and depends analytically on λ. In [3] the *Evans function* takes the form

$$D_S(\lambda)=\det\left[\mathbf{E}(\lambda)\right]\,,\quad \text{for any }\lambda\in\Lambda\,. \tag{24}$$

The determinant is independent of x; it is an analytic function of λ and $D_S(\lambda)=0$ if and only if (1) has a non-trivial bounded solution.

The proof of the following result is given in the next two sections.

Theorem. *For $\lambda \in \Lambda$, $D_S(\lambda) = 0$ if and only if $D_{AGJ}(\lambda) = 0$. More precisely, there exists a non-zero λ-dependent analytic function $\mathcal{C}(\lambda)$ such that $D_S(\lambda) = \mathcal{C}(\lambda) D_{AGJ}(\lambda)$.*

2. Multilinear ODEs and adjoint systems

The vector spaces $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{n-k}(\mathbb{C}^n)$ are isomorphic and, for an oriented vector space, the natural isomorphism is given by the Hodge star operator. The idea will be to restrict the system (1) to $\bigwedge^k(\mathbb{C}^n)$ and to restrict the adjoint system (19) to $\bigwedge^k((\mathbb{C}^n)^*)$, where $(\mathbb{C}^n)^*$ can be identifed with $\mathbb{C}^n$, and use the natural isomorphism $\bigwedge^k((\mathbb{C}^n)^*) \cong \bigwedge^{n-k}(\mathbb{C}^n)$. Then, to relate these two multilinear ODEs, an inner product is introduced on $\bigwedge^k(\mathbb{C}^n)$. This inner product provides a natural framework for defining the adjoint of a multilinear operator on $\bigwedge^k(\mathbb{C}^n)$.

On each of the vector spaces $\bigwedge^k(\mathbb{C}^n)$, $k = 1, \cdots, n$, there is a natural induced inner product. Let $U_1, \cdots, U_k$ and $V_1, \cdots, V_k$ be any vectors in $\mathbb{R}^n$, and let

$$U = U_1 \wedge \cdots \wedge U_k \in \bigwedge^k(\mathbb{C}^n) \quad \text{and} \quad V = V_1 \wedge \cdots \wedge V_k \in \bigwedge^k(\mathbb{C}^n) .$$

Then the inner product of U and V is defined by

$$[\![U, V]\!]_k = \det \begin{bmatrix} \langle U_1, V_1 \rangle & \cdots & \langle U_1, V_k \rangle \\ \vdots & \ddots & \vdots \\ \langle U_k, V_1 \rangle & \cdots & \langle U_k, V_k \rangle \end{bmatrix} . \tag{25}$$

The definition extends to any pair of elements of $\bigwedge^k(\mathbb{C}^n)$ (i.e. not necessarily decomposable) by linearity, and satisfies all the conditions of an inner product (cf. [7, §1.6]).

Using the inner product $[\![\cdot, \cdot]\!]_k$ on $\bigwedge^k(\mathbb{C}^n)$, a unitary basis for $\bigwedge^k(\mathbb{C}^n)$ can be constructed. Denote this basis by

$$\{\vartheta_1, \ldots, \vartheta_d\} \quad \text{with} \quad [\![\vartheta_i, \vartheta_j]\!]_k = \delta_{ij} \quad \text{and} \quad d = \dim \bigwedge^k(\mathbb{C}^n) = \frac{n!}{(n-k)!k!} . \tag{26}$$

A unitary basis for $\bigwedge^{n-k}(\mathbb{C}^n)$ can be constructed using the Hodge star operator (cf. [7, §1.7] and [6, Chapter V]). The Hodge star operator is an isomorphism between $\bigwedge^{n-k}(\mathbb{C}^n)$ and $\bigwedge^k(\mathbb{C}^n)$, and for any basis k-form $\vartheta_j \in \bigwedge^k(\mathbb{C}^n)$, the Hodge star of ϑ_j, denoted $\star\vartheta_j$, is an element of $\bigwedge^{n-k}(\mathbb{C}^n)$ and satisfies $\vartheta_j \wedge \star\vartheta_j = \mathcal{V}$. A unitary basis for $\bigwedge^{n-k}(\mathbb{C}^n)$ is then

$$\{\theta_1, \ldots, \theta_d\} \quad \text{with} \quad \theta_j = \star\vartheta_j \quad \text{and} \quad d = \dim \bigwedge^{n-k}(\mathbb{C}^n) = \dim \bigwedge^k(\mathbb{C}^n) , \tag{27}$$

and the basis elements satisfy

$$\vartheta_j \wedge \theta_k = \vartheta_j \wedge \star\vartheta_k = \delta_{jk}\, \mathcal{V} . \tag{28}$$

Explicit expressions for $\theta_1, \ldots, \theta_d$ can be obtained in terms of the oriented basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ (cf. [8, p. 174]), but will not be needed.

The multilinear functions $Y_+(x, \lambda)$ and $Y_-(x, \lambda)$ satisfy the differential equations obtained by restricting (1) to $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{n-k}(\mathbb{C}^n)$ respectively,

$$\frac{d}{dx} Y_+ = \mathbf{A}^{(k)}(x, \lambda) Y_+ \quad \text{and} \quad \frac{d}{dx} Y_- = \mathbf{A}^{(n-k)}(x, \lambda) Y_- , \tag{29}$$

where $\mathbf{A}^{(k)}(x, \lambda)$ and $\mathbf{A}^{(n-k)}(x, \lambda)$ are derivations associated with $\mathbf{A}(x, \lambda)$,

$$\mathbf{A}^{(k)} Y_+ = \sum_{j=1}^{k} U_1 \wedge \cdots \wedge \mathbf{A} U_j \wedge \cdots \wedge U_k \tag{30}$$

and

$$\mathbf{A}^{(n-k)} Y_- = \sum_{j=k+1}^{n} U_{k+1} \wedge \cdots \wedge \mathbf{A} U_j \wedge \cdots \wedge U_n . \tag{31}$$

(cf. [9]). The following result can be proved via direct calculation using the definition of the derivation $\mathbf{A}^{(k)}$ and the inner product $[\![\cdot,\cdot]\!]_k$ on $\bigwedge^k(\mathbb{C}^n)$.

Proposition 1. *Suppose $U, V \in \bigwedge^k(\mathbb{C}^n)$ are decomposable k-forms. Let $\mathbf{A}^{(k)} : \bigwedge^k(\mathbb{C}^n) \to \bigwedge^k(\mathbb{C}^n)$ be the induced derivation of $\mathbf{A} \in g\ell(n,\mathbb{C})$ on $\bigwedge^k(\mathbb{C}^n)$ as in (30). Then*

$$[\![V, \mathbf{A}^{(k)}U]\!]_k = [\![(\mathbf{A}^{(k)})^*V, U]\!]_k \,,$$

where, with $V = V_1 \wedge \cdots \wedge V_k$,

$$(\mathbf{A}^{(k)})^*V = \sum_{j=1}^{k} V_1 \wedge \cdots \wedge \mathbf{A}^*V_j \wedge \cdots \wedge V_k$$

and $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$ on $\mathbb{C}^n$ defined using the inner product $\langle\cdot,\cdot\rangle$.

Proposition 2. *Let $U(x) \in \bigwedge^{n-k}(\mathbb{C}^n)$ be an arbitrary solution of $\frac{d}{dx}U = \mathbf{A}^{(n-k)}(x,\lambda)U$. Then $\star U \in \bigwedge^k(\mathbb{C}^n)$ satisfies the differential equation*

$$\frac{d}{dx}(\star U) = \left[\overline{\tau(x,\lambda)}\mathbf{I}_k - (\mathbf{A}^{(k)}(x,\lambda))^*\right](\star U)\,, \quad \star U \in \textstyle\bigwedge^k(\mathbb{C}^n)\,, \tag{32}$$

where $\tau(x,\lambda) = \mathrm{Trace}(\mathbf{A}(x,\lambda))$, $\mathbf{I}_k$ is the identity on $\bigwedge^k(\mathbb{C}^n)$ and $(\mathbf{A}^{(k)}(x))^*$ is the adjoint of $\mathbf{A}^{(k)}(x)$ defined in Proposition 1.

Proof. Let $V(x) \in \bigwedge^k(\mathbb{C}^n)$ be any solution of $\frac{d}{dx}V = \mathbf{A}^{(k)}(x,\lambda)V$. Using the unitary bases for $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{n-k}(\mathbb{C}^n)$, expand V and U as

$$V(x,\lambda) = \sum_{j=1}^{d} v_j(x,\lambda)\vartheta_j \quad \text{and} \quad U(x,\lambda) = \sum_{m=1}^{d} u_m(x,\lambda)\theta_m\,. \tag{33}$$

Then

$$V \wedge U = \left(\sum_{j=1}^{d} v_j\vartheta_j\right) \bigwedge \left(\sum_{m=1}^{d} u_m\theta_m\right) = \sum_{j=1}^{d}\sum_{m=1}^{d} v_j u_m\, \vartheta_j \wedge \theta_m\,.$$

But $\vartheta_j \wedge \theta_m = \vartheta_j \wedge \star\vartheta_m = \delta_{jm}\mathcal{V}$ (cf. equation (28)), and so

$$U \wedge V = (-1)^{k(n-k)} V \wedge U = (-1)^{k(n-k)} \left(\sum_{j=1}^{d} v_j u_j\right) \mathcal{V}\,,$$

or

$$U \wedge V = [\![\star U, V]\!]_k\, \mathcal{V}\,, \tag{34}$$

since

$$\star U = (-1)^{k(n-k)} \sum_{j=1}^{d} \overline{u}_j \vartheta_j \in \textstyle\bigwedge^k(\mathbb{C}^n)\,, \tag{35}$$

where we have used the property $\star\star\,\vartheta_j = (-1)^{k(n-k)}\vartheta_j$ (cf. [7, p. 19]). Differentiate both sides of (34) with respect to x,

$$\frac{d}{dx}(U \wedge V) = \left([\![(\star U)_x, V]\!]_k + [\![\star U, (V)_x]\!]_k\right) \mathcal{V}\,. \tag{36}$$

The left-hand side of this expression can be simplifed using the Abel-Liouville Theorem,

$$\frac{d}{dx}(U \wedge V) = \tau(x,\lambda) U \wedge V = \tau(x,\lambda)[\![\star U, V]\!]_k\, \mathcal{V}\,,$$

where $\tau(x,\lambda) = \mathrm{Trace}(\mathbf{A}(x,\lambda))$. Therefore (36) reduces to

$$\left([\![(\star U)_x, V]\!]_k + [\![\star U, (V)_x]\!]_k - [\![\star U, \tau(x,\lambda)V]\!]_k\right) \mathcal{V} = 0\,.$$

Substituting the equation for $(V)_x$ from (26),

$$[\![(\star U)_x, V]\!]_k + [\![\star U, \mathbf{A}^{(k)} V]\!]_k - [\![\star U, \tau(x,\lambda) V]\!]_k = 0\,,$$

or

$$[\![(\star U)_x + (\mathbf{A}^{(k)})^*(\star U) - \overline{\tau(x,\lambda)}(\star U), V]\!]_k = 0\,, \tag{37}$$

using Proposition 1. The proof is completed by noting that Hodge star is independent of the unitary basis chosen in the orientation (cf. [10]) and (37) must hold for all $V \in \bigwedge^k(\mathbb{C}^n)$. □

3. Hodge duality and the Evans matrix

Define

$$\mathcal{Y}_-(x,\lambda) = e^{-\int_0^x \tau(s,\lambda)\,ds} Y_-(x,\lambda)\,. \tag{38}$$

The Hodge star theory gives that

$$\star\mathcal{Y}_-(x,\lambda) = e^{-\int_0^x \overline{\tau(s,\lambda)}\,ds}(\star Y_-(x,\lambda)) \in \bigwedge^k(\mathbb{C}^n)\,,$$

and Proposition 2 applied to (38) shows that $(\star\mathcal{Y}_-)$ satisfies the differential equation

$$\frac{d}{dx}(\star\mathcal{Y}_-) = [-\mathbf{A}^{(k)}(x,\lambda)^*](\star\mathcal{Y}_-)\,, \quad \star\mathcal{Y}_- \in \bigwedge^k(\mathbb{C}^n)\,. \tag{39}$$

Both $\mathcal{Y}_-(x,\lambda)$ and $\star\mathcal{Y}_-(x,\lambda)$ decay with the *maximal* decay rate as $x \to -\infty$. This can be seen as follows. Let $\sigma_+(\lambda)$ and $\sigma_-(\lambda)$ be as defined in equation (15). Then $\mathrm{Re}(\sigma_+(\lambda)) < 0$ and $\mathrm{Re}(\sigma_-(\lambda)) > 0$; moreover, $\mathrm{Re}(\sigma_+(\lambda))$ is strictly less than the real part of any other linear combination of k eigenvalues of $\mathbf{A}_\infty(\lambda)$.

Since $Y_-(x,\lambda)$ characterises the subspace of solutions which decay exponentially as $x \to -\infty$ it follows that

$$\lim_{x\to-\infty} e^{-\sigma_-(\lambda)x} Y_-(x,\lambda) = Y_-^\infty(\lambda) \in \bigwedge^{n-k}(\mathbb{C}^n)\,. \tag{40}$$

Now $\tau(x,\lambda) = \mathrm{Tr}(\mathbf{A}(x,\lambda) - \mathbf{A}_\infty(\lambda)) + \mathrm{Tr}(\mathbf{A}_\infty(\lambda)$, and since the trace of a matrix equals the sum of its eigenvalues,

$$\mathrm{Tr}(\mathbf{A}_\infty(\lambda)) = \sigma_+(\lambda) + \sigma_-(\lambda)\,.$$

By the hypothesis in equation (8), $\mathrm{Tr}(\mathbf{A}(x,\lambda) - \mathbf{A}_\infty(\lambda))$ is integrable. Hence

$$\lim_{x\to-\infty} e^{(\sigma_+(\lambda)+\sigma_-(\lambda))x}\, e^{-\int_0^x \tau(s,\lambda)\,ds} = \lim_{x\to-\infty} e^{-\int_0^x \mathrm{Tr}(\mathbf{A}(s,\lambda)-\mathbf{A}_\infty(\lambda))\,ds} = C_1(\lambda)\,. \tag{41}$$

Therefore

$$\lim_{x\to-\infty} e^{\sigma_+(\lambda)x}\,\mathcal{Y}_-(x,\lambda) = \lim_{x\to-\infty} e^{-\int_0^x \mathrm{Tr}(\mathbf{A}(s,\lambda)-\mathbf{A}_\infty(\lambda))\,ds}\, e^{\sigma_-(\lambda)x}\,\mathcal{Y}_-(x,\lambda) = \Upsilon(\lambda) \in \bigwedge^{n-k}(\mathbb{C}^n)\,, \tag{42}$$

and

$$\lim_{x\to-\infty} e^{\overline{\sigma_+(\lambda)}x}\,\star\mathcal{Y}_-(x,\lambda) = \star\Upsilon(\lambda) \in \bigwedge^k(\mathbb{C}^n)\,. \tag{43}$$

Since $\mathrm{Re}(\sigma_+(\lambda))$ has the smallest real part of any other linear combination of k eigenvalues of $\mathbf{A}_\infty(\lambda)$, it follows that $\star\mathcal{Y}_-(x,\lambda)$ decays exponentially at the maximal rate as $x \to -\infty$.

Using the Hodge identity (34) and the expression (38), the fundamental definition of the Evans function (16) takes the form

$$\widetilde{D}(\lambda) = e^{-\int_0^x \tau(s,\lambda)\,ds} Y_-(x,\lambda) \wedge Y_+(x,\lambda) = \mathcal{Y}_-(x,\lambda) \wedge Y_+(x,\lambda) = [\![\star\mathcal{Y}_-(x,\lambda), Y_+(x,\lambda)]\!]_k\,\mathcal{V}\,. \tag{44}$$

Comparision of this expression with (18) shows that $D_{AGJ}(\lambda)$ has an equivalent representation as an inner product of two elements of $\bigwedge^k(\mathbb{C}^n)$:

$$D_{AGJ}(\lambda) = [\![\star\mathcal{Y}_-(x,\lambda), Y_+(x,\lambda)]\!]_k\,. \tag{45}$$

By construction, both $Y_+(x,\lambda)$ and $Y_-(x,\lambda)$ are decomposable. But, is $\star\mathcal{Y}_-(x,\lambda)$ decomposable? To see the nature of this question more clearly, suppose $n = 100$ and $k = 2$. If $Y \in \bigwedge^{n-k}(\mathbb{C}^n)$ is non-zero and decomposable, it consists of the wedge product of 98 linearly independent vectors. For $\star Y \in \bigwedge^k(\mathbb{C}^n)$ to be non-zero and decomposable it must be representable as the wedge product of two linearly independent vectors!

Proposition 3. *Let $Y \in \bigwedge^{n-k}(\mathbb{C}^n)$ be any non-zero decomposable $(n-k)$-form. Then $\star Y \in \bigwedge^k(\mathbb{C}^n)$ is a non-zero decomposable k-form.*

In other words, Hodge star is a bijection which maps the non-zero decomposable elements in $\bigwedge^{n-k}(\mathbb{C}^n)$ onto the set of non-zero decomposable elements in $\bigwedge^k(\mathbb{C}^n)$. For a proof, see Theorem 1.8 and its proof on page 28 of Marcus [11].

A consequence of Proposition 3 is that $D_{AGJ}(\lambda)$ in (45) is an inner product between two decomposable elements of $\bigwedge^k(\mathbb{C}^n)$. Decomposability of $\star\mathcal{Y}_-(x,\lambda)$ implies that there exists k functions $\delta_1(x,\lambda),\ldots,\delta_k(x,\lambda)$ such that

$$\star\mathcal{Y}_-(x,\lambda) = \delta_1(x,\lambda)\wedge\cdots\wedge\delta_k(x,\lambda) \in \textstyle\bigwedge^k(\mathbb{C}^n)\,, \tag{46}$$

where each $\delta_j(x,\lambda)$ satisfies $\frac{d}{dx}\delta_j = -\mathbf{A}(x,\lambda)^*\delta_j$. The fact that each $\delta_j(x,\lambda)$ satisfies this equation is more subtle than it might appear. For example, functions $\delta_j(x,\lambda)$ satisfying $\frac{d}{dx}\delta_j = -\mathbf{A}(x,\lambda)^*\delta_j + f_j$ with f_j some linear combination of $\delta_1,\ldots,\delta_k$ also satisfies (39) and (46), but this inhomogeneous equation is not solvable unless each $f_j = 0$.

The functions $\overline{\delta_j(x,\lambda)}$ in (46) are analytic functions of λ. Since Λ excludes branch points, a basis for the space associated with the decomposable form $\star\mathcal{Y}_-(x,\lambda)$ which is also analytic can be found. However, in general a more careful proof is needed. Abstractly the question is: given an arbitrary decomposable k-form, depending analytically on a parameter λ, is it possible to construct an analytic basis for the k-dimensional space which the form characterises? This question is analogous to the problem solved by Kato [12, p. 99], of finding an analytic basis for a k-dimensional space defined by an analytic projection.

Using the functions $\delta_1(x,\lambda),\ldots,\delta_k(x,\lambda)$, the inner product in (45) takes the explicit form,

$$[\![\star\mathcal{Y}_-(x,\lambda), Y^+(x,\lambda)]\!]_k = \det\begin{bmatrix} \langle\delta_1(x,\lambda), U_1(x,\lambda)\rangle & \cdots & \langle\delta_1(x,\lambda), U_k(x,\lambda)\rangle \\ \vdots & \ddots & \vdots \\ \langle\delta_k(x,\lambda), U_1(x,\lambda)\rangle & \cdots & \langle\delta_k(x,\lambda), U_k(x,\lambda)\rangle \end{bmatrix}. \tag{47}$$

It remains to connect the functions $\delta_j(x,\lambda)$ with the set of adjoint functions (21). Given a decomposable element in $\bigwedge^k(\mathbb{C}^n)$, it is possible to retrieve a basis for the k-dimensional vector space which it characterises, but one cannot (for $k > 1$) construct the individual vectors of which it was formed as an exterior product. In fact, an algorithm for extracting a basis from a decomposable k-form is given on pages 95-96 of [8]. However, this explicit construction is not necessary. We can argue as follows. According to equation (43), $\star\mathcal{Y}_-(x,\lambda)$ decays exponentially as $x\to-\infty$ at the maximal possible rate. Therefore the set $\{\delta_1(x,\lambda),\ldots,\delta_k(x,\lambda)\}$ must span the space of solutions of the adjoint equation which are exponentially decaying as $x\to-\infty$. Since the space of solutions of the adjoint equation which decays exponentially as $x\to-\infty$ is k-dimensional, and the set $\delta_1,\ldots,\delta_k$ is also k-dimensional, it follows that the two sets

$$\{\delta_1(x,\lambda),\ldots,\delta_k(x,\lambda)\} \quad\text{and}\quad \{W_1(x,\lambda),\ldots,W_k(x,\lambda)\}$$

span the same space. Hence there exists an invertible $k\times k$ matrix $\mathbf{C}(\lambda)$ depending analytically on $\lambda\in\Lambda$ satisfying

$$[\delta_1(x,\lambda)\,|\cdots|\,\delta_k(x,\lambda)] = [W_1(x,\lambda)\,|\cdots|\,W_k(x,\lambda)]\overline{\mathbf{C}(\lambda)}\,. \tag{48}$$

Therefore, combining (45), (47) and (48) we find

$$D_{AGJ}(\lambda) = [\![\star\mathcal{Y}_-(x,\lambda), Y_+(x,\lambda)]\!]_k = \det[\mathbf{C}(\lambda)]\det[\mathbf{E}(\lambda)] = \det[\mathbf{C}(\lambda)]\,D_S(\lambda)\,, \tag{49}$$

using (24), since $\mathbf{E}(\lambda)$ is the Evans matrix (23). Let $\mathcal{C}(\lambda) = \det[\mathbf{C}(\lambda)]$. Then, $\mathcal{C}(\lambda)$ is a non-zero complex parameter and $D_{AGJ}(\lambda) = \mathcal{C}(\lambda)D_S(\lambda)$, completing the proof of the theorem.

4. Analyticity of the Evans function

The expression (49) raises an interesting question about analyticity. It is defined for all $\lambda \in \Lambda$ and this set – by hypothesis – excludes those values of λ where elements in the set $\{\mu_1(\lambda), \ldots, \mu_k(\lambda)\}$ coalesce. Coalescence of μ-eigenvalues produces branch points in the complex λ-plane which would appear to shrink the region of analyticity of the Evans function. However, in [2], the k-form $Y_+(x, \lambda)$ is introduced as a solution of the first differential equation in (29), which decays with maximal rate σ_+ if $x \to \infty$. Since σ_+ is a simple eigenvalue of $\lim_{x\to\infty} A^{(k)}(x, \lambda)$, this can be done in an analytic way, even if some elements in the set $\{\mu_1(\lambda), \ldots, \mu_k(\lambda)\}$ coalesce. Similarly, $Y_-(x, \lambda)$ is defined as a solution of the second differential equation in (29), analytic in λ, which decays with maximal rate σ_-, if $x \to -\infty$. In this way, the Evans function $D_{AGJ}(\lambda)$ is defined and analytic on a set Λ^+, which includes the set Λ and contains values of λ associated with coalescing eigenvalues μ_j as well.

So a natural question is, can the Evans function $D_S(\lambda)$ be continued in an analytic way to the set Λ^+? Since $D_{AGJ}(\lambda) = \mathcal{C}(\lambda) D_S(\lambda)$ for $\lambda \in \Lambda$, the product $\mathcal{C}(\lambda) D_S(\lambda)$ can be continued analytically to the set Λ^+. However, this does not imply that the individual functions $\mathcal{C}(\lambda)$ and $D_S(\lambda)$ can be continued analytically. On the other hand, the functions $Y_+(x, \lambda)$ and $Y_-(x, \lambda)$ are decomposable on the larger set Λ^+, implying that $\star\mathcal{Y}_-(x, \lambda)$ is also decomposable on the set Λ^+. An Evans matrix can be defined for the decomposable k-forms $Y_+(x, \lambda)$ and $\star\mathcal{Y}_-(x, \lambda)$. However, this raises again the question considered in §3: given an arbitrary decomposable k-form which depends analytically on λ for some set Λ^+, can a k-dimensional basis be recovered, with each basis vector analytic, for all $\lambda \in \Lambda^+$? We conjecture that this is true. If so then the Evans matrix can be analytically continued to all of Λ^+.

5. Concluding remarks

In summary, the main technical devices used were the Hodge star operator and the induced inner product on $\bigwedge^k(\mathbb{C}^n)$. Introduction of the Hodge theory implicitly includes the hypothesis that the space on which (1) is defined is an orientable inner product space; Hodge star then provides the natural isomorphism between $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{n-k}(\mathbb{C}^n)$. The inner product $[\![\cdot, \cdot]\!]_k$ provides a framework for Defining adjoint multilinear operators and shows that the determinant (24) can be characterised as an inner product.

In addition to its fundamental interest, the equivalence between the two forms of the Evans function (16) and (24) is of interest because some analysis is easier in one setting or the other. For example, the form (16) of the Evans function has important theoretical advantages and forms the basis for a topological framework for the analysis of (1) (cf. [2]). On the other hand, numerical analysis of the Evans function has advantages in the setting of (24) (cf. [3]), and the Evans matrix is a natural setting for developing derivative formulas (cf. [13]).

In [13], the Evans framework is generalised further in two directions. First $m_u(\Lambda) + m_s(\Lambda)$ can be less than n, allowing for purely imaginary eigenvalues of $\mathbf{A}_\infty(\lambda)$. This is of interest for systems with symmetry which give rise to zero eigenvalues in the spectrum of $\mathbf{A}_\infty(\lambda)$. Secondly, when the system is symplectic or multi-symplectic it is shown in [13] that the Evans matrix becomes the *symplectic Evans matrix*, where each entry of the Evans matrix corresponds to a restricted symplectic form.

References

[1] J. W. Evans, Indiana Univ. Math. J. **24** (1975) 1169-1190.

[2] J. Alexander, R. Gardner & C. K. R. T. Jones, J. reine Angew. Math. **410** (1990) 167-212.

[3] J. Swinton, Phys. Lett. A **163** (1992) 57-62.

[4] W. A. Coppel, *Stability and Asymptotic Behaviour of Differential Equations* (D. C. Heath & Co: Boston, 1965).

[5] M. S. P. Eastham, *The Asymptotic Solution of Linear Differential Systems* (Clarendon Press: Oxford, 1989).

[6] R. O. Wells, *Differential Analysis on Complex Manifolds* (Springer-Verlag: New York, 1979).

[7] R. W. R. DARLING, *Differential Forms and Connections*, (CUP: Cambridge, 1994).

[8] M. CRAMPIN & F. A. E. PIRANI, *Applicable Differential Geometry* (CUP: Cambridge, 1986).

[9] D. LONDON Lin. & Mult. Alg. **4** (1976) 179-189.

[10] M. MARCUS & H. ROBINSON, Lin. Algeb. & Appl. **10** (1975) 85-87.

[11] M. MARCUS, *Finite Dimensional Multilinear Algebra, Part II* (Marcel Dekker: New York, 1975).

[12] T. KATO, *Perturbation Theory for Linear Operators* (Springer-Verlag: Berlin, Second Edition, 1976).

[13] T. J. BRIDGES & G. DERKS, The symplectic Evans matrix, and the instability of solitary waves and fronts with symmetry, Preprint (1998).