THE SYMPLECTIC EVANS MATRIX AND SOLITARY WAVE INSTABILITY

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Many models for physical phenomena in oceanography, atmospheric dynamics, optical fibre transmission, nerve conduction, acoustical and gas dynamic flows are conservative translation-invariant evolution equations with a Hamiltonian structure. Solitary waves and fronts form an important class of solutions of such equations and the calculus of variations, critical point theory and symplectic structure have played a major role in the analysis of their stability and instability. For example, the characterisation of solitary waves as critical points of the Hamiltonian (energy) constrained to level sets of the momentum (or momentum and other constants of motion) leads to a powerful framework for proving nonlinear Lyapunov stability – when the second variation, evaluated at the constrained critical point, has a finite number of negative eigenvalues (e.g. Benjamin[2], Bona[3], Holm et al[14], Grillakis et al[12,13], Maddocks & Sachs[16] and references therein).

However, for many Hamiltonian evolution equations, particularly coupled systems of PDEs, even though the characterisation of a solitary wave or front solution as a constrained critical point is well-defined, the second variation is strongly indefinite and the relation between critical point type and stability is lost. In this case, an important first step is to study the linear stability and instability, that is, analyse the spectral problem associated with the linearisation about the solitary wave or front solution.

A dynamical systems approach for the analysis of spectral problems associated with the linearisation about a solitary wave or front was first introduced by Evans[10] in the context of the stability analysis of nerve impulses in mathematical biology. The Evans function framework was substantially generalised by Alexander, Gardner & Jones[1] to apply to a large class of parabolic PDEs. For more recent results and generalisations see e.g. Gardner & Zumbrun[11], Kapitula & Sandstede[15] and references therein.

Central to the Evans theory is the *Evans function*, $D(\lambda)$, a complex analytic function of the spectral parameter $\lambda \in \mathbb{C}$. Under suitable hypotheses, the Evans function has the property that, if $\lambda_0 \in \mathbb{C}$ has positive real part and $D(\lambda_0) = 0$, then λ_0 is an unstable eigenvalue associated with the linearisation about a solitary wave. One way to prove the existence of such unstable eigenvalues is to study the sign of $D(\lambda)$ for λ real when λ is near zero and when λ is large. When the initial-value problem for the PDE is well-posed one can expect that when λ is real and large, $D(\lambda)$ will be of one sign; that is, there would not exist unstable eigenvalues with arbitrarily large growth rate. Assume $D(\lambda) > 0$ for λ large; then a negative sign of the slope of the Evans function for λ near zero can be used to predict the existence of unstable eigenvalues along the real λ axis. The Hamiltonian setting provides a geometrical framework and therefore one can expect to get explicit information about the derivatives of $D(\lambda)$ near $\lambda = 0$ in this setting.

The connection between the Evans function framework and the stability analysis of solitary wave solutions of Hamiltonian evolution equations was first studied by PEGO & WEINSTEIN[17]. For three particular Hamiltonian PDEs they obtained the result that $D(\lambda)$ satisfies $D(0) = D'(0) = 0$, $\operatorname{sign} D''(0) = \operatorname{sign} \frac{\mathrm{d}I}{\mathrm{d}c}$, and $D(\lambda) \to 1$ as $\lambda \to +\infty$ along the real axis. In here, I is the value of the momentum level set and c is the speed of the solitary wave. The system of ODEs associated with the spectral problem had no special structure, requiring explicit calculations in parts of the proof and limiting application to the particular PDEs studied where the solitary wave was known explicitly.

The primary difficulty with an abstract Evans function framework for Hamiltonian evolution equations is that the classical Hamiltonian formulation provides a symplectic structure for time evolution, but much of the analysis of the Evans function is associated with a dynamical system in the x-variable. In BRIDGES & DERKS[7,8] an abstract formulation of the Evans function for Hamiltonian PDEs was proposed based on a multi-symplectic formulation of the PDE, where distinct symplectic operators are assigned for time and space. To be precise, a Hamiltonian system on a multi-symplectic structure will be written in the canonical form

$$\mathbf{M}Z_t + \mathbf{K}Z_x = \nabla S(Z)\,, \quad Z \in \mathbb{R}^{2n}\,, \quad x \in \mathbb{R}, \quad t \geq 0\,, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are skew-symmetric constant matrices, $S : \mathbb{R}^{2n} \to \mathbb{R}$ is some smooth function and ∇ is a gradient with respect to the standard inner product on $\mathbb{R}^{2n}$.

To demonstrate the multi-symplectification of a Hamiltonian PDE, we consider the (good) Boussinesq equation[4]:

$$u_{tt} = (f(u) - u_{xx})_{xx}, \quad x \in \mathbb{R},\, t > 0\,, \tag{2}$$

where $u(x,t)$ is a real-valued function of x and t and $f(\cdot)$ is some smooth real-valued function. This system can be written as a classical Hamiltonian system on an infinite-dimensional phase space. For example, let $q = (w, u)^T$ with $w_{xx} = u_t$, then (2) can be reformulated as

$$q_t = \mathbf{J}\frac{\delta H}{\delta q}, \text{ with } \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } H(w, u) = \frac{1}{2}\int (w_x^2 + u_x^2 + f(u))\, dx. \quad (3)$$

However, the phase space is infinite-dimensional and the spatial symplecticity is not explicit in this formulation, but it is implicit in the Hamiltonian function. To formulate this PDE as a multi-symplectic system on a *finite-dimensional phase space*, one can take the Legendre transform of the Hamiltonian function (3). With $v = u_x$ and $z = w_x$ the form (1) is recovered by taking

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where $Z = (u, v, w, z)^T$ and $S(Z) = \frac{1}{2}(u^2 - v^2 - z^2) - \frac{1}{3}u^3$.

Many models of physical phenomena, particularly in atmospheric dynamics and optics, are equivariant with respect to a Lie group symmetry. So it is assumed that the system (1) is equivariant with respect to a q-dimensional Abelian subgroup of the Euclidean group acting on $\mathbb{R}^{2n}$, denoted by $\mathcal{G}$, *as well as with respect to spatial translations.* The generators of the group $\mathcal{G}$ are spanned by $\xi_1, \ldots, \xi_q$. According to classical Noether theory for symplectic systems, the symplectic flow of a group generates an invariant function. However, in the multisymplectic setting, there is a flow associated with each symplectic structure which generates a family of functions (cf. BRIDGES[5]). Hence for each generator ξ_i, $i = 1, \ldots, q$, there are functionals P_i and Q_i such that $\mathbf{M}\xi_i(Z) = \nabla P_i(Z)$ and $\mathbf{K}\xi_i(Z) = \nabla Q_i(Z)$.

It is natural to include the symmetries in the definition of the solitary waves or fronts, i.e., a solitary wave/front is a solution of (1) of the form

$$Z(x, t) = G_{\theta(x,t)}\widetilde{Z}(x - ct),$$

where G_θ is the action $\mathcal{G}$ on $\mathbb{R}^{2n}$ and $\theta(x, t) = (a_1 t + b_1 x, \ldots, a_q t + b_q x)$. Substitution in the multi-symplectic framework (1) shows that the shape $\widetilde{Z}$ is a homoclinic or heteroclinic orbit of the Hamiltonian ODE

$$(\mathbf{K} - c\mathbf{M})\widetilde{Z}_x = \nabla V(\widetilde{Z}), \quad (4)$$

where $V(Z) = [S - \sum_{i=1}^{q}(a_i P_i + b_i Q_i)](Z)$. The shape of the wave $\widetilde{Z}$ is biasymptotic to an invariant manifold at infinity which is the $\mathcal{G}$-group orbit of a fixed point of (4).

The linearisation of (1) about this solitary wave reduces to a linear ODE of the form

$$U_x = \mathbf{A}(x,\lambda;\mathbf{p})\,U\,, \quad U \in \mathbb{C}^{2n}\,, \quad \lambda \in \mathbb{C}, \tag{5}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, $\mathbf{p}$ represents parameters a_i, b_i and c and $\mathbf{A}(x,\lambda;\mathbf{p}) := (\mathbf{K} - c\mathbf{M})^{-1}[D^2V(\widetilde{Z}(x;\mathbf{p})) - \lambda\,\mathbf{M}]$. Central to the Evans function theory are the systems at infinity, defined by

$$\mathbf{A}^{\pm}(\lambda;\mathbf{p}) = \lim_{x\to\pm\infty} \mathbf{A}(x,\lambda;\mathbf{p})\,.$$

Associated with this parameter dependent matrix are the subspaces

$$E^s_{\pm}(\lambda;\mathbf{p}) = \{\,\xi \in \mathbb{C}^{2n} \;:\; \lim_{x\to+\infty} e^{\mathbf{A}^{\pm}(\lambda;\mathbf{p})x}\xi = 0\,\}\,, \qquad \lambda \in \mathbb{C} \tag{6}$$

$$E^u_{\pm}(\lambda;\mathbf{p}) = \{\,\xi \in \mathbb{C}^{2n} \;:\; \lim_{x\to-\infty} e^{\mathbf{A}^{\pm}(\lambda;\mathbf{p})x}\xi = 0\,\}\,, \qquad \lambda \in \mathbb{C}, \tag{7}$$

and $E^c_{\pm}(\lambda;\mathbf{p})$, which is defined to be a complement of $E^s_{\pm}(\lambda;\mathbf{p}) \oplus E^u_{\pm}(\lambda;\mathbf{p})$ in $\mathbb{C}^{2n}$. For definiteness, the following properties on the dimension of the systems at infinity are taken: for fixed values of the parameters $\mathbf{p}$,

$$\dim E^s_{\pm}(0;\mathbf{p}) = \dim E^u_{\pm}(0;\mathbf{p}) = 1\,,$$

(hence $\dim E^c_{\pm}(0;\mathbf{p}) = 2n-2$) and, when $\lambda \neq 0$, there is some $1 \le p \le n$ such that

$$\min\{\,\dim E^s_{\pm}(\lambda;\mathbf{p}), \dim E^u_{\pm}(\lambda;\mathbf{p})\,\} = p\,,$$

for all $\lambda \in \Lambda$, where Λ is subset of $\mathbb{C}_+$, the complex half-plane with positive real part and $0 \in \overline{\Lambda}$. The symplectic structure forces the dimensions of $E^s_{\pm}(0;\mathbf{p})$ and $E^u_{\pm}(0;\mathbf{p})$ to be equal. The property $\dim E^{s,u}_{\pm}(0;\mathbf{p}) = 1$ is not essential and many of the results, such as the construction and definition of the symplectic Evans matrix, are independent of this property.

With these hypotheses on the systems at infinity the Evans function takes the geometric form $D(\lambda;\mathbf{p}) = \det(\mathbf{E}(\lambda;\mathbf{p}))$, where $\mathbf{E}(\lambda;\mathbf{p})$ is the $p \times p$ *symplectic Evans matrix.* Each entry of $\mathbf{E}(\lambda;\mathbf{p})$ is an Ω-symplectic form restricted to a pair of solutions of the linearised stability problem and its adjoint, where Ω is the symplectic form associated with $(\mathbf{K} - c\mathbf{M})$. Hodge duality[6] is the key to transforming the exterior-algebra definition of the Evans function[1] to the Evans matrix. It is follows that $D(0;\mathbf{p}) = \frac{\partial}{\partial\lambda}\big|_{\lambda=0} D(\lambda;\mathbf{p}) = 0$ and $\frac{\partial^2}{\partial\lambda^2}\Big|_{\lambda=0} D(\lambda;\mathbf{p})$ satisfies

$$\operatorname{sign} D''(0) = \Pi \operatorname{sign}\left(\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}c} - \mathcal{B}(c)\right)\,, \tag{8}$$

where $\Pi = \pm 1$ is a geometric sign associated with the shape of the wave, $\mathcal{I}(\widetilde{Z}) = \int_{-\infty}^{\infty} \langle M\widetilde{Z}_x, \widetilde{Z}\rangle \, \mathrm{d}x$ and $\mathcal{B}(c)$ is associated with the properties of the nonconstant manifold of states at infinity (for example, $\mathcal{B}(c) = 0$ for classical solitary waves). All these results combine to give a general instability criterion for a large class of solitary waves and fronts as stated in Theorem 1.

Theorem 1 *For fixed* $\mathbf{p}$*, let* $\lambda_\infty \in \Lambda \cap \mathbb{R}$ *be a positive value of* λ *and let* $d_\infty(\mathbf{p}) = D(\lambda_\infty; \mathbf{p})$*. Define* $Z_0^\pm(\mathbf{p}) = \lim_{x\to\pm\infty} \widetilde{Z}(x, \mathbf{p})$ *and*

$$\chi_{00}^-(\mathbf{p}) = \lim_{x\to\infty} [\, e^{2\beta(\mathbf{p})x} \Omega(\widetilde{Z}_x(-x; \mathbf{p}), DG_\gamma(Z_0^-(\mathbf{p}))^T \widetilde{Z}_x(x; \mathbf{p}))\,]^{-1}.$$

If

$$d_\infty(\mathbf{p})\, \chi_{00}^-(\mathbf{p}) \, \left[\tfrac{\mathrm{d}}{\mathrm{d}c}\mathcal{I}(\widetilde{Z}(x; \mathbf{p})) - \tfrac{1}{2}\,\omega(Z_0^+(\mathbf{p}), \partial_c Z_0^+(\mathbf{p}))\right] < 0,$$

then the solitary wave or front $G_{\theta(x,t)}\widetilde{Z}(x - ct; \mathbf{p})$ *is linearly spectrally unstable.*

Illustrations of the theory can be found for many examples: a generalised Korteweg-de Vries model from fluid mechanics[7], a Boussinesq model from oceanography[8], a class of nonlinear Schrödinger equations (both coupled and uncoupled) from optics[8], a complex nonlinear Klein-Gordon equation from atmospheric dynamics[8] and a generalised Kawahara equation from plasma waves[9].

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