A stability criterion for the non-linear wave equation with spatial inhomogeneity

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Abstract

In this paper the non-linear wave equation with a spatial inhomogeneity is considered. The inhomogeneity splits the unbounded spatial domain into three or more intervals, on each of which the non-linear wave equation is homogeneous. In such setting, there often exist multiple stationary fronts. In this paper we present a necessary and sufficient stability criterion in terms of the length of the middle interval(s) and the energy associated with the front in these interval(s). To prove this criterion, it is shown that critical points of length function and zeros of the linearisation have the same order. The criterion is illustrated with an example which shows the existence of bi-stability: two stable fronts, one of which is non-monotonic.

1 Introduction

The non-linear wave equation (sometimes called the non-linear Klein-Gordon equation)

$$u_{tt} = u_{xx} + V'(u), \quad t > 0, \quad x \in \mathbb{R},$$

models various systems. For instance, taking $V(u) = D(1 - \cos u)$, gives the sine-Gordon equation,

$$u_{tt} = u_{xx} - D \sin u,$$

which describes various physical and biological systems, including molecular systems, dislocation of crystals and DNA processes [1, 2, 4, 8, 9]. As an illustrative example: the sine-Gordon equation is a fundamental model for long Josephson junctions, two superconductors sandwiching a thin insulator [6, 7]. In the case of Josephson junctions, the factor D represents the Josephson tunnelling critical current. In an ideal uniform Josephson junction, this is a constant. But if there are magnetic variations, e.g. because of non-uniform thickness of the width of the insulator or if the insulator is comprised of materials with different magnetic properties next to each other, then the Josephson tunnelling critical current D will vary with the spatial variable x, leading to an inhomogeneous potential $V(u, x) = D(x)(1 - \cos u)$. If there is a defect in the form of a scratch or local thickening in the insulator then we model the inhomogeneity by a step function $D(x)$, with $D = 1$ outside the defect and $D \neq 1$ inside, see [5] and references therein.

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To make it easier to generalise this example to more general wave equations, we write the model for a Josephson junction with one defect as an equation on the disjoint open intervals I_l, I_m and I_r ($\mathbb{R} = \overline{\cup I_i}$):

$$u_{tt} = u_{xx} + \frac{\partial}{\partial u} V(u, x; I_l, I_m, I_r) - \alpha u_t. \tag{1}$$

Here $\alpha \geq 0$ is a constant damping coefficient and the potential $V(u, x; I_l, I_m, I_r)$ consists of three smooth (C^3) functions $V_i(u)$, defined on three disjoint open intervals I_i of the real spatial axis, such that $\mathbb{R} = \overline{\cup I_i}$. Without loss of generality, we can write $I_l = (-\infty, -L)$, $I_m = (-L, L)$ and $I_r = (L, \infty)$ for some L, by translating the x variable so that the origin occurs at the centre of the middle interval. So, the length of the inhomogeneity or 'defect' is $2L$. In this paper we study this equation in its generality, as well as the motivating example of Josephson junctions.

The existence and stability of stationary fronts or solitary waves (from now on we shall simply refer to both of these as stationary fronts) of (1) is studied in [3, 5]. In particular, spectral stability of the stationary fronts (which in this case implies non-linear stability, see [5] and references therein) is studied and a necessary and sufficient criterion is developed for the spectral operator to have an eigenvalue zero. The spectral operator can be related to a self-adjoint linearisation operator such that λ is an eigenvalue of the spectral operator if and only if $\Lambda = \lambda(\lambda + \alpha)$ is an eigenvalue of self-adjoint operator. The self-adjoint operator has real eigenvalues and a continuous spectrum on the negative half line bounded away from zero. So as the largest eigenvalue passes through zero a change of stability occurs. The self-adjoint operator is also a Sturm-Liouville operator, which implies that the discrete eigenvalues are simple and bounded above. It also means that the eigenfunction associated with the largest discrete eigenvalue will have no zeroes, providing a tool to identify the largest eigenvalue. In [5] a necessary and sufficient condition for the existence of an eigenvalue zero is derived, but this is only a necessary condition for a change of stability. The proof of the Theorem in [5] contains the construction of the eigenfunction and hence the absence of zeroes can be checked to verify that the eigenvalue zero is the largest eigenvalue. But to guarantee a change of stability, it also needs verification that the largest eigenvalue crosses through zero. In this paper we will consider this crossing question and derive a necessary and sufficient condition for the change of stability of a stationary front.

As this paper builds on [5], we start with a brief introduction to the relevant notation and results from this paper. Stationary fronts of (1) are solutions of the Hamiltonian ODE

$$0 = u_{xx} + \frac{\partial}{\partial u} V(u, x; I_l, I_m, I_r),$$

which satisfy $u_x(x) \to 0$ for $x \to \pm\infty$. We introduce the notation $p = u_x$, then the Hamiltonian $H(u,p) = \frac{1}{2}p^2 + V(u, x; I_l, I_m, I_r)$ is constant on each of the intervals I_i, $i = l, m, r$. In the middle interval, we define the Hamiltonian parameter $g = \frac{1}{2}p^2 + V_m(u)$ for $x \in I_m$. Writing u_l and u_r for the values of u at the boundaries between the different intervals ($x = -L$ and $x = L$) and p_l and p_r for the values of p at the same values of x, the matching conditions at the boundaries give that u_i and p_i can be parametrised by g:

$$\begin{array}{rcccl} \frac{1}{2}p_l^2 & = & g - V_m(u_l) & = & V_- - V_l(u_l); \\ \frac{1}{2}{p_r}^2 & = & g - V_m(u_r) & = & V_+ - V_r(u_r). \end{array} \tag{2}$$

Here V_- and V_+ are the asymptotic values of $V_l(u(x))$ and $V_r(u(x))$ respectively, that is $V_- := \lim_{x\to-\infty} V_l(u(x))$ and $V_+ := \lim_{x\to\infty} V_r(u(x))$. These limits are well defined as $u(x)$

is a front. The relations in (2) might lead to multi-valued functions u_l, etc. However, usually they are locally well-defined, except potentially at some isolated bifurcation points. To find those, we define the following bifurcation functions

$$\begin{aligned}\mathscr{B}_l(g) &= p_l(g)[V_m'(u_l(g)) - V_l'(u_l(g))],\\ \mathscr{B}_r(g) &= p_r(g)[V_r'(u_r(g)) - V_m'(u_r(g))].\end{aligned}$$

A bifurcation occurs if g satisfies $\mathscr{B}_l(g) = 0$ or $\mathscr{B}_r(g) = 0$. We define the bifurcation values g_{bif} to be any value of g (if it exists) where $\mathscr{B}_l(g) = 0$ or $\mathscr{B}_r(g) = 0$. These points are usually associated with the edge of the existence interval.

Finally, the fundamental theorem of calculus enables the length of the middle interval to be parametrised by the Hamiltonian parameter, g. For instance, if $u_x(x; g)$ has no zeroes in the middle interval then

$$2L(g) = \int_{u_l(g)}^{u_r(g)} \frac{du}{p(u, g)}$$

where $u_l(g)$ and $u_r(g)$ are the values of u where the front $u(x; g)$ enters resp. leaves the middle interval. If $u_x(x; g)$ has zeroes in the middle interval than the expression for $L(g)$ is more complicated but can still be expressed explicitly, see [5].

Now we can formulate the necessary and sufficient condition for the existence of an eigenvalue zero from [5].

Theorem 1.1 ([5, Theorem 4.5]). *Let the front $u_f(x; g)$ be a solution of (1), such that all zeroes of $\partial_x u_f(x; g)$ are simple and the length of the middle interval of $u_f(x; g)$ is part of a smooth length curve $L(g)$. The linearisation operator $\mathcal{L}(g) := D_{xx} + \frac{\partial^2}{\partial u^2} V(u_f(x; g), x; g)$, $x \in \cup I_i$, associated with $u_f(x; g)$ has an eigenvalue zero (with eigenfunction in $H^2(\mathbb{R})$) if and only if*

$$\mathscr{B}_l(g)\, \mathscr{B}_r(g) L'(g) = 0. \tag{3}$$

If $g \neq g_{\text{bif}}$ then there is an eigenvalue zero (with eigenfunction in $H^2(\mathbb{R})$) if and only if $L'(g) = 0$.

As the linear operator $\mathcal{L}(g)$ is a Sturm-Liouville operator, the eigenvalues are simple and bounded above, meaning that away from the bifurcation points, the curves of eigenvalues, $\Lambda(g)$ are well-defined and continuous. Thus if there exists an eigenvalue zero for $g = g_0 \neq g_{\text{bif}}$, and the associated eigenfunction $\Psi(x)$ has no zeroes (is constructed in the proof of Theorem 1.1, see [5]), then a change of stability (both spectral and non-linear) *may* occur as g passes from one side of g_0 to the other. Whether or not the change of stability will *actually* happen, depends on the degree of the zero of the largest eigenvalue $\Lambda(g)$ at $g = g_0$. Specifically, the largest eigenvalue may be strictly negative, touch zero for some value of g and then become strictly negative again, i.e. the eigenvalue $\Lambda(g)$ may have a non-simple zero and the front is stable for all g, see the left panel of Figure 1. In the examples in [5], no eigenvalues of this type were encountered; in all cases, the presence of an eigenvalue zero also led to a change of the sign in this eigenvalue. Thus the eigenvalue, as a function of g, locally looked like the right panel of Figure 1. This suggested that this would always be the case for the inhomogeneous non-linear wave equation (1). However this suggestion is false.

In the next section we shall present an example where the length function $L(g)$ has an inflection point. From Theorem 1.1 it follows that an inflection point of the length function $L(g)$ must correspond to an eigenvalue zero. We will show that in this example the inflection point of $L(g)$ corresponds to a non-simple root of the eigenvalue $\Lambda(g)$ and that a change

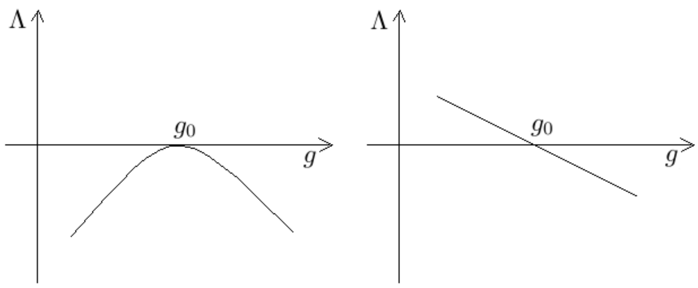

Figure 1: Two possible local behaviours for the eigenvalue $\Lambda(g)$ when $\Lambda(g_0) = 0$. In the left panel the eigenvalue $\Lambda(g_0)$ has a second order zero.

of stability does not occur as g moves from one side of g_0 to the other, see Figure 2. This result is generalised in section 3 leading to a necessary and sufficient condition for a change of stability. In section 4, we consider the wave equation with multiple middle intervals. It will be shown that the criterion for a wave equation with one middle interval can be used to derive a sufficient and necessary condition for the change of stability in such systems.

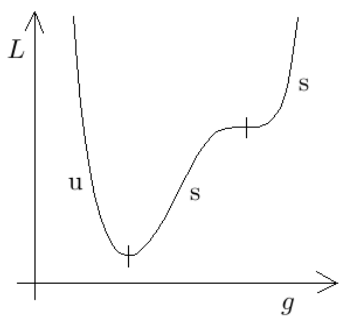

Figure 2: A length curve with an inflection point and an eigenvalue with a non-simple root (see left panel of Figure 1). The symbol 's' denotes a stable branch whilst the symbol 'u' denotes an unstable branch.

2 Motivating Example

In this section we present an example in which the potential in the inhomogeneous wave equation (1) is such that the length curve $L(g)$ has an inflection point at which no change of stability occurs. This example involves a one parameter family of potentials, leading to a two parameter curve of length curves (the Hamiltonian parameter g and the parameter for the potentials). We will use the full family of examples to make inferences about the stability of various branches of $L(g)$ and hence to show that the inflection point in the length curve does not correspond to a change of stability. Finally we illustrate the inferences that we made by confirming the prediction that there is a region in the potential parameter space for which there is bi-stable behaviour.

The example we consider has the sine-Gordon potential (with induced current and dissipation) in the left and right intervals ($|x| > L$) and has a potential of the form $V_m(u) = \frac{k}{2}(u - 2\pi - c)^2$ in the middle interval ($|x| < L$), where k and c are parameters. That is, we consider stationary fronts of the following equation

$$u_{tt} = u_{xx} + \frac{\partial V(u, x; L)}{\partial u} - \alpha u_t \quad \text{where} \quad V(u, x; L) := \begin{cases} \cos(u) + \gamma u, & |x| > L; \\ \frac{k}{2}(u - 2\pi - c)^2, & |x| < L. \end{cases} \tag{4}$$

The stationary fronts join the steady states $\arcsin(\gamma)$ as $x \to -\infty$ and $2\pi + \arcsin(\gamma)$ as $x \to \infty$. In Figure 3, typical length curves $L(g)$ are plotted for various values of c between 0 and -2.8, while γ and k are kept fixed at $\gamma = 0.1$ and $k = 1$. Recall that g represents the Hamiltonian in the middle interval, i.e., $g = \frac{1}{2}(u_x^2 + k(u - 2\pi - c)^2)$. The value of α is not relevant for the existence of stationary fronts nor for the length curve. It only comes into play once the (linear) stability of the stationary fronts is considered. However, for all values of $\alpha \geq 0$, the front is either always stable or always unstable. For $\alpha = 0$, the stable fronts are neutrally stable (purely imaginary eigenvalues in the linearisation). While for $\alpha > 0$, the same fronts are now attractors. Figure 3 illustrates that there are qualitative changes in the

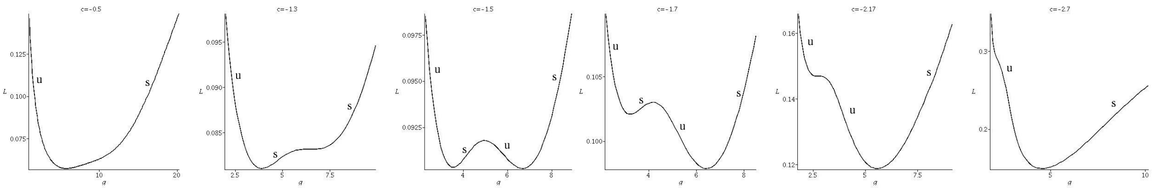

Figure 3: Typical changes in the length curve when $k = 1$ and $\gamma = 0.1$ and c is decreased (left to right) from $c = -0.5$ to $c = -2.7$. The symbol 's' denotes a stable branch whilst the symbol 'u' denotes an unstable branch.

length curves when c is varied. This qualitative change effects the number of turning points of $L(g)$, changing from one to three and back to one again as c is varied. We show this sketch for $k = 1$ as it provides the full picture of what happens as c is varied. For smaller k some of these panels (right most ones) are not seen due to a restricted g existence interval.

We have included the stability of the various branches of $L(g)$ in Figure 3. The stability of the two branches for smallest and largest values of c are found by numerical simulations with $\alpha = 0.1$ for one solution on each of the branches. Theorem 1.1 gives that an eigenvalue zero can only occur at turning or inflection points of $L(g)$, hence on one branch the stability can not change. The stability profiles for the other panels are then inferred from these two panels, using the fact that V is continuous in c, hence L will be continuous in c. For example, to get the stability of the branches for the second image from the left, note that the inflection point occurs away from the turning point so the stability of the branches in the locale of the turning point cannot change (no eigenvalues have crossed zero for g in the neighbourhood of the turning point). Furthermore, the stability at the outer parts of the branches (largest and smallest values of g) can not change as $L'(g) \neq 0$ at the outer parts. Therefore the only possibility for the stability of the branches is as shown in this second panel and hence the stability doesn't change at the inflection point (as g is varied and c is kept fixed). A similar argument can be used for the second image on the right. Now the stability in the middle images follows from matching the outer panels on the left and right. Going from the second to the third image from the left, continuity does not provide any conclusion about the stability of the third branch in first instance. Similarly we don't know the stability of the second branch in the third picture on the right. However, by comparing these two images and using continuity, we can conclude the stability of these branches.

To illustrate the inferences about the stability of the various branches, we observe that Figure 3 suggests that there are values of c with two stable stationary fronts. It is easier to visualise this bi-stable behaviour for smaller values of k, so we focus on $k = 0.6$. Recall that this change in k has the effect of removing some of the panels (from the right) in Figure 3. For $k = 0.6$ and $\gamma = 0.1$, there is an inflection point on the right branch when $c = \widehat{c} \approx 0.75$. If $c = -2$ (and $k = 0.6$, $\gamma = 0.1$) then the length curve has three turning points, see Figure 4, and for $L = 0.26$ there are four stationary fronts, two of which are stable. In the left panel

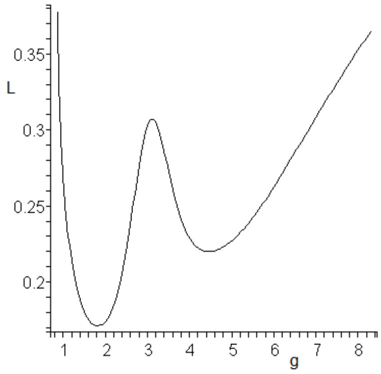

Figure 4: The curve $L(g)$ for $\gamma = 0.1$, $k = 0.6$ and $c = -2$ showing that there are four stationary fronts for $L = 0.26$.

of Figure 5 we show the four stationary fronts for $L = 0.26$. The red non-monotonic front corresponds to the right most branch in Figure 4. The fact that two of the fronts are stable is confirmed by simulating equation (4), starting with different initial conditions, see the middle and right panels of Figure 5. The initial conditions used are the (unique) stable front for a slightly larger length value ($L = 0.33$), which converges to the stable non-monotonic stationary front (right plot). And the other initial condition is the stationary front solution to the sine-Gordon equation, which converges to the other stable front (middle plot). It is interesting to see that the non-monotonic stationary front is indeed stable.

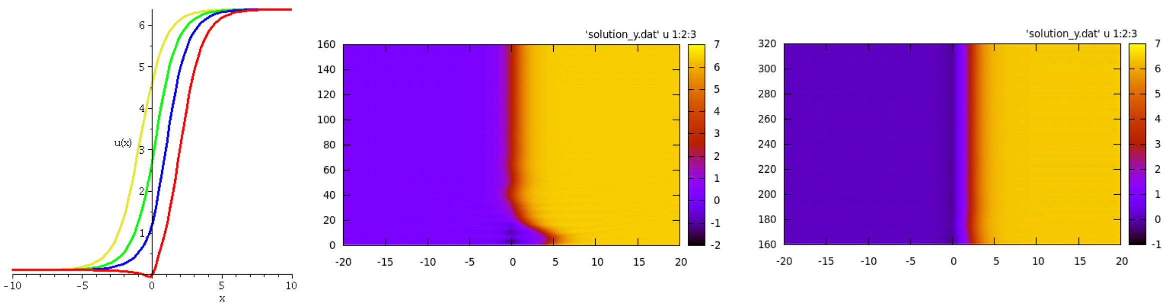

Figure 5: Left: the four stationary fronts for $\alpha = 0.1$, $k = 0.6$, $\gamma = 0.1$, $c = -2$ and $L = 0.26$. Middle and Right: Simulations of (4) converging to the two stable stationary fronts (second (yellow) and fourth (red) front in the left plot).

This simulation also confirms the inference that no change of stability occurs at the inflection point. In the next section we will show that this is generically true.

3 A Necessary and Sufficient Stability Criterion

In the example of the previous section we saw an inflection point of the length function $L(g)$ which corresponded to a non-simple zero of the eigenvalue function $\Lambda(g)$. That is: at the inflection point $g = \widehat{g}$, it holds that $L'(\widehat{g}) = 0 = \Lambda(\widehat{g})$ and also $L''(\widehat{g}) = 0 = \Lambda'(\widehat{g})$. Here we see a hint of the relationship between the order of a zero of $L'(g)$ and the order of the corresponding zero of $\Lambda(g)$. We make this explicit in the following lemma.

Lemma 3.1. *Away from the bifurcation point $g_{\rm bif}$, the order of a zero of $\Lambda(g)$ is the same as the order of the associated stationary point of $L(g)$. For instance if $L(g)$ has a stationary point at $g = g_0 \neq g_{\rm bif}$, with $L'(g_0) = 0 = L''(g_0)$ and $L'''(g_0) \neq 0$, then g_0 is a second order root of $\Lambda(g)$ (i.e., $\Lambda(g_0) = 0 = \Lambda'(g_0)$ and $\Lambda''(g_0) \neq 0$) and visa verse.*

Proof.
Let $L(g)$ have a stationary point at $g = g_0$, then Theorem 1.1 implies that the linearisation operator $\mathcal{L}$ associated with the front $u(x; g_0)$ has an eigenvalue $\Lambda(g_0) = 0$.

As a thought experiment (motivated by the previous section), consider smooth perturbations of the potential $V_m(u)$ which have the effect of perturbing the associated $L(g)$ curve. The perturbation may lead to a change in the number of stationary points of $L(g)$ (in the locality of g_0). As it is perturbations of the potentials which cause this change in $L(g)$, Theorem 1.1 still holds, meaning that the stationary points of $L(g)$ are still associated with zeroes of $\Lambda(g)$. Thus the original curve $\Lambda(g)$ will be smoothly perturbed in such a way that it has the same number of zeroes as the perturbed curve $L(g)$ has stationary points. Away from any bifurcation points ($g_{\rm bif}$) a smooth perturbation of $V_m(u)$ leads to a smooth perturbation of $L(g)$. If $L(g)$ has a turning point of first order at $g = g_0$, then any small smooth perturbation will lead to $L(g)$ still having exactly one turning point (in the locality of g_0). Similarly, if $\Lambda(g)$ has a simple zero (right panel in Figure 1) then any smooth perturbation to $\Lambda(g)$ will still (locally) have one zero. Whilst if $\Lambda(g)$ has a second order zero (left panel in Figure 1) a small smooth perturbation would generically result in either no zeroes of $\Lambda(g)$ or in two zeroes of $\Lambda(g)$, which is not consistent with a first order turning point of $L(g)$. Thus we can conclude that a first order turning point of $L(g)$ cannot be not associated with a second order zero of $\Lambda(g)$, and similarly, it cannot be associated with an even order zero of $\Lambda(g)$.

To make those ideas formal, we embed the middle potential $V_m(u)$ smoothly in a larger family $V_m(u, \epsilon)$, where ϵ is a small parameter and $V_m(u, 0) = V_m(u)$. For ϵ small and g near g_0, this embeds the family of fronts $u_f(x; g)$ in the family $u_f(x; g, \epsilon)$ with associated length curves $L(g, \epsilon)$ and the eigenvalue $\Lambda(g, \epsilon)$. As shown in [5], the length $L(g)$ is the sum of integrals of the form $\int_I \frac{du}{2\sqrt{g - V_m(u)}}$. Hence the embedding $V_m(u, \epsilon)$ can be chosen such that

$$\frac{\partial^2 L}{\partial\epsilon\partial g}(g_0, 0) \neq 0 \text{ and } \left\langle \frac{D}{D\epsilon}\left(\frac{\partial^2}{\partial u^2} V(u_f(g_0, \epsilon), \epsilon)\right)\bigg|_{\epsilon=0} \psi(g_0, 0)\, ,\, \psi(g_0, 0) \right\rangle_{L_2} \neq 0, \qquad (5)$$

implying that

$$\frac{\partial\Lambda}{\partial\epsilon}(g_0, 0) \neq 0.$$

To see this inequality, differentiating the eigenvalue equation $[\mathcal{L}(g_0, \epsilon) - \Lambda(g, \epsilon)]\psi(g_0, \epsilon) = 0$ with respect to ϵ, evaluating at $\epsilon = 0$, and taking the L_2 inner product with the eigenfunction $\psi(g_0, 0)$ shows that

$$\left\langle \left[\frac{D}{D\epsilon}\left(\frac{\partial^2}{\partial u^2} V(u_f(g_0, \epsilon), \epsilon)\right)\bigg|_{\epsilon=0} - \frac{\partial\Lambda}{\partial\epsilon}(g_0, 0)\right] \psi(g_0, 0)\, ,\, \psi(g_0, 0) \right\rangle_{L_2} = 0.$$

Hence the second inequality in (5) implies that $\frac{\partial\Lambda}{\partial\epsilon}(g_0, 0) \neq 0$.

From Theorem 1.1 we know that $\frac{\partial L}{\partial g}(g, \epsilon) = 0 \Leftrightarrow \Lambda(g, \epsilon) = 0$. To see that this implies that the order of the critical point of $L(g)$ and the order of the zero of $\Lambda(g)$ is the same at $g = g_0$, we start with showing the existence of a curve of critical points $g_L(\epsilon)$ with $\frac{\partial L}{\partial g}(g_L(\epsilon), \epsilon) = 0$ and $g_L(0) = g_0$.

First we consider the non-degenerate case: if $\frac{\partial L}{\partial g}(g_0,0)=0$ and $\frac{\partial^2 L}{\partial g^2}(g_0,0)\neq 0$ then the Taylor series for $\frac{\partial L}{\partial g}(g,\epsilon)$ about $(g_0,0)$ is

$$\frac{\partial L}{\partial g}(g,\epsilon)=(g-g_0)\frac{\partial^2 L}{\partial g^2}(g_0,0)+\epsilon\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)+O\left(|\epsilon+(g-g_0)|^2\right)$$

Since $\frac{\partial^2 L}{\partial g^2}(g_0,0)\neq 0$, the implicit function theorem gives that there exist a unique curve $g_L(\epsilon)$ for ϵ near zero such that $\frac{\partial L}{\partial g}(g_L(\epsilon),\epsilon)=0$ and $g_L(\epsilon)=g_0-\epsilon\left(\frac{\partial^2 L}{\partial g^2}(g_0,0)\right)^{-1}\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)+O(\epsilon^2)$. Differentiating $\frac{\partial L}{\partial g}(g_L(\epsilon),\epsilon)=0$ with respect to ϵ and evaluating at $\epsilon=0$ gives

$$g_L'(0)=-\left(\frac{\partial^2 L}{\partial g^2}(g_0,0)\right)^{-1}\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)\neq 0.$$

Since $\frac{\partial L}{\partial g}(g_L(\epsilon),\epsilon)=0$, Theorem 1.1 gives that $\Lambda(g_L(\epsilon),\epsilon)=0$ too. Differentiating this expression with respect to ϵ and evaluating at $\epsilon=0$ shows

$$0=\frac{\partial\Lambda}{\partial g}(g_0,0)g_L'(0)+\frac{\partial\Lambda}{\partial\epsilon}(g_0,0),\ \text{hence}\ \frac{\partial\Lambda}{\partial g}(g_0,0)=\frac{\partial\Lambda}{\partial\epsilon}(g_0,0)\left(\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)\right)^{-1}\frac{\partial^2 L}{\partial g^2}(g_0,0)\neq 0.$$

And we can conclude that g_0 is a first order zero of the eigenvalue $\Lambda(g)$, hence a first order turning point of $L(g)$ is associated with a simple zero of $\Lambda(g)$.

Next we consider the higher order critical points of $L(g)$. Let n be the smallest value ($n\geq 2$) such that $\frac{\partial^{n+1}L}{\partial g^{n+1}}(g_0,0)\neq 0$, that is $\frac{\partial^k L}{\partial g^k}(g_0,0)=0$ for $k=1,\dots,n$. Then a Taylor series gives

$$\frac{\partial L}{\partial g}(g,\epsilon)=\frac{1}{n!}(g-g_0)^n\frac{\partial^{n+1}L}{\partial g^{n+1}}(g_0,0)+\epsilon\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)+O\left(\epsilon^2+\epsilon(g-g_0)+(g-g_0)^{n+1}\right).$$

In order to be able to apply the implicit function theorem and get a smooth curve of critical points g_L, we replace ϵ with the new variable η such that

$$\eta^n=-\epsilon\left(\frac{\partial^{n+1}L}{\partial g^{n+1}}(g_0,0)\right)^{-1}\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0).$$

If n is even, then we restrict ϵ to only positive or only negative values. However, η is allowed to have both positive and negative values. We define

$$\widetilde{L}(g,\eta)=L\left(g,-\eta^n\left(\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)\right)^{-1}\frac{\partial^{n+1}L}{\partial g^{n+1}}(g_0,0)\right).$$

Now the implicit function theorem gives that there exist a unique curve $g_L(\eta)$ for η near zero such that $\frac{\partial\widetilde{L}}{\partial g}(g_L(\eta),\eta)=0$ and $g_L(\eta)=g_0+\eta+O(\eta^2)$, hence $g_L'(0)=1$. Now we can proceed as before. As $0=\frac{\partial\widetilde{L}}{\partial g}(g_L(\eta),\eta)=\frac{\partial L}{\partial g}\left(g,-\eta^n\left(\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)\right)^{-1}\frac{\partial^{n+1}L}{\partial g^{n+1}}(g_0,0)\right)\Big|_{g=g_L(\eta)}$, Theorem 1.1 gives that $\Lambda\left(g_L(\eta),-\eta^n\left(\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)\right)^{-1}\frac{\partial^{n+1}L}{\partial g^{n+1}}(g_0,0)\right)=0$. Differentiating this expression n times with respect to η and evaluating at $\eta=0$, gives that for $k=1,\dots,n-1$

$$\frac{\partial^k\Lambda}{\partial g^k}(g_0,0)=0\ \text{and}\ \frac{\partial^n\Lambda}{\partial g^n}(g_0,0)=n!\frac{\partial\Lambda}{\partial\epsilon}(g_0,0)\left(\frac{\partial^2 L}{\partial g\partial\epsilon}(g_0,0)\right)^{-1}\frac{\partial^{n+1}L}{\partial g^{n+1}}(g_0,0)\neq 0.$$

Thus we have shown that an n-th order critical point of $L(g)$ corresponds to a n-th order zero of $\Lambda(g)$. $\square$

Lemma 3.1 equates the order of zeroes of $L'(g)$ with the order of zeroes of $\Lambda(g)$. This, together with the results of [5] and Sturm-Liouville theory, gives a necessary and sufficient condition for a change of stability to occur for g away from any (existence) bifurcation point g_{bif}.

Theorem 3.2. *Let the front $u_f(x;g)$, $g \neq g_{\text{bif}}$, be a solution of* (1)*, such that all zeroes of $\partial_x u_f(x;g)$ are simple and the length of the middle interval of $u_f(x;g)$ is part of a smooth length curve $L(g)$. There is a change in stability of $u_f(x;g)$ at $g = \widehat{g} \neq g_{\text{bif}}$ if and only if*

i) $L'(\widehat{g}) = 0$;

ii) the order of the zero $\widehat{g}$ of $L'(g)$ is odd;

iii) the eigenfunction $\Psi(\widehat{g})$ has no zeroes.

If just i) and ii) are satisfied then the number of positive eigenvalues of the linearisation operator $\mathcal{L}(g)$ changes by one as g crosses $\widehat{g}$.

Proof.
As stated in Theorem 1.1, the results of [5] (in particular Theorem 4.5) gives that the linearisation operator $\mathcal{L}(g)$ has an eigenvalue zero for $g = \widehat{g}$ if and only if $L'(\widehat{g}) = 0$. From Lemma 3.1, we can now conclude that if the order of the zero $\widehat{g}$ of $L'(g)$ is odd then the sign of $\Lambda(g)$ changes as g passes from one side of $\widehat{g}$ to the other. That is, the number of positive eigenvalues of the linearisation operator $\mathcal{L}(g)$ changes by one. If the order of the zero $\widehat{g}$ of $L'(g)$ is even then there is the same number of positive eigenvalues either side of $\widehat{g}$. A change of stability occurs at an eigenvalue zero if it is the largest eigenvalue and a positive eigenvalue is lost/gained as g moves from one side of $\widehat{g}$ to the other. By Sturm-Liouville theory, an eigenvalue is the largest eigenvalue if and only if its associated eigenfunction $\Psi(\widehat{g})$ has no zeroes. □

Note that the eigenfunction $\Psi(\widehat{g})$ is constructed explicitly in [5], making it easy to check *iii)* and that non-linear stability can be concluded from linear stability.

4 Multiple Middle Intervals

So far we have considered an inhomogeneous wave equation with three intervals. A natural extension is an inhomogeneous wave equation with $N+2$ intervals, i.e.,

$$u_{tt} = u_{xx} + \frac{\partial}{\partial u} V(u, x; I_l, I_{m_1} \ldots, I_{m_N}, I_r) - \alpha u_t. \tag{6}$$

The potential $V(u, x; I_l, I_{m_1} \ldots, I_{m_N}, I_r)$ consists of $N+2$ smooth (C^3) functions $V_i(u)$, defined on $N+2$ disjoint open intervals I_i of the real spatial axis, such that $\mathbb{R} = \overline{\cup I_i}$. The N middle intervals have lengths $L_1, \ldots, L_N$ and associated Hamiltonians $g_1, \ldots, g_N$. In [5, Theorem 6.1], it is shown that, away from the existence bifurcation points, the linearisation about a front $u_f(g_1, \ldots, g_N)$ has an eigenvalue zero if and only if the determinant of the Jacobian $\frac{\partial(L_1,\ldots,L_N)}{\partial(g_1,\ldots,g_N)}$ vanishes. We will now derive a condition on the length functions that determines whether the eigenvalue zero is related to a change in the number of positive eigenvalues.

Let's first consider the case of two middle intervals, i.e., $N = 2$ and assume that there is some point $(\widehat{g}_1, \widehat{g}_2)$ away from the existence bifurcation points, such that the determinant of the Jacobian $\frac{\partial(L_1,L_2)}{\partial(g_1,g_2)}$ vanishes at $(g_1, g_2) = (\widehat{g}_1, \widehat{g}_2)$ and hence the linearisation about $u_f(\widehat{g}_1, \widehat{g}_2)$ has an eigenvalue zero. We define the values of the length functions at this point to be $L_1(\widehat{g}_1, \widehat{g}_2) = \widehat{L}_1$ and $L_2(\widehat{g}_1, \widehat{g}_2) = \widehat{L}_2$. In [5] it is shown that $\frac{\partial L_1}{\partial g_2}(g_1, g_2) = \frac{\partial L_2}{\partial g_1}(g_1, g_2) = \frac{1}{\mathcal{B}_1(g_1,g_2)} \neq 0$. Hence at $(g_1, g_2) = (\widehat{g}_1, \widehat{g}_2)$, the vanishing Jacobian implies that

$$\frac{\partial L_1}{\partial g_1}(\widehat{g}_1, \widehat{g}_2)\frac{\partial L_2}{\partial g_2}(\widehat{g}_1, \widehat{g}_2) = \frac{1}{\mathcal{B}_1(g_1, g_2)^2} \neq 0, \text{ thus both } \frac{\partial L_1}{\partial g_1}(\widehat{g}_1, \widehat{g}_2) \neq 0, \ \frac{\partial L_2}{\partial g_2}(\widehat{g}_1, \widehat{g}_2) \neq 0.$$

Looking at the equations $L_1(g_1, g_2) = \widehat{L}_1$ and $L_2(g_1, g_2) = \widehat{L}_2$, the inequalities above and the implicit function theorem give that there exist smooth curves $\widetilde{g}_1(g_2)$, for g_2 nearby $\widehat{g}_2$, and $\widetilde{g}_2(g_1)$, for g_1 nearby $\widehat{g}_1$, such that $L_1(\widetilde{g}_1(g_2), g_2) = \widehat{L}_1$, $\widetilde{g}_1(\widehat{g}_2) = \widehat{g}_1$ and $L_2(g_1, \widetilde{g}_2(g_1)) = \widehat{L}_2$, $\widetilde{g}_2(\widehat{g}_1) = \widehat{g}_2$. The associate wave fronts

$$u_f^1(g_1) := u_f(g_1, \widetilde{g}_2(g_1)) \text{ and } u_f^2(g_2) := u_f(\widetilde{g}_1(g_2), g_2)$$

solve the wave equation with one middle interval (with length $L = L_1 + L_2$ and middle potential V_m being the combination of V_{m_1} and V_{m_2}, each at their appropriate interval) and $u_f^1(\widehat{g}_1) = u_f(\widehat{g}_1, \widehat{g}_2) = u_f^2(\widehat{g}_2)$. So the results of the previous section can be used to determine whether or not the eigenvalue zero of the linearisation about the $u_f(\widehat{g}_1, \widehat{g}_2)$ signals a change in stability.

Corollary 4.1. *Let the front $u_f(x; g_1, g_2)$ be a solution of* (6) *with $N = 2$ (with (g_1, g_2) away from the existence bifurcation points), such that all zeroes of $\partial_x u_f(x; g_1, g_2)$ are simple and the lengths of the middle intervals of $u_f(x; g_1, g_2)$ form a smooth length surface $L(g_1, g_2)$. There is a change in stability of $u_f(x; g_1, g_2)$ at $(g_1, g_2) = (\widehat{g}_1, \widehat{g}_2)$ when varying g_1 or g_2 if and only if*

i) the determinant of the Jacobian $\det\left(\frac{\partial(L_1,L_2)}{\partial(g_1,g_2)}\right) = 0$, *implying that nearby $(\widehat{g}_1, \widehat{g}_2)$ there are curves $\widetilde{g}_1(g_2)$ and $\widetilde{g}_2(g_1)$ with $L_1(\widetilde{g}_1(g_2), g_2) = \widehat{L}_1$, $\widetilde{g}_1(\widehat{g}_2) = \widehat{g}_1$ and $L_2(g_1, \widetilde{g}_2(g_1)) = \widehat{L}_2$, $\widetilde{g}_2(\widehat{g}_1) = \widehat{g}_2$;*

ii) the order of the zero $\widehat{g}_1$ of $\frac{dL_1}{dg_1}(g_1, \widetilde{g}_2(g_1)))$ is odd or *the order of the zero $\widehat{g}_2$ of $\frac{dL_2}{dg_2}(\widetilde{g}_1(g_2), g_2)$ is odd;*

iii) the eigenfunction $\Psi(\widehat{g}_1, \widehat{g}_2)$ has no zeroes.

This result has been used implicitly in [5] to prove that the introduction of a defect in a 0-π Josephson junction can lead to the stabilisation of a *non-monotonic* stationary front (the fact that the largest eigenvalue becomes negative was verified numerically in [5]).

This result can be generalised to an arbitrary number of middle intervals. In the Lemma below, it will be shown that the vanishing of determinant of the Jacobian $\frac{\partial(L_1,\ldots,L_N)}{\partial(g_1,\ldots,g_N)}$ at $(\widehat{g}_1, \ldots, \widehat{g}_N)$ implies that there exist two curves $\widetilde{g}_j^1(g_1)$ and $\widetilde{g}_j^N(g_N)$, $j = 1, \ldots, N$, nearby $(\widehat{g}_1, \ldots, \widehat{g}_N)$, with $L_j(\widetilde{g}_1^i(g_i), \ldots, \widetilde{g}_N^i(g_i)) = L_j(\widehat{g}_1, \ldots, \widehat{g}_N)$, $i = 1, N$ and $j \in \{1, \ldots, N\}\backslash\{i\}$.

Lemma 4.2. *Let* $\det\left(\frac{\partial(L_1,\ldots,L_N)}{\partial(g_1,\ldots,g_N)}(\widehat{g}_1, \ldots, \widehat{g}_N)\right) = 0$ *and define $\widehat{L}_j = L_j(\widehat{g}_1, \ldots, \widehat{g}_N)$. Then there exist curves $\widetilde{g}_j^1(g_1)$, $j = 1, \ldots, N$, for g_1 nearby $\widehat{g}_1$, and $\widetilde{g}_j^N(g_N)$, $j = 1, \ldots, N$, for g_N nearby $\widehat{g}_N$, with $\widetilde{g}_j^i(\widehat{g}_i) = \widehat{g}_j$, $j = 1, \ldots, N$ and $L_j(\widetilde{g}_1^i(g_i), \ldots, \widetilde{g}_N^i(g_i)) = \widehat{L}_j$, $j \in \{1, \ldots, N\}\backslash\{i\}$, for $i = 1, N$.*

Proof. First we will show with a contradiction argument that if $\det\left(\frac{\partial(L_1,\ldots,L_N)}{\partial(g_1,\ldots,g_N)}\right) = 0$, then $\det\left(\frac{\partial(L_2,\ldots,L_N)}{\partial(g_2,\ldots,g_N)}\right) \neq 0$. In [5], it is shown that the Jacobian $\frac{\partial(L_1,\ldots,L_N)}{\partial(g_1,\ldots,g_N)}$ is tri-diagonal with on the diagonal the derivatives $\frac{\partial L_i}{\partial g_i}$, $i = 1, \ldots, N$, and on the off-diagonal the non-zero derivatives

$\mathcal{B}_i^{-1} = \frac{\partial L_{i+1}}{\partial g_i} = \frac{\partial L_i}{\partial g_{i+1}}$, $i = 1, \dots, N-1$. All other derivatives $\frac{\partial L_i}{\partial g_j} = 0$, $|j-i| > 1$. An expansion with respect to the first row gives

$$\det\left(\frac{\partial(L_1,\dots,L_N)}{\partial(g_1,\dots,g_N)}\right) = \frac{\partial L_1}{\partial g_1}\det\left(\frac{\partial(L_2,\dots,L_N)}{\partial(g_2,\dots,g_N)}\right) - \frac{1}{\mathcal{B}_1^2}\det\left(\frac{\partial(L_3,\dots,L_N)}{\partial(g_3,\dots,g_N)}\right). \quad (7)$$

Thus if both $\det\left(\frac{\partial(L_1,\dots,L_N)}{\partial(g_1,\dots,g_N)}\right) = 0$ and $\det\left(\frac{\partial(L_2,\dots,L_N)}{\partial(g_2,\dots,g_N)}\right) = 0$, then also $\det\left(\frac{\partial(L_3,\dots,L_N)}{\partial(g_3,\dots,g_N)}\right) = 0$. Using this in the equivalent expression to (7) for $\det\left(\frac{\partial(L_2,\dots,L_N)}{\partial(g_2,\dots,g_N)}\right)$, it follows that also $\det\left(\frac{\partial(L_4,\dots,L_N)}{\partial(g_4,\dots,g_N)}\right) = 0$. We can continue this argument and conclude that if both $\det\left(\frac{\partial(L_1,\dots,L_N)}{\partial(g_1,\dots,g_N)}\right) = 0$ and $\det\left(\frac{\partial(L_2,\dots,L_N)}{\partial(g_2,\dots,g_N)}\right) = 0$, then also $\det\left(\frac{\partial(L_j,\dots,L_N)}{\partial(g_j,\dots,g_N)}\right) = 0$, for $j = 3, \dots, N$. However, as we have seen in the case $N = 2$, $\det\left(\frac{\partial(L_{N-1},L_N)}{\partial(g_{N_1},g_N)}\right) = 0$, implies that $\frac{\partial L_N}{\partial g_N} \neq 0$, which contradicts the previous statement for $j = N$. So we conclude that if $\det\left(\frac{\partial(L_1,\dots,L_N)}{\partial(g_1,\dots,g_N)}\right) = 0$, then $\det\left(\frac{\partial(L_2,\dots,L_N)}{\partial(g_2,\dots,g_N)}\right) \neq 0$.

By definition, $L_j(\widehat{g}_1, \dots, \widehat{g}_N) = \widehat{L}_j$, $j = 2, \dots, N$. Since $\det\left(\frac{\partial(L_2,\dots,L_N)}{\partial(g_2,\dots,g_N)}\right) \neq 0$, the implicit function theorem implies that there are curves $\widetilde{g}_j^1(g_1)$, $j = 2, \dots, N$ for g_1 nearby $\widehat{g}_1$, such that $\widetilde{g}_j^1(\widehat{g}_1) = \widehat{g}_j$, and $L_j(g_1, \widetilde{g}_2^1(g_1), \dots, \widetilde{g}_N^1(g_1)) = \widehat{L}_j$, $j = 2, \dots, N$. If we define $\widetilde{g}_1^1(g_1) = g_1$, then we have derived the statement in the Lemma for $i = 1$. In a similar way, we can prove the case for $i = N$. □

Lemma 4.2 implies that we are again in the situation of the previous section and we can state the following corollary about a change in stability.

Corollary 4.3. *Let the front $u_f(x; g_1, \dots, g_N)$ be a solution of (6) (with $(g_1, \dots, g_N)$ away from the existence bifurcation points), such that all zeroes of $\partial_x u_f(x; g_1, \dots, g_N)$ are simple and the lengths of the middle intervals of $u_f(x; g_1, \dots, g_N)$ form a smooth length hyper-surface $L(g_1, \dots, g_N)$. There is a change in stability for the front $u_f(x; g_1, \dots, g_N)$ at $(g_1, \dots, g_N) = (\widehat{g}_1, \dots, \widehat{g}_N)$ when varying g_1 or g_N if and only if*

i) the determinant of the Jacobian $\det\left(\frac{\partial(L_1,\dots,L_N)}{\partial(g_1,\dots,g_)}(\widehat{g}_1, \dots, \widehat{g}_N)\right) = 0$*;*

ii) the order of the zero $\widehat{g}_1$ of $\frac{dL_1}{dg_1}(g_1, \widetilde{g}_2^1(g_1), \dots, \widetilde{g}_N^1(g_1))$ is odd or *the order of the zero $\widehat{g}_N$ of $\frac{dL_N}{dg_N}(\widetilde{g}_1^N(g_N), \dots, \widetilde{g}_{N-1}^N(g_N), g_N)$ is odd;*

iii) the eigenfunction $\Psi(\widehat{g}_1, \dots, \widehat{g}_N)$ has no zeroes.

5 Conclusion

If stationary fronts of the inhomogeneous nonlinear wave equation (1) are considered away from any bifurcation points, then Theorem 3.2 gives a necessary and sufficient condition for a branch of stationary fronts parametrised by the Hamiltonian g to change stability. And Corollary 4.3 gives such necessary and sufficient condition for a family of stationary fronts of the nonlinear wave equation (6) with N inhomogeneity intervals.

As illustrated in the example in section 2, in many specific applications, these conditions can be used to determine the stability of whole branches of solutions from the length curves. For instance if one knows that one branch of solutions is stable, maybe from an asymptotic analysis or otherwise, then the conditions imply a change of stability will occur when the length function has a turning point. If a branch was unstable, then looking at the eigenfunction $\Psi(g)$ at turning points and seeing if it has any zeroes, will immediately show whether the adjacent branch is stable or 'more unstable' (has an extra positive eigenvalue).

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