ERGODIC OPTIMIZATION FOR GENERIC CONTINUOUS FUNCTIONS

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Abstract. Given a real-valued continuous function f defined on the phase space of a dynamical system, an invariant measure is said to be *maximizing* if it maximises the integral of f over the set of all invariant measures. Extending results of Bousch, Jenkinson and Brémont, we show that the ergodic maximizing measures of a generic continuous function have the same properties as generic ergodic measures.

1. Introduction

Let $T\colon X \to X$ be a continuous transformation of a compact metric space. Let $\mathcal{M}_T$ denote the set of T-invariant Borel probability measures on X, and let $\mathcal{E}_T \subseteq \mathcal{M}_T$ denote the set of ergodic measures. We equip both of these sets with the weak-* topology. For each continuous $f\colon X \to \mathbb{R}$ we define the *maximum ergodic average*

$$\beta(f) = \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu = \sup_{x \in X} \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

and the set of all *maximizing measures* of f,

$$\mathcal{M}_{\max}(f) := \left\{ \mu \in \mathcal{M}_T \colon \int f \, d\mu = \beta(f) \right\}$$

The study of the functional β and sets $\mathcal{M}_{\max}(f)$ has been termed *ergodic optimisation*, and has attracted some recent research interest [2, 3, 4, 6, 5, 7, 9, 13, 16]. In this note we examine the behaviour of the set $\mathcal{M}_{\max}(f)$ when f ranges over a dense G_δ subset of $C(X)$.

We briefly summarise some existing results. In [4], T. Bousch and O. Jenkinson showed that in the case where $T\colon X \to X$ is an expanding map of the circle, there is a dense G_δ subset of $C(X)$ for which $\mathcal{M}_{\max}(f)$ is a singleton set containing a fully supported measure. In [5] J. Brémont showed under a more general hypothesis that, additionally, there is a dense G_δ subset of $C(X)$ for which the unique element of $\mathcal{M}_{\max}(f)$ has zero entropy. In the case of an expanding map of the circle, zero entropy and full support are *generic* properties of elements of $\mathcal{M}_T$ in the sense of Baire category. In this note we extend and unite these results by showing that the properties of the ergodic elements of $\mathcal{M}_{\max}(f)$ for generic $f \in C(X)$ are the same as the properties of generic elements of $\overline{\mathcal{E}_T}$. We prove the following:

Theorem 1.1. *Suppose that $\mathcal{U}$ is a dense open subset of $\overline{\mathcal{E}_T}$. Then the set*

$$U := \left\{ f \in C(X) \colon \overline{\mathcal{E}_T} \cap \mathcal{M}_{\max}(f) \subseteq \mathcal{U} \right\}$$

is open and dense in $C(X)$*. Conversely, if* $U \subseteq C(X)$ *is open and dense, then the set*

$$\mathcal{U} := \mathcal{E}_T \cap \bigcup_{f \in U} \mathcal{M}_{\max}(f)$$

is open and dense in $\mathcal{E}_T$*.*

Since $\mathcal{E}_T$ is a dense G_δ subset of the compact metric space $\overline{\mathcal{E}_T}$ it is a Baire space, and we may deduce

Corollary 1.2. *Suppose that* $\mathcal{U}$ *is a dense* G_δ *subset of* $\mathcal{E}_T$*. Then the set*

$$U := \{f \in C(X) \colon \mathcal{E}_T \cap \mathcal{M}_{\max}(f) \subseteq \mathcal{U}\}$$

a dense G_δ *subset of* $C(X)$*. Conversely, if* $U \subseteq C(X)$ *is a dense* G_δ*, then*

$$\mathcal{U} := \mathcal{E}_T \cap \bigcup_{f \in U} \mathcal{M}_{\max}(f)$$

is a dense G_δ *subset of* $\mathcal{E}_T$*.*

The following generalisation of results of Bousch-Jenkinson [4, Theorem C] and Brémont [5, Theorem 1.2] follows directly.

Corollary 1.3. *Suppose that* $T \colon X \to X$ *satisfies Bowen's specification property. Then there is a dense* G_δ *set* $Z \subseteq C(X)$ *such that for every* $f \in Z$*,* $\mathcal{M}_{\max}(f)$ *is a singleton set containing a measure of zero entropy which has support equal to* X *and is not strongly mixing.*

Proof. K. Sigmund proves in [15] that in this context $\overline{\mathcal{E}_T} = \mathcal{M}_T$ and a dense G_δ subset of $\mathcal{M}_T$ exists in which every measure has zero entropy and full support and is not strongly mixing. Applying Theorem 1.2 yields the result. □

We suggest the following further application of Theorem 1.1. Let $X = \mathbb{R}/\mathbb{Z}$, let $T \colon X \to X$ be the transformation $Tx = 2x \mod 1$. T. Bousch, O. Jenkinson and others have shown that elements of $\mathcal{E}_T$ which are supported in a semi-circle - termed *Sturmian measures* - arise as maximising measures for several classes of continuous functions [1, 2, 11]. It is easily seen that the set of Sturmian measures is a closed subset of $\overline{\mathcal{E}_T} = \mathcal{M}_T$ having empty interior, and so by Theorem 1.1 it follows that the set of continuous functions having a Sturmian maximising measure is nowhere dense in $C(X)$.

2. Preliminaries

We begin with the following simple result:

Lemma 2.1. *Let* $\mu \in \mathcal{M}_T$ *and* $\nu \in \mathcal{E}_T$*, and suppose that there is* $\kappa < 2$ *such that* $|\int \psi \, d\mu - \int \psi \, d\nu| \leq \kappa |\psi|_\infty$ *for every* $\psi \in C(X)$*. Then there exist* $\hat{\mu} \in \mathcal{M}_T$ *and* $\lambda \in (0,1)$ *such that* $\mu = (1-\lambda)\hat{\mu} + \lambda\nu$*.*

Proof. We begin by showing that μ and ν are not mutually singular. Let $\delta < (2-\kappa)/4$. If $\mu \perp \nu$, choose closed sets $K_1, K_2 \subseteq X$ with $\min\{\mu(K_1), \nu(K_2)\} > 1-\delta$ and $K_1 \cap K_2 = \emptyset$, and let $\psi \in C(X)$ with $|\psi|_\infty = 1$, $\psi^{-1}\{1\} = K_1$ and $\psi^{-1}\{-1\} = K_2$. We then have

$$\int \psi \, d\mu - \int \psi \, d\nu \geq \mu(K_1) - \mu(X \setminus K_1) + \nu(K_2) - \nu(X \setminus K_2) > 2 - 4\delta > \kappa$$

contradicting the hypothesis of the lemma.

In view of this we deduce from the Lebesgue decomposition theorem that there exist $\hat{\mu}, \hat{\nu} \in \mathcal{M}$ and $\lambda \in (0,1)$ such that $\mu = (1-\lambda)\hat{\mu} + \lambda\hat{\nu}$, where $\hat{\mu} \perp \nu$ and $\hat{\nu} \gg \nu$. A standard argument such as in e.g. [17] shows that necessarily $\hat{\mu}, \hat{\nu} \in \mathcal{M}_T$, and since ν is ergodic it follows that $\hat{\nu} = \nu$. □

The following key result is due to J. Brémont. A weaker version is implicit in [12, Proposition 1]; a similar result occurred in a preprint version of [14] but was not included in the published version. T. Bousch has remarked that this result resembles a theorem of I. Ekeland [8, Theorem 1.1].

Lemma 2.2 ([5]). *Let $f \in C(X)$ and $\mu \in \mathcal{M}_T$, and let $\beta(f) = \int f\,d\mu + \varepsilon\delta$ with $\varepsilon, \delta \geq 0$. Then there exists $g \in C(X)$ and $\nu \in \mathcal{M}_{\max}(g)$ such that $|f-g|_\infty \leq \delta$ and such that $\left|\int \psi\,d\mu - \int \psi\,d\nu\right| \leq \varepsilon|\psi|_\infty$ for every $\psi \in C(X)$.*

We deduce the following

Proposition 2.3. *Let $f \in C(X)$ and $\varepsilon > 0$. If $\nu \in \mathcal{E}_T$ has $\beta(f) - \int f\,d\nu < \varepsilon$ then there exists $g \in C(X)$ such that $|f-g|_\infty < \varepsilon$ and $\nu \in \mathcal{M}_{\max}(g)$.*

Proof. By the preceding lemma there exist $g \in C(X)$ and $\mu \in \mathcal{M}_{\max}(g)$ such that $|f-g|_\infty < \varepsilon$ and $|\int \psi\,d\mu - \int \psi\,d\nu| \leq |\psi|_\infty$ for every $\psi \in C(X)$. It follows from Lemma 2.1 that $\mu = (1-\lambda)\hat{\mu} + \lambda\nu$ for some $\lambda \in (0,1)$ and $\hat{\mu} \in \mathcal{M}_T$, whence $\int g\,d\nu = \beta(f)$ and $\nu \in \mathcal{M}_{\max}(g)$ as required. □

Finally we recall the following result due to O. Jenkinson [10].

Proposition 2.4. *Let $\mu \in \mathcal{E}_T$. Then there exists $h \in C(X)$ such that $\mathcal{M}_{\max}(h) = \{\mu\}$.*

3. Proof of Theorem 1.1

Theorem 1.1 may be deduced from the following four lemmas, some of which may be of individual interest.

Remarks. Lemma 3.1 is of a standard type, being similar to parts of the proofs of [7, Proposition 10], [9, Theorem 3.2] and [13, Theorem 1]. Lemma 3.3 generalises a result of J. Brémont [5, Proposition 2.1].

Lemma 3.1. *Let $\mathcal{U}$ be an open subset of $\overline{\mathcal{E}_T}$, and define*

$$U = \left\{ f \in C(X) \colon \mathcal{M}_{\max}(f) \cap \overline{\mathcal{E}_T} \subseteq \mathcal{U} \right\}.$$

Then U is open in $C(X)$.

Proof. Let $(f_n)_{n\geq 1}$ be a convergent sequence in $C(X) \setminus U$ with limit $f \in C(X)$, and take $\mu_n \in \mathcal{M}_{\max}(f_n) \cap \left(\overline{\mathcal{E}_T} \setminus \mathcal{U}\right)$ for each n. Since $\overline{\mathcal{E}_T} \setminus \mathcal{U}$ is weak-* compact we may take a subsequence n_r so that $\mu_{n_r} \to \mu \in \overline{\mathcal{E}_T} \setminus \mathcal{U}$. For every $\nu \in \mathcal{M}_T$ and $r \geq 1$ we have

$$\int f\,d\nu - |f - f_{n_r}|_\infty \leq \int f_{n_r}\,d\nu \leq \int f_{n_r}\,d\mu_{n_r} \leq \int f\,d\mu_{n_r} + |f - f_{n_r}|_\infty$$

yielding $\int f\,d\nu \leq \int f\,d\mu$. Since ν is arbitrary we deduce that $\mu \in \left(\mathcal{M}_{\max}(f) \cap \overline{\mathcal{E}_T}\right) \setminus \mathcal{U}$ and therefore $f \in C(X)\setminus U$. We conclude that $C(X)\setminus U$ is closed and consequently U is open. □

Lemma 3.2. *Let $U \subseteq C(X)$ be open, and define*

$$\mathcal{U} := \mathcal{E}_T \cap \bigcup_{f\in U} \mathcal{M}_{\max}(f).$$

Then $\mathcal{U}$ is open in $\mathcal{E}_T$.

Proof. Let $\mu \in \mathcal{U}$, and let $f \in U$ such that $\mu \in \mathcal{M}_{\max}(f)$. Choose $\varepsilon > 0$ such that $|f - g|_\infty < \varepsilon$ implies $g \in U$, and let $\mathcal{V} \subseteq \mathcal{E}_T$ be an open neighbourhood of μ small enough that $\beta(f) - \int f\,d\nu < \varepsilon$ for every $\nu \in \mathcal{V}$. If $\nu \in \mathcal{V}$, then it follows by Proposition 2.3 there exists $g \in U$ such that $\nu \in \mathcal{M}_{\max}(g)$. We conclude that $\mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{U}$ is an open subset of $\mathcal{E}_T$. □

Lemma 3.3. *Let $\mathcal{U}$ be a dense subset of $\mathcal{E}_T$, and let $U \subseteq C(X)$ be the set of all f such that $\mathcal{M}_{\max}(f)$ is a singleton set containing an element of $\mathcal{U}$. Then U is dense in $C(X)$.*

Proof. We must show that U intersects every nonempty open $V \subseteq C(X)$. Consider such a V, and define $\mathcal{V} := \mathcal{E}_T \cap \bigcup_{f\in V} \mathcal{M}_{\max}(f)$. By Lemma 3.2, $\mathcal{V}$ is open in $\mathcal{E}_T$, and it is clearly nonempty. Since $\mathcal{U}$ is dense in $\mathcal{E}_T$ we have $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. Let $\mu \in \mathcal{U} \cap \mathcal{V}$ and choose $f \in V$ with $\mu \in \mathcal{M}_{\max}(f)$. Since $\mu \in \mathcal{E}_T$ we may apply Proposition 2.4 to deduce that there exists $h \in C(X)$ with $\mathcal{M}_{\max}(h) = \{\mu\}$. For each $\delta > 0$ we clearly have $\mathcal{M}_{\max}(f + \delta h) = \{\mu\}$ so that $f + \delta h \in U$. Since V is open we conclude that $f + \delta h \in U \cap V$ for small enough $\delta > 0$. □

Lemma 3.4. *Let $U \subseteq C(X)$ and define $\mathcal{U} = \mathcal{E}_T \cap \bigcup_{f\in U} \mathcal{M}_{\max}(f)$. If U is dense in $C(X)$, then $\mathcal{U}$ is dense in $\mathcal{E}_T$.*

Proof. Clearly it suffices to show that $\mathcal{U}$ intersects every nonempty open subset of $\overline{\mathcal{E}_T}$. Let $\mathcal{V} \subseteq \overline{\mathcal{E}_T}$ be a such an open set. By Lemma 3.1 the set

$$V := \{f \in C(X) \colon \mathcal{M}_{\max}(f) \cap \overline{\mathcal{E}_T} \subseteq \mathcal{V}\}$$

is open in $C(X)$. Since $\mathcal{V}$ is nonempty and open in $\overline{\mathcal{E}_T}$ it contains an ergodic element μ. By Proposition 2.4 there exists $g \in C(X)$ such that $\mathcal{M}_{\max}(g) = \{\mu\} \subseteq \mathcal{V}$ and thus V is nonempty. Since U is dense in $C(X)$ it follows that $U \cap V \neq \emptyset$ and consequently $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. □

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