Lyapunov optimizing measures for C^1 expanding maps of the circle

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Abstract. For a generic C^1 expanding map of the circle, the Lyapunov maximizing measure is unique, fully supported, and has zero entropy.

1. Introduction

Let $T : \mathbb{T} \to \mathbb{T}$ be a C^1 expanding self-map of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $\mathcal{M}(T)$ denote the set of T-invariant Borel probability measures. For any $\mu \in \mathcal{M}(T)$, its Lyapunov exponent $\lambda_T(\mu)$ is defined by $\lambda_T(\mu) = \int \log |T'| \, d\mu$. Any $m \in \mathcal{M}(T)$ satisfying

$$\lambda_T(m) = \sup_{\mu \in \mathcal{M}(T)} \lambda_T(\mu) =: \lambda^+(T)$$

is called a *Lyapunov maximizing measure for* T. Similarly, any $m \in \mathcal{M}(T)$ satisfying $\lambda_T(m) = \inf_{\mu \in \mathcal{M}(T)} \lambda_T(\mu)$ is called a *Lyapunov minimizing measure for* T. Since T is C^1 and has no critical points, $\log |T'|$ is a continuous function on $\mathbb{T}$, so the functional $\mu \mapsto \lambda_T(\mu)$ is continuous with respect to the weak-$*$ topology on $\mathcal{M}(T)$. Since $\mathcal{M}(T)$ is compact for this topology, it follows that T has both a Lyapunov maximizing and a Lyapunov minimizing measure. In this note we shall be concerned with typical properties of such measures. For simplicity we shall only consider Lyapunov maximizing measures, though all our statements have obvious analogues in the context of Lyapunov minimizing measures; the proofs of these analogues involve only minor modifications of the arguments we present.

For any integer $r \geq 1$, the set $\mathcal{T}^r$ of C^r self-maps of $\mathbb{T}$ is complete with respect to the metric $d_r(S,T) = \max_{x \in \mathbb{T}} \sum_{k=0}^{r} |S^{(k)}(x) - T^{(k)}(x)|$. Let $\mathcal{E}^r$ denote the set of C^r expanding self-maps of $\mathbb{T}$, i.e. those $T \in \mathcal{T}^r$ such that $\min_{x \in \mathbb{T}} |T'(x)| > 1$. Since $\mathcal{E}^r$ is open in $\mathcal{T}^r$, it is a Baire space: that is, every countable intersection of open dense subsets of $\mathcal{E}^r$ is dense in $\mathcal{E}^r$.

Recall that any set containing a countable intersection of open dense subsets is said to be *residual*. If the set of those maps in $\mathcal{E}^r$ satisfying some given property is residual, we say that the property is a *generic* one in $\mathcal{E}^r$, and refer to a *generic* member of $\mathcal{E}^r$ (i.e. any $T \in \mathcal{E}^r$ having the property).

It is easy to construct smooth expanding maps whose Lyapunov maximizing measure is unique, and supported on a periodic orbit (e.g. $T(x) = 2x + \varepsilon \sin 2\pi x \pmod 1$ for small $\varepsilon > 0$). Conjecturally, such maps are generic in $\mathcal{E}^r$, whenever $r \geq 2$:

Conjecture 1. *For $r \geq 2$, a generic $T \in \mathcal{E}^r$ has a unique Lyapunov maximizing measure, and this measure is supported on a periodic orbit of T.*

The following result constitutes a weakened version of Conjecture 1:

Theorem 1. *For $r \geq 2$, the set of those $T \in \mathcal{E}^r$ with no fully supported Lyapunov maximizing measure is an open and dense subset of $\mathcal{E}^r$.*

It turns out that the situation for $r = 1$ is rather different. The main purpose of this article is to establish the following properties of Lyapunov maximizing measures for generic members of $\mathcal{E}^1$:

THEOREM 2. *A generic member of $\mathcal{E}^1$ has a unique Lyapunov maximizing measure. This measure is fully supported, and of zero entropy.*

Despite Theorem 2, it is an open problem to explicitly exhibit an expanding map whose Lyapunov maximizing measure is unique and fully supported.

The organisation of this article is as follows. After some preliminaries in §2, in §3 we prove that a generic map in $\mathcal{E}^1$ has a unique Lyapunov maximizing measure, and that this measure has zero entropy. In §4 we show that this unique Lyapunov maximizing measure is fully supported. The short proof of Theorem 1 is included as §5.

2. Preliminaries

The set $\mathcal{E}^1$ is the disjoint union of the set $\mathcal{E}^1_+$ of orientation-preserving C^1 expanding maps, and the set $\mathcal{E}^1_-$ of orientation-reversing C^1 expanding maps. Although all results stated in this paper are valid for the set $\mathcal{E}^1$, to economise space we often only prove the analogous result for the set $\mathcal{E}^1_+$, i.e. we assume that expanding maps T satisfy $T' > 1$. In all cases the full proof, covering both $T' > 1$ and $T' < -1$, involves only trivial modifications.

Let $C(\mathbb{T})$ denote the space of continuous real-valued functions defined on $\mathbb{T}$, equipped with the Banach norm $\|f\|_\infty = \sup_{x\in\mathbb{T}} |f(x)|$. For continuous $\varphi, \psi : \mathbb{T} \to \mathbb{T}$, define the metric $d_0(\varphi, \psi) := \max_{x\in\mathbb{T}} |\varphi(x) - \psi(x)|$. For each $T \in \mathcal{E}^1$ we define

$$\mathfrak{L}^+(T) = \left\{ \mu \in \mathcal{M}(T) \colon \int \log |T'| \, d\mu = \lambda^+(T) \right\}.$$

As noted previously, $\mathfrak{L}^+(T)$ is non-empty. As a weak-$*$ closed subset of $\mathcal{M}(T)$, it is also weak-$*$ compact.

The following is a result of Sigmund [**Sig**].

LEMMA 1. *If $T \in \mathcal{E}^1$, and $\mu \in \mathcal{M}(T)$ is such that its support supp(μ) is a proper subset of $\mathbb{T}$, then there is a sequence $\mu_i \in \mathcal{M}(T)$, converging to μ in the weak-$*$ topology, such that each supp(μ_i) is a periodic orbit disjoint from supp(μ).*

The following lemma may be deduced from the proof of [**KH**, Thm. 2.4.6]; for completeness we include a condensed proof.

LEMMA 2. *If $T, T_i \in \mathcal{E}^1$ such that $\inf_i \inf_x T_i'(x) > 1$ and $d_0(T_i, T) \to 0$ as $i \to \infty$, then there exists a sequence of homeomorphisms $\xi_i \colon \mathbb{T} \to \mathbb{T}$ such that $\xi_i \circ T_i = T \circ \xi_i$ for all sufficiently large i, and such that $d_0(\xi_i, id_{\mathbb{T}}) \to 0$ and $d_0(\xi_i^{-1}, id_{\mathbb{T}}) \to 0$ as $i \to \infty$.*

PROOF. If T has degree $k > 1$ then so does T_i for i sufficiently large, and we may assume this is the case for *all* i. Suppose $\inf_i \inf_x T_i'(x) \geq \gamma > 1$. Choose a fixed point p of T and a sequence of points $p(i) \to p$ such that each $p(i)$ is a fixed point of T_i. For each $i, n > 0$ define $A_n(i) = T_i^{-n} p(i) = \{a_n^0(i), a_n^1(i), \dots, a_n^{k^n-1}(i)\}$, where $a_n^0(i) = p(i)$ for each n, and the points $a_n^j(i)$ are listed in order around the circle; for each $n > 0$ define $A_n = T^{-n}p = \{a_n^j\}$ in the same manner. Note that if j, n are fixed then $a_n^j(i) \to a_n^j$ as $i \to \infty$. Since T_i and T are orientation-preserving k-fold self-coverings of the circle, $T_i a_{n+1}^{jk+\ell}(i) = a_n^\ell(i)$ and $T a_{n+1}^{jk+\ell} = a_n^\ell$ whenever $j, n > 0$ and $0 \leq \ell < k$ (see [**KH**] for a detailed description). Since T_i is expanding, $\sup_j |a_n^j(i) - a_n^{j+1}(i)| \leq \gamma^{-n}$, so for each i the set $\cup_{n>0} A_n(i)$ is dense in $\mathbb{T}$. Define $\xi_i(a_n^j(i)) = a_n^j$ for each i, j, n. For each i this defines a strictly monotone orientation-preserving map ξ_i between dense subsets of the circle $\mathbb{T}$, which can therefore be extended to a homeomorphism of $\mathbb{T}$. For fixed $i > 0$, it follows from the relations $T_i a_{n+1}^{jk+\ell}(i) = a_n^\ell(i)$ and

$Ta_{n+1}^{jk+\ell} = a_n^\ell$ that $\xi_i \circ T_i = T \circ \xi_i$ on the dense set $\cup_{n>0} A_n(i)$, and hence on the whole of $\mathbb{T}$ by continuity.

We now show that $\xi_i \to \mathrm{id}_{\mathbb{T}}$ and $\xi_i^{-1} \to \mathrm{id}_{\mathbb{T}}$. Given $\varepsilon > 0$, choose $N > 0$ large enough that $\gamma^{-N} < \varepsilon$, and choose M large enough that $\sup_j |a_N^j(i) - a_N^j| < \varepsilon$ whenever $i > M$. If $i > M$ and $x \in [a_N^j(i), a_N^{j+1}(i)]$, then $\xi_i(x) \in [a_N^j, a_N^{j+1}]$ and

$$\begin{aligned} |\xi_i(x) - x| &\le |\xi_i(x) - \xi_i(a_N^j(i))| + |\xi_i(a_N^j(i)) - a_N^j(i)| + |a_N^j(i) - x| \\ &\le |a_N^j - a_N^{j+1}| + |a_N^j - a_N^j(i)| + |a_N^j(i) - a_N^{j+1}(i)| < 3\varepsilon. \end{aligned}$$

Similarly if $x \in [a_N^j, a_N^{j+1}]$, then $\xi_i^{-1}(x) \in [a_N^j(i), a_N^{j+1}(i)]$ and one may show that $|\xi_i^{-1}(x) - x| < 3\varepsilon$ by the same method. □

LEMMA 3. *If $T_i, T \in \mathcal{E}^1$ such that $\inf_i \inf_x T_i'(x) > 1$ and $d_0(T_i, T) \to 0$ as $i \to \infty$, then*
(a) *Every $\mu \in \mathcal{M}(T)$ is the weak-$*$ limit of a sequence $\{\mu_i\}$ of T_i-invariant measures.*
(b) *If $m_i \in \mathcal{M}(T_i)$ then any weak-$*$ accumulation point of the sequence $\{m_i\}$ belongs to $\mathcal{M}(T)$.*

PROOF. For each i let $\xi_i \colon \mathbb{T} \to \mathbb{T}$ be the homeomorphism given by Lemma 2.

(a) Each $\mu_i := \mu \circ \xi_i^{-1}$ is T_i-invariant, and $\{\mu_i\}$ is weak-$*$ convergent to μ, since if $f : \mathbb{T} \to \mathbb{R}$ is Lipschitz (such functions are dense in $C(\mathbb{T})$) then $|\int f\, d\mu_i - \int f\, d\mu| = |\int (f \circ \xi_i - f)\, d\mu| \le \|f \circ \xi_i - f\|_\infty \le \mathrm{lip}(f) d_0(\xi_i, \mathrm{id}) \to 0$.

(b) Without loss of generality, suppose that $\{m_i\}$ is weak-$*$ convergent. Each $\nu_i := m_i \circ \xi_i$ is T-invariant, and if f is Lipschitz then $|\int f\, d\nu_i - \int f\, dm_i| = |\int f \circ \xi_i^{-1}\, dm_i - \int f\, dm_i| \le \mathrm{lip}(f) d_0(\xi_i^{-1}, \mathrm{id}) \to 0$, so the sequences $\{m_i\}$ and $\{\nu_i\}$ are weak-$*$ convergent, with the same limit. The limit of $\{\nu_i\}$ is T-invariant since $\mathcal{M}(T)$ is weak-$*$ closed. □

For $T \in \mathcal{E}^1$, its topological entropy $h_{\mathrm{top}}(T)$ is equal to the logarithm of the modulus of the degree of T. For $\mu \in \mathcal{M}(T)$, let $h(\mu; T)$ denote the entropy of μ with respect to T.

LEMMA 4. *Let $T_i, T \in \mathcal{E}^1$ and suppose that $\inf_i \inf_x T_i'(x) > 1$ and $d_0(T_i, T) \to 0$ as $i \to \infty$. If $\mu_i \in \mathcal{M}(T_i)$ for each i, and $\mu_i \to \mu \in \mathcal{M}(T)$, then*

$$\limsup_{i \to \infty} h(\mu_i; T_i) \le h(\mu; T).$$

PROOF. By Lemma 2 there exists a sequence of homeomorphisms $\xi_i \colon \mathbb{T} \to \mathbb{T}$ such that $T_i \circ \xi_i = \xi_i \circ T$ for i sufficiently large, and such that $d_0(\mathrm{id}_{\mathbb{T}}, \xi_i^{-1}) \to 0$. It follows that $\mu_i \circ \xi_i \in \mathcal{M}(T)$ and $h(\mu_i; T_i) = h(\mu_i \circ \xi_i; T)$, and that $\lim_i \mu_i = \lim_i \mu_i \circ \xi_i = \mu$. By the upper semi-continuity of the entropy map $m \mapsto h(m; T)$ (see e.g. [**New**]), it follows that

$$\limsup_{i \to \infty} h(\mu_i; T_i) = \limsup_{i \to \infty} h(\mu_i \circ \xi_i; T) \le h(\mu; T).$$

□

LEMMA 5. *Suppose $T_i \to T$ in $\mathcal{E}^1$. If m is any weak-$*$ accumulation point of a sequence $m_i \in \mathfrak{L}^+(T_i)$, then $m \in \mathfrak{L}^+(T)$.*

PROOF. Without loss of generality, suppose that $m_i \to m$. By Lemma 3 (b), $m \in \mathcal{M}(T)$. If $\mu \in \mathcal{M}(T)$ is arbitrary, then by Lemma 3 (a) there exists a sequence $\mu_i \in \mathcal{M}(T_i)$ such that $\mu_i \to \mu$. Therefore, writing $f = \log|T'|$ and $f_i = \log|T_i'|$,

$$\begin{aligned} \int f\, dm - \int f\, d\mu = &\left(\int f\, dm - \int f\, dm_i \right) \\ &+ \left(\int f\, dm_i - \int f_i\, dm_i \right) + \left(\int f_i\, dm_i - \int f_i\, d\mu_i \right) \\ &+ \left(\int f_i\, d\mu_i - \int f\, d\mu_i \right) + \left(\int f\, d\mu_i - \int f\, d\mu \right). \end{aligned}$$

The term $\int f_i\,dm_i - \int f_i\,d\mu_i$ is non-negative since m_i is Lyapunov maximizing for T_i, while the other four terms on the right-hand side of the above equation tend to zero as $i \to \infty$. Letting $i \to \infty$ gives $\int f\,dm \geq \int f\,d\mu$ as required. $\square$

LEMMA 6. *Let U be an open sub-interval of $\mathbb{T}$, and suppose $c \in U$. For any $A, B, \varepsilon > 0$ there exists a C^∞ function $\Delta_\varepsilon : \mathbb{T} \to \mathbb{R}$ such that*

$$\Delta_\varepsilon \equiv 0 \quad \text{on } \mathbb{T} \setminus U\,, \tag{1}$$

$$\Delta_\varepsilon(c) = 0\,, \tag{2}$$

$$\max_{x\in\mathbb{T}} |\Delta_\varepsilon(x)| \leq B\,, \tag{3}$$

$$\max_{x\in\mathbb{T}} \Delta'_\varepsilon(x) = \Delta'_\varepsilon(c) = (e^\varepsilon - 1)A\,, \tag{4}$$

$$\min_{x\in\mathbb{T}} \Delta'_\varepsilon(x) > -B\,. \tag{5}$$

PROOF. Without loss of generality suppose that $c = 0$, and that $(-b, b)$ is a ball of radius b contained in U. Define $\Delta_\varepsilon \equiv 0$ on $\mathbb{T} \setminus (-b, b) \supset \mathbb{T} \setminus U$, so (1) holds.

Define the C^∞ function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0\,. \end{cases}$$

Define Δ_ε on $(-b, b)$ by

$$\Delta_\varepsilon(x) = B\,h(x+b)\,h(b-x)\,g_\varepsilon(x)\,,$$

where

$$g_\varepsilon(x) = \tanh(b_\varepsilon x)\,,$$

and

$$b_\varepsilon = \frac{(e^\varepsilon - 1)A}{Bh(b)^2}\,.$$

Clearly the function Δ_ε is C^∞, and $\Delta_\varepsilon(0) = 0$, so (2) is satisfied.

If $x \in (-b, b)$ then $\max(x + b, b - x) < 1$ since necessarily $b < 1/2$. It follows that $\max(h(x + b), h(b - x)) < e^{-1}$, and in particular $h(x + b)h(b - x) < 1$. Moreover $|g_\varepsilon| < 1$, so $|\Delta_\varepsilon| < B$ on $(-b, b)$, and since Δ_ε is identically zero on $\mathbb{T} \setminus (-b, b)$ then (3) holds.

Now

$$\begin{aligned} \Delta'_\varepsilon(x) = {} & \frac{(e^\varepsilon - 1)A}{h(b)^2} \tanh'(b_\varepsilon x) h(x+b) h(b-x) \\ & + Bg_\varepsilon(x) \left[h'(x+b)h(b-x) - h(x+b)h'(b-x) \right]\,, \end{aligned} \tag{6}$$

and since $g_\varepsilon(0) = 0$ then $\max_{x\in\mathbb{T}} \Delta'_\varepsilon(x) = \Delta'_\varepsilon(0) = (e^\varepsilon - 1)A$, so (4) holds.

It remains to verify (5). Suppose $x \in (-b, 0)$, so that $g_\varepsilon(x) < 0$, and since h, h' and g'_ε are all positive on $(-b, b)$, from (6) we see that

$$\Delta'_\varepsilon(x) > Bg_\varepsilon(x)h'(x+b)h(b-x)\,. \tag{7}$$

Now $0 < b - x < 1$, so $h(b - x) < e^{-1} < 1$. Since $0 < x + b < 1$, a short calculation shows that $h'(x + b) < e^{-1} < 1$. Moreover $g_\varepsilon(x) > -1$, so (7) implies

$$\Delta'_\varepsilon(x) > -B\,. \tag{8}$$

An analogous calculation establishes (8) for $x \in (0, b)$, so (5) is proved. $\square$

LEMMA 7. *Let $T \in \mathcal{E}^1$, with z_0 a point of least period p under T. If $\kappa > 0$ and $0 < \varepsilon < 1$ then for every sufficiently small $\delta > 0$ there exists $T_\delta \in \mathcal{E}^1$ such that if*

$$g_\delta(x) := \varepsilon^{-1} \log\left(\frac{T'_\delta(x)}{T'(x)}\right)$$

then:

(i) $T^j z_0 = T_\delta^j z_0$ *for* $0 \le j \le p-1$
(ii) $d_1(T, T_\delta) \le C_T \varepsilon$
(iii) $d_0(T, T_\delta) \le \delta$
(iv) $\sup_x g_\delta(x) \le 1 + \frac{\kappa}{8}$
(v) $g_\delta(T_\delta^j z_0) = 1$ *for* $0 \le j \le p-1$
(vi) $g_\delta(x) = 0$ *whenever* $\min_{0 \le j < p} d(x, T_\delta^j z_0) > \delta$
(vii) $\inf_x T_\delta'(x) \ge \gamma > 1$.

Here $\gamma > 1$ *and* $C_T > 0$ *are constants depending only on* T.

Proof. For $j = 0, \dots, p-1$, let U_δ^j be the open interval centred at $T^j z_0$ with radius δ, where $\delta < \varepsilon < 1$ is chosen sufficiently small that $U_\delta^j \cap U_\delta^k = \emptyset$ when $j \ne k$.

For every $0 \le j < p$, let $\Delta_{\varepsilon,\delta}^j$ be the C^∞ function given by Lemma 6, with $A = T'(T^j z_0)$, $c = T^j z_0$, $U = U_\delta^j$ and $B = \delta$. Define

$$T_\delta = T + \sum_{j=0}^{p-1} \Delta_{\varepsilon,\delta}^j,$$

and note using (2) that $T_\delta(T^j z_0) = T(T^j z_0)$ for all $0 \le j \le p-1$ so that (i) is satisfied. Since $T_\delta \equiv T$ on $\mathbb{T} \setminus \bigcup_{j=0}^{p-1} U_\delta^j$ by (1), (vi) is satisfied also.

Using the pairwise disjointness of the intervals U_δ^j together with (3), (4) and (5) it follows easily that

$$d_0(T_\delta, T) \le \delta < \varepsilon,$$

$$\left\|T_\delta' - T'\right\|_\infty \le (e^\varepsilon - 1)\,\|T'\|_\infty \le 3\varepsilon\|T'\|_\infty$$

using that $\delta < \varepsilon < 1$, so properties (ii) and (iii) follow.

Using (4) and (i), for each $0 \le j \le p-1$ we have

$$\begin{aligned}
\varepsilon g_\delta(T_\delta^j z_0) &= \log\left(1 + \frac{(\Delta_{\varepsilon,\delta}^j)'(T^j z_0)}{T'(T^j z_0)}\right) \\
&= \log\left(1 + \frac{(e^\varepsilon - 1)T'(T^j z_0)}{T'(T^j z_0)}\right) = \varepsilon,
\end{aligned}$$

which gives (v). If δ is chosen small enough such that if $0 \le j \le p-1$ then $T'(x) > (1 + \frac{\varepsilon\kappa}{8})^{-1} T'(T^j z_0)$ whenever $x \in U_\delta^j$, then for each such x, using (4), we have

$$\begin{aligned}
\varepsilon g_\delta(x) &= \log\left(1 + \frac{(\Delta_{\varepsilon,\delta}^j)'(x)}{T'(x)}\right) \\
&< \log\left(1 + \left(1 + \frac{\varepsilon\kappa}{8}\right)\frac{(\Delta_{\varepsilon,\delta}^j)'(T^j z_0)}{T'(T^j z_0)}\right) \\
&= \log\left(1 + \left(1 + \frac{\varepsilon\kappa}{8}\right)(e^\varepsilon - 1)\right) \\
&< \varepsilon + \log\left(1 + \frac{\varepsilon\kappa}{8}\right) < \varepsilon\left(1 + \frac{\kappa}{8}\right),
\end{aligned}$$

which together with (vi) yields (iv). Lastly, if we define $\tilde{\gamma} = \inf_x T'(x) > 1$, then $T_\delta'(x) \ge \tilde{\gamma} - \delta$ for all $x \in \mathbb{T}$ by (5), and so it follows that $\inf_x T_\delta'(x) \ge (1 + \tilde{\gamma})/2 > 1$ for all sufficiently small δ, which is (vii). This clearly also implies that $T_\delta \in \mathcal{E}^1$ as required.

$\square$

3. Generic uniqueness and zero entropy

Let $\mathcal{M}$ be the set of all Borel probability measures on $\mathbb{T}$. Throughout this section, fix a metric ϱ on $\mathcal{M}$ which generates the weak-$*$ topology, and satisfies the property

$$\varrho((1-\beta)\mu+\beta\nu,\mu)\le\beta \tag{9}$$

for every $\mu,\nu\in\mathcal{M}$ and $\beta\in[0,1]$. For example if the sequence $f_k\in C(\mathbb{T})\setminus\{0\}$ is dense in $C(\mathbb{T})$, then

$$\varrho(\mu_1,\mu_2)=\sum_{k=1}^{\infty}\frac{\left|\int f_k\,d\mu_1-\int f_k\,d\mu_2\right|}{2^{k+1}\|f_k\|_\infty}$$

is such a metric.

For each $\kappa>0$, define

$$\mathcal{R}_\kappa=\left\{T\in\mathcal{E}^1\colon \operatorname{diam}_\varrho\mathfrak{L}^+(T)<\kappa\text{ and }\frac{\sup_{\mu\in\mathfrak{L}^+(T)}h(\mu;T)}{h_{\text{top}}(T)}<\kappa\right\}.$$

We will show that the set $\bigcap_{n=1}^{\infty}\mathcal{R}_{1/n}$ is residual in $\mathcal{E}^1$, thereby proving the entropy and uniqueness statements of Theorem 2. To establish this, it suffices to prove Lemma 8 and Proposition 1 below.

Lemma 8. *For each $\kappa>0$ the set $\mathcal{R}_\kappa$ is open in $\mathcal{E}^1$.*

Proof. Let $T\in\mathcal{R}_\kappa$, and suppose that $T_n\to T$ in $\mathcal{E}^1$. We then have $h_{\text{top}}(T_n)=h_{\text{top}}(T)$ for all large enough n. By Lemma 5, if μ_n^1,μ_n^2 are convergent sequences of measures such that $\mu_n^i\in\mathfrak{L}^+(T_n)$ for each n, then $\lim_n\mu_n^1,\lim_n\mu_n^2\in\mathfrak{L}^+(T)$ and hence $\varrho(\mu_n^1,\mu_n^2)<\kappa$ for all sufficiently large n. It follows that $\operatorname{diam}_\varrho\mathfrak{L}^+(T_n)<\kappa$ for all large enough n.

Using the compactness of $\mathfrak{L}^+(T_n)$ and the upper semi-continuity of the entropy map $m\mapsto h(m;T_n)$, we can for each n choose $\mu_n\in\mathfrak{L}^+(T_n)$ such that

$$h(\mu_n;T_n)=\sup_{\nu\in\mathfrak{L}^+(T_n)}h(\nu;T_n).$$

If μ is any weak-$*$ accumulation point of the measures μ_n, then $\mu\in\mathfrak{L}^+(T)$ by Lemma 5, hence $h(\mu;T)<\kappa h_{\text{top}}(T)$, since $T\in\mathcal{R}_\kappa$. Lemma 4 then implies that

$$\limsup_{n\to\infty}\sup_{\nu\in\mathfrak{L}^+(T_n)}h(\nu;T_n)\le h(\mu;T)<\kappa h_{top}(T)\,,$$

and we conclude that $T_n\in\mathcal{R}_\kappa$ for all sufficiently large n. □

Proposition 1. *For each $\kappa>0$, the set $\mathcal{R}_\kappa$ is dense in $\mathcal{E}^1$.*

Proof. Let $T\in\mathcal{E}^1$ and $\kappa>0$, $0<\varepsilon<1$. We will construct $T_\delta\in\mathcal{E}^1$ such that $d_1(T,T_\delta)\le C_T\varepsilon$ and $T_\delta\in\mathcal{R}_\kappa$; the map T_δ will be as in Lemma 7 for some sufficiently small $\delta>0$.

Recall that measures supported on periodic orbits are dense in $\mathcal{M}_T$ (see [**Sig**]). We may therefore choose a periodic point z_0, of least period p, say, with orbit $\mathcal{O}:=\{T^j(z_0):0\le j\le p-1\}$, such that

$$\frac{1}{p}\sum_{j=0}^{p-1}\log T'(T^jz_0)>\lambda^+(T)-\frac{\kappa\varepsilon}{8}.$$

Define

$$\mu=\frac{1}{p}\sum_{j=0}^{p-1}\delta_{T^jz_0},$$

and note that $\mu \in \mathcal{M}(T) \cap \mathcal{M}(T_\delta)$ for all $\delta > 0$ as a consequence of Lemma 7(i). Using Lemma 7(v), for each $\delta > 0$ we have

$$\lambda^+(T_\delta) \geq \int \log T'_\delta \, d\mu = \int \log T' \, d\mu + \varepsilon \int g_\delta \, d\mu > \lambda^+(T) + \varepsilon \left(1 - \frac{\kappa}{8}\right). \tag{10}$$

We claim that if $(\delta_k)_{k\geq 1}$ is a strictly decreasing sequence of positive reals, and ν is a weak-* limit of a sequence $(\nu_k)_{k\geq 1}$ such that each $\nu_k \in \mathfrak{L}^+(T_{\delta_k})$, then $\varrho(\nu, \mu) < \kappa/2$ and $h(\nu; T) < \kappa$. Note that by Lemma 7(vii), $\inf_k \inf_x T'_{\delta_k}(x) > 1$ and so Lemmas 3 and 4 may be applied in the limit $k \to \infty$.

Observe now that $\nu \in \mathcal{M}(T)$ as a consequence of Lemma 7(iii) and Lemma 3. For each $k > 0$,

$$\lambda^+(T_{\delta_k}) = \int \log T'_{\delta_k} \, d\nu_k = \int \log T' d\nu_k + \varepsilon \int g_{\delta_k} \, d\nu_k \, .$$

Evidently,

$$\lim_{k\to\infty} \int \log T' \, d\nu_k = \int \log T' \, d\nu \leq \lambda^+(T),$$

and so for all sufficiently large k,

$$\varepsilon \int g_{\delta_k} \, d\nu_k \geq \lambda^+(T_{\delta_k}) - \lambda^+(T) - \frac{\varepsilon\kappa}{8}.$$

It follows from this and (10) that for large enough k,

$$\varepsilon \int g_{\delta_k} \, d\nu_k \geq \varepsilon \left(1 - \frac{\kappa}{4}\right). \tag{11}$$

For each $k > 0$, let U_k denote the open δ_k-neighbourhood of $\mathcal{O}$. Using Lemma 7(iv) and (vi) together with (11), it follows that

$$\begin{aligned}\nu_k\left(U_k\right) \geq \left(1 + \frac{\kappa}{8}\right)^{-1} \int_{U_k} g_{\delta_k} \, d\nu_k &= \left(1 + \frac{\kappa}{8}\right)^{-1} \int g_{\delta_k} \, d\nu_k \\ &> \left(1 - \frac{\kappa}{4}\right)\left(1 - \frac{\kappa}{8}\right) > 1 - \frac{3}{8}\kappa\end{aligned}$$

for all large enough k. Since the sequence $(\delta_k)_{k\geq 1}$ is strictly decreasing, $U_{k+1} \subset U_k$ for every k. It follows that for every sufficiently large integer ℓ, and $k > \ell$,

$$\nu_k\left(\overline{U_\ell}\right) \geq \nu_k\left(\overline{U_k}\right) \geq \nu_k(U_k) \geq 1 - \frac{3}{8}\kappa.$$

Letting $k \to \infty$ then yields

$$\nu\left(\overline{U_\ell}\right) \geq \limsup_{k\to\infty} \nu_k\left(\overline{U_\ell}\right) \geq 1 - \frac{3}{8}\kappa.$$

Hence,

$$\nu(\mathcal{O}) = \nu\left(\bigcap_{\ell>0} \overline{U_\ell}\right) \geq 1 - \frac{3}{8}\kappa.$$

Since $\mathcal{O}$ is a uniquely ergodic set and $\nu \in \mathcal{M}(T)$, the ergodic decomposition theorem implies that $\nu = (1 - \frac{3}{8}\kappa)\mu + \frac{3}{8}\kappa\hat{\mu}$ for some $\hat{\mu} \in \mathcal{M}(T)$. In particular it follows using (9) that

$$\lim_{k\to\infty} \varrho(\nu_k, \mu) = \varrho(\nu, \mu) \leq \frac{3}{8}\kappa \tag{12}$$

and by virtue of Lemma 4,

$$\limsup_{k\to\infty} h(\nu_k; T_{\delta_k}) \leq h(\nu; T) \, .$$

Now μ is a periodic orbit measure, so $h(\mu; T) = 0$, and therefore

$$h(\nu; T) = \left(1 - \frac{3}{8}\kappa\right) h(\mu; T) + \frac{3}{8} h(\hat{\mu}; T) = \frac{3}{8}\kappa h(\hat{\mu}; T) \leq \frac{3}{8}\kappa h_{\text{top}}(T) \, , \tag{13}$$

by the variational principle.

So we have shown that (12) and (13) hold whenever ν is a weak-$*$ accumulation point as $\delta \to 0$ of the sets $\mathfrak{L}^+(T_\delta)$. Consequently, for all sufficiently small $\delta > 0$,

$$\operatorname{diam}_\varrho \mathfrak{L}^+(T_\delta) < \kappa\,,$$

and

$$\sup_{m \in \mathfrak{L}^+(T_\delta)} h(m; T_\delta) < \kappa h_{\text{top}}(T) = \kappa h_{\text{top}}(T_\delta)\,.$$

In other words, $T_\delta \in \mathcal{R}_\kappa$. Since $d_1(T, T_\delta) \le C_T \varepsilon$ for every $\delta > 0$, this completes the proof. $\square$

4. Generic full support

To establish the fully supported part of Theorem 2 we now prove the following:

PROPOSITION 2. *$\{T \in \mathcal{E}^1 : supp(\mu) = \mathbb{T}$ for all $\mu \in \mathfrak{L}^+(T)\}$ is residual in $\mathcal{E}^1$.*

PROOF. For any proper closed subset Y of $\mathbb{T}$, define

$$M^1(Y) := \left\{ T \in \mathcal{E}^1 : \text{ some } \mu \in \mathfrak{L}^+(T) \text{ has } \operatorname{supp}(\mu) \subseteq Y \right\}\,.$$

We claim that each such $M^1(Y)$ is a closed subset of $\mathcal{E}^1$ with empty interior. Once the claim is established, it will imply the proposition, since if $\{Y_i\}$ is an enumeration of those proper closed subintervals of $\mathbb{T}$ with rational endpoints, then $\cap_i M^1(Y_i)^c$ is equal to the set $\{T \in \mathcal{E}^1 : \operatorname{supp}(\mu) = \mathbb{T} \text{ for all } \mu \in \mathfrak{L}^+(T)\}$.

To prove the claim we first show that $M^1(Y)$ is closed. For this, suppose that $T_i \to T$ in $\mathcal{E}^1$, where each $T_i \in M^1(Y)$, and let $\mu_i \in \mathfrak{L}^+(T_i)$ be such that $\operatorname{supp}(\mu_i) \subseteq Y$. By Lemma 5, any weak-$*$ accumulation point μ of $\{\mu_i\}$ is Lyapunov maximizing for T. If $\mu_{i_n} \to \mu$ as $n \to \infty$ then since Y is closed, $\mu(Y) \ge \lim_{n\to\infty} \mu_{i_n}(Y) = 1$ (see e.g. [**Bil**, Thm. 2.1]). Therefore $\operatorname{supp}(\mu) \subseteq Y$, and hence $T \in M^1(Y)$, so $M^1(Y)$ is indeed closed.

Now we show that $M^1(Y)$ has empty interior in $\mathcal{E}^1$. If $M^1(Y)$ is empty then there is nothing to check, so suppose $T \in M^1(Y)$. Given $0 < \varepsilon < 1$, we will construct $\hat{T} \in \mathcal{E}^1 \setminus M^1(Y)$ such that $d_1(T, \hat{T}) < C_T \varepsilon$, where $C_T > 0$ is as in Lemma 7.

Note that any T-invariant measure carried[1] by Y is in fact carried by $\bigcap_{i=0}^\infty T^{-i} Y$. Let $0 < \varepsilon < 1$ be arbitrary. By Lemma 1 there is a T-periodic orbit $\{x_1, \dots, x_p\}$ (i.e. $T(x_i) = x_{i+1}$ for $1 \le i \le p-1$ and $T(x_p) = x_1$) which is disjoint from the T-invariant set $\bigcap_{i=0}^\infty T^{-i} Y$, with

$$\frac{1}{p} \sum_{i=1}^{p} \log T'(x_i) > \lambda^+(T) - \varepsilon\,. \tag{14}$$

Choose $N > 0$ such that the orbit $\{x_1, \dots, x_p\}$ is disjoint from the set $Y_N := \bigcap_{i=0}^N T^{-i} Y$, and for each $\delta > 0$ define

$$Y_{N,\delta} := \bigcup_{d_0(\hat{T}, T) \le \delta} \bigcap_{i=0}^{N} \hat{T}^{-i} Y\,,$$

where the union is taken over all $\hat{T} \in \mathcal{E}^1$ such that $d_0(\hat{T}, T) \le \delta$. Note that $\cap_{\delta > 0} Y_{N,\delta} = Y_N$. Now let $T_\delta \in \mathcal{E}^1$ be the map given by Lemma 7, with $z_0 = x_1$, $\kappa = 1$ and $\delta > 0$ chosen small enough that

$$\inf_{y \in Y_{N,\delta}} \inf_{1 \le i \le p} d(x_i, y) > \delta. \tag{15}$$

By Lemma 7(i), $\{x_1, \dots, x_p\}$ is a periodic orbit for T_δ as well as for T, and Lemma 7(iii),(vi) together with (15) ensures that $T_\delta \equiv T$ on $Y_{N,\delta}$. It follows that any $\mu \in \mathcal{M}(T_\delta)$ which is carried by Y must be carried by $\bigcap_{i=0}^N T_\delta^{-i} Y$, and hence by $Y_{N,\delta}$. Moreover, a probability measure carried by $Y_{N,\delta}$ is T-invariant if and only if it is T_δ-invariant.

[1]We say a probability measure is *carried* by a set whenever its support is contained in that set.

We now claim that any T_δ-invariant probability measure carried by Y cannot be Lyapunov maximizing for T_δ, thus $T_\delta \notin M^1(Y)$. Indeed if $\nu \in \mathcal{M}(T_\delta)$ is carried by Y then it is carried by $Y_{N,\delta}$, hence

$$\int \log T_\delta' \, d\nu = \int \log T' \, d\nu + \varepsilon \int g_\delta \, d\nu = \int \log T' \, d\nu \leq \lambda^+(T) \,, \tag{16}$$

since $g_\delta \equiv 0$ on $\operatorname{supp}(\nu) \subseteq Y_{N,\delta}$, and because ν is also T-invariant.

The inequality (14), together with Lemma 7(v), gives

$$\lambda^+(T) < \frac{1}{p} \sum_{i=1}^{p} \log T'(x_i) + \varepsilon = \frac{1}{p} \sum_{i=1}^{p} \log T_\delta'(x_i) \,,$$

so combining with (16) yields

$$\int \log T_\delta' \, d\nu \leq \lambda^+(T) < \frac{1}{p} \sum_{i=1}^{p} \log T_\delta'(x_i) \,,$$

and since $\{x_1, \ldots, x_p\}$ is T_δ-periodic we see that ν is not Lyapunov maximizing for T_δ. By Lemma 7(ii) we have $d_1(T_\delta, T) \leq C_T \varepsilon$; since ε was chosen arbitrarily in $(0,1)$, it follows that $M^1(Y)$ has empty interior in $\mathcal{E}^1$. □

Remark 1. The proof of Proposition 2 resembles the proof that, for a fixed expanding map T, a generic function $f \in C(\mathbb{T})$ has all its maximizing measures of full support (see [**BJ**]); by a maximizing measure we mean an $m \in \mathcal{M}(T)$ such that $\int f \, dm = \sup_{\mu \in \mathcal{M}(T)} \int f \, d\mu$.

5. Generic properties in higher differentiability classes

For completeness we now include the short proof of Theorem 1, concerning Lyapunov maximizing measures for generic members of $\mathcal{E}^r$, $r \geq 2$.

Proof of Theorem 1. If $T \in \mathcal{E}^r$ then $\log |T'|$ is Hölder, so by [**CLT**, Thm. 4] there exists $\varphi \in C(\mathbb{T})$ such that $\log |T'| + \varphi - \varphi \circ T \leq \lambda^+(T)$, therefore the Lyapunov maximizing measures for T are precisely those whose support lies in the set $Z = \{x \in \mathbb{T} : (\log |T'| + \varphi - \varphi \circ T)(x) = \lambda^+(T)\}$. Thus T has a fully supported Lyapunov maximizing measure if and only if $Z = \mathbb{T}$, in which case all the T-invariant measures have identical Lyapunov exponent $\lambda^+(T)$ (necessarily equal to $\log k$ where T has degree $\pm k$).

The set $\mathcal{F}^r$ of those $T \in \mathcal{E}^r$ for which all invariant measures have identical Lyapunov exponent is easily shown to be closed in $\mathcal{E}^r$, using Lemma 5. It also has empty interior: any $T \in \mathcal{F}^r$ can be $\mathcal{E}^r$-perturbed to a map where the Lyapunov exponent of some fixed point measure differs from that of some period-2 orbit measure, say. □

Remark 2.

(a) In fact Theorem 1 is also true, with an almost identical proof, if $\mathcal{E}^r$ is replaced by $\mathcal{E}^{r,\alpha}$, consisting of those $T \in \mathcal{E}^r$ whose r-th derivative is α-Hölder, for $r \geq 1$, $\alpha \in (0,1]$.

(b) Contreras, Lopes & Thieullen [**CLT**, Thm. 2] have proved a related result, in the spirit of Conjecture 1, concerning the subspace $\mathcal{E}^{1,\alpha+} := \cup_{\beta > \alpha} \mathcal{E}^{1,\beta}$ of $\mathcal{E}^{1,\alpha}$: the set of those $T \in \mathcal{E}^{1,\alpha+}$ with a unique Lyapunov maximizing measure, supported on a periodic orbit, is open and dense in $\mathcal{E}^{1,\alpha+}$ (with respect to the topology of $\mathcal{E}^{1,\alpha}$).

(c) By analogy with Conjecture 1, it seems likely that for all $r \geq 1$, $\alpha \in (0,1]$, a generic $T \in \mathcal{E}^{r,\alpha}$ has a unique Lyapunov maximizing measure, supported on a periodic orbit.

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