# THE MAÑÉ-CONZE-GUIVARC'H LEMMA FOR INTERMITTENT MAPS OF THE CIRCLE

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Abstract. We study the existence of solutions $g$ to the functional inequality $f \leq g \circ T - g + \beta$ where $f$ is a prescribed continuous function, $T$ is a weakly expanding transformation of the circle having an indifferent fixed point, and $\beta$ is the maximum ergodic average of $f$. Using a method due to T. Bousch we show that continuous solutions $g$ always exist when the Hölder exponent of $f$ is close to 1. In the converse direction, we construct explicit examples of continuous functions $f$ with low Hölder exponent for which no continuous solution $g$ exists. We give sharp estimates on the best possible Hölder regularity of a solution $g$ given the Hölder regularity of $f$.

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## 1. Introduction

Let $T\colon X \to X$ be a discrete dynamical system, and let $\mathcal{M}_T$ be the set of all Borel probability measures which are invariant under the map $T$. For a given continuous function $f\colon X \to \mathbb{R}$, define the maximum ergodic average $\beta(f)$ by

$$\beta(f) = \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu,$$

and say that $\nu \in \mathcal{M}_T$ is a *maximising measure* for $f$ if it satisfies $\int f \, d\nu = \beta(f)$. The study of maximising measures has recently become the focus of significant research interest. While early articles of T. Bousch and O. Jenkinson [2, 14] were motivated by abstract questions relating to the geometric structure of the set of measures $\mathcal{M}_T$, questions relating to maximising measures have also appeared in research into chaotic control [13, 25], Livšic-type theorems [6], thermodynamic formalism [9, 15, 16], Tetris heaps [7], and the Lagarias-Wang finiteness conjecture in linear algebra [7].

This article is concerned with a key technical tool arising in the study of maximising measures, which we call the *Mañé-Conze-Guivarc'h lemma*. A lemma of this type takes the following form: given a continuous function $f\colon X \to \mathbb{R}$ with some prescribed regularity, under suitable dynamical hypotheses, we show that there exists a continuous function $g\colon X \to \mathbb{R}$ with the property that $f \leq g \circ T - g + \beta(f)$. This relation is equivalent to the statement that there exists continuous $g$ such that $\sup(f + g - g \circ T) = \beta(f)$. Conze and Guivarc'h's version of this lemma may be found in the unpublished manuscript [10]. It has been noted that theorems of a similar character occur in the field of optimal control, e.g. [1, 17]; this relationship is examined in T. Bousch's recent preprint [5].

We briefly describe the immediate implications of this result. Firstly, rewrite the aforementioned inequality in the form $f = g \circ T - g + \beta(f) - r$, where $r$ is continuous and satisfies $r \geq 0$. We then obtain $\int f\,d\nu = \beta(f) - \int r\,d\nu$ for every $\nu \in \mathcal{M}_T$, and so $\nu$ is maximising for $f$ if and only if $\int r\,d\mu = 0$. Since $r(x) \geq 0$ for all $x$, we conclude that the maximising measures of $f$ are precisely those invariant measures $\nu$ whose support lies in the compact set $r^{-1}(0)$. This leads to the *subordination principle* described by T. Bousch [3]: if invariant measures $\mu, \nu$ satisfy $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$ and $\mu$ is a maximising measure for $f$, then the 'subordinate' measure $\nu$ is maximising also. It has been shown that this subordination principle can fail to hold when the regularity of $f$ is relaxed [6].

A particularly interesting application of the Mañé-Conze-Guivarc'h lemma is a recent result of T. Bousch [4] which shows that for dynamical systems satisfying a Mañé-Conze-Guivarc'h lemma, measures supported on periodic orbits are the only maximising measures which persist under Lipschitz perturbations of the observable $f$. A similar result which was previously shown by G. Yuan and B. R. Hunt under more restrictive dynamical assumptions [25]. Mañé-Conze-Guivarc'h type lemmas have also been found useful in circumstances not *a priori* related to maximising measures [20].

When $T\colon X \to X$ is an expanding map, a subshift of finite type, or an Anosov diffeomorphism, and $f\colon X \to \mathbb{R}$ is Hölder continuous, it is known that we can always find $g\colon X \to \mathbb{R}$ Hölder continuous such that $f \leq g \circ T - g + \beta(f)$ is satisfied [3, 11, 19, 22]. The purpose of the present article is to examine the extension of this result to a simple class of non-uniformly hyperbolic dynamical systems on the circle, namely the case in which $T$ is uniformly expanding except in the neighbourhood of a weakly repelling fixed point.

Previously, it was shown by R. Souza [23] that for an expanding map $T\colon [0,1] \to [0,1]$ with a weakly repelling fixed point, a Mañé-Conze-Guivarc'h lemma may be proved when $f$ is Hölder continuous and monotone in some neighbourhood of the indifferent fixed point $z$, and additionally satisfies $\int f\,d\nu_- < f(z) < \int f\,d\nu_+$ for some $\nu_-, \nu_+ \in \mathcal{M}_T$. Prior to the research described in this article, S. Branton has shown that when $f$ is Lipschitz continuous, Souza's conditions may be removed [8]. In this article, using a different method to S. Branton, we study the case in which $f$ is Hölder, and prove a complementary result which shows that solutions can fail to exist in certain cases when $f$ is Hölder continuous with exponent close to 0.

Let $\mathbb{T} = \mathbb{R} \mod \mathbb{Z}$ with metric $d$ inherited from the standard metric on $\mathbb{R}$. The precise class of maps $T\colon \mathbb{T} \to \mathbb{T}$ which we study is as follows:

**Definition 1.1.** *For each $\alpha > 0$ we say that a continuous function $T\colon \mathbb{T} \to \mathbb{T}$ is an expanding map of Manneville-Pomeau type $\alpha$ if it fixes $0$, is differentiable with derivative greater than $1$ in the interval $\mathbb{T} \setminus \{0\}$, and satisfies*

$$T'(x) = 1 + \xi x^\alpha + o(x^\alpha) \text{ as } x \to 0^+,$$

$$\liminf_{x \to 1^-} T'(x) > 1$$

*for some $\xi > 0$.*

The archetypal map $T$ represented by this definition is the *Manneville-Pomeau map* defined by $x \mapsto x + x^{1+\alpha} \mod 1$. Expanding maps of Manneville-Pomeau type are studied in, for example, [12, 18, 24].

For each $\gamma \in (0,1]$, let $H_\gamma$ denote the space of all $\gamma$-Hölder continuous real-valued functions on the circle $\mathbb{T}$, and define $|f|_\gamma = \sup_{x \neq y} |f(x) - f(y)|/d(x,y)^\gamma$ for $f \in H_\gamma$. The set $H_\gamma$ is a Banach space when equipped with the norm $\|\cdot\|_\gamma$ given by $\|f\|_\gamma := |f|_\infty + |f|_\gamma$. Using a method based on Young towers, S. Branton proved the following:

**Theorem** ([8]). *Let $T\colon \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville-Pomeau type $\alpha \in (0,1)$. Then for every $f \in H_1$ and $\delta \in (0, 1-\alpha)$ there exists $g \in H_{1-\alpha-\delta}$ such that $f \leq g \circ T - g + \beta(f)$.*

We are able to establish:

**Theorem 1.** *Let $T\colon \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville-Pomeau type $\alpha \in (0,1)$ and suppose that $\alpha < \gamma \leq 1$. Then for every $f \in H_\gamma$ there exists $g \in H_{\gamma-\alpha}$ such that $f \leq g \circ T - g + \beta(f)$. In addition, the function $g$ satisfies the functional equation*

$$g(x) + \beta(f) = \max_{Ty=x} \left[ f(y) + g(y) \right].$$

Further, we are able to show that Theorem 1 is sharp both in the regularity of $f$ and in the regularity of $g$.

**Theorem 2.** *Let $T\colon \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville-Pomeau type $\alpha \in (0,1)$ and suppose that $0 < \alpha < \gamma \leq 1$. Then the following hold:*

- *(a) There exists $f \in H_\gamma$ such that if $f \leq g \circ T - g + \beta(f)$ for $g \in H_\theta$, then $\theta \leq \gamma - \alpha$.*
- *(b) There exists $f \in H_\alpha$ such that $f \leq g \circ T - g + \beta(f)$ is not satisfied for any continuous function $g$.*

In a recent article, T. Bousch proves the following theorem, which extends a result of Yuan and Hunt [25]:

**Theorem** ([4]). *Let $T\colon X \to X$ be a continuous surjection of a compact metric space. Suppose that for all $f \in H_1$, there exists $g \in H_1$ such that $f \leq g \circ T - g + \beta(f)$ and $|g|_1 \leq C|f|_1$ for some $C > 0$ independent of $f$. Suppose also that $\mu \in \mathcal{M}_T$ is a maximising measure for every element of some nonempty open set $U \subset H_1$. Then $\mu$ is supported on a periodic orbit of $T$.*

We remark that while uniformly expanding dynamical systems have been shown to satisfy the hypotheses of this theorem (see [3, 11, 22]), Theorem 2(a) demonstrates that the required hypotheses do not hold for maps of Pomeau-Manneville type.

## 2. Proof of Theorem 1

We use a fixed point method occurring in the work of Bousch [2, 4]. We begin with the following lemma:

**Lemma 2.1.** *Let $T$ be of Manneville-Pomeau type $\alpha$, and let $z_1, z_2 \in \mathbb{T}$ with $d(z_1, z_2)$ sufficiently small. Then*

$$d(Tz_1, Tz_2) \geq d(z_1, z_2) \left(1 + C_0 d(z_1, z_2)^\alpha\right)$$

*for some constant $C_0$ depending only on $T$.*

*Proof.* We consider separately the cases in which the shortest arc connecting $z_1$ and $z_2$ does, or does not, contain 0.

We begin with the latter case. Choose representatives $a_1, a_2 \in [0,1)$ of $z_1, z_2 \in \mathbb{T}$ respectively, assuming without loss of generality that $0 \leq a_1 \leq a_2 < 1$. If $d(z_1, z_2)$ is small enough then

$$\begin{aligned} d(Tz_1, Tz_2) &= \int_{z_1}^{z_2} |T'(s)|\, ds \geq \int_{a_1}^{a_2} 1 + \rho_0 s^\alpha \, ds \\ &\geq (a_2 - a_1) + \rho_1 (a_2 - a_1)^{1+\alpha} = d(z_1, z_2) + \rho_1 d(z_1, z_2)^{1+\alpha} \end{aligned}$$

for some small $\rho_0, \rho_1 > 0$ not depending on $z_1$ and $z_2$. This completes the proof in this case.

Now suppose that 0 lies in the arc connecting $z_1$ and $z_2$, with the triple $(z_1, 0, z_2)$ being positively oriented. Arguing as previously we have $d(Tz_2, 0) \geq d(z_2, 0) + \rho_1 d(z_2, 0)^{1+\alpha}$. Since $T$ has derivative bounded away from 1 in any small interval of the form $(-\delta, 0)$, there is $\rho_2 > 0$ such that $d(Tz_1, 0) \geq (1 + \rho_2) d(z_1, 0)$ when $d(z_1, 0)$ is small enough. Combining these estimates yields

$$d(Tz_1, Tz_2) = d(Tz_1, 0) + d(0, Tz_2) \geq d(z_1, z_2) + \rho_1 d(z_2, 0)^{1+\alpha} + \rho_2 d(z_1, 0).$$

If we take $C_0 = \min\{\rho_1/2^{1+\alpha}, \rho_2/2\}$ then by separating the cases $d(z_1, 0) \geq d(z_2, 0)$ and $d(z_1, 0) \leq d(z_2, 0)$ we obtain

$$\rho_1 d(z_2, 0)^{1+\alpha} + \rho_2 d(z_1, 0) \geq C_0 d(z_1, z_2)^{1+\alpha}$$

for every sufficiently close choice of $z_1$ and $z_2$ separated by 0. Combining the above two inequalities completes the proof. □

**Lemma 2.2.** *Let $T$ be of Manneville-Pomeau type $\alpha$, and let $\gamma \in (\alpha, 1]$. Then there exists $C_\gamma > 0$ with the following property: for every $x_1, x_2, y_1 \in \mathbb{T}$ with $Ty_1 = x_1$, we may choose $y_2 \in T^{-1}\{x_2\}$ such that*

$$d(y_1, y_2)^{\gamma-\alpha} + C_\gamma d(y_1, y_2)^\gamma \leq d(x_1, x_2)^{\gamma-\alpha} \tag{1}$$

*Proof.* Given $x_1, x_2, y_1 \in \mathbb{T}$ with $Ty_1 = x_1$, we claim that there exists $y_2 \in T^{-1}\{x_2\}$ such that

$$d(y_1, y_2)(1 + \rho_3 d(y_1, y_2)^\alpha) \leq d(x_1, x_2) \tag{2}$$

for some $\rho_3 > 0$ independent of $x_1, x_2, y_1$. Taking $\rho_4 = (1 + \rho_3)^{\gamma-\alpha} - 1 > 0$ we have $(1 + \rho_3 t)^{\gamma-\alpha} \geq 1 + \rho_4 t$ for all $t \in [0,1]$. Applying this to (2) yields (1) with $C_\gamma = \rho_4$.

We begin by noting that $T$ expands sufficiently long intervals by a uniform factor: for every $\delta > 0$, there exists $K_\delta > 0$ such that if $d(x_1, x_2) \geq \delta$, then $y_2$ may be chosen with

$$(1 + K_\delta) d(y_1, y_2) \leq d(x_1, x_2).$$

Thus given some fixed $\delta > 0$, (2) holds for every case in which $d(x_1, x_2) \geq \delta$ by taking $\rho_3 \leq K_\delta$. On the other hand, if $d(x_1, x_2) < \delta$ for some sufficiently small fixed $\delta > 0$, we may choose $y_2 \in T^{-1}\{x_2\}$ with $d(y_1, y_2) \leq d(x_1, x_2) < \delta$ and apply Lemma 2.1 to obtain

$$d(y_1, y_2)(1 + C_0 d(y_1, y_2)^\alpha) \leq d(x_1, x_2)$$

so that taking $\rho_3 = \min\{K_\delta, C_0\}$ completes the proof. □

We now prove Theorem 1. Let $\gamma \in (\alpha, 1]$, and define a subset of $C(\mathbb{T})$ by

$$K = \left\{ g \in H_{\gamma-\alpha} \colon |g|_{\gamma-\alpha} \leq C_\gamma^{-1} |f|_\gamma \right\},$$

where $C_\gamma > 0$ is as in Lemma 2.2. Let $K_0 = K/\mathbb{R}$, the set of equivalence classes of element of $K$ modulo addition of a constant. Clearly $K_0$ is compact with respect to uniform distance. For each $g \in K$, define $L_f g \in C(\mathbb{T})$ by $(L_f g)(x) = \max_{Ty=x}(f + g)(y)$. We assert that $L_f$ is a continuous transformation of $K$ with respect to uniform distance.

Given $x_1, x_2 \in \mathbb{T}$, choose $y_1 \in T^{-1}x_1$ such that $(L_f g)(x_1) = (f+g)(y_1)$. Invoking Lemma 2.2 we may choose $y_2 \in T^{-1}x_2$ such that (1) holds and therefore

$$\begin{aligned}(L_f g)(x_1) - (L_f g)(x_2) &\leq (f+g)(y_1) - (f+g)(y_2)\\ &\leq |f|_\gamma d(y_1, y_2)^\gamma + |g|_{\gamma-\alpha} d(y_1, y_2)^{\gamma-\alpha}\\ &\leq C_\gamma^{-1} |f|_\gamma d(x_1, x_2)^{\gamma-\alpha}.\end{aligned}$$

We conclude that $|L_f g|_{\gamma-\alpha} \leq C_\gamma^{-1}|f|_\gamma$ for all $g \in K$ and therefore $L_f K \subseteq K$. A simple argument shows that $|L_f g_1 - L_f g_2|_\infty \leq |g_1 - g_2|_\infty$ for $g_1, g_2 \in K$ so that $L_f$ is a continuous tranformation of $K$. It follows that $L_f$ induces a continuous transformation of $K_0$. By the Schauder-Tychonoff theorem there therefore exists $h \in K$ such that $L_f h = h \mod \mathbb{R}$. Let $b \in \mathbb{R}$ be chosen such that $h(x) = b + \max_{Ty=x}(f+h)(y)$ for all $x \in \mathbb{T}$; a simple argument as in [2] shows that $b = \beta(f)$. The proof of Theorem 1 is complete.

## 3. Proof of Theorem 2

In this section we will take the liberty of using the fundamental domain $[0,1)$ as a model for $\mathbb{T}$ and treating $T$ as a map $[0,1) \to [0,1)$ in the obvious fashion. Let $u_1 = \min\{u \in (0,1) \colon Tu = 0\}$, and define a sequence $(u_n)_{n\geq 1}$ in $[0,1)$ by $u_n := \min\{u \in (0,1) \colon Tu = u_{n-1}\}$. We require two simple lemmas:

**Lemma 3.1.** *There is $C_1 > 1$ such that for all $n \geq 1$,*

$$C_1^{-1} n^{-1-1/\alpha} \leq u_n - u_{n+1} \leq C_1 n^{-1-1/\alpha}$$

*and*

$$C_1^{-1} n^{-1/\alpha} \leq u_n \leq C_1 n^{-1/\alpha}.$$

*Proof.* This follows from the relation $Tu_n - u_n = \xi u_n^{1+\alpha} + o(u_n)^{1+\alpha}$ in a fairly straightforward fashion, see e.g. [24]. ☐

**Lemma 3.2.** *Let $f \colon [0,1) \to \mathbb{R}$. Suppose that $f(0) = 0$, that there is $C > 0$ such that for all $\kappa \in (0,1)$,*

$$|f(\kappa)| \leq C\kappa^{\gamma_1}$$

*and*

$$\sup_{\substack{x,y\in[\kappa,1]\\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|} \leq C\kappa^{-\gamma_2},$$

*where $\gamma_1, \gamma_2 > 0$ and $\gamma_1 + \gamma_2 \geq 1$. Then $f$ is $\frac{\gamma_1}{\gamma_1+\gamma_2}$-Hölder continuous throughout $[0,1)$.*

*Proof.* Let $0 \leq x < y < 1$ and let $\lambda = y^{-\gamma_1-\gamma_2}(y-x)$ and $\gamma = \frac{\gamma_1}{\gamma_1+\gamma_2}$. If $\lambda > 1/2$ then $y^{\gamma_1+\gamma_2} < 2(y-x)$ and hence

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq 2Cy^{\gamma_1} < 2^{1+\gamma}C|y-x|^\gamma.$$

If otherwise then $y-x=\lambda y^{\gamma_1+\gamma_2} \leq \lambda y \leq y/2$ so that $0<y\leq 2x$ and hence

$$|f(x)-f(y)| \leq Cx^{-\gamma_2}(y-x)^{1-\gamma}(y-x)^{\gamma} = C\lambda^{1-\gamma}\left(\frac{y}{x}\right)^{\gamma_2}(y-x)^{\gamma} \leq 2^{\gamma-1+\gamma_2}C(y-x)^{\gamma}$$

as required. □

3.1. **Proof of part (a).** Given $0<\alpha<\gamma\leq 1$, let $K_\gamma = C_1\sum_{n=1}^\infty n^{-\gamma/\alpha} < \infty$. Define $f(x)=x^\gamma$ for all $x\in[0,u_3]$, $f(x)=-K$ for all $x\in[u_2,u_1]$, and define $f$ by linear interpolation in the intervals $[u_3,u_2]$ and $[u_1,1)$ subject to the constraint $\lim_{x\to 1} f(x)=0$ which ensures that $f$ yields a continuous function $\mathbb{T}\to\mathbb{R}$. Note that $f(x)\leq u_k^\gamma$ when $u_{k+1}\leq x\leq u_k$ and that $f\in H_\gamma$.

We claim that $\beta(f)=0$. Since the Dirac measure $\delta_0$ is invariant and $f(0)=0$ it is clear that $\beta(f)\geq 0$. By a lemma of Y. Peres [21] there exists $x\in\mathbb{T}$ such that $\sum_{j=0}^{n-1} f(T^jx)\geq n\beta(f)$ for all $n\geq 0$, and so to prove that $\beta(f)\leq 0$ it is sufficient for us to show that for each $x\in[0,1]$ we may find $v(x)>0$ such that $\sum_{j=0}^{v(x)-1} f(T^jx)\leq 0$.

If $x=0$ or $x\in[u_2,1)$ then clearly we may take $v(x)=1$. Otherwise we have $x\in U_r$ for some $r\geq 2$. Applying Lemma 3.1 we have

$$\sum_{j=0}^{r} f(T^jx) \leq \sum_{j=0}^{r-1}\left(T^jx\right)^\gamma - K \leq \sum_{k=1}^{r} u_k^\gamma - K \leq C_1\sum_{k=1}^{\infty} k^{-\gamma/\alpha} - K = 0$$

so that taking $v(x)=r+1$ proves the claim.

Now suppose that $f\leq g\circ T - g + \beta(f)$ where $g\in H_\theta$. For every $n>0$ and $r\geq 3$, we have

$$g(u_{n+r}) + \sum_{j=0}^{n-1} f(T^ju_{n+r}) \leq g(T^nu_{n+r}),$$

and therefore

$$g(u_r) \geq \sum_{k=r+1}^{r+n} f(u_k) + g(u_{n+r}) \geq C_1^{-1}\sum_{k=r+1}^{r+n} k^{-\gamma/\alpha} + g(u_{n+r}).$$

Taking the limit as $n\to\infty$ it follows that

$$g(u_r) \geq C_1^{-1}\sum_{k=r+1}^{\infty} k^{-\gamma/\alpha} + g(0) \geq \tilde{C}r^{1-\gamma/\alpha} + g(0),$$

and therefore

$$\tilde{C}r^{-1-\gamma/\alpha} \leq |g(0)-g(u_r)| \leq |g|_\theta u_r^\theta \leq |g|_\theta C_1^\theta r^{-\theta/\alpha}$$

for every $r\geq 3$. We deduce that $\theta\leq\gamma-\alpha$. □

3.2. **Proof of part (b).** Define $f(0)=0$, $f(x)=0$ for all $x\in[u_1,1)$, and for each $n\geq 0$

$$f(u_{2^{4n}}) = f(u_{2^{4n+2}}) = 0$$

$$f(u_{2^{4n+1}}) = -2^{-4n}$$

$$f(u_{2^{4n+3}}) = \tau 2^{-4n},$$

where $\tau\in(0,1)$ is a real number to be fixed later. Extend $f$ to the whole of $[0,1)$ by interpolating linearly in each interval $[u_{2^{4n+k+1}}, u_{2^{4n+k}}]$.

We will show that $f$ is $\alpha$-Hölder. Suppose that $u_{2^{4n+4}} \leq \kappa \leq u_{2^{4n}}$ for some $n \geq 0$; then,

$$|f(\kappa)| < 2^{-4n} \leq C_1^\alpha u_{2^{4n}}^\alpha \leq C_1^\alpha \kappa^\alpha. \tag{3}$$

We must estimate the Lipschitz norm of $f$ in the interval $[\kappa, 1)$. We will require the simple lower bound

$$\begin{aligned} u_{2^{r+1}} - u_{2^r} = \sum_{\ell=0}^{2^r-1} u_{2^r+\ell+1} - u_{2^r+\ell} &\geq \sum_{k=2^r}^{2^{r+1}-1} C_1^{-1} k^{-1-1/\alpha} \\ &\geq \tilde{C}\left(2^{-r/\alpha} - 2^{-(r+1)/\alpha}\right) \geq \tilde{C} 2^{-r/\alpha} \end{aligned}$$

for all $r > 0$, where we have used Lemma 3.1. It follows that when $u_{2^{4n+4}} \leq \kappa \leq u_{2^{4n}}$, the gradient of $f$ in $[\kappa, 1)$ is bounded by

$$\sup_{\substack{0 \leq k \leq n \\ 0 \leq \ell < 4}} \frac{2^{-4k}}{|u_{2^{4k+\ell+1}} - u_{2^{4k+\ell}}|} \leq \sup_{\substack{0 \leq k \leq n \\ 0 \leq \ell < 4}} \frac{2^{-4k}}{\tilde{C} 2^{-(4k+\ell)/\alpha}} = \tilde{C} 2^{-4k+4k/\alpha} \leq \tilde{C} \kappa^{\alpha-1}. \tag{4}$$

Combining estimates (3) and (4) with Lemma 3.2 we deduce that $f \in H_\alpha$.

We next compute $\beta(f)$. Since $f(0) = 0$ and the Dirac measure $\delta_0$ is $T$-invariant, we have $\beta(f) \geq 0$. To prove that $\beta(f) = 0$ we proceed as in part (a) by showing that for each $x \in [0,1)$, there is $v(x) > 0$ such that $\sum_{j=0}^{v(x)-1} f(T^j x) \leq 0$.

If $x \geq u_2$, or $x = 0$, or if $u_{2^{4n+2}} \leq x \leq u_{2^{4n}}$ for some $n > 0$, then $f(x) \leq 0$ and we may take $v(x) = 1$. We may therefore restrict our attention to the case in which $u_{2^{4n+4}} < x < u_{2^{4n+2}}$ for some $n \geq 0$. Assuming this, suppose that

$$u_{2^{4n+2}+k+1} \leq x \leq u_{2^{4n+2}+k},$$

where $0 \leq k < 2^{4n+4} - 2^{4n+2}$. We choose $v(x) = k + 2^{4n+1} + 2$. Firstly we note that

$$\sum_{j=0}^{k} f(T^j x) \leq \tau k 2^{-4n} \leq 12\tau. \tag{5}$$

Using the monotonicity of $f$ in $[u_{2^{4n+1}}, u_{2^{4n}}]$, we obtain

$$\begin{aligned} \sum_{j=k+1}^{k+2^{4n+1}+1} f(T^j x) \leq \sum_{\ell=0}^{2^{4n+1}} f(u_{2^{4n+1}+\ell}) &= -\sum_{\ell=1}^{2^{4n+1}} 2^{-4n} \frac{|u_{2^{4n+1}} - u_{2^{4n+1}+\ell}|}{|u_{2^{4n+1}} - u_{2^{4n+2}}|} \\ &\leq -\sum_{\ell=1}^{2^{4n+1}} 2^{-4n} u_{2^{4n+1}}^{-1} \left(u_{2^{4n+1}} - u_{2^{4n+1}+\ell}\right) \\ &\leq -C_1^{-1} 2^{-1/\alpha - 4n + 4n/\alpha} \sum_{\ell=1}^{2^{4n+1}} \sum_{j=0}^{\ell-1} \left(u_{2^{4n+1}+j} - u_{2^{4n+1}+j+1}\right) \\ &\leq -C_1^{-2} 2^{-1/\alpha - 4n + 4n/\alpha} \sum_{\ell=1}^{2^{4n+1}} \ell 2^{-(4n+2)(1+1/\alpha)} \\ &\leq -\frac{1}{C_1^2 2^{2+3/\alpha}} 2^{-8n} \sum_{\ell=1}^{2^{4n+1}} \ell \leq -\frac{1}{C_1^2 2^{5+2/\alpha}} = -\varepsilon < 0, \end{aligned}$$

say, where we have twice used Lemma 3.1. Combining this with (5) we deduce that $\sum_{j=0}^{v(x)-1} f(T^j x) \leq \max\{0, 12\tau - \varepsilon\}$ for each $x \in [0,1)$ and so if $\tau$ is taken smaller than $\varepsilon/12$ then $\beta(f) = 0$.

Our final task is to show that the relation $f \leq g \circ T - g + \beta(f)$ is impossible for continuous $g$. Following the method of the preceding estimate, for each $n > 0$ we have

$$\begin{aligned}\sum_{\ell=2^{4n+2}}^{2^{4n+3}} f(u_\ell) &\geq \tau \sum_{\ell=1}^{2^{4n+2}} 2^{-4n} \frac{|u_{2^{4n+2}} - u_{2^{4n+2}+\ell}|}{|u_{2^{4n+2}} - u_{2^{4n+3}}|} \\ &\geq \tau \tilde{C} 2^{-4n+4n/\alpha} \sum_{\ell=1}^{2^{4n+2}} \sum_{j=0}^{\ell-1} \left(u_{2^{4n+2}+j} - u_{2^{4n+2}+j+1}\right) \\ &\geq \tau \tilde{C} 2^{-8n} \sum_{\ell=1}^{2^{4n+2}} \ell \geq \delta_\tau > 0,\end{aligned}$$

say. Suppose now that $f \leq g \circ T - g + \beta(f)$ is satisfied. Then for each $n > 0$ we have

$$g(u_{2^{4n+2}}) \geq g(u_{2^{4n+3}}) + \sum_{j=0}^{2^{4n+3}-2^{4n+2}} f(T^j u_{2^{4n+3}}) \geq g(u_{2^{4n+3}}) + \delta_\tau.$$

If $g$ is continuous at 0, then letting $n \to \infty$ yields

$$g(0) \geq g(0) + \delta_\tau > g(0),$$

a contradiction. □

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