Matrix equations for the relations between two-port parameters

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Abstract Algorithms are presented for linking together the extensive set of two-port parameter matrices in widespread use. These relations are given in a form readily suited to implementation in a computer program.
1. **Introduction**

A complete list of the relations between two-port parameters [1] has been published. It was thought that a useful complement would be a list of matrix equations which can be used to calculate any parameter matrix from any other matrix, and this paper presents such a list. In order to implement the algorithms in a computer program, it is only necessary to provide two subroutines: one for inverting a complex $2 \times 2$ matrix and one for multiplying a pair of complex $2 \times 2$ matrices.

2. **Notation**

The following notation is used below:

\[
M = \begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}
\]

where $M$ is any of the matrices $Z, Y, H, G, A, B, S, T, U$ [2,3]. These in turn are defined in terms of the quantities indicated in the figure.
Z represents the impedance matrix and Y the admittance matrix:

\[
\begin{bmatrix}
  v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix} =
\begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
\]

The h-parameter matrix is \( H \) and \( G \) is its inverse:

\[
\begin{bmatrix}
v_1 \\
i_2
\end{bmatrix} =
\begin{bmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
v_2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
i_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
i_2
\end{bmatrix}
\]

The elements of \( A \) are known as the general circuit constants and \( B \) is the inverse of \( A \), so that

\[
\begin{bmatrix}
v_1 \\
i_1
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
v_2 \\
-i_2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
v_2 \\
-i_2
\end{bmatrix} =
\begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
i_1
\end{bmatrix}
\]

These six matrices are referred to as the circuit parameter matrices since they relate voltages and currents.

In order to make our results of as general application as possible, we have allowed for unequal real characteristic impedances \( Z_{01}, Z_{02} \), by using the following definitions for \( a_1, a_2, b_1, b_2 \) (the forward and backward wave amplitudes) in terms of \( v_1, v_2, i_1, i_2 \) (the voltages and currents):

\[
a_n \triangleq q_n v_n + r_n i_n \quad b_n \triangleq q_n v_n - r_n i_n
\]

where

\[
q_n \triangleq \frac{1}{2\sqrt{Z_{0n}}} \quad r_n \triangleq \frac{\sqrt{Z_{0n}}}{2} \quad n = 1, 2
\]

The scattering matrix is denoted by \( S \) and is defined as follows:

\[
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} =
\begin{bmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
\]

We denote the scattering transfer matrix by \( T \) and the inverse of \( T \) by \( U \), i.e.

\[
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix} =
\begin{bmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{bmatrix}
\begin{bmatrix}
a_2 \\
b_2
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
b_2 \\
a_2
\end{bmatrix} =
\begin{bmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22}
\end{bmatrix}
\begin{bmatrix}
b_1 \\
a_1
\end{bmatrix}
\]
These three matrices comprise the wave parameter matrices, so called because they relate the forward and backward wave amplitudes.

It is also convenient to use the following constant matrices:

\[
I \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad J \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

\[
I_1 \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad I_2 \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad I_3 \triangleq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad I_4 \triangleq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

Some other constant matrices involving \(Z_{01}\) and \(Z_{02}\) are also defined in the appropriate sections.

Many of the constant matrices can be expressed in terms of each other, for example \(I_1 = I - I_4\). However, it was thought that clarity would be enhanced with the present arrangement. The set of supplementary matrices is therefore not minimal.

We present below sufficient of the relations between the various two-port parameters to allow the calculation of all relationships by using matrix inversion and multiplication only. Taking as an example equation (1), which gives \(S(T)\), \(S(U)\) can easily be found by substituting \(U^{-1}\) for \(T\) everywhere in that equation. Likewise, the first part of equation (3) gives \(Z(H)\), from which \(Z(G)\) can be immediately calculated by substituting \(G^{-1}\) for \(H\); furthermore, \(Y(H)\) can be found by inverting \(Z(H)\), and \(Y(G)\) by inverting \(Z(G)\).

3. Wave parameter relations

\[
S = (TI_4 - I_1)^{-1}(I_3 - TI_2) \quad (1)
\]

\[
T = (I_1S + I_3)(I_2 + I_4S)^{-1} \quad (2)
\]

4. Circuit parameter relations

\[
Z = (HI_4 - I_1)^{-1}(I_4 - HI_1) = (AI_2 - I_1)^{-1}(I_3 + AI_4) \quad (3)
\]

\[
H = (I_1Z + I_4)(I_1 + I_4Z)^{-1} = (AI_4 + I_1)^{-1}(AI_2 - I_3) \quad (4)
\]

\[
A = (I_1Z + I_3)(I_2Z - I_4)^{-1} = (I_1H + I_3)(I_2 - I_4H)^{-1} \quad (5)
\]
5. Mixed parameter relations

\[
Q_1 \triangleq \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \quad R_1 \triangleq \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}
\]

\[
S = (Q_1Z - R_1)(Q_1Z + R_1)^{-1}
\]

\[
Z = Q_1^{-1}(I - S)^{-1}(I + S)R_1
\]  

\[
Q_2 \triangleq \begin{bmatrix} q_1 & 0 \\ 0 & r_2 \end{bmatrix} \quad R_2 \triangleq \begin{bmatrix} r_1 & 0 \\ 0 & q_2 \end{bmatrix}
\]

\[
S = J(Q_2H - R_2)(Q_2H + R_2)^{-1}
\]

\[
H = Q_2^{-1}(J - S)^{-1}(J + S)R_2
\]  

\[
Q_3 \triangleq \begin{bmatrix} q_1 & r_1 \\ 0 & 0 \end{bmatrix} \quad R_3 \triangleq \begin{bmatrix} 0 & 0 \\ q_2 & r_2 \end{bmatrix}
\]

\[
S = (Q_3JA + R_3)(Q_3A + R_3J)^{-1}
\]

\[
A = (SQ_3 - Q_3J)^{-1}(R_3 - SR_3J)
\]  

\[
Q \triangleq \begin{bmatrix} q_1 & -r_1 \\ q_1 & r_1 \end{bmatrix} \quad R \triangleq \begin{bmatrix} q_2 & -r_2 \\ q_2 & r_2 \end{bmatrix}
\]

\[
T = Q(I_1Z + I_3)(I_2Z - I_4)^{-1}R^{-1}
\]

\[
= Q(I_1H + I_3)(I_2 - I_4H)^{-1}R^{-1} = QAR^{-1}
\]  

\[
Z = (TRI_2 - QI_1)^{-1}(TRI_4 + QI_3)
\]

\[
H = (TRI_4 + QI_1)^{-1}(TRI_2 - QI_3)
\]

\[
A = Q^{-1}TR
\]  

5
6. References

