The power series method for fast solution of $\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \mathbf{x}^2 = \alpha + \beta \sin \mathbf{t}$

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We give details of the power series method for fast numerical solution of the differential equation

$$\ddot{x} + \gamma\dot{x} + x^2 = \alpha + \beta \sin t, \tag{1}$$

whose solution contains movable singularities. The method is based on a modified version of that described in [1], the modification coming about because we assume that the singularities all have the order suggested by leading order analysis.
At a movable singularity, leading order analysis [2] can be used to determine the order of the singularities in solutions of (1) in the complex time plane. By making the ansatz

$$A(z) = a(z - z_0)^\rho, \tag{2}$$

where $z_0 \in \mathbb{C}$ is the location of a movable singularity and a and ρ are real constants to be determined, the form of a solution of (1) at the singularity can be established. By substituting (2) into the differential equation (1), ρ can be determined by balancing the most singular terms, which gives $\rho = -2$. Furthermore, since we require real solutions to (1) for real initial data, we assume that the whole solution can be represented as a series of the form

$$A(z) = \sum_{n=0}^{\infty} \frac{a_n}{(z - z_n)^2} + \frac{a_n}{(z - \bar{z}_n)^2}, \tag{3}$$

where the singularity positions, z_n and $\bar{z}_n$, constitute a complex conjugate pair. In brief, the proposed fast solution method enables us to calculate the solution to (1) at a time t_{M+1} given the solution and its derivative at t_M, where $t_{M+1} - t_M$ is not necessarily small, using a form of numerically-implemented analytical continuation. In summary, the method consists of the following steps:

1. Use the differential equation to calculate, recursively, the coefficients in the power series expansion of the solution in powers of $(t - t_M)$, given the initial data $x(t_M)$ and $\dot{x}(t_M)$;

2. Taking into account only the pair of poles nearest to $t = \text{Re}(z) = t_M$ in equation (3), these being at $z = z_0$ and $z = \bar{z}_0$, expand the truncated $A(z)$, $A_1(z) = a_0\left((z-z_0)^{-1} + (z-\bar{z}_0)^{-1}\right)$, in a Taylor series in powers of $(t - t_M)$, and compare coefficients with the series found in item (1).

We shall show shortly that the high-order coefficients of both series are asymptotically equal, and from this the position of the nearest pair of poles to $t = t_M$ can be estimated as follows. Since both these poles lie on a circle of radius r_M centred at $t = t_M$, the solution to (1) can be estimated within a disc of radius $\mu\, r_M$, $\mu \in [0, 1)$, with centre t_M, by summing the power series obtained in item (1) above, and its derivative. In practice, we use $\mu = 1/2$, and so, from the solution at $t = t_M$, we estimate the solution at $t = t_{M+1} = t_M + r_M/2$. In practice, $r_M \sim 1$ can be quite large, and so the solution can be advanced in large steps — compared to standard numerical methods like Runge-Kutta — using this method.
Details are now given. The radius of convergence of $A_1(z)$ is determined as follows. The Maclaurin series for $A_1(z)$ with $a_0 = 1$, since with hindsight, a_0 will cancel out, is given by

$$A_1(z) = \sum_{n=0}^{\infty} b_n z^n$$

where, for further simplification, we introduce coefficients d_n, which are the b_n rescaled, and which are given by

$$d_n \equiv \frac{b_n}{n+1} = \frac{1}{{z_0}^{n+2}} + \frac{1}{\bar{z}_0^{n+2}}.$$

Expressing the position of the singularity z_0 in polar form, $z_0 = r\exp(i\theta)$, and its conjugate $\bar{z}_0 = r\exp(-i\theta)$, and using standard trigonometrical relations, we obtain

$$d_n = \frac{2}{r^{n+2}}\left[\cos(n\theta)\cos(2\theta) - \sin(n\theta)\sin(2\theta)\right], \tag{4}$$

$$d_{n-1} = \frac{2}{r^{n+1}}\left[\cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta\right], \tag{5}$$

$$d_{n-2} = \frac{2}{r^{n}}\cos(n\theta). \tag{6}$$

By using (6) to eliminate $\cos(n\theta)$ from (4) and (5), and then eliminating $\sin(n\theta)$ between the two remaining equations, the recursion relation

$$d_n = \frac{2\cos\theta}{r} d_{n-1} - \frac{1}{r^2} d_{n-2}. \tag{7}$$

is obtained. Note that it is here where a_0 would have cancelled out as each of the d_n, d_{n-1} and d_{n-2} terms would have been multiplied by a_0. The solution to the differential equation (1), expanded about the point $t = t_M$, is given by

$$x(t) = \sum_{n=0}^{\infty} c_n (t - t_M)^n, \tag{8}$$

which has the initial conditions, $x(t_M) = c_0$ and $\dot{x}(t_M) = c_1$ built in to it. We now show that as $n \to \infty$, $c_n \sim b_n$. This is accomplished by truncating (3) at $n = 2$ which gives

$$A_2(z) = \frac{a_0}{(z - z_0)^2} + \frac{a_0}{(z - \bar{z}_0)^2} + \frac{a_1}{(z - z_1)^2} + \frac{a_1}{(z - \bar{z}_1)^2}, \tag{9}$$

where $|z_1| > |z_0|$. By substituting the polar form of the position of the singularities $z_k = r_k \exp(i\theta_k)$ and their conjugates $\bar{z}_k = r_k \exp(-i\theta_k)$, $k = 0, 1$, into (9) and using

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n,$$

(9) becomes

$$A_2(z) = \sum_{n=0}^{\infty} e_n z^n,$$

with

$$\frac{e_n}{n+1} = \frac{2a_0 \cos(n+2)\theta_0}{{r_0}^{n+2}} + \frac{2a_1 \cos(n+2)\theta_1}{{r_1}^{n+2}}.$$

Now since $r_1 > r_0$, $\lim_{n\to\infty} e_n = 2(n+1)a_0 \cos(n+2)\theta_0 / {r_0}^{n+2}$, and hence $e_n \sim a_0 d_n$: hence, only the nearest pair of poles influences the high-order terms in the power series solution of (1), and so c_n also obeys the recursion relation (7) for large enough n.

Now, using the Frobenius method to find the coefficients in the power series (8) gives

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)c_{n+2} + \gamma(n+1)c_{n+1} + h_n\right] (t - t_M)^n =$$

$$\alpha - \beta \sin t_M \left[\sum_{n=0}^{\infty} \frac{(-1)^n (t - t_M)^{2n}}{(2n)!} \right] + \beta \cos t_M \left[\sum_{n=0}^{\infty} \frac{(-1)^n (t - t_M)^{2n+1}}{(2n+1)!} \right], \tag{10}$$

where

$$h_n = \sum_{j=0}^{n} c_{n-j} c_j.$$

By equating coefficients of powers of $(t - t_M)$ in (10) and using the initial conditions $c_0 = x(t_M)$ and $c_1 = \dot{x}(t_M)$, as many c_n, $n > 1$ as required can be found.

Finally, since c_n and d_n asymptotically obey the same recursion formula for large enough n, we now compute $c_K, \ldots, c_L$ and estimate r and θ by least squares fitting this subset of the c_n to the recursion relation (7). Typical values used in practice were $K \sim 20, L \sim 50$. Hence, r_M, the distance from t_M to the nearest singularity in the complex plane, can be found, and the solution can be advanced in large steps of, say $r_M/2$. The method is illustrated in figure 1.

If for any reason the power series method fails, the Runge-Kutta method can be used to advance the solution a small distance along the t-axis. For instance, it is known [3] that

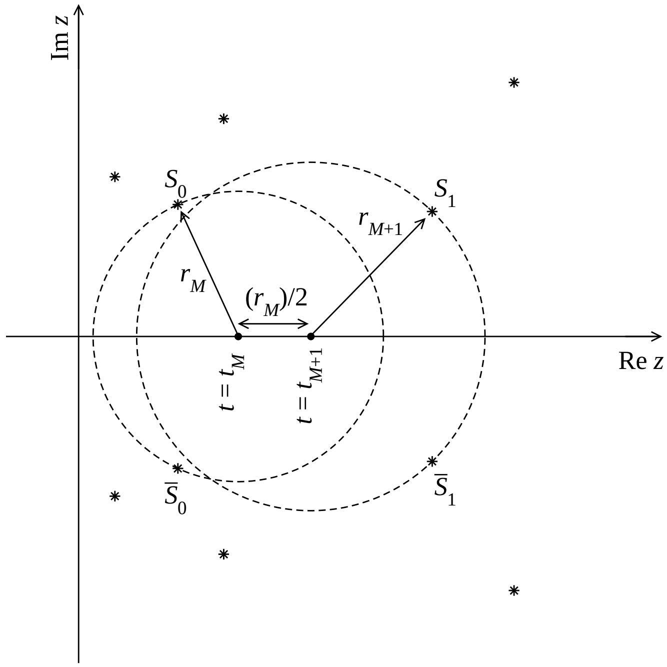

Figure 1: How the solution is advanced from t_M to t_{M+1}. The asterisks denote conjugate pairs of singularities. The nearest pair to point $t = t_M$ is denoted by $S_0, \bar{S}_0$ and the nearest to $t = t_{M+1}$, by $S_1, \bar{S}_1$. These are at distances r_M and r_{M+1} from t_M and t_{M+1} respectively. The disc of convergence for each point is shown as a dashed circle and within this disc, the power series for the solution converges. We assume convergence using a small number of terms (typically about 50) anywhere within a disc of radius $r_M/2$, where $r_M/2$ is the size of the step we can take — usually, $r_M/2$ is very much larger than the steps used in standard numerical methods for ODE solution.

the solutions to equation (1) can possess singularities on the real axis, leading to finite-time blow up, and if such a singularity were to be encountered, the power series method would fail. In this case, the solution is not meaningful for times greater than that at which blow up occurs. Occasionally however, *two* pairs of singularities happen to be close to the boundary of the disc of convergence, and then the calculation of r_M gives meaningless answers, because the recursion relation in the form (7) no longer holds. In this case, using Runge-Kutta to advance the solution for a few small timesteps gets around the difficulty.

References

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[3] Michele V. Bartuccelli, Guido Gentile and Jonathan H. B. Deane, Globally and locally attractive solutions for quasi-periodically forced systems, (to be submitted, 2006).