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Invariant sets for the varactor equation

By M. V. Bartuccelli[1], J. H. B. Deane[1,*], G. Gentile[2] and L. Marsh[1]

[1]*Department of Mathematics and Statistics, University of Surrey, Guildford GU2 7XH, UK*
[2]*Dipartimento di Matematica, Università di Roma Tre, Roma 00146, Italy*

The differential equation $\ddot{x}+\gamma\dot{x}+x^{\mu}=f(t)$ with $f(t)$ positive, periodic and continuous is studied. After describing some physical applications of this equation, we construct a variety of invariant sets for it, thereby partitioning the phase plane into regions in which solutions grow without bound and also those in which bounded periodic solutions exist.

Keywords: invariant sets; nonlinear circuit dynamics

1. Introduction

We construct invariant sets for the differential equation

$$\ddot{x}+\gamma\dot{x}+x^{\mu}=f(t), \tag{1.1}$$

where $\gamma>0$, $\mu>1$ and $f(t)$ is a continuous, bounded, positive, non-constant, periodic function with finite period τ and mean $\langle f\rangle=\tau^{-1}\int_0^{\tau}f(t)\mathrm{d}t$. In order that solutions remain real, when μ is not an integer we assume that equation (1.1) only applies for $x\geq 0$. It will be convenient to write the bounds on $f(t)$ as $\max_{t\in\mathbb{R}} f(t)=F^{\mu}$ and $\min_{t\in\mathbb{R}} f(t)=f^{\mu}$ with $F>f>0$. Since $f(t)$ is non-constant and continuous, $\langle f\rangle<F^{\mu}$. We re-write the differential equation as

$$\left.\begin{aligned}\dot{x}&=y,\\ \dot{y}&=f(t)-\gamma y-x^{\mu}.\end{aligned}\right\} \tag{1.2}$$

This equation with constant $f(t)$ is relatively trivial. The case of non-constant $f(t)$ arises in at least three different contexts.

(i) A simple electronic circuit, shown in figure 1 and known as the resistor–inductor–varactor circuit, is described, after linear rescaling (Deane & Marsh 2004; Marsh in preparation), by equation (1.1) provided that $x>0$, $\forall\, t$. The varactor is a particular type of diode, which is a nonlinear electronic device analogous to a nonlinear spring—one for which Hooke's law is modified to read 'applied force is proportional to x^{μ}', where x is the extension and typically $\mu\in[1.5, 2.5]$. We present results for the representative value $\mu=2$ and also for the more general case $\mu>1$. The mechanical analogies of the resistor and the inductor are, respectively, a source of linear damping and a constant mass. The full model for this circuit, i.e. one in which the

* Author for correspondence (j.deane@eim.surrey.ac.uk).

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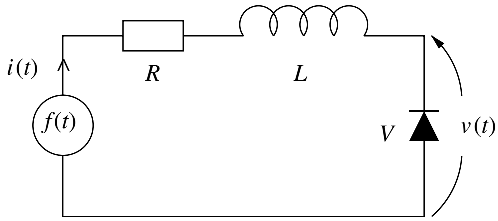

Figure 1. The resistor–inductor–varactor (R, L, V, respectively) circuit whose dynamics are described by equation (1.1). The state variables are the varactor voltage, $v(t)$, and the current, $i(t)$, which can be transformed into $x(t)$ and $\dot{x}(t)$, respectively.

restriction $x>0$ is removed, possesses a nonlinearity of a different form, $c_1\exp(c_2|x|)$, c_1, c_2 constants, for $x\leq 0$, and has been extensively studied—see for instance Azzouz *et al.* (1983) and Matsumoto *et al.* (1984).

(ii) Studies of ship roll and capsize have led to investigations of the behaviour of the ODE $\ddot{u}+\gamma\dot{u}+u-u^2=F\sin\omega t$ (Thompson 1997). Substituting $u=-x+1/2$ gives $\ddot{x}+\gamma\dot{x}+x^2=1/4-F\sin\omega t$, which is equation (1.1) with $f(t)=1/4-F\sin\omega t$ and $\mu=2$.

(iii) Stationary wave solutions of a perturbed Korteweg-de Vries (KdV) equation are described by a special case of equation (1.1) with $\gamma=0$ and $\mu=2$. Following (Blyuss 2002), we start with a perturbed KdV equation $u_\tau+cu_\xi+\beta u_{\xi\xi\xi}=f(u,\xi-V\tau)_\xi$, where $f(u,\ \xi-V\tau)$ is taken to be $f_0\cos\omega(\xi-V\tau)$, and subscripts refer to partial differentiation. The standard transformation to a moving frame, $\xi'\mapsto\xi-V\tau$, $\tau'\mapsto\tau$, results in $\beta u_{\xi'\xi'}-vu+u^2/2=f_0\cos\omega\xi'+\mathrm{C}$ in the steady state ($u_{\tau'}=0$), with $v=V\pm c$ and C a constant of integration. Finally, letting $u=2\beta x+v$ and re-naming ξ' as t, we again obtain equation (1.1) with $f(t)=(v^2+2\mathrm{C}+2f_0\cos\omega t)/4\beta^2)$, $\gamma=0$ and $\mu=2$.

2. Invariant sets, $\mu=2$

We define an invariant set, $S\subset\mathbb{R}^2$, as a subset of the phase plane such that solutions starting from an initial condition in S remain in S for all time. We use the term 'absorbing set' for an invariant set of finite area, with the intention that any bounded limit cycle solutions of equation (1.1) can be shown to lie within such a set, and two such sets, $\mathcal{A}_1$ and $\mathcal{A}_2$, are constructed.

In order to construct invariant sets for equation (1.1) we need to prove certain inequalities. To this end, we first assume that $\gamma=0$, $\mu=2$ and $f(t)=A$, a constant. Then, with $y=\dot{x}$, equation (1.1) becomes

$$y\frac{\mathrm{d}y}{\mathrm{d}x}+x^2=A, \tag{2.1}$$

which can be integrated to give

$$y^2=y_0^2+\frac{2}{3}(x_0-x)\left[\left(x+\frac{1}{2}x_0\right)^2+\frac{3}{4}x_0-3A\right], \tag{2.2}$$

where x_0, y_0 are the initial conditions. This elliptic curve plays an important role in the construction of an invariant set, $\mathcal{B}$, of initial conditions for solutions all of which eventually grow without bound.

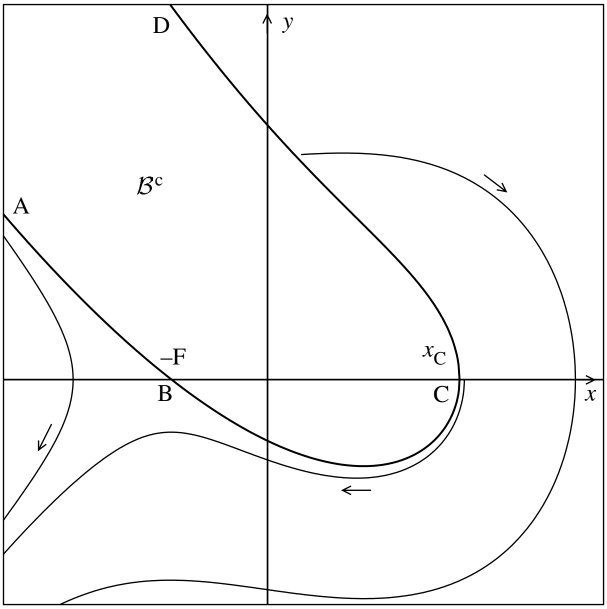

Figure 2. The invariant set $\mathcal{B}$, whose complement, $\mathcal{B}^c$, is inside the thick curve ABCD, where B is $(-F,\ 0)$, C is $(x_C,\ 0)$ and A and D are at infinity. Points on the boundary ABCD belong to $\mathcal{B}$. The thin curves are various numerical solutions to equation (1.1), with $\gamma=0.01$, $\mu=2$, and $f(t)=(5+3\sin t)/2$ so $F=2$, $f=1$. As expected, all the solutions that start in $\mathcal{B}$ remain in $\mathcal{B}$.

Depending on the parameters appearing in equation (2.2), the expression for y^2 can have one, two or three real roots; when there are two, one of these must be a pair of repeated roots, and the condition for this is easily seen from equation (2.2) to be $x_0=\pm 2\sqrt{A}$, $y_0=0$. When this is satisfied, the curve $y(x)$ is known as the separatrix since it separates the two qualitatively different types of behaviour (solutions that grow without bound, and those that are bounded and periodic) displayed by equation (2.1).

(*a*) *Construction of set* $\mathcal{B}$

We now construct $\mathcal{B}$, shown in figure 2, whose boundary consists of three curves, AB, BC and CD, where points A and D are at infinity. The technique used for all boundaries is essentially as follows. We define the two-dimensional vector fields

$$\boldsymbol{\phi}(t) = (y, f(t)-x^2-\gamma y), \quad \boldsymbol{\phi}_F = (y, F^2-x^2-\gamma y) \quad \text{and} \quad \boldsymbol{\phi}_f = (y, f^2-x^2-\gamma y).$$

The importance of the second and third fields is that, for any initial condition $(x, y)=(x(t_0),\ y(t_0))$, the direction of $\boldsymbol{\phi}(t)$ is such that $\boldsymbol{\phi}(t)=\mu_1\boldsymbol{\phi}_F+\mu_2\boldsymbol{\phi}_f$, where μ_1, μ_2 are non-negative scalars which sum to one. (Put loosely, $\boldsymbol{\phi}(t)$ 'lies between' $\boldsymbol{\phi}_f$ and $\boldsymbol{\phi}_F$). This simple observation allows us to project the three-dimensional system equation (1.2) onto the x, y phase plane.

If a given curve in the plane is defined by $G(x, y)=0$, then two normals to it are $\boldsymbol{n}=\pm(\partial G/\partial x,\ \partial G/\partial y)$. The choice of sign determines whether the normal is inward or outward pointing. To prove that the flow is always in a particular direction across a curve defined by G, we then only have to show that $\boldsymbol{n}\cdot\boldsymbol{\phi}$, which is proportional to the cosine of the angle between the normal to the curve and the vector field, $\boldsymbol{\phi}$, is of a given sign at all points on the curve. Hence it will be unnecessary to normalize either $\boldsymbol{n}$ or $\boldsymbol{\phi}$, as only the sign of the dot product is important.

(i) *Boundary* AB

Lemma 2.1. *Let curve* AB *be defined by* $G_{\mathrm{AB}}(x, y)=y^2-\lambda^2(2F-x)(x+F)^2=0$ *for* $x\in(-\infty, -F]$ *with* $y\geq 0$. *Then* $\boldsymbol{\phi}(t)$ *is into* AB *in the direction of decreasing* y, *along its entire length and for all time, provided that* $\lambda^2\leq 2/3$.

Proof. Since $y\geq 0$, curve AB is $y=-\lambda(x+F)\sqrt{2F-x}\geq 0$ for $x\in(-\infty,-F]$. The required normal to AB, pointing in the direction of negative y, is $\boldsymbol{n}=(-3\lambda^2(x^2-F^2),-2y)$. At any point (x, y), the y-component of $\boldsymbol{\phi}_F$ is greater than that of $\boldsymbol{\phi}_f$, so to show that for all time the flow is through AB in the direction of negative y, we should prove that $\boldsymbol{n}\cdot\boldsymbol{\phi}_F\geq 0$ for $x\in(-\infty,-F]$. The fact that $\boldsymbol{n}\cdot\boldsymbol{\phi}_f\geq 0$ then automatically follows, and hence positivity of the dot product for all time. Now,

$$\boldsymbol{n}\cdot\boldsymbol{\phi}_F = y[(2-3\lambda^2)(x^2-F^2) + 2\gamma y]\geq 0 \quad \text{for } x\in(-\infty,-F],$$

since $y\geq 0$, provided that $(2-3\lambda^2)\geq 0$. ■

In order to make $\mathcal{B}$ as large as possible we should maximize the area between AB and the x-axis and hence take $\lambda=\sqrt{2/3}$.

(ii) *Boundary* BC

Lemma 2.2. *Let curve* BC *be defined by* $G_{\mathrm{BC}}(x,y)=y^2-(2/3)(x_{\mathrm{C}}-x)(x+F)^2=0$ *for* $x\in[-F,x_{\mathrm{C}}]$ *with* $y\leq 0$. *Then* $\boldsymbol{\phi}(t)$ *is into* BC *in the direction of decreasing* y, *along its entire length and for all time, provided that*

$$x_{\mathrm{C}}\geq 2F+\frac{3}{4}\gamma^2\left[1+\sqrt{1+\frac{8F}{\gamma^2}}\right].$$

Proof. The required normal must be directed towards negative y and is therefore $\boldsymbol{n}=((2/3)(x+F)(3x+F-2x_{\mathrm{C}}),2y)$. For the same reason used in the construction of AB, we need only prove that $\boldsymbol{n}\cdot\boldsymbol{\phi}_F\geq 0$ for $x\in[-F,x_{\mathrm{C}}]$. This dot product is $-(2/3)y(x+F)[2x_{\mathrm{C}}-4F-\gamma\sqrt{6(x_{\mathrm{C}}-x)}]$ and since $y\leq 0$, it is non-negative provided that $2x_{\mathrm{C}}-4F\geq\gamma\sqrt{6(x_{\mathrm{C}}-x)}$ for $x\in[-F,x_{\mathrm{C}}]$. Furthermore, since $\gamma>0$, the inequality can only make sense if $x_{\mathrm{C}}>2F$. Squaring both sides of the inequality and solving for x_{C} we find that

$$x_{\mathrm{C}}\geq 2F+\frac{3}{4}\gamma^2\left[1+\sqrt{1+\frac{8F}{\gamma^2}}\right] \quad\text{or}\quad x_{\mathrm{C}}\leq 2F+\frac{3}{4}\gamma^2\left[1-\sqrt{1+\frac{8F}{\gamma^2}}\right].$$

Since $F>0$, the square root is greater than unity, so the second solution has $x_{\mathrm{C}}<2F$ and can therefore be rejected. ■

Since our objective is to make $\mathcal{B}$ as large as possible, we should choose the x_C that minimizes

$$\int_{-F}^{x_C} |y| \mathrm{d}x = \frac{4\sqrt{6}}{45}(x_C + F)^{5/2},$$

and as this is monotonically increasing with $x_C > -F$, we take the minimal value

$$x_C = 2F + \frac{3}{4}\gamma^2 \left[1 + \sqrt{1 + \frac{8F}{\gamma^2}}\right]. \tag{2.3}$$

(iii) *Boundary* CD

Lemma 2.3. *Let curve* CD *be defined by* $G_{CD}(x, y) = y^2 - (x_C - x)[(x_C - x)^2 + b^2] = 0$ *for* $x \in (-\infty, x_C)$ *with* $y \geq 0$. *Then* $\boldsymbol{\phi}(t)$ *is into* CD *in the direction of increasing y, along its entire length and for all time, provided that*

$$b^2(2x_C - \gamma^2) \geq 4x_C(x_C^2 - f^2),$$

where x_C *is defined in lemma 2.2. There always exist real values of b such that this inequality holds.*

Proof. For this curve, the normal we require is in the direction of increasing y and so $\boldsymbol{n} = (3(x_C - x)^2 + b^2, 2y)$. We need to consider $\boldsymbol{\phi}_f$ this time since $\boldsymbol{n} \cdot \boldsymbol{\phi}_f \geq 0$ implies $\boldsymbol{n} \cdot \boldsymbol{\phi}_F \geq 0$. The dot product is $y[u^2 + 4x_C u + 2(f^2 - x_C^2) + b^2 - 2\gamma y]$ where $u = (x_C - x) \in [0, \infty)$. Since $y \geq 0$, for the dot product to be non-negative we need

$$u^2 + 4x_C u + 2(f^2 - x_C^2) + b^2 \geq 2\gamma y \quad \text{for } u \geq 0, \tag{2.4}$$

and since γ is also positive, this inequality is only feasible over the required range of u if (i) $2(f^2 - x_C^2) + b^2 \geq 0$. With this assumed, we can square both sides of inequality (2.4), substitute for y in terms of x, and simplify to obtain

$$u^4 + 4(2x_C - \gamma^2)u^3 + 2(6x_C^2 + 2f^2 + b^2)u^2 + 4[b^2(2x_C - \gamma^2) + 4x_C(f^2 - x_C^2)]u \\ + [2(f^2 - x_C^2) + b^2]^2 \geq 0 \quad \text{for } u \geq 0. \tag{2.5}$$

Now, as well as (i), we need all the coefficients of the powers of u in this inequality to be non-negative; the coefficients of u^4, u^2 u^0 are obviously so, and those of u^3 and u are so too, provided (ii) $2x_C - \gamma^2 \geq 0$ and (iii) $b^2(2x_C - \gamma^2) + 4x_C(f^2 - x_C^2) \geq 0$. We now show that the definition of x_C in equation (2.3) implies (ii), and that (i) is implied by (iii).

From the definition of x_C in equation (2.3), we have $2x_C = 4F + (3/2)\gamma^2 q$ with $q = 1 + (1 + 8F/\gamma^2)^{1/2} > 2$; hence, $2x_C > 3\gamma^2$, so (ii) is automatically satisfied.

Dividing (iii) by $2x_C > 0$ gives $2(f^2 - x_C^2) + b^2 \geq (b\gamma)^2/(2x_C)$ and since the right-hand side is positive, (iii) implies (i).

In the light of (ii), it now becomes clear that there always exists b large enough that inequality (2.5) is true; and so the lemma is proved. ■

In order to maximize the area of $\mathcal{B}$, we should choose the minimal value of b, which, from (iii), is given by

$$b^2 = \frac{4x_C(x_C^2 - f^2)}{2x_C - \gamma^2}. \tag{2.6}$$

We have therefore proved the following.

Theorem 2.4. *Define* $\mathcal{B}=\mathcal{B}_{\rm AB}\cup\mathcal{B}_{\rm BC}\cup\mathcal{B}_{\rm CD}\cup\{(x,y)|x\geq x_{\rm C}\}$ *with*

$$\mathcal{B}_{\rm AB}=\left\{(x,y)|y\leq-(x+F)\sqrt{2(2F-x)/3},\quad x\leq-F\right\},$$
$$\mathcal{B}_{\rm BC}=\left\{(x,y)|y\leq-(x+F)\sqrt{2(x_{\rm C}-x)/3},\quad -F\leq x\leq x_{\rm C}\right\},$$
$$\mathcal{B}_{\rm CD}=\left\{(x,y)|y\geq\sqrt{(x_{\rm C}-x)\left[(x_{\rm C}-x)^2+b^2\right]},\quad x\leq x_{\rm C}\right\},$$

where $x_{\rm C}$ *is given by equation (2.3) and b by equation (2.6). Then* $\mathcal{B}$ *is an invariant set such that all solutions to equation (2.1) starting within* $\mathcal{B}$ *remain in* $\mathcal{B}$ *for all time.*

(*b*) *Dynamics within* $\mathcal{B}$

In fact, we can say more about solutions to the differential equation which are confined to $\mathcal{B}$: all solutions in $\mathcal{B}$ eventually enter a subset, $\mathcal{B}_7$, which is itself invariant, and once here, grow without limit. The proof, which is somewhat lengthy, is contained in lemmas 2.5–2.7, 2.9–2.13.

It is first necessary to split $\mathcal{B}$ into seven subsets and these are illustrated in figure 3. Also shown there are the two parabolas P_f: $y=(f^2-x^2)/\gamma$ and P_F: $y=(F^2-x^2)\gamma$, the importance of these being that for $y>\mathrm{P}_F$, $\dot{y}<0$; for $y<\mathrm{P}_f$, $\dot{y}>0$; with $\dot{y}$ being either positive, negative or zero only in the region between P_F and P_f.

We first define $\mathcal{B}_7$ and show that it is invariant.

Lemma 2.5. *Define*

$$\mathcal{B}_7=\{(x,y)|x\leq\xi,(F^2-x^2)/\gamma\leq y\leq 0\},$$

where $\xi<-F$ *is the least real root of* $h(x)=2x(F^2-x^2)+\gamma^2(f^2-F^2)=0$. *Then* $\mathcal{B}_7$ *is invariant.*

Proof. First, note that $h'(\pm F/\sqrt{3})=0$ and that $h''(\pm F/\sqrt{3})=\mp 4\sqrt{3}F$; hence, $x=-F/\sqrt{3}$ is a local minimum of $h(x)$ and so $h(x)$ decreases monotonically from $+\infty$, for increasing $x\in(-\infty,-F/\sqrt{3}]$. Furthermore, $h(-F)=\gamma^2(f^2-F^2)<0$ and so there is exactly one real root of $h(x)$, $\xi\in(-\infty,-F)$.

We now consider the three boundaries of $\mathcal{B}_7$ separately. The boundary P_F for $x\leq\xi$ can be treated by the same technique used for constructing $\mathcal{B}$. The required normal to P_F is $\boldsymbol{n}=(2x/\gamma,\ 1)$ and the appropriate choice for $\boldsymbol{\phi}$ is $\boldsymbol{\phi}_f$, giving $\boldsymbol{n}\cdot\boldsymbol{\phi}_f=h(x)\geq 0$ for $x\in(-\infty,\xi]$. Hence, the flow is through this part of P_F and is towards negative x, for all time.

Along the boundary $x=\xi$, $0\geq y\geq(F^2-\xi^2)/\gamma$, and so $\dot{x}\leq 0$; thus, the flow is through this boundary towards negative x for all time here also.

The remaining boundary is $y=0$, $x\leq\xi$, along which $\dot{y}<0$ since $\xi<-F$, and so the flow is through this boundary in the negative y direction for all time. Therefore, $\mathcal{B}_7$ is an invariant set. ■

We now prove several lemmas that, in combination, enable us to show that all solutions starting in $\mathcal{B}$ end up in $\mathcal{B}_7$.

Lemma 2.6. *All solutions* $\boldsymbol{x}(t)=(x(t),\ y(t))$ *initially in* $\mathcal{B}_1$ *or* $\mathcal{B}_2$ *enter* $\mathcal{B}_4$ *in finite time.*

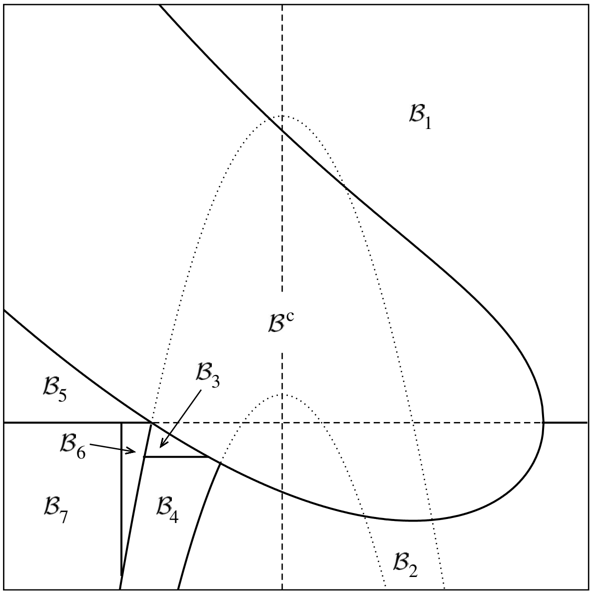

Figure 3. The subsets of $\mathcal{B}$ used in the proof of theorem 2.15. Thick, continuous lines are the boundaries between the named subsets of $\mathcal{B}$; the dashed straight lines are the x and y axes; and the upper/lower dotted curves are the parabolas P_F/P_f, respectively.

Proof. The points O–W are shown in figure 4, which is referred to throughout this proof. The curve OPQRS is a typical orbit starting in $\mathcal{B}_1$ and entering $\mathcal{B}_4$ in finite time and we now prove that all solutions starting in $\mathcal{B}_1$ or $\mathcal{B}_2$ behave in this way.

Consider first any initial point $(x(t_0), y(t_0))$ in the subset of $\mathcal{B}_1$ with $x \le 3F/2$. The line $x = 3F/2$ lies to the left of point C, whose x-co-ordinate $x_C > 2F$. Hence, for all x in this region, $y \ge y_T$, the (strictly positive) y-co-ordinate of point T, since solutions cannot cross CD, and so $x(t) \ge x(t_0) + (t - t_0)y_T$. Thus, any solution crosses the line $x = 3F/2$ in finite time.

Consider now a solution starting at point P on the line $x = 3F/2$; $\dot{y} = f(t) - \gamma y - x^2 \le -\gamma y - 5F^2/4$ since, while the solution remains in $\mathcal{B}_1$, $y \ge 0$ and so $x \ge 3F/2$. Solving this linear differential inequality, we obtain $y(t) \le e^{-\gamma t}[y_P + 5F^2/4\gamma] - 5F^2/(4\gamma)$ and so the solution reaches point Q on the x-axis in finite time.

The x-co-ordinate of $Q > x_C > 2F$ and so starting from Q, we have $\dot{y} \le -\gamma y - 3F^2$, and so $y(t) \le 3F^2(e^{-\gamma t} - 1)/\gamma$. Hence, the solution must cross the line $y = Y_1 < 0$ in finite time, provided that Y_1 satisfies the condition (a) that $Y_1 > -3F^2/\gamma$, so that $\lim_{t\to\infty} y(t) < Y_1$. Three additional conditions on Y_1 are required, these being that (b) Y_1 is sufficiently small that the x-co-ordinate of $R \ge 2F$, so that $\dot{y} \le -\gamma y - 3F^2$ remains true while the solution moves from Q to R; (c) the line $y = Y_1$ does not intersect the parabola P_F: $y = (F^2 - x^2)/\gamma$ for $x > 0$ in $\mathcal{B}_2$, so ensuring that $\dot{y} < 0$ everywhere in the subset of $\mathcal{B}_2$ in which $y \ge Y_1$; and (d) $Y_1 > Y_U$, the y-co-ordinate of U, the point at which curve BC and the parabola P_f intersect for $x < 0$. Such a Y_1 can always be chosen because Q and C are both to the right of the point $x = F$ where P_F intersects the x-axis. With these conditions on Y_1, point R is reached in finite time. By the definition of Y_1, once R

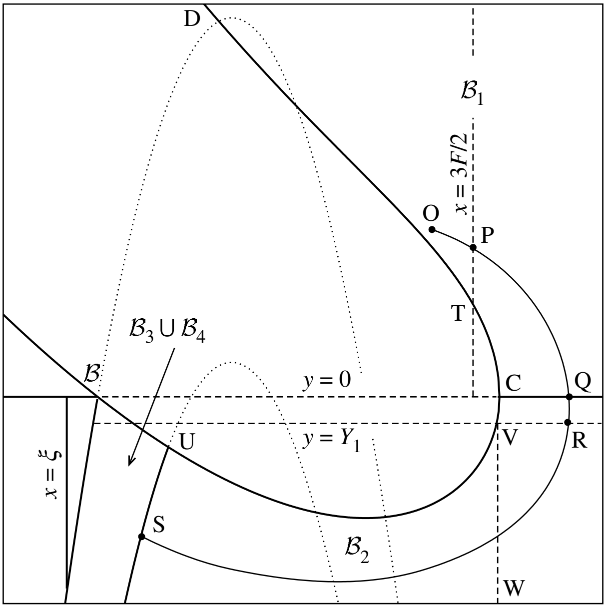

Figure 4. Definitions of the subsets of $\mathcal{B}$ and the points and lines used in the proof of lemma 2.6. The parabolas P_F and P_f are shown as well as a typical solution (OPQRS) to equation (1.1) with $f(t)=(5+3\sin t)/2$. Boundaries between subsets of $\mathcal{B}$ are shown as thick lines.

has been reached, then line VW must also be crossed in finite time, this being the vertical line from the point of intersection of BC with $y=Y_1$. This is because $\dot{x}=y\leq Y_1$ so $x(t)$ decreases at least linearly with time; and, by the definition of Y_1, $\dot{y}<0$ all along VR—so VR cannot be re-crossed.

From the crossing of VW, $\dot{x}\leq Y_1$ still applies and so x decreases at least linearly with time until S is reached. The part of $\mathcal{B}_2$ in the vicinity of S is below P_f and so $\dot{x}<0$ and $\dot{y}\geq 0$; hence the solution must cross P_f at some point S and hence enter $\mathcal{B}_4$. Since each of the steps described takes finite time, we conclude that point S is reached from any point in $\mathcal{B}_1$ or $\mathcal{B}_2$ in finite time. ∎

We now consider the fate of solutions starting in the rest of $\mathcal{B}$.

Lemma 2.7. *A solution initially at point* B, $(-F, 0)$ *eventually moves further into* $\mathcal{B}_3$.

Proof. Consider the case that $\boldsymbol{x}(t_0)$ is point $\mathrm{B}=(-F,0)\in\mathcal{B}_3$ at some time t_0; then either (i) $f(t_0)<F^2$ or (ii) $f(t_0)\equiv F^2$. From the differential equation, we have in case (i) that $\dot{x}(t_0)=0$ but $\dot{y}(t_0)<0$ and so the solution moves vertically away from B in the direction of decreasing y, and so further into $\mathcal{B}_3$. In case (ii), we have $\dot{x}(t)=0$ and $\dot{y}(t)=0$ for $t\in[t_0, t_1)$ where t_1 is the time at which $f(t_1)$ is first less than F^2; t_1 is necessarily finite by the conditions imposed on $f(t)$. Hence, the solution remains at point B only until $t=t_1$, at which time $\dot{y}<0$ and the subsequent behaviour is the same as in case (i). ∎

To proceed further, we need the following definition:

Definition 2.8. *A double crossing is said to occur when the solution* $\boldsymbol{x}(t)$ *crosses both the lines* $y=Y_0$ *and* $y=2Y_0$*, in either order. Here,* $Y_0<0$ *and is sufficiently*

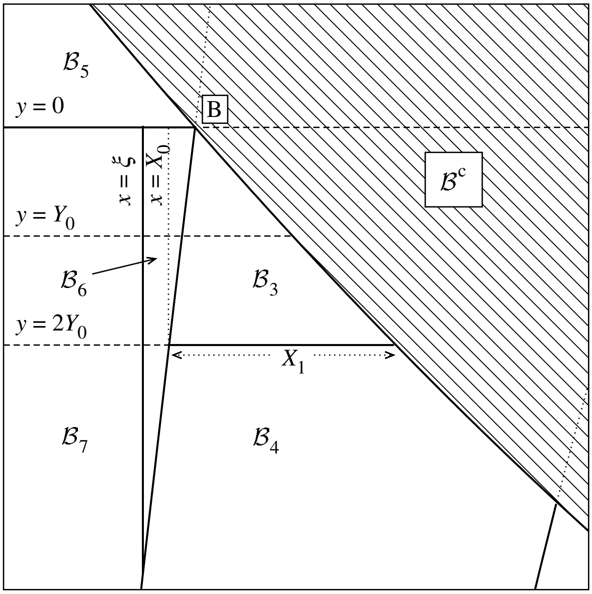

Figure 5. Magnification of $\mathcal{B}_3$ and its surroundings, including part of $\mathcal{B}^c$. The point B is $\boldsymbol{x}=(-F, 0)$. The x-axis and two horizontal lines at $y=Y_0, 2Y_0$ are also shown, where $Y_0<0$.

small that (1) the intersection of $y=2Y_0$ with P_f for $x<0$ takes place within $\mathcal{B}^c$, and that (2) $X_0>\xi$, with X_0 the negative root of $F^2-X_0^2=2\gamma Y_0$ (see figure 5).

The existence of the k-th double crossing implies that $\exists\ a_k, b_k \in \mathbb{R}$ with $b_k>a_k$ such that either $y(a_k)=Y_0$ and $y(b_k)=2Y_0$ (falling) or $y(a_k)=2Y_0$ and $y(b_k)=Y_0$ (rising).

Now, by continuity, any pair of successive double crossings must consist either of a falling double crossing followed by a rising one or vice versa. Then we have the following.

Lemma 2.9. *An infinite sequence of double crossings confined to $\mathcal{B}_3 \cup \mathcal{B}_4$ cannot occur, in either (i) finite or (ii) infinite time.*

Proof. First, we consider the sets that solutions leaving $\mathcal{B}_3$ can enter: by continuity, these must be $\mathcal{B}^c$, $\mathcal{B}_5$, $\mathcal{B}_4$ or $\mathcal{B}_6$. The first two are impossible, $\mathcal{B}^c$ by definition and $\mathcal{B}_5$ because $\mathcal{B}_3$ and $\mathcal{B}_5$ only meet at point B, and passing through this in the direction of increasing y is impossible by lemma 2.7.

The other two transitions *can* take place; $\mathcal{B}_3$ to $\mathcal{B}_6$ because $\dot{x}<0$, $\dot{y}\leq 0$ along their common boundary, and $\mathcal{B}_3$ to $\mathcal{B}_4$ because $\dot{y}$ can have any sign along the common boundary.

To prove the lemma, first observe that $\forall \boldsymbol{x}=(x,y)\in \mathcal{B}_3 \cup \mathcal{B}_4$ we have $f^2 - x^2 \leq \gamma y \leq F^2 - x^2$ so $-f^2 \geq -\gamma y - x^2 \geq -F^2$. Now, $\dot{y} = f(t) - \gamma y - x^2$ so

$$F^2 - f^2 \geq \dot{y} \geq f^2 - F^2. \tag{2.7}$$

By hypothesis, $\boldsymbol{x}(t) \in \mathcal{B}_3 \cup \mathcal{B}_4$ for all $t\geq 0$, so, by equation (2.7), $|\dot{y}| \leq F^2 - f^2 \equiv v$ with $v>0$.

Consider first case (i), in which infinitely many double crossings occur within $\mathcal{B}_3 \cup \mathcal{B}_4$ during the time interval $[0, T]$ with $T>0$ finite. Define the sequence of crossing times $\{I_k\}_{k\in\mathbb{N}}$ where $I_k=[a_k,b_k]$ with $a_k\geq b_{k-1}$, so that $\cup_{k\in\mathbb{N}}I_k\subseteq[0,T]$; then $y(t)\in[2Y_0,Y_0]$ if $t\in\cup_{k\in\mathbb{N}}I_k$. Hence,

$$\begin{aligned} vT &\geq \int_0^T |\dot{y}(t)|\,\mathrm{d}t \geq \sum_{k\in\mathbb{N}}\int_{I_k}|\dot{y}(t)|\,\mathrm{d}t \geq \sum_{k\in\mathbb{N}}\left|\int_{I_k}\dot{y}(t)\,\mathrm{d}t\right| \\ &= \sum_{k\in\mathbb{N}}|y(b_k)-y(a_k)| = \sum_{k\in\mathbb{N}}|Y_0| = \infty, \end{aligned}$$

and since v and T are both finite, the latter by hypothesis, this leads to a contradiction and so an infinite sequence of double crossings within $\mathcal{B}_3\cup\mathcal{B}_4$ cannot occur in finite time.

Now consider case (ii), in which infinitely many double crossings occur during infinite time. We introduce the sequence $\{I_k\}_{k\in\mathbb{N}}$ as before but now $\lim_{k\to\infty} b_k=\infty$. In order that $\boldsymbol{x}(t)$ remains in $\mathcal{B}_3\cup\mathcal{B}_4$, we must have $\lim_{k\to\infty}x(b_k)>X_0$ (see figure 5). From the differential equation we have

$$\int_{b_1}^{b_k}\dot{y}(t)\mathrm{d}t = y(b_k)-y(b_1) = \int_{b_1}^{b_k}(f(t)-\gamma y-x^2)\mathrm{d}t. \tag{2.8}$$

Now, $\forall k\in\mathbb{N}$, $\exists\, N=N(k)$ such that $N\tau\leq b_k\leq(N+1)\tau$ and, by hypothesis, $\lim_{k\to\infty}N(k)=\infty$. Furthermore, given positive real constants c and c', we have, by the continuity of curve BC, that $x(t)<-F+c|Y_0|<0$ and so

$$-x(t)^2<-F^2+c'|Y_0|. \tag{2.9}$$

Using this in equation (2.8) results in

$$y(b_k)-y(b_1)\leq N\tau\langle f\rangle+\int_{N\tau}^{b_k}f(t)\mathrm{d}t-\gamma\int_{b_1}^{b_k}y(t)\mathrm{d}t+N\tau(-F^2+c'|Y_0|)-\int_{N\tau}^{b_k}x(t)^2\mathrm{d}t.$$

Also, from $\dot{x}=y$ we have that $\int_{b_1}^{b_k}y(t)\mathrm{d}t=x(b_k)-x(b_1)$ is bounded, as is $\int_{N\tau}^{b_k}(f(t)-x(t)^2)\mathrm{d}t$, since $x(b_k)\geq X_0$, and so

$$y(b_k)-y(b_1)\leq N\tau(\langle f\rangle-F^2+c'|Y_0|)+K,$$

where K is a finite constant. Remembering that $f(t)$ is continuous and periodic and that $f(t)\leq F^2$, $\forall\, t$, the coefficient of $N\tau$ in the above can be made negative by choosing $|Y_0|$ sufficiently small. Hence, finally, we have proved that $\limsup_{k\to\infty} y(b_k)<K'-\infty$ for a constant K', which is in contradiction with our initial hypothesis that $y(b_k)=Y_0$ or $2Y_0$. Hence, an infinite number of double crossings, confined to $\mathcal{B}_3\cup\mathcal{B}_4$, cannot occur either in finite or infinite time. ■

Lemma 2.10. *No solution can remain in $\mathcal{B}_3$ indefinitely.*

Proof. From lemma 2.9 we now have the result that an infinite number of double crossings cannot occur, and so $\exists\ t_0<\infty$ such that $\forall\, t>t_0$, while $x(t)\in\mathcal{B}_3$, either (i) $y(t)<Y_0$ or (ii) $y(t)>2Y_0$—there are no further double crossings after $t=t_0$. We can now prove the lemma as follows.

We consider case (i) first. Here, $\dot{x}(t)<Y_0$ for all $t>t_0$, and since $Y_0<0$, $\lim_{t\to\infty}x(t)=-\infty$. Hence, in case (i), no solution can remain in $\mathcal{B}_3$ indefinitely.

Turning now to case (ii), we have $\dot{y}=f(t)-\gamma y-x^2$ and so, with $N\in\mathbb{N}$,

$$\int_{t_0}^{t_0+N\tau}\dot{y}(t)\mathrm{d}t = y(N\tau+t_0)-y(t_0)\leq N\tau\langle f\rangle-\gamma\int_{t_0}^{t_0+N\tau}y(t)\mathrm{d}t+N\tau(-F^2+c'|Y_0|),$$

where the last term on the RHS is obtained from equation (2.9). Since $y(t)<0$ for all $t>t_0$, we have

$$0<-\gamma\int_{t_0}^{t_0+N\tau}y(t)\mathrm{d}t=\gamma\left|\int_{t_0}^{t_0+N\tau}y(t)\mathrm{d}t\right|=\gamma|x(t_0+N\tau)-x(t_0)|\leq\gamma X_1,$$

where X_1 is shown in figure 5. Thus,

$$y(N\tau+t_0)\leq y(t_0)+N\tau(\langle f\rangle-F^2+c'|Y_0|)+\gamma X_1,$$

and since $|Y_0|$ can be made as small as desired, we have $\lim_{N\to\infty}y(N\tau+t_0)=-\infty$. Hence, in case (ii) too, no solution can remain in $\mathcal{B}_3$ indefinitely. ■

Lemma 2.11. *All solutions initially in $\mathcal{B}_3$, $\mathcal{B}_4$ with $x>\xi$, or $\mathcal{B}_6$, cross the line $x=\xi$ in the direction of decreasing x, in finite time.*

Proof. Consider first solutions initially in $\mathcal{B}_3$. By lemma 2.10, solutions here must eventually leave $\mathcal{B}_3$. As shown in the proof of lemma 2.9, solutions leaving $\mathcal{B}_3$ can only enter $\mathcal{B}_4$ or $\mathcal{B}_6$.

To show that all solutions cross the line $x=\xi$ in the direction of decreasing x, consider first a solution that leaves $\mathcal{B}_3$ and enters $\mathcal{B}_6$ at a point (x_0, y_0) at a time t_0. Then we need to prove two facts: (i) $y<y_0$ for all $t>t_0$, and (ii) $y_0<0$.

We prove (i) as follows. While the solution remains in $\mathcal{B}_6$, $\dot{y}<0$, so $\dot{x}=y<y_0$, and so the lemma is plainly true for a solution that remains in $\mathcal{B}_6$. The solution may, however, re-cross P_F and enter $\mathcal{B}_3$ or $\mathcal{B}_4$, in both of which $\dot{y}$ can have either sign. If the point at which the solution re-crossed P_F is (x_1, y_1) then clearly $y_1<y_0$. Suppose now a further crossing occurs, from $\mathcal{B}_3\cup\mathcal{B}_4$ back into $\mathcal{B}_6$, this time at a point (x_2, y_2); then $y_2<y_1<y_0$ despite the ambiguity of the sign of $\dot{y}$ in $\mathcal{B}_3\cup\mathcal{B}_4$. This is because $y<0$ always, so x decreases at least linearly in time, so $x_2<x_1<x_0$; but the crossing points are all on P_F and so $y_2<y_1<y_0$. Hence, we conclude that $y<y_0$ for all $t>t_0$.

The proof of (ii) relies on lemma 2.7. A solution at point B can initially only move vertically downwards away from B, and hence cannot enter $\mathcal{B}_6$ from point B. Hence, any solution that enters $\mathcal{B}_6$ from $\mathcal{B}_3$ does so at $y=y_0<0$. Thereafter, the argument used in case (i) applies.

Consider now a solution that leaves $\mathcal{B}_3$ and enters $\mathcal{B}_4$. According to lemma 2.10, although solutions may oscillate between $\mathcal{B}_3$ and $\mathcal{B}_4$, they cannot do so indefinitely and so there must be a last crossing into $\mathcal{B}_4$, let us say at time t_0. Thereafter, as before, $y<y_0=2Y_0$, whether or not the solution subsequently remains in $\mathcal{B}_4$ or moves into $\mathcal{B}_6$.

Hence, we conclude that in all cases, $\dot{x}=y<y_0<0$ for all $t>t_0$, and so all solutions eventually cross the line $x=\xi$. ■

The previous lemma shows that solutions with $x>\xi$ eventually cross the line $x=\xi$. Hence, solutions eventually enter $\mathcal{B}_7$ (invariant), or $\mathcal{B}_4$ with $x<\xi$, and we now deal with the fate of solutions in the latter case.

Lemma 2.12. *All solutions initially in* $\mathcal{B}_4$ *with* $x \leq \xi$ *enter* $\mathcal{B}_7$ *in finite time and remain there.*

Proof. We prove this lemma by contradiction: we assume that $\boldsymbol{x} \in \mathcal{B}_4$ with $x \leq \xi$ for all t and find an inconsistency. Set $t=0$ when $x=\xi$ to simplify notation, and write $y(0)=y_0$. Then $\dot{x}(t) \leq (F^2-\xi^2)/\gamma$ by the assumption that $\boldsymbol{x}(t)$ is between P_F and P_f. Integrating gives

$$x(t) \leq at + \xi, \tag{2.10}$$

with $a=(F^2-\xi^2)/\gamma$ a negative constant. Also, by equation (2.7) we have $|\dot{y}| \leq F^2-f^2=v$ and so integrating from 0 to t gives

$$-vt + y_0 \leq y(t) \leq vt + y_0. \tag{2.11}$$

Now, as long as $\boldsymbol{x}(t)$ remains in $\mathcal{B}_4$, we must have $(f^2-x^2)/\gamma \leq y \leq (F^2-x^2)/\gamma$, but, by equation (2.10), $y \leq (F^2-x^2)/\gamma \leq [F^2-(at+\xi)^2]/\gamma$. Hence, in order to remain in $\mathcal{B}_4$ for all $t>0$, $y(t)$ has to be bounded above by a function that decreases as $-t^2$, and, from equation (2.11), bounded below by one that goes as $-t$. These two requirements are mutually incompatible, and so no solutions can remain in $\mathcal{B}_4$ indefinitely. ■

Lemma 2.13. *All solutions initially in* $\mathcal{B}_5$ *enter* $\mathcal{B}_7$ *in finite time.*

Proof. For all $\boldsymbol{x} \in \mathcal{B}_5$, $x<-F$ and $y \geq 0$. Solving the differential equation from initial condition $x(0)<-F$, $y(0) \geq 0$, we have

$$\int_0^{n\tau} \dot{y}\,\mathrm{d}t = y(n\tau)-y(0) = n\tau\langle f\rangle - \gamma\int_0^{n\tau} y(t)\mathrm{d}t - \int_0^{n\tau} x^2\mathrm{d}t,$$

and, while the solution is in $\mathcal{B}_5$, $-x^2<-F^2$, so

$$y(n\tau) < y(0) + n\tau(\langle f\rangle - F^2) + \gamma(F + x(0)),$$

where we have used $x(n\tau)<-F$. Since, as before, $\langle f\rangle - F^2<0$, the second term above ensures that y decreases with time and so the solution eventually crosses the x-axis at $x<-F$, and so enters $\mathcal{B}_6$ or $\mathcal{B}_7$; if $\mathcal{B}_6$, then lemmas 2.11 and 2.12 apply, and in either case $\mathcal{B}_7$ is entered in finite time. ■

Lemma 2.14. *All solutions in* $\mathcal{B}_7$ *grow without limit as* $t \to \infty$.

Proof. For all $\boldsymbol{x} \in \mathcal{B}_7$, $\dot{x}<0$ and $\dot{y}<0$. Furthermore, by lemma 2.5, $\mathcal{B}_7$ is invariant, and so both x and y tend to $-\infty$ in $\mathcal{B}_7$. ■

Theorem 2.15. *All solutions initially in* $\mathcal{B}$ *remain in* $\mathcal{B}$, *eventually entering* $\mathcal{B}_7$, *where they remain and grow without limit.*

Proof. The invariance of $\mathcal{B}$ is proved in theorem 2.4. The rest of the theorem is a direct consequence of lemmas 2.5–2.7, 2.9–2.14. ■

(*c*) *Set* $\mathcal{C}$

As an easy corollary to theorems 2.4 and 2.15, we can now construct a set $\mathcal{C}$ which is such that any finite area absorbing set $\mathcal{A} \subset \mathcal{C}$.

Corollary 2.16. *Any finite area absorbing set* $\mathcal{A} \subset \mathcal{C}$, *where*

$$\mathcal{C} = \overline{\mathcal{B}^{\mathrm{c}} \cap \{(x,y)|x \geq -F\}}.$$

Proof. The boundaries of set $\mathcal{C}$ consist of curve BC, the part of curve CD with $x \geq -F$ and the line $x = -F$ between $y=0$ and $y=\sqrt{(x_C+F)[(x_C+F)^2+b^2]}$, the latter being the intersection of this line with CD. Since this line has $y=\dot{x}\geq 0$ along its entire length, flow must be into $\mathcal{C}$ along this boundary. Furthermore, if $\mathcal{A}$ is any absorbing set, then $\mathcal{A}\cap\mathcal{B}=\emptyset$ since, by theorem 2.15, any initial conditions in $\mathcal{B}$ lead to solutions that grow without bound. ■

We can now visualize how good an approximation set $\mathcal{B}^c$ is to the set of initial conditions which lead to solutions that do not blow up. We make the following definition:

Definition 2.17. *Let $\boldsymbol{X}(x_0, y_0, t_0; t)$ be the solution of equation (1.1) for given t_0, with initial conditions $x(t_0)=x_0$, $y(t_0)=y_0$, and define*

$$\mathcal{F}_{t_0} = \left\{(x_0, y_0) \mid \lim_{t\to\infty} \boldsymbol{X}(x_0, y_0, t_0; t) \in \mathcal{B}^c\right\}.$$

Then

$$\mathcal{F} = \bigcup_{t_0\in[0,\tau)} \mathcal{F}_{t_0}.$$

Clearly, $\mathcal{F}\subseteq\mathcal{B}^c$. For an illustration of a numerical approximation to this set, see figure 6.

(*d*) *Absorbing set $\mathcal{A}_1$*

We now construct a polygonal absorbing set. The underlying method for this construction is the same as for $\mathcal{B}$, but by contrast to that case, there are now constraints on the parameters F, f and γ additional to $F>f>0$ and $\gamma>0$.

Three preliminary observations are in order. First, any absorbing set must lie partly above and partly below the x-axis: a set entirely above the x-axis would always have $\dot{x}=y>0$ and no such set whose boundary is a closed curve could be absorbing over its whole boundary.

Second, where possible, we choose the sides of the absorbing sets to be parallel to either the x or the y-axis, since this simplifies the proof that the flow is into that side. On the other hand, a rectangular set is not possible: for instance, the right-hand vertical boundary above the x-axis would necessarily entail flow out of the set, since $y=\dot{x}>0$. The same applies to the left-hand vertical boundary below the x-axis. Hence, there must be boundaries not parallel to either axis.

Third, as pointed out previously, $\dot{y}$ can only be zero if $\boldsymbol{x}=(x, y)\in I$, where I is the region between P_f and P_F. This suggests the possibility of basing the non-vertical/horizontal boundaries on scaled versions of these parabolas, and this is the case for $\mathcal{A}_1$.

Theorem 2.18. *Let $\gamma^2\geq 8F$ and define the set $\mathcal{A}_1$ as the closed hexagon* GHIJKL*, whose vertices are*

$$\mathrm{G} = \left(f, \frac{F^2-f^2}{\gamma}\right),$$

$$\mathrm{H} = \left(\sqrt{F^2+(f^2-F^2)/\lambda_1}, \left(\frac{F^2-f^2}{\gamma}\right)\right),$$

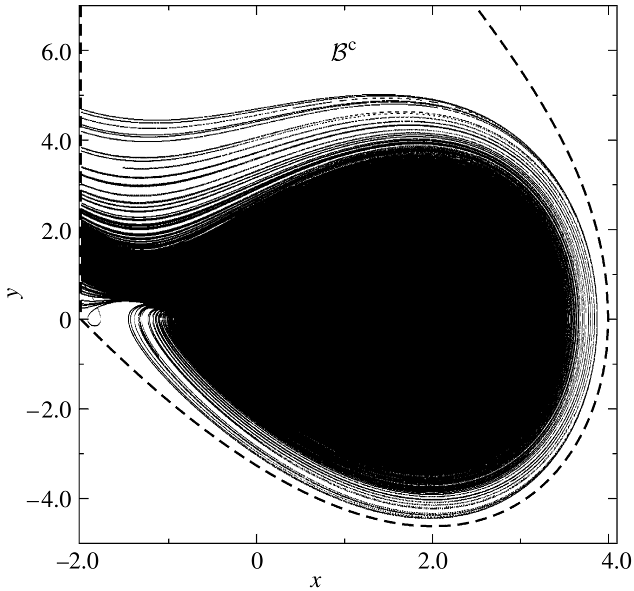

Figure 6. A numerical approximation to set $\mathcal{F}$ (black)—see definition 2.17—for $x \geq -F = -2$, showing how it fits within $\mathcal{B}^c$, part of whose boundary is shown as a thick dashed line. Here, $\gamma = 0.01$, $\mu = 2$ and $f(t) = (5 + 3\sin t)/2$.

$$\mathrm{I} = (F, 0),$$

$$\mathrm{J} = \left(F, \frac{f^2 - F^2}{\gamma}\right),$$

$$\mathrm{K} = \left(\sqrt{f^2 + \frac{F^2 - f^2}{\lambda_2}}, \frac{f^2 - F^2}{\gamma}\right),$$

$$\mathrm{L} = (f, 0),$$

and whose edges are straight lines except that

$$\mathrm{HI} : y = \lambda_1 \left(\frac{F^2 - x^2}{\gamma}\right) \quad \textit{and} \quad \mathrm{KL} : y = \lambda_2 \left(\frac{f^2 - x^2}{\gamma}\right).$$

Here

$$\lambda_1 = \frac{\gamma^2}{4F}\left(1 - \sqrt{1 - \frac{8F}{\gamma^2}}\right),$$

and λ_2 is such that

$$h(\lambda_2) = 4f^2\lambda_2^4 + 4(F^2 - f^2)\lambda_2^3 - \gamma^4(\lambda_2 - 1)^2 = 0.$$

Then at least one $\lambda_2 \in (1, 2)$ exists and the set $\mathcal{A}_1$ as defined is an absorbing set for all t.

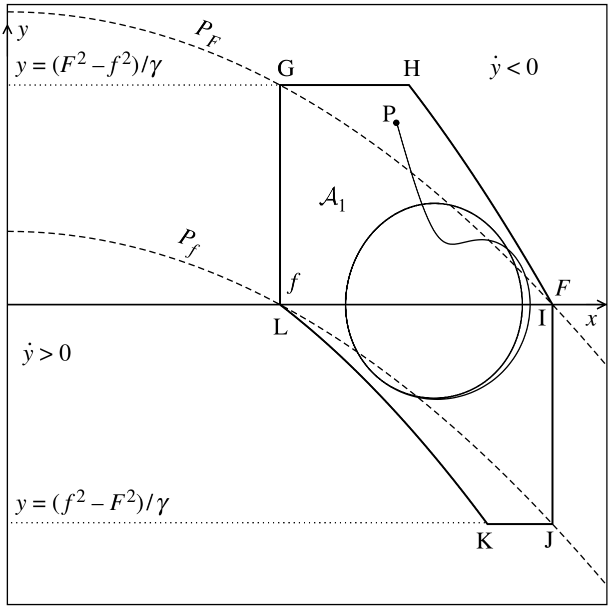

Figure 7. The invariant set $\mathcal{A}_1$, whose vertices are GHIJKL. The dashed lines are the parabolas between which lies region I in which the sign of $\dot{y}$ is indeterminate; above P_F, $\dot{y} < 0$ and below P_f, $\dot{y} > 0$. The continuous line starting at point P is a numerical solution to equation (1.1), with F=2, f=1, γ=4.01, which can be seen to lie within $\mathcal{A}_1$.

Proof. To show that $\mathcal{A}_1$, illustrated in figure 7, is an absorbing set, we need to prove that $\boldsymbol{n}\cdot\boldsymbol{\phi}(t) \geq 0$ over the entire boundary of $\mathcal{A}_1$ and for all t, with the appropriate choice for $\boldsymbol{\phi}$, and where $\boldsymbol{n}$ is the inward normal to the boundary.

This is trivial for the horizontal and vertical portions of the boundary, i.e. GH, IJ, JK and LG. Taking GH as a horizontal example, we have $x \geq f$ and $\boldsymbol{n}=(0, -1)$ is an inward-pointing normal. By the same reasoning as used in the construction of $\mathcal{B}$, we need to prove that $\boldsymbol{n}\cdot\boldsymbol{\phi}_F = x^2 - f^2 \geq 0$ for $x \geq f$, which is clearly true. Taking IJ as a vertical example, we have $y \leq 0$ and an inward-pointing normal is $\boldsymbol{n}=(-1, 0)$. Then $\boldsymbol{n}\cdot\boldsymbol{\phi}_f = \boldsymbol{n}\cdot\boldsymbol{\phi}_F = -y \geq 0$, also obviously true.

We define HI as the curve $G_{\mathrm{HI}}(x, y) = y - \lambda_1(F^2 - x^2)/\gamma = 0$, from which the inward normal $\boldsymbol{n} = (-2\lambda_1 x/\gamma, -1)$. Here, real $\lambda_1 > 1$ is to be found. The appropriate choice for $\boldsymbol{\phi}$ is $\boldsymbol{\phi}_F$ and so we need

$$\boldsymbol{n}\cdot\boldsymbol{\phi}_F = (F^2 - x^2)(-2\lambda_1^2 x/\gamma^2 + \lambda_1 - 1) \geq 0 \quad \text{for } x \in [x_{\mathrm{H}}, F], \tag{2.12}$$

where x_{H} is the x-co-ordinate of H. Now, $F^2 - x^2 \geq 0$ so we require $-2\lambda_1^2 x/\gamma^2 + \lambda_1 - 1 \geq 0$. This is guaranteed for all $x \in [x_{\mathrm{H}}, F]$ if $-2\lambda_1^2 F/\gamma^2 + \lambda_1 - 1 \geq 0$, which is a quadratic in λ_1 in which λ_1^2 has a negative coefficient; hence inequality (2.12) is satisfied for $\lambda_1 \in [\Lambda_1^-, \Lambda_1^+]$, where $\Lambda_1^{\pm} = \gamma^2(1 \pm \sqrt{1 - 8F/\gamma^2})/4F$. Letting $r = 8F/\gamma^2 > 0$, so for λ_1 to be real, $r \in (0,1]$, we have $\Lambda_1^- = 2(1 - \sqrt{1-r})/r = 2/(1 + \sqrt{1-r})$, which is clearly monotonically increasing from 1, at r=0, to 2, at r=1. Hence, for the smallest possible $\mathcal{A}_1$ we choose $\lambda_1 = \gamma^2(1 - \sqrt{1 - 8F/\gamma^2})/4F$, which is guaranteed to be positive, as, by construction, it must be. Similarly, we define KL by $G_{\mathrm{KL}}(x, y) = y - \lambda_2(f^2 - x^2)/\gamma = 0$ and so $\boldsymbol{n} = (2\lambda_2 x/\gamma, 1)$ is the required normal, where $\lambda_2 > 1$ is to be found. The inequality to be satisfied this time is $\boldsymbol{n}\cdot\boldsymbol{\phi}_f = (f^2 - x^2)(2\lambda_2^2 x/\gamma^2 - \lambda_2 + 1) \geq 0$ for $x \in [f, x_{\mathrm{K}}]$, where x_{K} is the x-coordinate of K. The first bracket is clearly non-positive, so

$$2\lambda_2^2 x_K - \lambda_2\gamma^2 + \gamma^2 \leq 0, \tag{2.13}$$

guarantees flow through KL in the direction of increasing y. Additionally, we have $y_{\mathrm{K}} = y_{\mathrm{J}} = (f^2 - F^2)/\gamma$ and so $\lambda_2(f^2 - x_{\mathrm{K}}^2) = f^2 - F^2$. Eliminating x_{K} between this and equation (2.13) results in

$$h(\lambda_2) = 4f^2\lambda_2^4 + 4(F^2 - f^2)\lambda_2^3 - \gamma^4(\lambda_2 - 1)^2 = 0. \tag{2.14}$$

Now, $h(1) = 4F^2 > 0$ and $h(2) = 32(F^2 + f^2) - \gamma^4$; but $\gamma^2 \geq 8F$ so $h(2) \leq 32(f^2 - F^2) < 0$, and so, by the Intermediate Value Theorem, there is at least one real root of $h(\lambda_2)$ such that $\lambda_2 \in (1,2)$. ■

3. Absorbing set $\mathcal{A}_2$, $\mu > 1$

We now construct an absorbing set, $\mathcal{A}_2$ for all $\mu > 1$. The condition $\mu > 1$ is required for the nonlinear function in the differential equation, x^μ, to be Lipschitz, thereby guaranteeing uniqueness of solutions (Hirsch & Smale 1974). We redefine the vector field and its bounds as

$$\boldsymbol{\phi}(t) = (y, f(t) - x^\mu - \gamma y), \quad \boldsymbol{\phi}_F = (y, F^\mu - x^\mu - \gamma y) \quad \text{and} \quad \boldsymbol{\phi}_f = (y, f^\mu - x^\mu - \gamma y).$$

By analogy with the construction of $\mathcal{A}_1$, we observe that the set I in which $\dot{y}$ has indefinite sign is now $I = \{(x, y) | (f^\mu - x^\mu) \leq \gamma y \leq (F^\mu - x^\mu)\}$. We can then construct a hexagonal absorbing set $\mathcal{A}_2$ whose vertices are MNOPQR—see figure 8. In the course of the construction, we make several assumptions whose purpose is to make the problem tractable, these being (i) MN, QP are horizontal, MR and OP are vertical; (ii) the x-co-ordinates of N and Q are $x_{\mathrm{N}} = F$ and $x_{\mathrm{Q}} = f$ respectively; (iii) the gradients of lines NO and RQ are both $-\gamma$; and (iv) M lies on P_F: $y = (F^\mu - x^\mu)/\gamma$ and P lies on P_f: $y = (f^\mu - x^\mu)/\gamma$. Constraint (iv) arises in an attempt to make $\mathcal{A}_2$ as small as possible: for instance, M cannot be below P_F because $\dot{y}$ could then be positive, which would allow flow out of MN. With these assumptions in place, the co-ordinates of all the vertices of $\mathcal{A}_2$ can be expressed in terms of x_{M}, as will be seen from the following.

Theorem 3.1. *Let $\mu > 1$, $f^\mu + f\gamma^2 \geq (F + F^\mu/\gamma^2)^\mu$ and define the set $\mathcal{A}_2$ as the closed hexagon* MNOPQR *whose edges are straight lines and whose vertices are*

$$\mathrm{M} = (x_{\mathrm{M}}, \gamma\varDelta),$$

$$\mathrm{N} = (F, \gamma\varDelta),$$

$$\mathrm{O} = (F + \varDelta, 0),$$

$$\mathrm{P} = (F + \varDelta, [f^\mu - (F + \varDelta)^\mu]/\gamma),$$

$$\mathrm{Q} = (f, [f^\mu - (F + \varDelta)^\mu]/\gamma),$$

$$\mathrm{R} = (x_{\mathrm{M}}, 0),$$

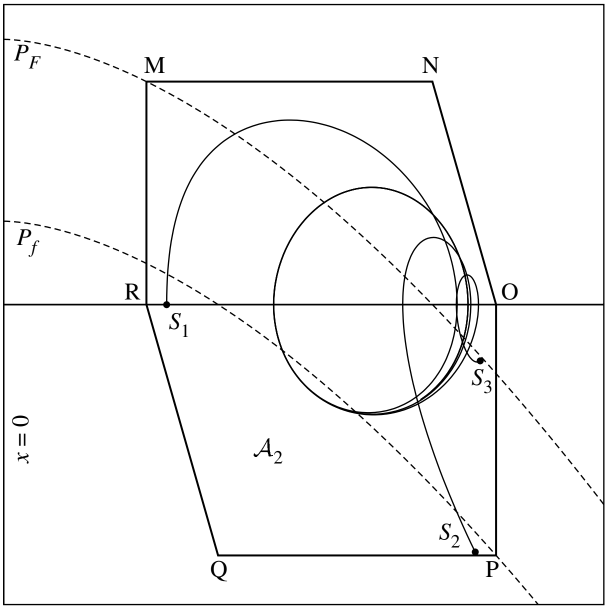

Figure 8. The invariant set $\mathcal{A}_2$, whose vertices are MNOPQR. The continuous lines starting at points S_1, S_2 and S_3 are numerical solutions to equation (1.1), with $\mu=1.67$ and $f(t)=(5+3\sin t)/2$ (so $F^{\mu}=4$, $f^{\mu}=1$), $\gamma=3$. All the solutions can be seen to be attracted to the same period-1 limit cycle and both transients and this limit cycle lie within $\mathcal{A}_2$. The curves P_F: $y=(F^{\mu}-x^{\mu})/\gamma$ and P_f: $y=(f^{\mu}-x^{\mu})/\gamma$ are shown as dashed lines.

where $\Delta=(F^{\mu}-x_{\mathrm{M}}^{\mu})/\gamma^2$ *and* x_{M} *is the smallest real root of*

$$\left(\frac{F^{\mu}-x_{\mathrm{M}}^{\mu}}{\gamma^2}+F\right)^{\mu}+\gamma^2(x_{\mathrm{M}}-f)-f^{\mu}=0, \tag{3.1}$$

such that $0\leq x_{\mathrm{M}}<f$. *Then at least one such* x_{M} *exists and the set* $\mathcal{A}_2$ *is an absorbing set for all* t *and all* $\mu>1$.

Proof. To show that $\mathcal{A}_2$, see figure 8, is an absorbing set, as before, we need to prove that $\boldsymbol{n}\cdot\boldsymbol{\phi}(t)\geq 0$ over the entire boundary of $\mathcal{A}_2$ for all t, with the appropriate choice for $\boldsymbol{\phi}$ and where $\boldsymbol{n}$ is the inward normal to the boundary. This is trivial for the horizontal and vertical portions of the boundary, i.e. OP, PQ and RM (see the proof of theorem 2.18).

We define NO as the straight line $G_{\mathrm{NO}}(x, y)=y-\lambda_3(x_{\mathrm{O}}-x)$ where x_{O} is the x-coordinate of point O and $\lambda_3>0$, from which the inward normal $\boldsymbol{n}=(-\lambda_3,-1)$. The appropriate choice for $\boldsymbol{\phi}$ is $\boldsymbol{\phi}_F$ here, so we need

$$\boldsymbol{n}\cdot\boldsymbol{\phi}_F=\lambda_3(x_{\mathrm{O}}-x)(-\lambda_3+\gamma)+(x^{\mu}-F^{\mu})\geq 0 \quad \text{for } x\in[F,x_{\mathrm{O}}], \tag{3.2}$$

where x_{N} has been set equal to F: with hindsight, we shall see that this choice greatly simplifies the construction. Here $\lambda_3(x_{\mathrm{O}}-x)\geq 0$ and $(x^{\mu}-F^{\mu})\geq 0$ if, and only if, $\mu\geq 0$, so we require $(-\lambda_3+\gamma)\geq 0$, and therefore choose $\lambda_3=\gamma$, which guarantees that the boundary NO has $\boldsymbol{n}\cdot\boldsymbol{\phi}(t)\geq 0$. This choice for λ_3 also simplifies the construction.

Similarly, we define QR by $G_{\mathrm{QR}}(x, y)=y-\lambda_4(x_{\mathrm{R}}-x)$, where x_{R} is the x-coordinate of R and $\lambda_4>0$, and so $\boldsymbol{n}=(\lambda_4, 1)$ is the required normal. The inequality to be obeyed is now $\boldsymbol{n}\cdot\boldsymbol{\phi}_f=\lambda_4(x_{\rm R}-x)(\lambda_4-\gamma)+(f^\mu-x^\mu)\geq 0$ for $x\in[x_{\rm R},f]$, where $x_{\rm Q}$ has been set equal to f, again for simplicity. Here $(f^\mu-x^\mu)\geq 0$ if $\mu\geq 0$ but $\lambda_4(x_{\rm R}-x)$ is non-positive, so we need $(\lambda_4-\gamma)\leq 0$. In choosing $\lambda_4=\gamma$ we guarantee that the boundary QR has $\boldsymbol{n}\cdot\boldsymbol{\phi}(t)\geq 0$.

Having determined the constants λ_3 and λ_4 which guarantee that $\boldsymbol{n}\cdot\boldsymbol{\phi}(t)\geq 0$ over the entire boundary of $\mathcal{A}_2$, we now need to find its vertices. In addition to $x_{\rm N}=F$, $x_{\rm Q}=f$ and $y_{\rm O}=y_{\rm R}=0$, we know the following:

$$\left.\begin{aligned} y_{\rm M} &= (F^\mu-x_{\rm M}^\mu)/\gamma = y_{\rm N} = \gamma(x_{\rm O}-F),\\ y_{\rm P} &= (f^\mu-x_{\rm P})/\gamma = y_{\rm Q} = \gamma(x_{\rm R}-f),\\ x_{\rm M} &= x_{\rm R},\\ x_{\rm O} &= x_{\rm P}, \end{aligned}\right\}$$

which results in the simultaneous equations

$$F^\mu-x_{\rm M}^\mu = \gamma^2(x_{\rm O}-F) \quad\text{and}\quad f^\mu-x_{\rm O}^\mu=\gamma^2(x_{\rm M}-f), \tag{3.3}$$

between which $x_{\rm O}$ can be eliminated to give equation (3.1). When this can be solved, it gives all the vertices of MNOPQR in terms of γ, μ, F, f and $x_{\rm M}$. The conditions under which a solution $x_{\rm M}$ of equation (3.1) exists satisfying $0\leq x_{\rm M}<f$ and $x_{\rm O}>F$ are easily derived. Consider equation (3.3), solved for $x_{\rm O}$ in terms of $x_{\rm M}$, giving

$$x_{\rm O} = F+(F^\mu-x_{\rm M}^\mu)/\gamma^2 \quad \ldots\text{curve (a)},$$

$$x_{\rm O} = [f^\mu+\gamma^2(f-x_{\rm M})]^{1/\mu} \quad \ldots\text{curve (b)},$$

respectively. At $x_{\rm M}=f$, we have $x_{\rm O}=F+(F^\mu-f^\mu)/\gamma^2$ for curve (a) and $x_{\rm O}=f$ for curve (b), and so (a) is above (b) at $x_{\rm M}=f$, provided $\mu>1$. Therefore, to guarantee at least one solution of equation (3.3), we need (b) to be above (a) at $x_{\rm M}=0$, and so we must have

$$f^\mu+\gamma^2 f\geq\left(\frac{f+F^\mu}{\gamma^2}\right)^\mu,$$

which is the condition given in the theorem. ■

(*a*) *Comparison of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$

We can now compare the restrictions on the parameters imposed in theorems 2.18 and 3.1 when $\mu=2$. These are that $\gamma^2\geq 8F$ (set $\mathcal{A}_1$) and that $f\gamma^4(\gamma^2+f)\geq F^2(\gamma^2+F)^2$ (set $\mathcal{A}_2$). Let $\gamma^2=8F$ so that the second inequality becomes $64F^2f(8F+f)=81F^4$. Solving for f in terms of F gives $f=(\sqrt{1105}/8-4)F\approx 0.1552F$. Hence we conclude that, subject to $F>f>0$, if $f<(\sqrt{1105}/8-4)F$, the constraint on γ is weaker for set $\mathcal{A}_1$; otherwise, the constraint on γ is weaker for set $\mathcal{A}_2$.

4. Conclusions

We have constructed a variety of invariant sets for the differential equation (1.1), the sets being one for which solutions grow without bound; one which must contain any bounded limit cycles, both of these sets requiring no constraints other than those given in the first paragraph of the paper. We have additionally found two absorbing sets, both of which require additional parameter constraints (effectively, large dissipation), one of which is valid for all $\mu > 1$. We have also described three areas of physical interest in which the differential equation arises.

We have had little to say here about the dynamics of this equation, since our aim was to construct some important invariant sets. Numerical results presented in Deane & Marsh (2004), Marsh (in preparation) indicate the presence of co-existing periodic attractors but not chaos. There are also some results on dynamics for large γ and $\mu = 2$ in Gentile *et al.* (2005), in particular a study of the analyticity properties of the orbit with the same period as $f(t)$ that, numerically at least, appears to be the only bounded periodic solution to equation (1.1) in this case. Additionally, there are in principle ways to understand which periodic orbits should occur (Bartuccelli *et al.* 2004), in particular when the dissipation and the oscillatory part of $f(t)$ are both small.

At least two interesting open questions remain. The first concerns the basin of attraction of periodic solutions, which clearly must lie in $\mathcal{B}^c$, but the construction of this set does not exclude the possibility that this basin has an infinite 'tail' lying between curves AB and CD as $x \to -\infty$. The second concerns the rate at which solutions that grow without bound approach infinity: do they do so in finite or infinite time? In the unperturbed version of the differential equation ($\gamma = 0$, $\mu = 2$, $f(t)$ constant), exact solutions exist and these can be expressed in terms of Weierstrass elliptic functions, which do indeed blow up in finite time, but it is not clear whether this property is inherited by solutions to equation (1.1).

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