Fluid Mechanics Formula Sheet

1 Cartesian coordinates $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$

For any function $f(\mathbf{x})$ and any vector field $\mathbf{F}(\mathbf{x}) = f(\mathbf{x})\mathbf{e}_x + g(\mathbf{x})\mathbf{e}_y + h(\mathbf{x})\mathbf{e}_z$, we have the following.

$$\begin{aligned}
\operatorname{grad} f &= \nabla f = f_{,x}\mathbf{e}_x + f_{,y}\mathbf{e}_y + f_{,z}\mathbf{e}_z. \\
\operatorname{curl} \mathbf{F} &= \nabla\times\mathbf{F} = (h_{,y} - g_{,z})\mathbf{e}_x + (f_{,z} - h_{,x})\mathbf{e}_y + (g_{,x} - f_{,y})\mathbf{e}_z. \\
\operatorname{div} \mathbf{F} &= \nabla\cdot\mathbf{F} = f_{,x} + g_{,y} + h_{,z}. \\
\nabla^2 f &= \nabla\cdot(\nabla f) = f_{,xx} + f_{,yy} + f_{,zz} \quad \text{is the Laplacian of } f. \\
\nabla^2\mathbf{F} &= (\nabla^2 f)\mathbf{e}_x + (\nabla^2 g)\mathbf{e}_y + (\nabla^2 h)\mathbf{e}_z \quad \text{is the Laplacian of } \mathbf{F}.
\end{aligned}$$

Here $,x$ denotes $\partial/\partial x$, etc.

2 Useful identities

$$\begin{aligned}
\nabla\times(\nabla f) &= \mathbf{0}. \\
\nabla\cdot(\nabla\times\mathbf{F}) &= 0. \\
\nabla^2\mathbf{F} &= \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F}). \\
\nabla\cdot(f\mathbf{F}) &= f\nabla\cdot\mathbf{F} + (\mathbf{F}\cdot\nabla)f. \\
(\mathbf{F}\cdot\nabla)\mathbf{F} &= (\nabla\times\mathbf{F})\times\mathbf{F} + \nabla(\tfrac{1}{2}\mathbf{F}\cdot\mathbf{F}).
\end{aligned}$$

The material derivative is $D/Dt = \partial/\partial t + (\mathbf{u}\cdot\nabla)$.

3 Divergence Theorem

If V is a closed region with boundary ∂V then

$$\int_V \nabla\cdot\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n}\cdot\mathbf{F}\,\mathrm{d}S$$

where $\mathbf{n}$ is the outward normal at each $\mathbf{x}\in\partial V$. Consequences:

$$\begin{aligned}
i) &\quad \textstyle\int_V \nabla f\,\mathrm{d}\mathbf{x} = \displaystyle\int_{\partial V} \mathbf{n} f\,\mathrm{d}S \\
ii) &\quad \textstyle\int_V \nabla\times\mathbf{F}\,\mathrm{d}\mathbf{x} = \displaystyle\int_{\partial V} \mathbf{n}\times\mathbf{F}\,\mathrm{d}S \\
iii) &\quad \textstyle\int_V \nabla^2 f\,\mathrm{d}\mathbf{x} = \displaystyle\int_{\partial V} (\mathbf{n}\cdot\nabla) f\,\mathrm{d}S.
\end{aligned}$$

4 Orthogonal Coordinates (a_1, a_2, a_3)

Write $\mathbf{x} = x(a_1, a_2, a_3)\mathbf{e}_x + y(a_1, a_2, a_3)\mathbf{e}_y + z(a_1, a_2, a_3)\mathbf{e}_z$.

The coordinates are orthogonal if $\frac{\partial \mathbf{x}}{\partial a_i} \cdot \frac{\partial \mathbf{x}}{\partial a_j} = 0$ for $i \neq j$. Then let

$$h_i = \left| \frac{\partial \mathbf{x}}{\partial a_i} \right| = \left(\left(\frac{\partial x}{\partial a_i} \right)^2 + \left(\frac{\partial y}{\partial a_i} \right)^2 + \left(\frac{\partial z}{\partial a_i} \right)^2 \right)^{1/2}$$

and let $\mathbf{e}_i = \frac{1}{h_i} \frac{\partial \mathbf{x}}{\partial a_1}$ be the unit vector corresponding to a_i. Grad, curl and div are given by the following formulae.

$$\begin{aligned}
\nabla f &= \tfrac{1}{h_1} f_{,1}\mathbf{e}_1 + \tfrac{1}{h_2} f_{,2}\mathbf{e}_2 + \tfrac{1}{h_3} f_{,3}\mathbf{e}_3. \\
\nabla \times \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial a_1 & \partial/\partial a_2 & \partial/\partial a_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}. \\
\nabla \cdot \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \Big((h_2, h_3 F_1)_{,1} + (h_1 h_3 F_2)_{,2} + (h_1 h_2 F_3)_{,3} \Big).
\end{aligned}$$

Here $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ and $,i$ denotes $\partial/\partial a_i$.

Cylindrical Polars $a_1 = r \geq 0, \quad a_2 = \theta \in (-\pi, \pi], \quad a_3 = z.$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z \text{ is unchanged.}$$

$$\therefore\ h_1 = 1, \quad h_2 = r, \quad h_3 = 1.$$

$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_x + \sin\theta\,\mathbf{e}_y, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_y, \quad \mathbf{e}_z \text{ is unchanged.}$$

Also $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$.

Spherical Polars $a_1 = r \geq 0, \quad a_2 = \theta \in [0, \pi], \quad a_3 = \phi \in (-\pi, \pi].$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

$$\therefore \quad h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta.$$

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z. \\
\mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{e}_x + \cos\theta\sin\phi\,\mathbf{e}_y - \sin\theta\,\mathbf{e}_z. \\
\mathbf{e}_\phi &= -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y.
\end{aligned}$$

Also $\mathbf{x} = r\mathbf{e}_r$.

5 Equations of Motion

Continuity equation: $\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0$.
[This reduces to $\nabla \cdot \mathbf{u} = 0$ when the density ρ is constant.]
Navier-Stokes equation for incompressible constant-density flow:

$$\rho\big(\mathbf{u}_{,t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\big) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g}.$$

[Omit $\mathbf{g}$ if gravitational effects are unimportant. For inviscid flow ($\mu = 0$) the Navier-Stokes equation becomes the Euler equation.]
Conditions at a boundary moving with velocity $\mathbf{U}$:

$$\begin{aligned} \mathbf{u} - \mathbf{U} &= \mathbf{0}, \quad \text{viscous flow } (\mu > 0); \\ \mathbf{n} \cdot (\mathbf{u} - \mathbf{U}) &= 0, \quad \text{inviscid flow } (\mu = 0). \end{aligned}$$

Stress vector on surface with outward unit normal $\mathbf{n}$:-

$$\mathbf{t} = -p\mathbf{n} + \mu\big[2(\mathbf{n} \cdot \nabla)\mathbf{u} + \mathbf{n} \times (\nabla \times \mathbf{u})\big].$$

6 Useful theorems on fluid flow

`Reynolds Transport Theorem`:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G(\mathbf{x}, t)\mathrm{d}\mathbf{x} = \int_{V(t)} \left\{\frac{DG(\mathbf{x}, t)}{Dt} + G(\mathbf{x}, t)\nabla \cdot \mathbf{u}\right\}\mathrm{d}\mathbf{x}$$

`Bernoulli Theorems` (for inviscid flow only) come from the identity

$$\mathbf{u}_{,t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H$$

where

$$H = p/\rho + \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u} + \chi.$$

Note: $\mathbf{g} = -\nabla\chi$; if $\mathbf{g} = -g\mathbf{e}_z$ then $\chi = gz$.
The main results are:

i) $(\mathbf{u} \cdot \nabla)H = 0$ (H is constant on each streamline) in steady flow;

ii) H is constant in steady irrotational flow;

iii) $\phi_{,t} + H$ is constant in unsteady potential flow.

`Milne-Thomson's Circle Theorem` (2-D potential flow only)
If all singularities of $f(\zeta)$ lie in the region $|\zeta| > a$ then

$$\Phi = f(\zeta) + \overline{f(a^2/\overline{\zeta})}$$

is the complex potential of a flow that has: i) the same singularities in $|\zeta| > a$ as $f(\zeta)$; ii) $|\zeta| = a$ as a streamline.

7 Streamfunctions (incompressible flow)

A vector streamfunction $\mathbf{\Psi}$ is a solution of $\mathbf{u} = \nabla \times \mathbf{\Psi}$.
For a 2-D flow with $\mathbf{u} = u(x, y, t)\mathbf{e}_x + v(x, y, t)\mathbf{e}_y$, let $\mathbf{\Psi} = \Psi(x, y, t)\mathbf{e}_z$. Then

$$u = \Psi_{,y}, \quad v = -\Psi_{,x}.$$

In orthogonal coordinates with $\mathbf{u} = u(a_1, a_2, t)\mathbf{e}_1 + v(a_1, a_2, t)\mathbf{e}_z$, let $\mathbf{\Psi} = \frac{1}{h_3}\psi(a_1, a_2, t)\mathbf{e}_3$. Then

$$u = \frac{1}{h_1 h_3}\psi_{,2}, \quad v = -\frac{1}{h_2 h_3}\psi_{,1}.$$

The function ψ is called the streamfunction.

8 Potential flow

A potential flow has $\mathbf{u} = \nabla\varphi$ for some function $\varphi(\mathbf{x}, t)$. Every potential flow is irrotational ($\nabla \times \mathbf{u} = \mathbf{0}$.)

`Complex potential` (steady 2-D inviscid incompressible flow only)
If $\mathbf{u} = u(x, y, t)\mathbf{e}_x + v(x, y, t)\mathbf{e}_y$ is a potential flow then φ and ψ satisfy the Cauchy-Riemann equations:-

$$\varphi_{,x} = \psi_{,y}, \quad \varphi_{,y} = -\psi_{,x}.$$

Consequently the complex potential $\Phi = \varphi + i\psi$ is a function of $\zeta = x + iy$ only.
A point (x_0, y_0) is a singularity of Φ if $|\Phi| \to \infty$ as ζ approaches $\zeta_0 = x_0 + iy_0$.