Introduction

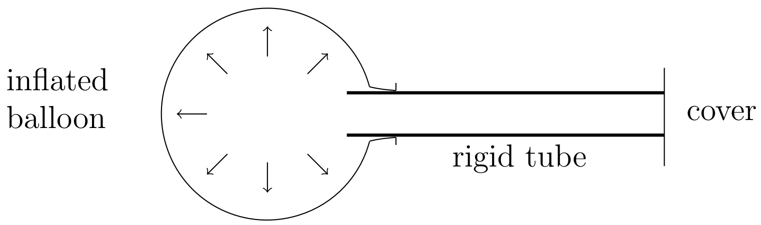
Def$^{\text{n}}$ A *fluid* is a liquid or a gas.

Examples Air, steam, water, blood, golden syrup.

All real fluids have a *viscosity* μ, which is a measure of how difficult it is to deform the fluid. The above examples are placed in order of increasing viscosity. For instance, it is easier to stir a glass of water than a glass of golden syrup.

Def$^{\text{n}}$ *Fluid mechanics* is the study of the motion of fluids.

An experiment Attach a balloon to a long rigid cylindrical tube and inflate the balloon through the tube. Then cover the open end of the tube, as shown.



The elasticity of the balloon generates a force that is trying to shrink the balloon to its original shape. However the air cannot escape because of the cover.

Newton's 2^{nd} Law,

$$\text{Force} = \text{mass} \times \text{acceleration} \qquad (F = ma),$$

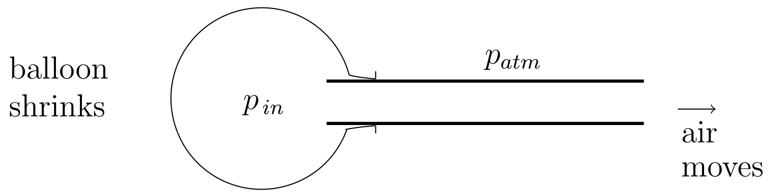
tells us that the *total* force on the balloon is zero, because the balloon is stationary.

Therefore the force generated by the elasticity (which tries to shrink the balloon) is balanced by a force pushing in the opposite direction, as shown by the arrows. This force (per unit area) is called the *pressure difference* between the air inside and the air outside the balloon.

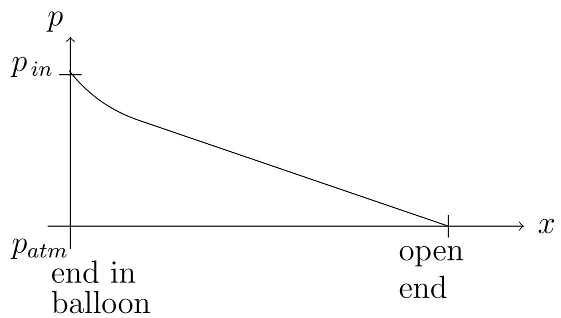
All fluids have a pressure, denoted p, which may vary with time and with position in the fluid. In the above situation, the air in the tube and balloon has a higher pressure (which we shall denote by p_{in}) than the outside air, whose pressure is atmospheric (denoted p_{atm}). The pressure difference, $p_{in} - p_{atm}$, is positive; this generates a force pointing from the higher-pressure region to the lower-pressure region.

Within either region, the pressure is *uniform* (it has the same value everywhere), so there is no net force due to pressure differences.

The force due to the difference in pressure between the inner and outer regions can be used to move air once the cover is removed.



The air pressure varies continuously as a function of distance x along the tube as shown by the following graph:-



This graph is a snapshot taken at a particular time. As time goes on, p_{in} decreases but the graph has roughly the same shape.

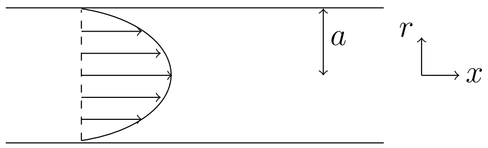
Within the tube a pressure gradient $\frac{\partial p}{\partial x}$ is set up, and fluid flows *down* the gradient. Over much of the tube, the gradient is uniform, i.e. pressure varies linearly with x.

Later we shall show that the fluid velocity u, in a cylindrical tube of radius a with a steady (time-independent) uniform pressure gradient $\frac{\partial p}{\partial x}$ is

$$u = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)(a^2 - r^2)$$

in cylindrical polar coordinates (r, θ, x). Here u is the velocity in the x-direction, so it is positive whenever $\frac{\partial p}{\partial x}$ is negative (and vice versa). Note that u does not depend on x when the pressure gradient is uniform.

Here is a pictorial representation of the flow:-



The fluid sticks to the wall ($u = 0$ when $r = a$) and has its maximum speed in the centre of the pipe.

The total rate at which fluid leaves the pipe is called the *volume flux*. It is the integral of u over the end of the pipe:

$$
\begin{aligned}
Q &= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} u\, r \,\mathrm{d}\theta \,\mathrm{d}r \\
&= \int_{r=0}^{a} \frac{2\pi r}{4\mu} \left(-\frac{\partial p}{\partial x}\right) (a^2 - r^2) \,\mathrm{d}r \\
&= \frac{\pi}{2\mu} \left(-\frac{\partial p}{\partial x}\right) \int_{r=0}^{a} (a^2 r - r^3) \,\mathrm{d}r \\
&= \frac{\pi}{2\mu} \left(-\frac{\partial p}{\partial x}\right) \left[\frac{a^2 r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{a} \\
&= \frac{\pi a^4}{8\mu} \left(-\frac{\partial p}{\partial x}\right).
\end{aligned}
$$

To increase the volume flux Q, do one of the following:-

i) increase the size of the pressure gradient $-\frac{\partial p}{\partial x}$ (push harder);

ii) increase the pipe radius a (a larger pipe lets air through faster);

iii) use a fluid with a lower viscosity than air;

conversely, water or golden syrup will flow more slowly than air.

Derivation of the equations of fluid motion

The continuum approximation

Throughout this module, we shall regard a fluid as a continuum - i.e. something whose properties are continuous functions of position. This approach neglects fluctuations that occur on a molecular scale.

For instance, within a *small* volume of fluid $V_{\mathbf{x}}$ centred on the point $\mathbf{x}$, the average velocity of the fluid molecules in V is

$$\mathbf{u} = \frac{\text{sum of the individual velocities of molecules in } V_{\mathbf{x}}}{\text{no. of molecules in } V_{\mathbf{x}}}.$$

Neighbouring particles may have wildly differing velocities (e.g. if they have recently collided and bounced off each other). Therefore $V_{\mathbf{x}}$ should be taken to be sufficiently small that $\mathbf{u}$ can be regarded as a function of $\mathbf{x}$, but sufficiently large that small changes in $\mathbf{x}$ produce small changes in $\mathbf{u}$. The fluid in $V_{\mathbf{x}}$ is sometimes referred to as a *fluid particle*.

It is a fundamental assumption of fluid mechanics that such averaging is possible, so that every fluid property may be written as a continuous function of position $\mathbf{x}$ and time t.

Def$^{\mathbf{n}}$ Any property that is independent of t will be called *steady* (or constant-in-time).

Def$^{\mathbf{n}}$ Any property that is independent of $\mathbf{x}$ will be called *uniform* (or constant-in-space).

Def$^{\mathbf{n}}$ Any property that is steady and uniform will be called *constant*.

Conservation of mass

In Newtonian mechanics, the total mass of a system is steady - mass cannot be created or destroyed. We must find a way to incorporate this condition into fluid mechanics.

Consider a fluid particle whose position at time t is $\mathbf{x}$. If $\mathbf{u}(\mathbf{x}, t)$ denotes the known fluid velocity at that position and time then the motion of the fluid particle is determined by the equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t).$$

If $\mathbf{u}$ is known everywhere in the fluid and at all times, the trajectory $\mathbf{x}(t)$ for a fluid particle whose initial position is $\mathbf{X}$ (at time $t = 0$) is the solution of the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}(t), t), \qquad \mathbf{x}(0) = \mathbf{X}.$$

Example In Cartesian coordinates $\mathbf{x} = (x, y, z)$, a *stagnation-point* flow is a steady flow of the form

$$\mathbf{u}(\mathbf{x}) = (u, v, w)$$

where $\quad u = \alpha x, \quad v = -\alpha y, \quad w = 0 \qquad (\alpha \text{ is constant}).$

So the initial-value problem for the trajectory starting at $\mathbf{X} = (X, Y, Z)$ is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = 0,$$

subject to

$$(x(0), y(0), z(0)) = (X, Y, Z).$$

The solution is

$$(x(t), y(t), z(t)) = (Xe^{\alpha t},\, Ye^{-\alpha t},\, Z).$$

The trajectories are the hyperbolae $xy = const$, together with the x and y axes.

More generally, we can write the position $\mathbf{x}$ of a fluid particle as a function of its initial position $\mathbf{X}$ and t:-

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t).$$

Note that $\mathbf{x}$ must be an invertible function of $\mathbf{X}$ (for each $t \geq 0$), because we can always trace back in time to find the initial position of a particle. Hence the Jacobian,

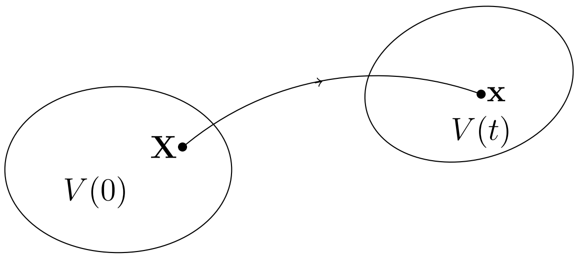
$$J(X, Y, Z, t) = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix},$$

is nonzero. Here (x, y, z) are regarded as functions of (X, Y, Z, t).

Theorem The Jacobian $J(\mathbf{X}, t) = J(X, Y, Z, t)$ satisfies the identity

$$\frac{\partial}{\partial t}\Big(J(\mathbf{X}, t)\Big) = (\nabla \cdot \mathbf{u}) J(\mathbf{x}, t).$$

This result can be used to show how fluid properties change with time. Let $V(t)$ denote an arbitrary blob of fluid as it moves in time. At $t = 0$, the fluid occupies the region $V(0)$ as shown.



Let $G(\mathbf{x}, t)$ be any fluid property, and let

$$\widetilde{G}(\mathbf{X}, t) = G(\mathbf{x}(\mathbf{X}, t), t).$$

Then, changing variables from $\mathbf{x}$ to $\mathbf{X}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} G(\mathbf{x}, t)\, \mathrm{d}\mathbf{x} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{V(0)} \widetilde{G}(\mathbf{X}, t) J(\mathbf{X}, t)\, \mathrm{d}\mathbf{X} \\ &= \int_{V(0)} \left\{ \frac{\partial \widetilde{G}(\mathbf{X}, t)}{\partial t} J(\mathbf{X}, t) + \widetilde{G}(\mathbf{X}, t) \frac{\partial J(\mathbf{X}, t)}{\partial t} \right\} \mathrm{d}\mathbf{X} \\ &= \int_{V(0)} \left\{ \frac{\partial \widetilde{G}(\mathbf{X}, t)}{\partial t} + \widetilde{G}(\mathbf{X}, t) \nabla \cdot \mathbf{u} \right\} J(\mathbf{X}, t)\, \mathrm{d}\mathbf{X}. \end{aligned}$$

Note that

$$
\begin{aligned}
\frac{\partial \widetilde{G}(\mathbf{X},t)}{\partial t} &= \frac{\partial G(\mathbf{x},t)}{\partial t} + \frac{\partial x(\mathbf{X},t)}{\partial t}\frac{\partial G(\mathbf{x},t)}{\partial x} + \frac{\partial y(\mathbf{X},t)}{\partial t}\frac{\partial G(\mathbf{x},t)}{\partial y} + \frac{\partial z(\mathbf{X},t)}{\partial t}\frac{\partial G(\mathbf{x},t)}{\partial z} \\
&= \frac{\partial G}{\partial t} + u\frac{\partial G}{\partial x} + v\frac{\partial G}{\partial y} + w\frac{\partial G}{\partial z} = \frac{\partial G}{\partial t} + (\mathbf{u}\cdot\nabla)G.
\end{aligned}
$$

For brevity, we introduce the *material derivative*

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla.$$

Therefore we have shown that $\frac{\partial \widetilde{G}}{\partial t} = \frac{DG}{Dt}$.

This is the rate at which G changes from the point of view of the fluid particle whose initial position was $\mathbf{X}$.

By rewriting our previous result in terms of $\mathbf{x}$ we obtain *Reynolds' Transport Theorem*:-

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \int_{V(t)} \left\{\frac{DG(\mathbf{x},t)}{Dt} + G(\mathbf{x},t)\nabla\cdot\mathbf{u}\right\}\mathrm{d}\mathbf{x}.$$

Def^n The *density* ρ of a fluid is the mass per unit volume; in practice, this is averaged over very small volumes $V_{\mathbf{X}}$, so we can write

$$\rho(\mathbf{x},t) = \frac{\text{mass in } V_{\mathbf{x}}}{\text{volume of } V_{\mathbf{x}}}.$$

Therefore the total mass in the blob occupying $V(t)$ is $\int_{V(t)} \rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$.

The principle of *conservation of mass* implies that the mass in a given fluid blob does not change with time.

Therefore

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&= \int_{V(t)} \left\{\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u}\right\}\mathrm{d}\mathbf{x}
\end{aligned}
$$

by Reynolds' Transport Theorem. As $V(t)$ is an *arbitrary* fluid blob

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0.$$

This is the *continuity equation*, which expresses conservation of mass throughout the fluid.

The continuity equation can be combined with Reynolds' Transport Theorem to give a useful result:-

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\rho F(\mathbf{x},t)\,\mathrm{d}\mathbf{x} &= \int_{V(t)}\left\{F(\mathbf{x},t)\frac{D\rho}{Dt}+\rho\frac{DF(\mathbf{x},t)}{Dt}+\rho F(\mathbf{x},t)\nabla\cdot\mathbf{u}\right\}\mathrm{d}\mathbf{x}\\ &= \int_{V(t)}\rho\frac{DF(\mathbf{x},t)}{Dt}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Incompressible flow

A fluid is *incompressible* if its density ρ is unaffected by fluctuations in pressure. Most fluids are somewhat compressible. However, it can be shown that if the flow driven by such fluctuations is much slower than the speed of sound in the fluid, the fluid behaves as if it were incompressible. We shall restrict attention to incompressible flows throughout the module.

The density of an incompressible fluid may vary as a result of temperature, salinity, or other factors. However, we shall not study flows in which such factors occur. Consequently, throughout the module, we assume that ρ is a constant.

> The continuity equation,
>
> $$\frac{D\rho}{Dt}+\rho\nabla\cdot\mathbf{u}=0,$$
>
> reduces to the following condition for an incompressible flow with constant density:-
>
> $$\nabla\cdot\mathbf{u}=0.$$

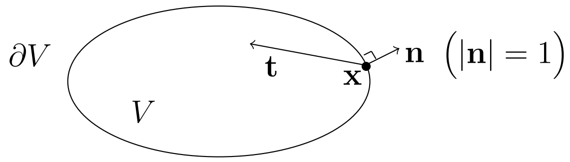
For an incompressible flow, this condition implies that

$$\frac{\partial J}{\partial t}=0;$$

in other words, the volume of a given blob of fluid does not change as the blob moves along in the flow. Because the density is constant, conservation of mass implies conservation of the volume of a given blob of fluid as it moves.

Stress

Consider a blob of fluid V whose boundary ∂V is a simple smooth closed surface. At each point $\mathbf{x}$ on the surface, there is a unique *outward unit normal* vector $\mathbf{n}$, as shown.



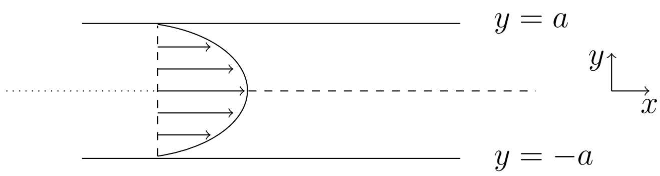
Force is transmitted from the surrounding fluid to the blob across the surface ∂V. The force per unit area is a vector $\mathbf{t}$, called the *stress*.

The pressure p is one component of the stress; this is directed perpendicularly to the surface ∂V at $\mathbf{x}$, from the surrounding fluid towards V. If pressure is the only force acting on the fluid then $\mathbf{t} = -p\mathbf{n}$.

In an incompressible *Newtonian* fluid, viscosity μ is responsible for the remaining components of $\mathbf{t}$. Hence $\mathbf{t} + p\mathbf{n}$ is proportional to μ. Viscosity resists deformation of the fluid; it produces stress only when the velocity varies with position.

For instance, a cup of tea in a railway carriage may be moving at 100 mph, but it experiences no viscous stress because the tea is all moving at the same velocity. If the tea is stirred or wobbled, the velocity will not remain uniform, and viscous stress will act.

To illustrate viscous stress, consider the flow in a channel of width $2a$ that is driven by a constant pressure gradient $\frac{\mathrm{d}p}{\mathrm{d}x}$.

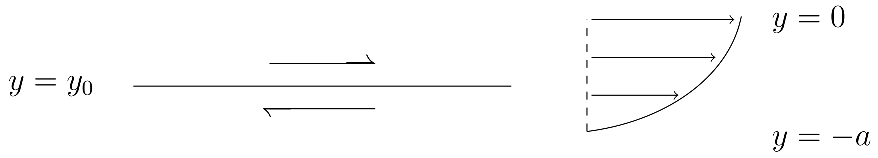


This is very similar to flow in a cylindrical pipe:-

$$\mathbf{u} = u(y)\mathbf{e}_x$$

where $\quad u(y) = \frac{1}{2\mu}(-\frac{\mathrm{d}p}{\mathrm{d}x})(a^2 - y^2)$.

In the region $y < 0$, the velocity increases with increasing y. This produces a positive tangential stress in the x-direction on the upper part of each surface $y = y_0$ in this region. This is balanced by an equal stress in the negative x-direction on the lower part of the surface.



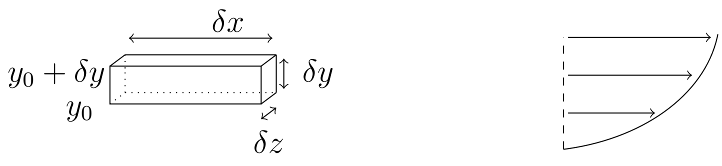
In a Newtonian fluid, the stress on the upper part of the surface in the positive x-direction is

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}y}(y_0).$$

The same formula holds in the region $y > 0$, where $\frac{\partial u}{\partial y}$ is negative.

Now consider the total force produced by this tangential stress (or *shear stress*) on a small cube of fluid occupying the region

$$V = [x_0,\ x_0 + \delta x] \times [y_0,\ y_0 + \delta y] \times [z_0,\ z_0 + \delta z].$$



The force produced by the shear stress on the upper surface $y = y_0 + \delta y$ is $\mu \frac{\mathrm{d}u}{\mathrm{d}y}(y_0 + \delta y)\delta x \delta z$. Similarly the force on the lower surface $y = y_0$ is $-\mu \frac{\mathrm{d}u}{\mathrm{d}y}(y_0)\delta x \delta z$. So the total force produced by the shear stress on the cube is

$$\mu \left(\frac{\mathrm{d}u}{\mathrm{d}y}(y_0 + \delta y) - \frac{\mathrm{d}u}{\mathrm{d}y}(y_0) \right) \delta x \delta z \approx \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}(y_0)\delta x \delta y \delta z.$$

Therefore the force per unit volume due to viscous stress is $\mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}(y_0)$.

For a general flow of an incompressible Newtonian fluid, it can be shown that the total stress on the outer side of a surface at a point where the outward unit normal is $\mathbf{n}$ is

$$\mathbf{t} = \underbrace{-p\mathbf{n}}_{\text{pressure}} \quad + \quad \underbrace{\mu[2(\mathbf{n}\cdot\nabla)\mathbf{u} + \mathbf{n}\times(\nabla\times\mathbf{u})]}_{\text{viscous stress}}.$$

Exercise Show that this expression reduces to

$$\mathbf{t} = -p\mathbf{e}_y + \tau\mathbf{e}_x$$

for the upper part of $y = y_0$ in the channel flow.

Exercise In this exercise, we use Cartesian coordinates (x_1, x_2, x_3) in place of (x, y, z), and write the components of $\mathbf{t}$ and $\mathbf{n}$ as $t_i = \mathbf{e}_i \cdot \mathbf{t}$ and $n_j = \mathbf{e}_j \cdot \mathbf{n}$ for $i, j \in \{1, 2, 3\}$. We also introduce the Kronecker delta δ_{ij}, which is 1 if $i = j$ and 0 otherwise.

Show that $t_i = \sum_{j=1}^{3} T_{ij} n_j$, where

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3.$$

(Many books use this result as the starting-point for thinking about stress. The coefficients T_{ij} are called components of the *stress tensor*. The difficulty with this approach is that it relies on the coordinates being Cartesian.)

A useful divergence

Let V be a volume bounded by a simple closed surface ∂V, and let $\mathbf{F}$ be an arbitrary vector. The Divergence Theorem states that

$$\int_V \nabla\cdot\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n}\cdot\mathbf{F}\,\mathrm{d}S$$

where $\mathbf{n}$ is the unit normal and $\mathrm{d}S$ is the area element.
Here $\int_V$ is a triple integral, whereas $\int_{\partial V}$ is a double integral.

The divergence theorem can be used to prove the following results:-

i) If ϕ is a function then

a) $\int_V \nabla\phi\,\mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n}\phi\,\mathrm{d}S = \int_{\partial V} \phi\mathbf{n}\,\mathrm{d}S$

b) $\int_V \nabla^2\phi\,\mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n}\cdot\nabla\phi\,\mathrm{d}S$.

ii) If $\mathbf{F}$ is a vector then

a) $\int_V \nabla^2 \mathbf{F} \, \mathrm{d}\mathbf{x} = \int_{\partial V} (n \cdot \nabla) \mathbf{F} \, \mathrm{d}S$

b) $\int_V \nabla \times \mathbf{F} \, \mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n} \times \mathbf{F} \, \mathrm{d}S$.

These useful identities can be proved by using Cartesian coordinates (x_1, x_2, x_3).

Newton's 2^{nd} Law revisited

Newton's 2^{nd} law is usually stated to be

$$\text{Mass } \times \text{Acceleration } = \text{Force}.$$

However, it can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{Momentum}) = \text{Force}$$

where

$$\text{Momentum} = \text{Mass} \times \text{Velocity}.$$

(Indeed if the mass is nonconstant, one *must* use this form of Newton's 2^{nd} Law.)

The total momentum of a blob of fluid occuppying a volume $V(t)$ is

$$\int_{V(t)} \rho \mathbf{u} \, \mathrm{d}\mathbf{x}.$$

The total force on the surface $\partial V(t)$ of the blob is obtained by integrating the stress $\mathbf{t}$ over the surface. If this is the only force on the blob then Newton's 2^{nd} Law amounts to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \mathbf{u} \, \mathrm{d}\mathbf{x} = \int_{\partial V(t)} \mathbf{t} \, \mathrm{d}S.$$

Sometimes, gravitational force is important in fluid flows. This accelerates each fluid particle in the blob by an amount $\mathbf{g}$. (On the Earth's surface, $\mathbf{g}$ points towards the centre of the Earth, and its magnitude is roughly $9.81\mathrm{m/s}^2$.)

If gravity affects the fluid flow then Newton's 2^{nd} Law gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \mathbf{u} \, \mathrm{d}\mathbf{x} = \int_{\partial V(t)} \mathbf{t} \, \mathrm{d}S + \int_{V(t)} \rho \mathbf{g} \, \mathrm{d}\mathbf{x}.$$

The identities that we derived earlier show that for an incompressible fluid with constant viscosity μ

$$
\begin{aligned}
\int_{\partial V(t)} \mathbf{t}\,\mathrm{d}S &= \int_{\partial V(t)} \{-p\mathbf{n} + \mu[2(\mathbf{n}\cdot\nabla)\mathbf{u} + \mathbf{n}\times(\nabla\times\mathbf{u})]\}\,\mathrm{d}S \\
&= \int_{V(t)} \{-\nabla p + \mu[2\nabla^2\mathbf{u} + \nabla\times(\nabla\times\mathbf{u})]\}\,\mathrm{d}\mathbf{x} \\
&= \int_{V(t)} \{-\nabla p + \mu\nabla^2\mathbf{u}\}\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Therefore, if gravity does not affect the flow,

$$
\int_{V(t)} \rho\frac{D\mathbf{u}}{Dt}\,\mathrm{d}\mathbf{x} = \int_{V(t)} \{-\nabla p + \mu\nabla^2\mathbf{u}\}\,\mathrm{d}\mathbf{x}.
$$

As $V(t)$ is an *arbitrary* volume, it follows that

$$
\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u}.
$$

This equation, together with the incompressibility condition

$$
\nabla\cdot\mathbf{u} = 0
$$

is called the *Navier-Stokes* equation.

Similarly, if gravity affects a fluid flow, the Navier-Stokes equation has an extra term:-

$$
\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g}.
$$

In Cartesian coordinates (x, y, z), the Navier-Stokes equation (without gravity) for the velocity $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$ amounts to

$$\begin{aligned}
\rho(u_{,t} + uu_{,x} + vu_{,y} + wu_{,z}) &= -p_{,x} + \mu(u_{,xx} + u_{,yy} + u_{,zz}) \\
\rho(v_{,t} + uv_{,x} + vv_{,y} + wv_{,z}) &= -p_{,y} + \mu(v_{,xx} + v_{,yy} + v_{,zz}) \\
\rho(w_{,t} + uw_{,x} + vw_{,y} + ww_{,z}) &= -p_{,z} + \mu(w_{,xx} + w_{,yy} + w_{,zz}) \\
u_{,x} + v_{,y} + w_{,z} &= 0,
\end{aligned}$$

where $u_{,t} = \frac{\partial u}{\partial t}$, etc. Note that we are using the coordinates (x, y, z) and t as independent variables. We have four equations and the four unknowns u, v, w, p; the density ρ and viscosity μ are known constants for a given fluid.

Boundary conditions

At any solid boundary, conditions must be imposed in order to determine the motion of fluid near the boundary. These are of two types:-

i) *Normal to the boundary.* Fluid cannot penetrate a solid boundary. If the boundary is stationary, with a unit normal $\mathbf{n}$, then $\mathbf{n} \cdot \mathbf{u} = 0$.
If the boundary is moving with velocity $\mathbf{U}$ then

$$\mathbf{n} \cdot (\mathbf{u} - \mathbf{U}) = 0.$$

ii) *Tangential to the boundary.* Viscosity causes fluids to stick to the boundary. Therefore if $\mu > 0$ (as is the case for all real fluids), the tangential velocity of the fluid coincides with that of the boundary. If the boundary is stationary then $\mathbf{n} \times \mathbf{u} = \mathbf{0}$; if it moves with velocity $\mathbf{U}$ then $\mathbf{n} \times (\mathbf{u} - \mathbf{U}) = \mathbf{0}$.

Putting together these two conditions, we obtain the boundary condition.

$$\mathbf{u} = \begin{cases} 0 & \text{for a stationary boundary,} \\ \mathbf{U} & \text{for a moving boundary.} \end{cases}$$

Some simplifications

The Navier-Stokes equation is generally too complicated to solve exactly, but some types of flow can be written down by looking for solutions that do not depend on one or more variables. The main types are as follows.

i) *Steady flows.* Here u, v, w, p are functions of (x, y, z) only; all t-derivatives are set to zero.

ii) *Two-dimensional flows.* Here there is no flow in one of the directions and all variables are independent of that direction's coordinate. For instance, $w = 0$ and u, v, p are functions of (x, y, t) only.

iii) *Unidirectional flows.* Here the velocity is entirely in one direction. For instance, we might have $v = w = 0$. In this case, the Navier-Stokes equation reduces to

$$\begin{aligned}\rho(u_{,t} + uu_{,x}) &= -p_{,x} + \mu(u_{,xx} + u_{,yy} + u_{,zz})\\ 0 &= -p_{,y}\\ 0 &= -p_{,z}\\ u_{,x} &= 0.\end{aligned}$$

The last three of these equations imply that $p = p(x, t)$ and $u = u(y, z, t)$. Therefore

$$\rho u_{,t} = -p_{,x} + \mu(u_{,yy} + u_{,zz}).$$

Differentiating this with respect to x gives $0 = -p_{,xx}$, so $-p_{,x}$ is a function of t only.

Some unidirectional two-dimensional flows

For unidirectional two-dimensional flows without gravity, we must solve the following equation for $u(y,t)$:-

$$\rho u_{,t} = G(t) + \mu u_{,yy},$$

where $\quad G(t) = -p_{,x}$ is the pressure gradient

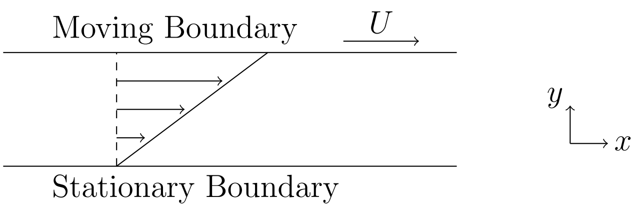
Note: this implies that the pressure is

$$p = p_0(t) - G(t)x$$

where $p_0(t)$ is an arbitrary function.

Couette Flow

Consider the motion of fluid in a channel of width a, whose upper boundary is moving at a constant velocity $U\mathbf{e}_x$, and whose lower boundary is stationary, as shown.



The fluid velocity at the moving boundary is $u = U$; at the stationary boundary, $u = 0$. The fluid motion is due entirely to the motion of the upper boundary, which is transmitted to the fluid by the viscous shear stress. There is no pressure gradient, so $G(t) = 0$. As the boundary conditions are steady, it makes sense to look for a steady solution $u = u(y)$.

Therefore we must solve

$$\mu u''(y) = 0, \qquad u(0) = 0,\ u(a) = U.$$

For any real fluid, $\mu > 0$, so $u''(y) = 0$, and the solution is

$$u(y) = \frac{Uy}{a}.$$

Note If the fluid viscosity were zero (i.e. an *inviscid* fluid), the tangential boundary condition would not apply; *any* steady unidirectional flow would be a solution of the equation of motion, because the fluid layers could slip over one another. Viscosity prevents this type of non-uniqueness occurring.

Channel Flow (plane Poisuille flow)

Channel flow (or plane Poiseuille flow) is very similar to steady flow in a pipe. The boundaries at $y = \pm a$ are stationary, and the flow is driven by a *steady* pressure gradient G.

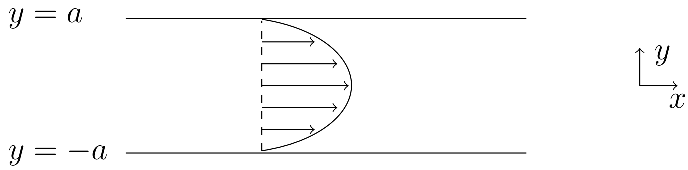
As the problem is steady, we again look for a steady solution $u = u(y)$. We must solve

$$\begin{aligned} G + \mu u''(y) &= 0, \qquad u(\pm a) = 0. \\ \therefore \quad u''(y) &= -\frac{G}{\mu}. \\ \therefore \quad u(y) &= -\frac{G}{2\mu}y^2 + Ay + B, \quad \text{where } A, B \text{ are constants.} \\ \therefore \quad u(a) &= -\frac{G}{2\mu}a^2 + Aa + B = 0, \\ u(-a) &= -\frac{G}{2\mu}a^2 - Aa + B = 0. \end{aligned}$$

$$\therefore A = 0, \quad B = \frac{G}{2\mu}a^2.$$

Hence steady flow in a channel with stationary boundaries at $y = \pm a$ is

$$u(y) = \frac{G}{2\mu}(a^2 - y^2).$$



The velocity profile is as shown.

Flow due to an impulsively moved plane boundary

Consider what happens to fluid in the region $y \in [0, \infty)$ if a solid boundary $y = 0$ begins to move at time $t = 0$. If the fluid is initially at rest and if the boundary moves at a constant velocity $U\mathbf{e}_x$ for all $t > 0$, we must solve the initial-value problem

$$\rho u_{,t} = \mu u_{,yy},$$

subject to

$$u(y,0) = 0 \quad \text{for } y > 0,$$

$$u(0,t) = U \quad \text{for } t > 0.$$

Note that the equation of motion is first-order in time and second-order in y. So far, we have one initial condition and one boundary condition; a second boundary condition is needed to make the problem well-posed.

This condition is

$$u(y,t) \to 0 \text{ as } y \to \infty$$

(If this were not so, the plate would have had to do an infinite amount of work per unit length to give an infinite volume of fluid an infinite total momentum - this never happens.)

For simplicity, let $\nu = \frac{\mu}{\rho}$; this quantity is called the *kinematic viscosity*. Then we must solve

$$u_{,t} = \nu u_{,yy} \quad \text{for } y > 0,\ t > 0,$$

subject to

$$u(y,0) = 0 \ \text{ for } y > 0,$$

$$u(0,t) = U \ \text{ for } t > 0,$$

$$u(y,t) \to 0 \text{ as } y \to \infty \ \text{ for } t > 0.$$

Note that this problem does not change if we replace y by ky and t by $k^2 t$ for any $k > 0$. Therefore we may seek a solution that is unchanged by this transformation; such solutions are called *similarity* solutions.

Note that u and $\frac{y}{\sqrt{t}}$ are unchanged by the transformation, so we must look for a solution in which u is a function of $\frac{y}{\sqrt{t}}$. For convenience, let

$$\eta = \frac{y}{2\sqrt{\nu t}}$$

and seek a solution $u = f(\eta)$. By the chain rule,

$$\begin{aligned} u_{,t} &= -\frac{y}{4\sqrt{\nu t^3}} f'(\eta) = -\frac{\eta}{2t} f'(\eta), \\ u_{,y} &= \frac{1}{2\sqrt{\nu t}} f'(\eta), \\ u_{,yy} &= \frac{1}{4\nu t} f''(\eta). \end{aligned}$$

Therefore $u_{,t} = \nu u_{,yy}$ is equivalent to

$$-2\eta f'(\eta) = f''(\eta).$$
$$\therefore \quad f'(\eta) = Ae^{-\eta^2}.$$
$$\text{Hence} \quad f(\eta) = A\int_{s=0}^{\eta} e^{-s^2}\,\mathrm{d}s + B.$$

The condition $u(0,t) = U$ for $t > 0$ implies that $f(0) = U$, so $B = U$.

The remaining conditions imply that $f(\infty) = 0$.

It can be shown that
$$\int_{s=0}^{\infty} e^{-s^2}\,\mathrm{d}s = \sqrt{\pi}/2.$$
Therefore $f(\infty) = A\sqrt{\pi}/2 + U = 0. \quad \therefore A = -\frac{2U}{\sqrt{\pi}}$.

Hence $u = f(\frac{y}{2\sqrt{\nu t}}) = U\left(1 - \frac{2}{\sqrt{\pi}}\int_{s=0}^{\frac{y}{2\sqrt{\nu t}}} e^{-s^2}\,\mathrm{d}s\right)$.

The term in brackets is called the *complementary error function* and is denoted $\mathrm{erfc}(\frac{y}{2\sqrt{\nu t}})$. Therefore
$$u = U\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right).$$
The velocity profile at various times is shown below.

As t increases, the motion spreads outwards from the moving boundary into the fluid. This is typical behaviour in a fluid whose boundaries suddenly begin to move at a different speed to the rest of the fluid.

The Navier-Stokes equation in orthogonal coordinates

The Navier-Stokes equation for incompressible flows (without gravity) can be written in a coordinate-independent way in terms of the differential operators div, grad and curl, as follows:-

$$\begin{aligned} \rho(\mathbf{u}_{,t} + (\mathbf{u}\cdot\nabla)\mathbf{u}) &= -\nabla p + \mu\nabla^2\mathbf{u}, \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned}$$

Alternatively, we can use the identities

$$\boxed{\begin{aligned} \nabla^2\mathbf{u} &= \nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u}), \\ \mathbf{u}\cdot\nabla\mathbf{u} &= (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right), \end{aligned}}$$

to write the Navier-Stokes equation in the form

$$\begin{aligned} \rho\left[\mathbf{u_{,t}} + (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\right] &= -\nabla p - \mu\nabla\times(\nabla\times\mathbf{u}), \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned}$$

The alternative form is useful for writing down the Navier-Stokes equation in non-Cartesian coordinates, because we need only use formulae for ∇f, $\nabla\times\mathbf{G}$ and $\nabla\cdot\mathbf{G}$, where f is a function and G is a vector. The following theorem gives these formulae for an arbitrary *orthogonal* coordinate system a_1, a_2, a_3 with orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Examples

i) Cylindrical polar coordinates:

$a_1 = r,\ a_2 = \theta,\ a_3 = z,$

$\mathbf{e}_1 = \mathbf{e}_r,\ \mathbf{e}_2 = \mathbf{e}_\theta,\ \mathbf{e}_3 = \mathbf{e}_z.$

ii) Spherical polar coordinates:

$a_1 = r,\ a_2 = \theta,\ a_3 = \phi,$

$\mathbf{e}_1 = \mathbf{e}_r,\ \mathbf{e}_2 = \mathbf{e}_\theta,\ \mathbf{e}_3 = \mathbf{e}_\phi.$

The vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ has components x, y, z that can be written in terms of the new coordinates a_1, a_2 and a_3. A coordinate system is orthogonal if the vectors $\frac{\partial \mathbf{x}}{\partial a_i}$, $i = 1, 2, 3$, are orthogonal to one another. For any orthogonal system, let

$$h_i = \left|\frac{\partial \mathbf{x}}{\partial a_i}\right| = \left[\left(\frac{\partial x}{\partial a_i}\right)^2 + \left(\frac{\partial y}{\partial a_i}\right)^2 + \left(\frac{\partial z}{\partial a_i}\right)^2\right]^{\frac{1}{2}}.$$

Then the orthonormal basis vectors are $\mathbf{e}_i = \frac{1}{h_i}\frac{\partial \mathbf{x}}{\partial a_i}$, $\quad i = 1, 2, 3.$

Examples(cont.)

i) For cylindrical polar coordinates,

$$\begin{aligned}
& x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \\
\therefore \quad & h_1 = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right]^{\frac{1}{2}} = [\cos^2\theta + \sin^2\theta]^{\frac{1}{2}} = 1. \\
& h_2 = \left[\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2\right]^{\frac{1}{2}} = [r^2\sin^2\theta + r^2\cos^2\theta]^{\frac{1}{2}} = r. \\
& h_3 = 1.
\end{aligned}$$

ii) For spherical polar coordinates,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

$$\therefore \quad h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta.$$

Having established the method for calculating h_1, h_2, h_3, we can now calculate div, grad and curl in arbitrary orthogonal coordinates, using the following result.

Theorem For the orthogonal coordinates a_1, a_2, a_3 (in the notation used above):-

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial a_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial a_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial a_3}\mathbf{e}_3,$$

$$\nabla \times \mathbf{G} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial a_1} & \frac{\partial}{\partial a_2} & \frac{\partial}{\partial a_3} \\ h_1 G_1 & h_2 G_2 & h_3 G_3 \end{vmatrix},$$

$$\nabla \cdot \mathbf{G} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 G_1)}{\partial a_1} + \frac{\partial(h_1 h_3 G_2)}{\partial a_2} + \frac{\partial(h_1 h_2 G_3)}{\partial a_3}\right],$$

for all functions f and all vectors

$$\mathbf{G} = G_1 \mathbf{e}_1 + G_2 \mathbf{e}_2 + G_3 \mathbf{e}_3.$$

Theorem The Navier-Stokes equation (without gravity) has the following components in cylindrical polar coordinates.

Write $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$. Then

$$\begin{aligned}
u_{,t} + (\mathbf{u}\cdot\nabla)u - \frac{1}{r}v^2 &= -\frac{1}{\rho}p_{,r} + \nu[\nabla^2 u - \frac{1}{r^2}u - \frac{2}{r^2}v_{,\theta}], \\
v_{,t} + (\mathbf{u}\cdot\nabla)v + \frac{1}{r}uv &= -\frac{1}{\rho r}p_{,\theta} + \nu[\nabla^2 v + \frac{2}{r^2}u_{,\theta} - \frac{1}{r^2}v], \\
w_{,t} + \mathbf{u}\cdot\nabla w &= -\frac{1}{\rho}p_{,z} + \nu\nabla^2 w, \\
u_{,r} + \frac{1}{r}u + \frac{1}{r}v_{,\theta} + w_{,z} &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{u}\cdot\nabla &= u\frac{\partial}{\partial r} + \frac{1}{r}v\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial z}, \\
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \quad \text{for functions.}
\end{aligned}$$

Some simple flows in cylindrical polar coordinates

Poiseuille flow

Steady unidirectional pipe flow is named after the physiologist Poiseuille, who investigated blood flow by modelling arteries as pipes.

If $\mathbf{u} = w\mathbf{e}_z$ then the Navier-Stokes equation reduces to

$$w_{,t} = -\frac{1}{\rho}p_{,z} + \nu\nabla^2 w,$$

$$p_{,r} = p_{,\theta} = w_{,z} = 0.$$

In particular, for steady flow that is *axisymmetric* (independent of θ),

$w = w(r)$ is determined by

$$G + \mu\nabla^2 w = G + \mu(w_{,rr} + \frac{1}{r}w_{,r}) = 0,$$

where $G = -p_{,z}$ is constant.

This can be rearranged as

$$\frac{1}{r}(rw_{,r})_{,r} = -\frac{G}{\mu}.$$

$$\therefore \quad rw_{,r} = -\frac{Gr^2}{2\mu} + A.$$

$$\therefore \quad w = -\frac{Gr^2}{4\mu} + A\ln r + B.$$

The fluid velocity is finite everywhere, so $A = 0$ (otherwise w is infinite at $r = 0$).

The velocity vanishes at the solid wall of the pipe, so if the pipe radius is a, we need

$$B = \frac{Ga^2}{4\mu}.$$

$$\therefore \quad w = \frac{G}{4\mu}(a^2 - r^2).$$

Flow along an annular pipe

Suppose that the pipe has an annular cross-section, with an inner radius a and an outer radius $b > a$. Steady flow along such a pipe satisfies the same equations as previously, so

$$w = -\frac{Gr^2}{4\mu} + A\ln r + B.$$

However the boundary conditions are now

$$w = 0 \qquad \text{at } r = a, b.$$

$$\therefore\ 0 = w(a) = -\frac{Ga^2}{4\mu} + A\ln a + B.$$
$$0 = w(b) = -\frac{Gb^2}{4\mu} + A\ln b + B.$$

$$\therefore \quad A = \frac{G}{4\mu}\frac{b^2 - a^2}{\ln\left(\frac{b}{a}\right)}, \qquad B = \frac{Gb^2}{4\mu} - \frac{G}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)}\ln b.$$

$$\therefore \quad w(r) = \frac{G}{4\mu}\left[b^2 - r^2 - \frac{b^2 - a^2}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{b}{r}\right)\right].$$

Taylor-Couette Flow

Consider an annular pipe, as in the last example, where the boundary cylinders rotate about the z-axis with constant angular velocities Ω_a, Ω_b respectively.

If there is no other flow, the fluid will

move in the θ-direction only:-

$$\mathbf{u} = v\mathbf{e}_\theta.$$

Then the Navier-Stokes equation reduces to

$$r^2 v'' + rv' - v = 0$$

where $v = v(r)$. (This will be proved in an exercise.)

Exercise Prove this result

The solution of this ODE is

$$v = Ar + \frac{B}{r},$$

and the boundary conditions are

$$v(a) = a\Omega_a, \qquad v(b) = b\,\Omega_b.$$

$$\therefore \quad A = \frac{b^2\Omega_b - a^2\Omega_a}{b^2 - a^2}, \qquad B = \frac{(\Omega_a - \Omega_b)a^2b^2}{b^2 - a^2}.$$

In principle, this solution is valid for any angular velocities Ω_a, Ω_b. In practice, other types of flow are seen when at least one of Ω_a, Ω_b is sufficiently large. This phenomenon is called *instability*.

The Reynolds number and instability

The Navier-Stokes equation,

$$\mathbf{u}_{,t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u},$$

$$\nabla\cdot\mathbf{u} = 0,$$

involves quantities with dimensions. If M denotes mass, L denotes length and T denotes time, the dimensions of these quantities are as follows:-

$$[\mathbf{u}] = \frac{L}{T}, \quad [p] = \frac{M}{LT^2}, \quad [\rho] = \frac{M}{L^3}, \quad [\nu] = \frac{L^2}{T},$$

$$\left[\frac{\partial}{\partial t}\right] = \frac{1}{T}, \quad [\nabla] = \frac{1}{L}, \quad [\nabla^2] = \frac{1}{L^2}.$$

Note that each term in the equation has the same dimensions (LT^{-2}). This must always be true for any equation, as one cannot add quantities that have different dimensions. It is helpful to nondimensionalize the equations by introducing reference quantities as follows.

Let U_0 be a 'typical' value of the fluid speed, and let L_0 be a 'typical lengthscale'. Then, provided that $\frac{L_0}{U_0}$ is a typical timescale for the flow, we can nondimensionalize the Navier-Stokes equation by writing

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U_0} \quad \tilde{p} = \frac{p}{(\rho U_0^2)}, \quad \tilde{t} = \frac{U_0 t}{L_0}, \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{L_0}.$$

This reduces the Navier-Stokes equation to

$$\tilde{u}_{,\tilde{t}} + \tilde{\mathbf{u}}\cdot\widetilde{\nabla}\tilde{\mathbf{u}} = -\widetilde{\nabla}\tilde{p} + \frac{1}{Re}\widetilde{\nabla}^2\tilde{\mathbf{u}},$$

$$\widetilde{\nabla}\cdot\tilde{\mathbf{u}} = 0,$$

where $Re = \frac{U_0 L_0}{\nu}$ is called the *Reynolds number*. This nondimensionalization is particularly useful for *steady* flows. If the flow is unsteady, $\frac{L_0}{U_0}$ may not be a typical timescale; for instance, the flow driven by a pressure gradient or boundary oscillating with frequency ω will have a timescale $\frac{1}{\omega}$, so dimensionless quantities using that timescale will also be important.

Note that

$Re << 1$ if fluid is very viscous, slow-moving, or on a small lengthscale,

$Re >> 1$ if fluid has low viscosity, is fast-moving, or has a large lengthscale.

Note that the Navier-Stokes equation has a nonlinear term. When $Re << 1$, this term is commonly unimportant relative to the viscous term. Conversely, when $Re >> 1$, the nonlinear term cannot be neglected.

An important consequence of nonlinearity is the existence of multiple solutions. As Re is increased, the fluid flow may change abruptly from one type of solution to another. The original solution may still be a solution, but it is no longer seen because it has become *unstable*.

Osborne Reynolds demonstrated instability in pipe flow, with $L_0 = a$ (pipe radius), $U_0 =$ maximum velocity.

When Re is small enough, the flow is unidirectional steady (Poiseuille) flow. For very large Re, Poiseuille flow becomes unstable, and the fluid exhibits unsteady non-unidirectional flow. At very high Re, the flow is turbulent.

Similarly, Taylor-Couette flow becomes unstable as Re increases, exhibiting a variety of types of steady and unsteady flow.

The Euler Equation

For very large values of *Re*, the viscous term becomes small *except* in regions where $\left|\widetilde{\nabla}^2\tilde{\mathbf{u}}\right|$ is large. By neglecting such regions, the Navier-Stokes equation can be approximated by the *Euler equation*

$$\boxed{\begin{aligned} \mathbf{u}_{,t} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\frac{1}{\rho}\nabla p, \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned}}$$

(If gravity is significant, $\mathbf{g}$ should be added to the right-hand side.) As the viscous term has been dropped, a fluid satisfying the Euler equation is said to be *inviscid*. No real fluid is inviscid, but the approximation is good for describing the motion of real fluids in regions where viscous effects are small.

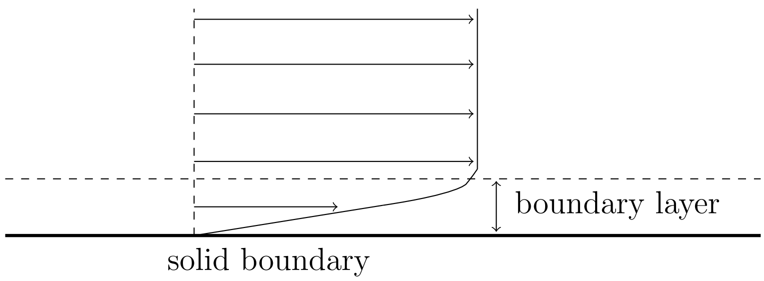
Boundary conditions

The Euler equation is *first-order* in the spatial derivatives, whereas the Navier-Stokes equation is second-order. Therefore we cannot expect to be able to satisfy all of the same boundary conditions as previously.

Clearly, no fluid can pass through a solid boundary, so $\mathbf{n}\cdot\mathbf{u} = 0$ at a stationary boundary as before. However, the tangential boundary condition $\mathbf{n}\times\mathbf{u} = \mathbf{0}$ is *not* satisfied. By eliminating the viscous term, we have effectively set $\mu = 0$, so the fluid cannot stick to the boundary. Similarly if the boundary is moving with velocity $\mathbf{U}$, the only boundary condition that can be applied is

$$\boxed{\mathbf{n}\cdot(\mathbf{u}-\mathbf{U}) = 0.}$$

When $Re >> 1$, viscous effects are significant only if $|\nabla^2\mathbf{u}| >> 1$. There are two common places where this occurs: i) in turbulence, where the flow is very disordered, and the direction of flow changes within short length scales; ii) in thin *boundary layers*, which exist to ensure that the no-slip condition $\mathbf{n}\times\mathbf{u} = \mathbf{0}$ is satisfied.



Bernoulli theorems

It is helpful to use the identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right)$$

to rewrite the Euler momentum equation (with gravity) as

$$\mathbf{u}_{,t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) + \mathbf{g}.$$

Note that if $\mathbf{g} = -g\mathbf{e}_z$ then we can write $\mathbf{g} = -\nabla \chi$ where $\chi = gz$. Furthermore the density ρ is constant. Hence

$$\mathbf{u}_{,t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H$$

where

$$\boxed{H = \frac{p}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \chi.}$$

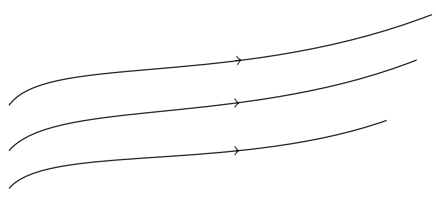
This result has several important consequences.

a) **Steady flow**

If the flow is steady (i.e. $\mathbf{u}_{,t} = \mathbf{0}$) then we obtain *Bernoulli's theorem*:

$$\mathbf{u} \cdot \nabla H = -\mathbf{u} \cdot [(\nabla \times \mathbf{u}) \times \mathbf{u}] = 0.$$

Note that $\mathbf{u} \cdot \nabla$ is $|\mathbf{u}|$ times the derivative in the direction of $\mathbf{u}$. So H is constant on any curve whose tangent points in the same direction as $\mathbf{u}$; such curves are called *streamlines*. The streamlines give an instantaneous snapshot of the flow.



For steady flows (viscous or inviscid), particles move along the streamlines, because the streamlines do not change with time. In unsteady flows, this is not true.

Although H is constant on every streamline, the constant may vary from streamline to streamline.

b) **Steady irrotational flow**

A flow is *irrotational* if the *vorticity*, $\omega = \nabla \times \mathbf{u}$, is zero everywhere. If the flow is steady and irrotational then

$$\nabla H = -\mathbf{u}_{,t} - (\nabla \times \mathbf{u}) \times \mathbf{u} = 0.$$

Therfore H is constant; it is the same for *every* streamline.

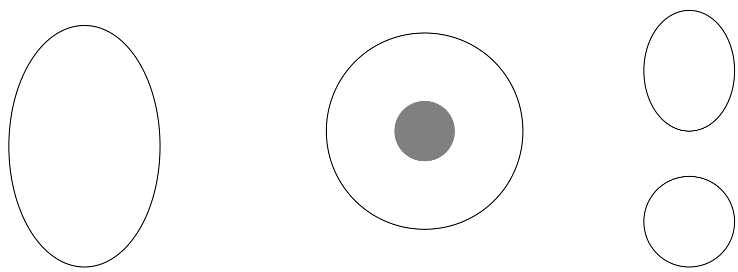
c) **Potential flow**

A flow is called "potential flow" if there exists a smooth function φ such that $\mathbf{u} = \nabla\varphi$. For any potential flow, the vorticity is zero:

$$\omega = \nabla \times \mathbf{u} = \nabla \times (\nabla\varphi) = \mathbf{0}.$$

So every potential flow is irrotational. Whether or not the converse is true depends upon the domain in which the fluid flows.

Def^n A domain V is *connected* if any two points in V can be connected by a curve lying wholly within V.



i) ii) iii)

In the above examples, i) and ii) are connected, but iii) is not (it consists of two 'pieces').

Def^n A domain V is *simply-connected* if any simple closed curve (loop) in V can be shrunk continuously to a point $\mathbf{x} \in V$, remaining in V throughout.

In the above examples, only i) is simply-connected.

A theorem from algebraic topology states that if $V \subset \mathbb{R}^3$ is simply-connected then

$$\nabla \times \mathbf{u} = 0 \;\Rightarrow\; \exists\, \varphi \text{ s.t. } \mathbf{u} = \nabla \varphi.$$

So in a simply-connected domain, every irrotational flow is a potential flow, and vice-versa.

The momentum equation

$$\mathbf{u}_{,t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H$$

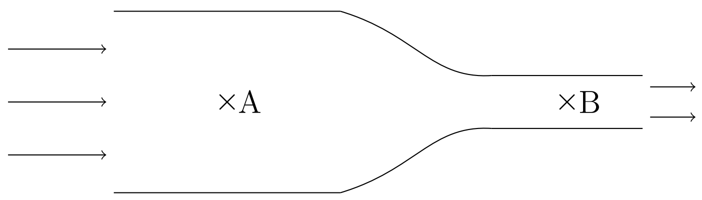
reduces, for a potential flow, to

$$\nabla(\varphi_{,t} + H) = \mathbf{0}.$$

Consequently $\varphi_{,t} + H$ is constant throughout the flow.

Inviscid flow in a constriction

Consider what happens when the inviscid fluid flows through a pipe whose radius decreases from b to $a < b$ at a constriction, as shown.



Assume that the flow is unidirectional at z_0 and z_1, i.e.

$$\mathbf{u} = \begin{cases} u_0 \mathbf{e}_z & \text{at } z_0 \\ u_1 \mathbf{e}_z & \text{at } z_1 \end{cases}$$

where u_0, u_1 are constants.

Note: in inviscid unidirectional flow, the fluid does not stick to the walls, so $\mathbf{u}$ does not vary with r in a straight pipe.

Conservation of mass, coupled with the Divergence Theorem gives

$$0 = \int_V \nabla \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n} \cdot \mathbf{u} \, \mathrm{d}S$$

for any volume V with boundary ∂V. If V is the volume of the pipe between $z = z_0$ and $z = z_1$, then this gives

$$\int_{r=0}^{a} \int_{\theta=0}^{2\pi} (\mathbf{u} \cdot \mathbf{e}_z)\Big|_{z=z_1} r \, \mathrm{d}\theta \, \mathrm{d}r + \int_{r=0}^{b} \int_{\theta=0}^{2\pi} (\mathbf{u} \cdot (-\mathbf{e}_z))\Big|_{z=z_0} r \, \mathrm{d}\theta \, \mathrm{d}r$$

$$= \pi a^2 u_1 - \pi b^2 u_0.$$

This is because $\mathbf{n} \cdot \mathbf{u} = 0$ on the solid pipe wall, so the only contributions to the integral over ∂V come from the ends $z = z_0$ and $z = z_1$. Hence

$$u_1 = \left(\frac{b}{a}\right)^2 u_0.$$

Bernoulli's Theorem, applied to the streamline $r = 0$, tells us that H is constant on the streamline. So

$$\left[\frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right]\Bigg|_{z_0} = \left[\frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right]\Bigg|_{z_1}.$$

(There is no contribution from gravity as the streamline is horizontal.)

Consequently if p_0, p_1 are the pressures at z_0, z_1 respectively (which are the same for all r as the flow is unidirectional there),

$$\frac{p_0}{\rho} + \frac{1}{2}u_0^2 = \frac{p_1}{\rho} + \frac{1}{2}u_1^2 = \frac{p_1}{\rho} + \frac{1}{2}u_0^2 \left(\frac{b}{a}\right)^4.$$

Therefore

$$p_1 = p_0 + \frac{\rho u_0^2}{2}\left[1 - \left(\frac{b}{a}\right)^4\right].$$

As $b > a$, it follows that $p_1 < p_0$. Indeed, if u_0 or $\frac{b}{a}$ is too large, p_1 would appear to be negative! This cannot happen in practice (pressures are non-negative). Therefore for a given $\frac{b}{a}$, the condition $p_1 \geq 0$ limits the velocity u_0 upstream of the constriction. At the limiting velocity,

$$u_0 = \left(\frac{2p_0}{\rho\left[\left(\frac{b}{a}\right)^4 - 1\right]}\right)^{\frac{1}{2}},$$

the flow is said to be "choked".

If the pipe is made of a deformable elastic material, choking creates a vacuum that sucks the pipe wall inwards, which closes the constriction altogether and stops the flow (so u_0 becomes zero). The upstream pressure then reopens the pipe, allowing fluid to flow again, whereupon the cycle repeats. This creates *flutter*, which is a form of instability where the wall wobbles open and shut rapidly, making a 'raspberry' sound. Such sounds are also heard in arteries when they are compressed during blood pressure monitoring - in this context, they are called *Korotkoff sounds.*

Flow from a tap under gravity

If water flows downwards through air from a tap at $z = 0$ with velocity $\mathbf{u} = -u_0\mathbf{e}_z$, what is the shape of the water column?

Consider the column between $z = 0$ and $z = -h$, where the radius is R.

As in the previous example, conservation of mass gives $\mathbf{u} = -u_1\mathbf{e}_z$ at $z = -h$, where

$$u_1 = u_0\frac{a^2}{R^2}.$$

Assuming that the water column is steady and that its radius changes slowly with z, apply Bernoulli's Theorem to a streamline on the water's surface, where the pressure is p_{atm}. Then

$$\left[\frac{p}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz\right]\Bigg|_{z=0} = \left[\frac{p}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz\right]\Bigg|_{z=-h}.$$

$$\therefore \quad \frac{p_{atm}}{\rho} + \frac{1}{2}u_0^2 = \frac{p_{atm}}{\rho} + \frac{1}{2}u_1^2 - gh.$$

$$\therefore \quad \frac{1}{2}u_0^2 = \frac{1}{2}u_0^2\left(\frac{a}{R}\right)^4 - gh.$$

Solving for R gives

$$R = a\left[\frac{u_0^2}{u_0^2 + 2gh}\right]^{\frac{1}{4}}.$$

If h is large enough, so that $\frac{u_0^2}{2gh} \ll 1$, then

$$R \approx a\left(\frac{u_0^2}{2gh}\right)^{\frac{1}{4}}.$$

In practice, the solution breaks down when R is small, because surface tension (which has been neglected) causes an instability that breaks up the liquid column into droplets.

At the opposite extreme, if the flow out of the tap is too fast, turbulence is produced, so viscosity cannot be neglected. Moreover tiny air bubbles can occur within the liquid due to the very low pressures that are associated with fast flows.

Perhaps the main message of Bernoulli's Theorem is: "increasing the fluid velocity along a streamline decreases the pressure, and vice versa." Another way of looking at this is:

kinetic energy $\left(\frac{1}{2}\rho \mathbf{u} \cdot \mathbf{u}\right)$+potential energy (pressure, gravitational potential energy)

is constant in inviscid flows.

The streamfunction

In any incompressible flow (whether the fluid is viscous or inviscid), the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0$$

is generally easier to solve than the momentum equation.

For instance

$$\mathbf{u} = \nabla \times \boldsymbol{\Psi}$$

is a solution for *any* vector $\boldsymbol{\Psi}$. A result from algebraic topology (see MS319 Introduction to Manifolds and Topology) states that this is the general solution if (but not only if) the domain is 'starshaped'. A starshaped domain D is one that has a point $\mathbf{0}$ that can be reached from every other point $\mathbf{x}$ in the domain by going along a straight line that lies within the domain. So

$$\mathbf{x} \in D \;\Rightarrow\; s\mathbf{x} \in D \qquad \forall s \in [0, 1].$$

Some starshaped domains.

We could substitute $\nabla \times \boldsymbol{\Psi}$ for $\mathbf{u}$ in the momentum equation, and try to find $\boldsymbol{\Psi}$, but we are left with the problem of finding an unknown vector, so nothing has been gained.

However if $\mathbf{u}$ is independent of one spatial variable then it turns out that $\boldsymbol{\Psi}$ depends on only *one* function which is called the streamfunction. The streamfunction greatly simplifies the problem of finding $\mathbf{u}$.

Two-dimensional flow

If the flow is two-dimensional, i.e.

$$\mathbf{u} = u(x, y, t)\mathbf{e}_x + v(x, y, t)\mathbf{e}_y,$$

the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ reduces to

$$u_{,x} + v_{,y} = 0.$$

If the two-dimensional projection of the domain onto the (x, y) plane is simply-connected then $\mathbf{u} = \nabla \times \mathbf{\Psi}$ where $\mathbf{\Psi} = \psi(x, y, t)\mathbf{e}_z$. In components,

$$u = \psi_{,y}, \quad v = -\psi_{,x}.$$

Here $\psi(x, y, t)$ is the streamfunction. Both compoments of $\mathbf{u}$ can be obtained from it. Note that if any function of t is added to ψ then u, v do not change; this is the only nonuniqueness in the definition of ψ.

Example In any two-dimensional unidirectional flow $\mathbf{u} = u(y, t)\mathbf{e}_x$, the streamfunction is

$$\psi(y, t) = \int^y u(y', t)\,\mathrm{d}y'$$

(up to an arbitrary function of t).

Note that

$$\mathbf{u} \cdot \nabla\psi = u\psi_{,x} + v\psi_{,y} = \psi_{,y}\psi_{,x} - \psi_{,x}\psi_{,y} = 0,$$

so the streamfunction is constant on each streamline at any given instant. Consequently the streamlines are precisely the lines of constant ψ, z. In particular, in unidirectional two-dimensional flows, the streamlines are the lines $y = \text{const.}$, $z = \text{const}$. The set of all streamlines corresponding to a particular value of ψ is called a *streamsurface*.

Substituting $\mathbf{u} = \psi_{,y}\mathbf{e}_x - \psi_{,x}\mathbf{e}_y$ into the momentum equation gives two simultaneous equations for $\psi(x, y, t)$ and $p(x, y, t)$:-

$$\begin{aligned}
\psi_{,yt} + \psi_{,y}\psi_{,xy} - \psi_{,x}\psi_{,yy} &= -\frac{1}{\rho}p_{,x} + \nu[\psi_{,xxy} + \psi_{,yyy}] \\
-\psi_{,xt} - \psi_{,y}\psi_{,xx} + \psi_{,x}\psi_{,xy} &= -\frac{1}{\rho}p_{,y} - \nu[\psi_{,xxx} + \psi_{,xyy}].
\end{aligned}$$

It is often helpful to eliminate p by differentiating the first equation with respect to y, differentiating the second equation with respect to x, and eliminating the common term $-\frac{1}{\rho}p_{,xy}$ to obtain

$$(\nabla^2\psi)_{,t} + \psi_{,y}(\nabla^2\psi)_{,x} - \psi_{,x}(\nabla^2\psi)_{,y} = \nu\nabla^2(\nabla^2\psi),$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. This reduces the Navier-Stokes equation to a single differential equation for the streamfunction ψ.

Exercise Check that the above equation is correct.

Streamfunctions in orthogonal coordinates

In general orthogonal coordinates (a_1, a_2, a_3), suppose that $\mathbf{u}$ is independent of one of the coordinates. We shall look at the case when $\mathbf{u}$ is independent of a_3, so

$$\mathbf{u} = u(a_1, a_2, t)\mathbf{e}_1 + v(a_1, a_2, t)\mathbf{e}_2.$$

The cases when $\mathbf{u}$ is independent of a_1 or a_2 are similar. From the formula for $\nabla \cdot \mathbf{u}$ in orthogonal coordinates

$$\nabla \cdot \mathbf{u} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial a_1}(h_2h_3u) + \frac{\partial}{\partial a_2}(h_1h_3v)\right] = 0.$$

If the projection of the domain onto (a_1, a_2) space is simply-connected then we can solve this equation as follows:-

$$u = \frac{1}{h_2h_3}\psi_{,2}\,, \quad v = -\frac{1}{h_1h_3}\psi_{,1}\,,$$

where $\psi_{,i}$ denotes $\frac{\partial\psi}{\partial a_i}$.

Exercise Show that $\mathbf{\Psi} = \frac{\psi}{h_3}\mathbf{e}_3$.

Note The construction of ψ really only requires that $\mathbf{u}\cdot\mathbf{e}_3 = 0$, for then $\nabla \cdot \mathbf{u}$ contains only two terms. We could therefore allow u, v to depend on a_3 provided that every surface $a_3 =$ const. is simply-connected.

As in two-dimensional flow, the surfaces of constant ψ are streamsurfaces because

$$\mathbf{u}\cdot\nabla\psi = \frac{u}{h_1}\psi_{,1} + \frac{v}{h_2}\psi_{,2} = \frac{1}{h_1h_2h_3}[\psi_{,2}\psi_{,1} - \psi_{,1}\psi_{,2}] = 0.$$

Exercise Eliminate p from the momentum equation by taking the curl of the momentum equation; show that this yields

$$(\nabla\times\mathbf{u})_{,t} + \nabla\times[(\nabla\times\mathbf{u})\times\mathbf{u}] = -\nu\nabla\times(\nabla\times(\nabla\times\mathbf{u})).$$

For any *potential flow* the streamsurfaces are orthogonal to the surfaces on which the potential φ is constant, because

$$(\nabla\varphi)\cdot(\nabla\psi) = \mathbf{u}\cdot\nabla\psi = 0.$$

In Cartesian coordinates (for which $h_1 = h_2 = h_3 = 1$), the potential and streamfunction satisfy the *Cauchy-Riemann equations*

$$\varphi_{,x} = \psi_{,y}, \quad \varphi_{,y} = -\psi_{,x}.$$

Therefore, for two-dimensional potential flows, the potential and streamfunction each satisfy the 2-D Laplace equation:-

$$\nabla^2\varphi = 0, \quad \nabla^2\psi = 0.$$

Proof

$$\nabla^2\varphi = \frac{\partial}{\partial x}(\varphi_{,x}) + \frac{\partial}{\partial y}(\varphi_{,y}) = \frac{\partial}{\partial x}(\psi_{,y}) + \frac{\partial}{\partial y}(-\psi_{,x}) = 0.$$

$$\nabla^2\psi = \frac{\partial}{\partial x}(\psi_{,x}) + \frac{\partial}{\partial y}(\psi_{,y}) = \frac{\partial}{\partial x}(-\varphi_{,y}) + \frac{\partial}{\partial y}(\varphi_{,x}) = 0.$$

If we introduce the *complex potential*

$$\Phi = \varphi + i\psi$$

and use $\zeta = x + iy$ and $\overline{\zeta} = x - iy$ as coordinates instead of (x, y) then the first-order partial derivatives are

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}}, \quad \frac{\partial}{\partial y} = i\frac{\partial}{\partial \zeta} - i\frac{\partial}{\partial \overline{\zeta}}.$$

Hence

$$\Phi_{,\zeta} + \Phi_{,\overline{\zeta}} = \varphi_{,x} + i\psi_{,x} = \psi_{,y} - i\varphi_{,y} \quad \text{(using Cauchy-Riemann)}$$

$$\Phi_{,\zeta} - \Phi_{,\overline{\zeta}} = -i(\varphi_{,y} + i\psi_{,y}) = \psi_{,y} - i\varphi_{,y}.$$

Therefore $\Phi_{,\overline{\zeta}} = 0$. Consequently the complex potential can be written as a function of ζ only;-

$$\Phi = f(\zeta).$$

Complex variable theory tells us that f is analytic - it coincides with its Taylor series about any point. Given an analytic function $f(\zeta)$, we can recover the streamfunction and potential as follows:-

$$\varphi(x, y) = \Re\{f(x + iy)\}, \quad \psi(x, y) = \Im\{f(x + iy)\}.$$

Example For $\Phi = \frac{1}{2}\alpha\zeta^2$, we obtain

$$\varphi(x, y) = \frac{1}{2}\alpha(x^2 - y^2), \quad \psi(x, y) = \alpha xy.$$

Therefore $u = \alpha x, \quad v = -\alpha y$. The streamlines are the hyperbolae $xy =$ const. This is the stagnation-point flow that we have met earlier.

2-D potential flow in cylindrical polar coordinates

Instead of using Cartesian coordinates (x, y, z), we can write the complex potential $\Phi = f(\zeta)$ in terms of cyclindrical polar coordinates (r, θ, z):-

$$\zeta = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

This representation is useful for flows with cylindrical geometry. The velocity can be recovered from

$$\mathbf{u} = \nabla\varphi = \varphi_{,r}\mathbf{e}_r + \frac{1}{r}\varphi_{,\theta}\mathbf{e}_\theta,$$

and the streamlines are the curves $\psi = \text{const.}, \quad z = \text{const.}$

Example

The *line vortex* of strength Γ has the complex potential

$$\Phi = -\frac{i\Gamma}{2\pi}\ln\zeta$$

$$\therefore \Phi = -\frac{i\Gamma}{2\pi}(\ln(r) + i\theta).$$

Therefore

$$\varphi = \frac{\Gamma\theta}{2\pi}, \quad \psi = -\frac{\Gamma}{2\pi}\ln r.$$

The velocity is $\mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta$, and the streamlines are the circles $r = \text{const.}$, $z = \text{const.}$

The centre of the vortex is at the point $(x, y) = (0, 0)$.

To obtain a line vortex centred at $(x, y) = (x_0, y_0)$ use

$$\Phi = -\frac{i\Gamma}{2\pi}\ln(\zeta - \zeta_0), \quad \text{where } \zeta_0 = x_0 + iy_0.$$

Def$^{\mathbf{n}}$. The *circulation* of a flow around a closed curve C is the integral of (the component of velocity tangential to the curve) around C (going anticlockwise).

For the line vortex centred at $(0,0)$, the circulation around any circle $r =$ const. is

$$\int_{\theta=0}^{2\pi} \mathbf{u}\cdot\mathbf{e}_\theta \, r \, \mathrm{d}\theta = \int_{\theta=0}^{2\pi} \frac{\Gamma}{2\pi r} r \, \mathrm{d}\theta = \int_{\theta=0}^{2\pi} \frac{\Gamma}{2\pi} \, \mathrm{d}\theta = \Gamma.$$

So Γ is a measure of how rapidly the fluid circulates around the streamlines.

Note that the modulus of the complex potential for the line vortex centred at $(0,0)$ tends to infinity as $r \to 0$; it is undefined at $r = 0$. Points at which the complex potential becomes infinite are called *singularities*.

Examples i) The only singularity of the line vortex with complex potential

$$\Phi = -\frac{i\Gamma}{2\pi}\ln(\zeta - \zeta_0)$$

occurs at $\zeta = \zeta_0$.

ii) Consider the flow with complex potential

$$\Phi = U_0\zeta.$$

Clearly

$$\varphi = U_0 x \quad \text{and} \qquad \psi = U_0 y.$$

Therefore $u = U_0,\ v = 0$ and the streamlines are the lines $y =$ const.,

$z =$ const.

This complex potential corresponds to *uniform flow* with velocity $U_0\mathbf{e}_x$. Its only singularity occurs "at infinity", because as $|\zeta| \to \infty,\ |\Phi| \to \infty$ also.

The second of these examples is at the heart of the theory of flight (using a fixed wing). From the viewpoint of an aeroplane's wing, the air is rushing past. The velocity far ahead of the wing is uniform.

Potential flow past a wing section such as the one shown above can be calculated using a technique from complex analysis called *conformal mapping*. However, this goes beyond the bounds of this module, so we shall retrict attention to flow past a "wing" of circular cross-section; let a denote the radius of the circle.

To do this, we need the following theorem.

Milne-Thomson's Circle Theorem
If all of the singularities of $f(\zeta)$ lie outside the circle $|\zeta| = a$ (i.e. in the region $|\zeta| > a$) then

$$\Phi = f(\zeta) + \overline{f\left(a^2/\overline{\zeta}\right)}$$

is the complex potential of a flow that has:-
i) the same singularities in $|\zeta| > a$ as $f(\zeta)$ has,
ii) the circle $|\zeta| = a$ as a streamsurface.

Proof i) If the singularities of $f(\zeta)$ are in $|\zeta| > a$ then the singularities of $f(a^2/\overline{\zeta})$ are in $\left|a^2/\overline{\zeta}\right| > a$ i.e. in $|\zeta| < a$ (because $\left|\overline{\zeta}\right| = |\zeta|$). Consequently the only singularities of Φ in the region $|\zeta| > a$ are the singularities of $f(\zeta)$.

ii) On the circle $|\zeta| = a$, we can replace $a^2/\overline{\zeta}$ by ζ (because $\zeta\overline{\zeta} = a^2$). Therefore, on this circle,

$$\psi = \Im\{\Phi\} = \Im\{f(\zeta) + \overline{f(\zeta)}\} = 0.$$

As Ψ is constant on the circle, the circle is a streamsurface.

Now let us apply Milne-Thomson's Circle Theorem to the uniform flow with complex potential $f(\zeta) = U_0\zeta$. This gives a complex potential for flow past a cylindrical "wing":-

$$\Phi = U_0\zeta + (\overline{U_0 a^2/\overline{\zeta}}\,) = U_0(\zeta + a^2/\zeta).$$

As $|\zeta|$ increases (i.e. far from the wing), the complex potential tends to the uniform flow case.

In terms of cylindrical polar coordinates,

$$\Phi = U_0\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) = U_0\left(r + \frac{a^2}{r}\right)\cos\theta + iU_0\left(r - \frac{a^2}{r}\right)\sin\theta.$$

So

$$\varphi = U_0\left(r + \frac{a^2}{r}\right)\cos\theta, \qquad \psi = U_0\left(r - \frac{a^2}{r}\right)\sin\theta.$$

Therefore

$$\mathbf{u} = U_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta\,\mathbf{e}_r - U_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\,\mathbf{e}_\theta.$$

The flow is symmetric about the cylinder, as shown.

Exercise Show that the circulation around the streamline $|\zeta| = a$, $z =$ const. is zero.

Circulation can be included by adding a line vortex at $\zeta = 0$ to give the complex potential

$$\Phi = U_0\left(\zeta + a^2/\zeta\right) - \frac{i\Gamma}{2\pi}\ln\zeta.$$

Then the flow is

$$\mathbf{u} = U_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta\,\mathbf{e}_r + \left[-U_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right]\mathbf{e}_\theta.$$

As before, the flow tends to the uniform flow

$$\mathbf{u} = U_0\cos\theta\,\mathbf{e}_r - U_0\sin\theta\,\mathbf{e}_\theta = U_0\,\mathbf{e}_x$$

as $r \to \infty$.

If $\Gamma < 0$, i.e. the circulation around the cyclinder is negative, the flow produced by the vortex is clockwise. Consequently, the velocity on the upper half of the cylinder is increased by the vortex, whereas it is decreased on the lower half of the cylinder.

Bernoulli's Theorem tells us that

$$\frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} = \text{const. on } r = a.$$

Therefore the pressure on the lower half is greater than on the upper half. The total difference in force in the y-direction is called the *lift*; this force is responsible for the fact that aeroplanes fly. The following exercise shows that uniform motion *and* circulation are needed to generate lift.

Exercise Integrate the y-component of the stress due to pressure over the cylinder and show that the lift is $-\rho U_0 \Gamma$.

Stokes Flow

Inviscid flow is calculated by ignoring viscosity: this can be a good approximation only if $Re >> 1$. By contrast, if $Re << 1$, the viscous term $\nu\nabla^2\mathbf{u}$ is typically much greater in magnitude than the "inertia" $\frac{D\mathbf{u}}{Dt}$. Consequently it is reasonable to approximate low Reynolds number flows by solutions to

$$\mathbf{0} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0.$$

Such solutions are called *Stokes flows*.

Notes 1. As the momentum equation involves second-order spatial derivatives, the same boundary conditions are applicable as for the Navier-Stokes equations:-

$$\mathbf{u} = \mathbf{U}$$

on any solid boundary moving with velocity $\mathbf{U}$. (For stationary solid boundaries, this reduces to $\mathbf{u} = 0$.)

2. If fluid is enclosed in a region V with a stationary solid boundary ∂V then motion occurs if and only if $\nabla p \neq \mathbf{u}$.

Proof Suppose that $\nabla p = \mathbf{0}$. Then $\nabla^2\mathbf{u} = \mathbf{0}$ and $\nabla\cdot\mathbf{u} = 0$.

Note that $\nabla^2\left(\frac{1}{2}f^2\right) = f(\nabla^2 f) + (\nabla f)\cdot(\nabla f)$ for any function f. If $\mathbf{u} = u\,\mathbf{e}_x + v\,\mathbf{e}_y + w\,\mathbf{e}_z$ in Cartesian coordinates then

$$\begin{aligned}
0 &= \int_V \mathbf{u}\cdot(\nabla^2\mathbf{u})\,\mathrm{d}\mathbf{x} = \int_V (u\nabla^2 u + v\nabla^2 v + w\nabla^2 w)\,\mathrm{d}\mathbf{x} \\
&= \int_V \left\{\nabla^2\left(\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2}w^2\right) - (\nabla u)\cdot(\nabla u) - (\nabla v)\cdot(\nabla v) - (\nabla w)\cdot(\nabla w)\right\}\mathrm{d}\mathbf{x} \\
&= \int_{\partial V} \frac{1}{2}\mathbf{n}\cdot\nabla(u^2+v^2+w^2)\,\mathrm{d}S - \int_V \{|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\}\,\mathrm{d}\mathbf{x} \\
&= \int_{\partial V} \{u\mathbf{n}\cdot\nabla u + v\mathbf{n}\cdot\nabla v + w\mathbf{n}\cdot\nabla w\}\,\mathrm{d}S - \int_V \{|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\}\,\mathrm{d}\mathbf{x} \\
&= -\int_V \{|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\}\,\mathrm{d}\mathbf{x},
\end{aligned}$$

because $u = v = w = 0$ on the solid stationary boundary ∂V.

Consequently $\nabla u = \nabla v = \nabla w = \mathbf{0}$, so u, v, w are constants. As each of these is zero on ∂V, they are zero in V. Hence $\mathbf{u} = \mathbf{0}$.

Uniqueness of Stokes flows

The above argument can be modified to show that in a bounded region for any given boundary conditions, there is a unique solution of the equations of

Stokes flow.

To prove this, suppose that $\mathbf{u}$ and $\mathbf{u}'$ are two solutions of the Stokes flow equations for the *same* boundary conditions on $\mathbf{u}, \mathbf{u}'$, and that p and p' are the corresponding pressures.

Then let $\mathbf{u}'' = \mathbf{u} - \mathbf{u}'$ and $p'' = p - p'$. As the Stokes flow equations are linear, it follows that

$$0 = -\frac{1}{\rho}\nabla p'' + \nu\nabla^2\mathbf{u}'', \qquad \nabla \cdot \mathbf{u}'' = 0.$$

Furthermore $\mathbf{u}'' = \mathbf{0}$ on ∂V. Hence

$$\begin{aligned}
0 &= \int_V \mathbf{u}'' \cdot \left(-\frac{1}{\rho}\nabla p'' + \nu\nabla^2\mathbf{u}''\right) \mathrm{d}\mathbf{x} \\
&= \frac{1}{\rho}\int_V \{p''\nabla \cdot \mathbf{u}'' - \nabla \cdot (p''\mathbf{u}'')\}\ \mathrm{d}\mathbf{x} + \nu\int_V \mathbf{u}'' \cdot (\nabla^2\mathbf{u}'')\,\mathrm{d}\mathbf{x} \\
&= -\frac{1}{\rho}\int_{\partial V} \mathbf{n} \cdot (p''\mathbf{u}'')\,\mathrm{d}S + \nu\int_V \mathbf{u}'' \cdot (\nabla^2\mathbf{u}'')\,\mathrm{d}\mathbf{x} \\
&= \nu\int_V \mathbf{u}'' \cdot (\nabla^2\mathbf{u}'')\,\mathrm{d}\mathbf{x},
\end{aligned}$$

because $\mathbf{u}'' = \mathbf{0}$ on ∂V. We can now use exactly the same argument as in Note 2 above to show that $\mathbf{u}'' = \mathbf{0}$ throughout V. Hence $\mathbf{u}' = \mathbf{u}$ and so the flow is unique. Note that $\nabla p'' = 0$, so p'' is constant. Hence p is unique up to an arbitrary constant which does not appear in ∇p.

This proof has been extended to show that if Re is non-zero but "sufficiently small", the solution to the Navier-Stokes equation is unique. For larger Re, several solutions may exist, which is why instabilities can occur as the flow changes from one solution to another.

Reversibility of Stokes flows

Given a boundary condition $\mathbf{u} = \mathbf{f}(\mathbf{x})$ on ∂V, we can calculate the unique Stokes flow $\mathbf{u} = \mathbf{F}(\mathbf{x})$ in V that satisfies this condition, together with the pressure $p = P(x)$. If the boundary condition is reversed, so that $\mathbf{u} = -f(x)$ on ∂V then $\mathbf{u} = -\mathbf{F}(\mathbf{x}), \quad p = c - P(\mathbf{x})$ is a solution of the Stokes flow equations for any constant c. By the uniqueness theorem, this is the *only* solution of the Stokes flow equations with this boundary condition.

Consequently, if a given boundary condition is applied for a time τ and then reversed for a further time τ, every fluid particle will return to its starting position. This is shown dramatically in G.I.Taylor's stirring experiment.

Microorganisms such as bacteria and spermatozoa are very small and so the Reynolds numbers of the flows that they produce by swimming are close to zero. Hence they cannot swim using a motion that is reversible (i.e. looks the same on a film played forwards or backwards). Examples of swimming strokes that are not reversible are shown below:-

1. Corkscrew motion

As the corkscrew-shaped tail rotates, the organism is propelled forwards.

2. Breast stroke

Main stroke Recovery

Boundary Layers

For steady flow past a solid object (such as a wing), we know that the inviscid theory cannot be correct everywhere, because an inviscid fluid does not stick to the object. However, when $Re >> 1$, it is common for there to exist a thin region called a *boundary layer*, in which the velocity varies rapidly to satisfy the boundary condition. Outside the boundary layer, the flow is effectively inviscid.

The boundary layer on a flat plate

The simplest example is the *Blasius boundary layer* on a flat plate aligned with a uniform two-dimensional flow $\mathbf{u} = U_0\mathbf{e}_x$. The plate occupies the region $\{(x, y, z) : x \geq 0, y = 0\}$ as shown.

Outside the boundary layer, the inviscid flow $\mathbf{u} = U_0\mathbf{e}_x$ is unchanged. So the boundary layer exists to enable the boundary condition $\mathbf{u} = \mathbf{0}$ on the plate to be satisfied. The velocity profile in the boundary layer looks something like this, for each fixed x.

Note that we should expect the thickness $\delta(x)$ of the boundary layer to grow with x because, as x increases, any fluid particle passing the plate at x has spent an increasing amount of time in flow that has been disturbed by the plate. (We know that for an implusively-started flow, the disturbance moves outwards with time in the absence of any reason for it not to do so.)

It can be shown that, within any steady two-dimensional boundary layer, the Navier-Stokes equations can be approximated by the *boundary-layer equations:*

$$
\begin{aligned}
uu_{,x} + vu_{,y} &= -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu u_{,yy}, \\
u_{,x} + v_{,y} &= 0.
\end{aligned}
$$

To leading order, p is a function of x only, which is the same as the pressure in the outer 'inviscid' flow.

For flow past a flat plate, the inviscid flow is uniform, so $\frac{\mathrm{d}p}{\mathrm{d}x} = 0$. Then in terms of the streamfunction ψ, the boundary-layer equations reduce to

$$\psi_{,y}\psi_{,xy} - \psi_{,x}\psi_{,yy} = \nu\psi_{,yyy}.$$

Note that the scaling transformations

$$(x, y, \psi) \mapsto (\alpha\beta x, \alpha y, \beta\psi), \quad \alpha\beta \neq 0,$$

leave this equation invariant, which suggests that we should seek a similarity solution. As there are *two* scaling parameters α, β, there exists a family of such solutions; however, only one of these solutions leaves $u = \psi_{,y}$ invariant - for this solution, the velocity profile is the same (up to a rescaling in the y-direction) for every x.

The scalings map $\psi_{,y}$ to $\frac{\beta}{\alpha}\psi_{,y}$, so we require that $\beta = \alpha$ if u is to be invariant. Then

$$(x, y, \psi) \mapsto (\alpha^2 x, \alpha y, \alpha\psi).$$

The invariants are $\psi/\sqrt{x}$ and $y/\sqrt{x}$, and so we could seek a solution of the form

$$\psi/\sqrt{x} = F(y/\sqrt{x}).$$

However, one should not take functions of dimensional objects (what is the square root of a metre?), so instead we write

$$\psi = (2\nu U_0 x)^{\frac{1}{2}} f(\eta), \quad \text{where } \eta = \frac{y}{(2\nu x/u_0)^{\frac{1}{2}}}.$$

This is in dimensionless form; the factor of 2 is there for convenience only.

Exercise Show that the streamfunction equation reduces to the ODE

$$f''' + ff'' = 0,$$

which is called the *Blasius equation*. Show also that

$$u = U_0 f'(\eta), \quad v = U_0 \left(\frac{\nu}{2U_0 x}\right)^{\frac{1}{2}} (\eta f'(\eta) - f(\eta)).$$

The exercise shows that the boundary conditions are

$$f(0) = f'(0) = 0, \quad f'(\eta) = 1.$$

The last condition ensures that $u \to U_0$ as y leaves the boundary layer.

The Blasius equation can be solved numerically, but not (at present) exactly. The ratio u/U_0 is 0.97 when $\eta = 3$ and 0.999936 when $\eta = 5$. As

$$y = \left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} \eta$$

it follows that the boundary layer thickness is proportional to $\sqrt{\nu x/U_0}$.

Separation of the boundary layer

For many objects, the boundary layer does not remain close to the boundary everywhere, because an adverse pressure gradient lifts the layer away from the surface.

Example For a cylinder that accelerates from rest in a stationary fluid, the following sequence is common

a) *Re* small

b) *Re* increasing -separation occurs

c) *Re* increasing further - steady flow unstable

Karman vortex street shed into fluid

d) Re very large turbulence occurs.

Aeroplane wings are designed to provide lift whilst avoiding boundary-layer separation. However, if the wing is tilted too steeply to the oncoming flow, separation and turbulence will occur; this is called *aerodynamic stall*.

a) attached boundary layer

b) stall.