How to use Lie symmetries to find discrete symmetries

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Abstract

This paper describes a new method for calculating all discrete point symmetries of a given differential equation, using the equation's Lie point symmetries.

1. Introduction

To analyse bifurcations of a given nonlinear dynamical system, it is vital to know the discrete and continuous symmetries of the system, [5]. Commonly, the Lie point symmetries of the system are known, having been found using Lie's method, [1,8,10]. Some discrete symmetries (such as reflections) may be found by inspection or by using an ansatz, [4]. However, it is not generally possible to calculate all discrete point symmetries, which are determined by a system of nonlinear partial differential equations. In some instances, it is possible to use computer algebra to reduce this system to a differential Gröbner basis, which may be solved more easily than the original system, [9].

Discrete symmetries are useful for increasing the efficiency of numerical methods for solving differential equations, by reducing either the computational domain or (for spectral methods) the space of trial functions. They also arise as hidden symmetries of some boundary value problems [3]. Therefore it is important to be able to find discrete symmetries in a systematic manner.

This paper describes a new method that can be used to calculate all discrete point symmetries of a given differential equation, provided that its Lie point symmetries are known. The method also extends readily to other types of symmetry whose infinitesimal generators form a Lie algebra, such as contact symmetries.

2. Determining equations for discrete symmetries

For simplicity, we first consider the problem of determining all discrete point symmetries of the ODE

$$y^{(n)} = \omega(x, y, y', \dots, y^{(n-1)}), \qquad n \geq 2. \tag{2.1}$$

Suppose that the Lie algebra, $\mathcal{L}$, of all infinitesimal generators of one-parameter (local) Lie groups of point symmetries of the ODE has a basis $\{X_i\}_{i=1}^N$. We use the notation

$$\Gamma_i(\epsilon) : (x, y) \mapsto \left(e^{\epsilon X_i} x, e^{\epsilon X_i} y\right) \tag{2.2}$$

to denote the one-parameter Lie group generated by

$$X_i = \xi_i(x, y)\partial_x + \eta_i(x, y)\partial_y. \tag{2.3}$$

If a point transformation

$$\Gamma : (x, y) \mapsto (\hat{x}(x, y), \hat{y}(x, y)) \tag{2.4}$$

is a symmetry of the ODE, then the point transformation

$$\hat{\Gamma}_i(\epsilon) = \Gamma\Gamma_i(\epsilon)\Gamma^{-1} \tag{2.5}$$

is also a symmetry, for each ϵ in some neighbourhood of zero. Note that

$$\hat{\Gamma}_i(\epsilon) : (\hat{x}, \hat{y}) \mapsto \left(e^{\epsilon \hat{X}_i} \hat{x}, e^{\epsilon \hat{X}_i} \hat{y}\right), \tag{2.6}$$

where

$$\hat{X}_i = \xi_i(\hat{x}, \hat{y})\partial_{\hat{x}} + \eta_i(\hat{x}, \hat{y})\partial_{\hat{y}}. \tag{2.7}$$

Hence, for each i, the symmetries $\hat{\Gamma}_i(\epsilon)$ constitute a one-parameter local Lie group, whose infinitesimal generator is $\hat{X}_i$. Therefore

$$\hat{X}_i \in \mathcal{L}, \qquad i = 1 \dots N. \tag{2.8}$$

The generators $\{\hat{X}_i\}_{i=1}^N$ are simply the basis generators $\{X_i\}_{i=1}^N$ with (x, y) replaced by $(\hat{x}, \hat{y})$. Thus they are linearly independent and form a basis for $\mathcal{L}$, and so each X_i can be written as a linear combination of the $\hat{X}_i$'s. Furthermore, the structure constants are preserved by the transformation $X_i \to \hat{X}_i$. If

$$[X_i, X_j] = c_{ij}^k X_k, \tag{2.9}$$

then

$$[\hat{X}_i, \hat{X}_j] = c_{ij}^k \hat{X}_k. \tag{2.10}$$

These results generalize to partial differential equations, and are summarized as follows.

Lemma 1 *Every point symmetry Γ of a differential equation induces an automorphism of the Lie algebra, $\mathcal{L}$, of generators of one-parameter local Lie groups of point symmetries of the differential equation. For each such Γ, there exists a constant non-singular $N \times N$ matrix (b_i^l) such that*

$$X_i = b_i^l \hat{X}_l. \tag{2.11}$$

This automorphism preserves all structure constants.

This lemma implies that every point symmetry Γ of an ODE satisfies the set of partial differential equations (PDEs)

$$X_i \hat{x} = b_i^l \hat{X}_l \hat{x} = b_i^l \xi_l(\hat{x}, \hat{y}), \qquad i = 1, \ldots, N \tag{2.12}$$

$$X_i \hat{y} = b_i^l \hat{X}_l \hat{y} = b_i^l \eta_l(\hat{x}, \hat{y}), \qquad i = 1, \ldots, N. \tag{2.13}$$

These can be solved by the method of characteristics to obtain $(\hat{x}, \hat{y})$ in terms of x, y, b_i^l and some unknown constants (or functions) of integration. The lemma provides a necessary, but not sufficient, condition for Γ to be a symmetry. However, it is simple to differentiate the functions $\hat{x}(x, y)$, $\hat{y}(x, y)$ obtained above, and to determine which of these yield point symmetries. A simple illustration of the method is provided by the ODE

$$y'' = \frac{y'}{x} + \frac{4y^2}{x^3}, \tag{2.14}$$

which Reid and co-workers used to demonstrate an algorithm for computing a differential Gröbner basis, [9]. The Lie point symmetries of this ODE are scalings, generated by

$$X_1 = x\partial_x + y\partial_y. \tag{2.15}$$

It is convenient to use canonical coordinates

$$r = \frac{y}{x}, \qquad s = \ln|x|, \tag{2.16}$$

in terms of which $X_1 = \partial_s$ and (2.14) is equivalent to

$$\frac{d^2r}{ds^2} = 4r^2 + r. \tag{2.17}$$

Then the PDEs (2.12), (2.13) are equivalent to

$$\hat{r}_s = 0, \qquad \hat{s}_s = b_1^1 \neq 0, \tag{2.18}$$

whose general solution is

$$\hat{r} = f(r), \qquad \hat{s} = b_1^1 s + g(r). \tag{2.19}$$

Here f, g are arbitrary functions of r, subject to the condition $\frac{df}{dr} \neq 0$. Thus the lemma has enabled us to construct an appropriate ansatz; all point symmetries are necessarily of the above form. Now we use the symmetry condition,

$$\frac{d^2\hat{r}}{d\hat{s}^2} = 4\hat{r}^2 + \hat{r} \qquad \text{when (2.17) holds,} \tag{2.20}$$

to determine b_1^1 and the functions f and g. This is achieved by first equating powers of r' to split the symmetry condition into an over-determined system of ODEs, and then solving this system. The procedure is much the same as that of determining the Lie point symmetries of the ODE (2.17). The general solution is

$$\begin{aligned}(\hat{r}, \hat{s}) \;\in\; \Big\{&(r, s+c), (r, -(s+c)), (-r - \frac{1}{4}, i(s+c)),\\ &(-r - \frac{1}{4}, -i(s+c))\Big\},\end{aligned} \tag{2.21}$$

where c is an arbitrary constant. The one-parameter Lie group generated by X consists of translations in s. These symmetries can be factored out by taking $c = 0$ in (2.21), leaving a discrete group of symmetries that are inequivalent under the action of the one-parameter group. The discrete group is isomorphic to the cyclic group of order four, Z_4, and is generated by

$$\Gamma_1 : (r, s) \mapsto (-r - \frac{1}{4}, is). \tag{2.22}$$

Reverting to the original (x, y) co-ordinates, the discrete symmetries are generated by

$$\Gamma_1 : (x, y) \mapsto \left(x^i, -x^{i-1}y - \frac{x^i}{4}\right). \tag{2.23}$$

The above example demonstrates that discrete point symmetries can be found systematically provided that the Lie algebra of Lie point symmetry generators can be calculated. The procedure is straightforward, and does not require computer algebra. However, the method cannot be used if a given ODE has no Lie symmetries. Then the best hope of solving the symmetry condition is to construct a differential Gröbner basis, [9]. Further examples of the basic technique are given in [6].

3. The simplified method

If $\mathcal{L}$ is not abelian, some of the structure constants c_{ij}^k are nonzero. By combining the commutation relations (2.9) and (2.10) with the result (2.11), the following constraint is obtained:

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n. \tag{3.1}$$

Furthermore, the adjoint action of one-parameter Lie groups can be used to factor out some equivalent symmetries before the appropriate ansatz is applied to the symmetry condition for a particular ODE. For each infinitesimal generator X_j, we construct the adjoint action

$$\begin{aligned} \mathrm{Ad}(\exp(\epsilon_j X_j))X_i &= X_i - \epsilon_j[X_j, X_i] + \frac{\epsilon_j^2}{2!}[X_j, [X_j, X_i]] \ldots \\ &= a_i^p(\epsilon_j, j)X_p. \end{aligned} \tag{3.2}$$

Then the system (2.11) is equivalent to

$$X_i = \tilde{b}_i^l \hat{X}_l \tag{3.3}$$

under the group generated by X_j, where

$$\tilde{b}_i^l = a_i^p(\epsilon_j, j) b_p^l. \tag{3.4}$$

By combining this result with (3.1), which is invariant under the equivalence transformation, we can simplify the matrix (b_i^l) considerably.

For example, consider the Lie algebra $\mathbf{a}(1)$, which commonly occurs in autonomous ODEs with scaling symmetries. Choose a basis $\{X_1, X_2\}$ such that

$$[X_1, X_2] = X_1. \tag{3.5}$$

Then the only non-zero structure constants are

$$c_{12}^1 = -c_{21}^1 = 1, \tag{3.6}$$

and therefore (3.1) yields

$$b_1^1 b_2^2 - b_1^2 b_2^1 = b_1^1, \qquad 0 = b_1^2. \tag{3.7}$$

Hence

$$(b_i^l) = \begin{bmatrix} b_1^1 & 0 \\ b_2^1 & 1 \end{bmatrix}, \qquad \text{where } b_1^1 \neq 0. \tag{3.8}$$

Now use the adjoint action of X_1 to set $b_2^1 = 0$; then apply the adjoint action of X_2 to rescale b_1^1 by a positive factor. Thus there are only two possibilities to consider:

$$(b_i^l) = \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.9}$$

The simplification described above is carried out without reference to any particular ODE, so the results apply to all ODEs having this Lie algebra. Further applications to particular ODEs are to be found in [6]. For example, it is proved that every ODE whose Lie algebra is $\mathbf{sl}(2)$ admits at least four inequivalent discrete point symmetries. Such ODEs include the much-studied Chazy equation, [2].

4. Discrete symmetries of some PDEs

The methods described above apply equally to PDEs and ODEs. Some physically-important examples are listed below. Details of

the calculations are omitted, for brevity; they can be found in [7].
The group of inequivalent discrete symmetries of the *heat equation*,

$$u_t = u_{xx}, \tag{4.1}$$

is isomorphic to Z_4, and is generated by

$$\Gamma_1 : (x, t, u) \mapsto \left(\frac{x}{2t}, \frac{-1}{4t}, \sqrt{2it} \exp\left\{ \frac{x^2}{4t} \right\} u \right). \tag{4.2}$$

These symmetries can be combined with complex scalings of u to yield real-valued symmetries.
The *Harry-Dym equation*,

$$u_t = u^3 u_{xxx}, \tag{4.3}$$

has inequivalent discrete symmetries isomorphic to $Z_2 \times Z_2 \times Z_3$. They are generated by

$$\Gamma_1 : (x, t, u) \mapsto (x, -t, -u), \tag{4.4}$$

$$\Gamma_2 : (x, t, u) \mapsto \left(\frac{-1}{x}, t, \frac{u}{x^2} \right), \tag{4.5}$$

$$\Gamma_3 : (x, t, u) \mapsto \left(x, t, \exp\left\{ \frac{2\pi i}{3} \right\} u \right). \tag{4.6}$$

The inequivalent discrete symmetries of the *Euler-Poisson-Darboux equation*,

$$u_{tt} - u_{xx} = \frac{p(p+1)}{t^2} u, \tag{4.7}$$

are generated by the reflections

$$\Gamma_1 : (x, t, u) \mapsto (-x, t, u), \tag{4.8}$$

$$\Gamma_2 : (x, t, u) \mapsto (x, -t, u), \tag{4.9}$$

$$\Gamma_3 : (x, t, u) \mapsto (x, t, -u), \tag{4.10}$$

and the generalized inversion

$$\Gamma_4 : (x, t, u) \mapsto \left(\frac{x}{t^2 - x^2}, \frac{t}{t^2 - x^2}, u \right). \tag{4.11}$$

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