Elementary differential equations

Notice: this material must not be used as a substitute for attending the lectures

1 Differential Equations

An **ordinary differential equation**, often abbreviated to ODE, is an equation containing an unknown function and some of its derivatives. It has to be solved to find the unknown function.

Examples of ODEs are

$$\frac{dy}{dx} = \cos x, \qquad \frac{dy}{dx} = 1 - \frac{y}{x}, \qquad y'' - 2y' + 2y = 0$$

where $y' = dy/dx$ and $y'' = d^2y/dx^2$. In each of these equations y is the unknown function and the idea is to find y in terms of x.

We shall shortly explain some of the methods available for solving simple ODEs, but first let us introduce some physical problems which give rise to ODEs.

1.1 Physical problems giving rise to ODEs

(i) Imagine water draining out from a large cylindrical tank through a small hole or tap at the bottom. Then the depth of water remaining in the tank is constantly changing, call it $h(t)$ where t is time. From the theory of fluid mechanics (Bernoulli's equation in particular) it can be shown that $h(t)$ must satisfy the differential equation

$$\frac{dh}{dt} = -k\sqrt{h}$$

where k is some constant depending on the tap radius, the tank radius and the acceleration due to gravity.

(ii) The differential equation for a simple pendulum of length l is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

which is to be solved for the angle $\theta(t)$.

(iii) Let $y(x)$ be the equation for the curve described by a heavy chain or rope hanging under gravity. It can be shown that $y(x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

You may already be aware that the curve in question is called a *catenary* and is described by the hyperbolic function cosh.

(iv) The differential equation

$$m\frac{d^2s}{dt^2} - (\mu\alpha - \beta)\left(\frac{ds}{dt}\right)^2 = T - \mu mg$$

has been used to model an aircraft taking off. Here, $s = s(t)$ is the distance travelled along the runway at time t, m is the aircraft's mass, g is acceleration due to gravity, T is the thrust from the engines, μ is the coefficient of friction, β is called the aerodynamic drag parameter and α is the lift parameter. The idea is to solve the differential equation to find s as a function of t.

1.2 Basic terminology

The **order** of a differential equation is the order of the highest order derivative appearing. For example,

$$\frac{dy}{dx} + 2y = 1 \quad \text{is first order}$$

$$\left(\frac{dy}{dx}\right)^2 + 2y = 1 \quad \text{is first order}$$

$$\frac{d^2y}{dt^2} + \omega^2 y = \sin 2t \quad \text{is second order}$$

$$y\frac{dy}{dt} = t^2 \quad \text{is first order}$$

$$\frac{d^3y}{dx^3} - y = 0 \quad \text{is third order}$$

A **solution** of a differential equation is a function that satisfies it for all values of the independent variable.

For example $y = Ce^{2x}$ is a solution of $\frac{dy}{dx} = 2y$. We can verify this by substituting in. When $y = Ce^{2x}$,

$$\frac{dy}{dx} = 2Ce^{2x} = 2y$$

and therefore the differential equation is satisfied.

Another example: $y = x^2 - 5x$ satisfies $x\frac{dy}{dx} - y = x^2$. We shall again check this by verification. When $y = x^2 - 5x$, $\frac{dy}{dx} = 2x - 5$ and so

$$x\frac{dy}{dx} - y = x(2x - 5) - (x^2 - 5x) = x^2$$

i.e. the differential equation is satisfied.

Of course, this is not how we normally "solve" differential equations. We do not simply pluck a function out of thin air and then show that it fits. We want systematic ways of actually *finding* the solution. There are many different methods available, and the appropriate method depends on what kind of differential equation you are dealing with. The first kind of differential equation we shall treat is called a *separable* differential equation.

2 Separable equations

If a differential equation can be put into the form

$$g(y)\,dy = f(x)\,dx$$

then it is said to be *separable*. This is because the variables are separated, all y's on one side and all x's on the other. To solve such an equation we integrate both sides:

$$\int g(y)\,dy = \int f(x)\,dx + c$$

and then get y in terms of x if possible.

Not all first order equations can be put into this form. You need to be able to recognise those that can. It is a question of building up experience.

2.1 Example

Solve

$$9y\frac{dy}{dx} + 4x = 0$$

Solution. Rearranging gives

$$9y\,dy = -4x\,dx$$

Integrating both sides:

$$\int 9y\,dy = -\int 4x\,dx$$

i.e.

$$\frac{9y^2}{2} = -2x^2 + c$$

Hence the solution is

$$y = \sqrt{\frac{2c}{9} - \frac{4x^2}{9}}$$

or, since $2c/9$ is a constant which we could call d,

$$y = \sqrt{d - \frac{4x^2}{9}}$$

This simple example illustrates two important points:

(i) You **must** add the constant c **when you do the integration** and not at any other stage. Some people think it is enough simply to add a constant to the final answer. Doing so will give you a wrong answer. This kind of error must be avoided at the outset.

(ii) As the above example shows, we often combine constants together, i.e. $2c/9$ is a constant and has become d. Some people do this even when a constant is combined with a variable, e.g. if they have a term cx they decide to call it d. This is genuine rubbish and will give a wrong answer. However, something like $2cx$ can be replaced by dx.

The above example has also illustrated a more general point, in that solutions of first order differential equations usually involve one arbitrary constant. Its value can be determined if we know y at one value of x, as the next example illustrates.

2.2 Example

Solve

$$\frac{dy}{dt} + 5t^4y^2 = 0 \qquad \text{subject to } y = 1 \text{ when } t = 0$$

Solution. The differential equation can be rearranged to $dy = -5t^4y^2\,dt$ or

$$\frac{1}{y^2}dy = -5t^4\,dt$$

Integrating both sides,

$$-\frac{1}{y} = -t^5 + c$$

recalling that we add the constant c **when we do the integration**. Hence

$$y = \frac{1}{t^5 - c}$$

But, when $t = 0$, $y = 1$. Substituting this in,

$$1 = \frac{1}{0 - c} = -\frac{1}{c}$$

so that $c = -1$. Hence, the solution of the differential equation is

$$y = \frac{1}{t^5 + 1}$$

2.3 Example

Solve

$$\frac{dy}{dx} = -2xy \quad \text{subject to } y = 2 \text{ when } x = 0 \text{ (i.e. } y(0) = 2)$$

Solution. Rearranging gives

$$\frac{1}{y}dy = -2x\,dx$$

so that

$$\ln y = -x^2 + c$$

Hence

$$y = e^{-x^2+c} = e^{-x^2}e^c = Ae^{-x^2}$$

Note that, since c is a constant, e^c is another constant which we have called A. Since $y = 2$ when $x = 0$ we find that $A = 2$. Hence the solution is

$$y = 2e^{-x^2}$$

2.4 Example

Solve

$$\frac{dy}{dx} = \cos x \tan y$$

Solution. Rearranging gives $\cot y\, dy = \cos x\, dx$ or

$$\frac{\cos y}{\sin y}\, dy = \cos x\, dx$$

Integrating both sides gives

$$\ln(\sin y) = \sin x + c$$

so that

$$\sin y = e^{\sin x + c} = e^{\sin x} e^c = Ae^{\sin x}$$

Hence

$$y = \sin^{-1}(Ae^{\sin x})$$

is the solution of the differential equation.

2.5 Example

Solve

$$e^x \frac{dy}{dx} = 2(x+3)y^3, \qquad y(0) = \tfrac{1}{4}$$

Solution. Rearranging,

$$\frac{1}{y^3}\, dy = 2(x+3)e^{-x}\, dx$$

Integrating both sides (and using integration by parts on the right hand side)

$$\frac{y^{-2}}{-2} = 2\left[-(x+3)e^{-x} + \int e^{-x}\, dx + c\right]$$

i.e.

$$\frac{1}{-2y^2} = 2\left[-(x+3)e^{-x} - e^{-x} + c\right]$$

But $y = 1/4$ when $x = 0$ so that

$$\frac{1}{-2(\frac{1}{4})^2} = 2[-3 - 1 + c]$$

which gives $c = 0$. Hence

$$\frac{1}{-2y^2} = 2[-xe^{-x} - 4e^{-x}]$$

or, rearranging,

$$y = \frac{e^{x/2}}{2\sqrt{x+4}}$$

which is the solution.

2.6 Solution of the water tank problem

We shall find the solution of the water tank problem described in Section 1.1. Let the initial depth of the water be H, so that

$$h(0) = H$$

Recall that the differential equation for this problem is

$$\frac{dh}{dt} = -k\sqrt{h}$$

where k is a constant. Separating the variables gives

$$h^{-1/2}\, dh = -k\, dt$$

Integrating,

$$2h^{1/2} = -kt + c$$

But $h(0) = H$ so $c = 2H^{1/2}$. Hence

$$2h^{1/2} = -kt + 2H^{1/2}$$

which gives the depth of the water at time t to be

$$h(t) = \left(\frac{2H^{1/2} - kt}{2}\right)^2$$

In particular the tank will be empty when $h = 0$, i.e., at time $t = 2H^{1/2}/k$.

2.7 Example: population growth

A model for the growth of some population $y(t)$ is the differential equation

$$\frac{dy}{dt} = ky(M - y)$$

where k and M are constants. The equation is separable since it can be written in the form

$$\frac{1}{y(M-y)}\, dy = k\, dt \tag{2.1}$$

To integrate the left hand side we shall have to write it in partial fractions. The partial fraction decomposition is

$$\frac{1}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y} = \frac{A(M-y) + By}{y(M-y)}$$

i.e. $1 = A(M - y) + By$. Setting $y = 0$ gives $1 = AM$ so that $A = 1/M$. Setting $y = M$ gives $1 = MB$ so that $B = 1/M$. Therefore (2.1) becomes

$$\left(\frac{1/M}{y} + \frac{1/M}{M-y}\right) dy = k\, dt$$

Integrating,

$$\frac{1}{M}\ln y - \frac{1}{M}\ln(M-y) = kt + C$$

Therefore,

$$\ln\frac{y}{M-y} = Mkt + D$$

where $D = MC$ is constant. Hence

$$\frac{y}{M-y} = e^{Mkt+D} = e^{Mkt}e^{D} = Ee^{Mkt}$$

where E is a constant. Solving for y,

$$y(t) = \frac{MEe^{Mkt}}{1+Ee^{Mkt}}$$

Note that, as $t \to \infty$, $y(t) \to M$ so the population levels off. This formula has proved to be a very good fit to data on the growth of the population of the United States.

2.8 Example: Newton's Law of Cooling

A copper ball is heated to 100^oC. At time $t = 0$ it is placed in water that is maintained at 30^oC. After 3 minutes the temperature of the ball is 60^oC. Find the time at which the temperature of the ball will be 31^oC. The relevant physical law is Newton's law of cooling, which states that the rate of change of temperature is proportional to the temperature difference between (in this example) the ball and the water.

Solution. Let $T(t)$ be the temperature of the ball. Newton's law of cooling tells us that

$$\frac{dT}{dt} = -k(T-30)$$

where k is the proportionality constant and the minus sign is because the ball is cooling down. This is a separable differential equation since

$$\frac{1}{T-30}\,dT = -k\,dt$$

Integrating,

$$\ln(T-30) = -kt + A$$

so that

$$T - 30 = e^{-kt+A} = e^{-kt}e^{A} = Be^{-kt}$$

Now $T = 100$ when $t = 0$ so $B = 70$. Hence

$$T = 30 + 70e^{-kt}$$

Also $T = 60$ when $t = 3$ (minutes) so

$$60 = 30 + 70e^{-3k}$$

which gives $k = 0.2824$. Hence

$$T = 30 + 70e^{-0.2824t}$$

We want t when $T = 31$. This gives

$$31 = 30 + 70e^{-0.2824t}$$

Solving for t gives $t = 15.04$ minutes.

3 The integrating factor method

This method is for differential equations of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{3.2}$$

or equations which can be rearranged to look like this. *Note in particular that the coefficient of dy/dx must be 1.*

We define the **Integrating Factor**, commonly abbreviated I.F. as

$$\text{Integrating Factor (I.F.)} = e^{\int P(x)\,dx}$$

The method for solving a differential equation in the form (3.2) is as follows:

(i) work out the integrating factor $e^{\int P(x)\,dx}$

(ii) multiply the differential equation through by the integrating factor

(iii) rewrite the left hand side as $\frac{d}{dx}$(something)

(iv) integrate and solve

The "something" in (iii) is always the integrating factor times y.

3.1 Example

Solve

$$\frac{dy}{dx} - y = e^{2x}$$

Solution. This example is already in the form of (3.2) with $P(x) = -1$ and $Q(x) = e^{2x}$. The integrating factor I.F. is

$$\text{I.F.} = e^{\int (-1)\,dx} = e^{-x}$$

(you do not need to add a constant of integration when working out the I.F.). Multiplying the differential equation through by the integrating factor gives

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{2x} = e^{x}$$

This can be rewritten as

$$\frac{d}{dx}((\text{I.F.})\, y\,) = e^x$$

i.e.

$$\frac{d}{dx}(e^{-x}y) = e^x$$

Integrating both sides gives

$$e^{-x}y = e^x + A$$

so that the solution of the differential equation is

$$y = e^{2x} + Ae^x$$

Note that, as in the separation of variables method, you must add the constant of integration **when you integrate** and not at any other stage. It does not usually appear in the final answer as an additive constant.

3.2 Example

Solve

$$\frac{dy}{dx} + (\tan x)\, y = \sin 2x \qquad y(0) = 1$$

Solution. The integrating factor is given by

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\ln(\sec x)} = \sec x$$

Multiplying the differential equation through by the Integrating factor,

$$\sec x \frac{dy}{dx} + \tan x \sec x\; y = \sin 2x \sec x$$

i.e.

$$\frac{d}{dx}((\sec x)\, y) = \sin 2x \sec x$$

Integrating,

$$\begin{aligned}
(\sec x)\, y &= \int \sin 2x \sec x\, dx \qquad \text{but } \sin 2x = 2\sin x \cos x \\
&= 2\int \sin x \cos x \left(\frac{1}{\cos x}\right)\, dx \\
&= 2\int \sin x\, dx \\
&= -2\cos x + A
\end{aligned}$$

Hence

$$y = -\frac{2\cos x}{\sec x} + \frac{A}{\sec x}$$

i.e.

$$y = -2\cos^2 x + A\cos x$$

But $y(0) = 1$. Hence $A = 3$. Therefore

$$y = -2\cos^2 x + 3\cos x$$

3.3 Example

Solve

$$t\frac{dy}{dt} = y + t^3$$

Solution. This differential equation is not yet in the right from for applying the integrating factor method. But it can be rewritten in the form

$$\frac{dy}{dt} - \frac{1}{t}y = t^2$$

which, apart from the change of notation, is in the form of (3.2) with $P(t) = -1/t$ and $Q(t) = t^2$. The integrating factor is

$$\text{I.F.} = e^{\int -\frac{1}{t}\,dt} = e^{-\ln t} = e^{\ln(t^{-1})} = t^{-1} = \frac{1}{t}$$

The differential equation becomes

$$\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y = t$$

i.e.

$$\frac{d}{dt}\left(\frac{1}{t}y\right) = t$$

Integrating,

$$\frac{1}{t}y = \int t\,dt = \frac{t^2}{2} + A$$

Hence

$$y = \frac{t^3}{2} + At$$

3.4 Example

Solve

$$\frac{dy}{dx} + 2xy = 4x \qquad y(0) = 3$$

Solution. The integrating factor is

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

The differential equation becomes

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 4xe^{x^2}$$

i.e.

$$\frac{d}{dx}(e^{x^2}y) = 4xe^{x^2}$$

Integrating,

$$\begin{aligned} e^{x^2} y &= \int 4xe^{x^2}\,dx \\ &= 2e^{x^2} + A \quad \text{(by inspection, or on substituting } u = x^2\text{)} \end{aligned}$$

Therefore

$$y = 2 + Ae^{-x^2}$$

But $y(0) = 3$. Hence $A = 1$ and the solution of the differential equation is

$$y = 2 + e^{-x^2}$$

3.5 Practical example 1

This concerns a body falling under gravity with air resistance. Letting $x(t)$ be the downward displacement at time t from the point of release, the differential equation is

$$\frac{d^2x}{dt^2} = g - k\frac{dx}{dt}$$

The last term is the air resistance term, and g denotes acceleration due to gravity. The differential equation is to be solved subject to the conditions

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 0$$

The first of these conditions reflects the fact that the particle is initially at the point and release, and the second that it is released from rest, so that the initial velocity is zero.

Letting $v = \frac{dx}{dt}$ then the equation can be rewritten as

$$\frac{dv}{dt} = g - kv$$

or

$$\frac{dv}{dt} + kv = g$$

The integrating factor is $e^{\int k\,dt} = e^{kt}$. Therefore, the differential equation becomes

$$e^{kt}\frac{dv}{dt} + ke^{kt}v = ge^{kt}$$

or

$$\frac{d}{dt}(e^{kt}\,v) = ge^{kt}$$

Integrating,

$$e^{kt}\,v = \frac{ge^{kt}}{k} + A$$

so that

$$v = \frac{g}{k} + Ae^{-kt}$$

Now $\dot{x}(0) = 0$ so $v(0) = 0$. This gives $0 = g/k + A$ so that $A = -g/k$. Hence

$$v = \frac{g}{k} - \frac{g}{k}e^{-kt} \tag{3.3}$$

But $v = dx/dt$ so

$$\frac{dx}{dt} = \frac{g}{k} - \frac{g}{k}e^{-kt}$$

Integrating,

$$x = \frac{gt}{k} + \frac{g}{k^2}e^{-kt} + B$$

But $x(0) = 0$. Hence $0 = g/k^2 + B$ so that $B = -g/k^2$. Thus the solution is

$$x(t) = \frac{gt}{k} + \frac{g}{k^2}e^{-kt} - \frac{g}{k^2}$$

From the expression (3.3) for the velocity v, we see that

$$v(t) \to \frac{g}{k} \qquad \text{as } t \to \infty$$

We say that g/k is the *terminal velocity*. For a free fall parachutist, the terminal velocity is about 120 mph (before the shute is opened!)

3.6 Practical Example 2

This concerns indoor temperature oscillations driven by outdoor temperature oscillations. Assume we are somewhere in the world where the temperature is very warm by day, and still quite warm at night, and shows this pattern every day of the year. Then the outside temperature $A(t)$ is a periodic function of time t. We shall take it to be

$$A(t) = 80 + 10\cos\omega t \qquad \text{with } \omega = \pi/12$$

which ranges between 70 and 90^oF. The reason ω has the value stated is so that the function has a period of 24 hours (recall that the period of $\cos\omega t$ is $2\pi/\omega$).

What we want to do, is find an expression for the indoor temperature $u(t)$ and to make some comments on how it responds to changes in the outdoor temperature. We shall apply Newton's law of cooling to deduce that the indoor temperature $u(t)$ satisfies the differential equation

$$\frac{du}{dt} = -k(u - A(t))$$

where k is some constant depending on the insulation. We can rewrite this differential equation as

$$\frac{du}{dt} + ku = k(80 + 10\cos\omega t)$$

which can be solved by using the integrating factor method.

The integrating factor is $e^{\int k\,dt} = e^{kt}$. The differential equation becomes

$$e^{kt}\frac{du}{dt} + ke^{kt}u = k(80e^{kt} + 10e^{kt}\cos\omega t)$$

or

$$\frac{d}{dt}(e^{kt}\,u) = k(80e^{kt} + 10e^{kt}\cos\omega t)$$

Integrating,

$$e^{kt}\,u = k\left(80\int e^{kt}\,dt + 10\int e^{kt}\cos\omega t\,dt\right)$$

One can easily check, by integrating by parts twice, that

$$\int e^{kt}\cos\omega t\,dt = \frac{\omega}{\omega^2+k^2}e^{kt}\sin\omega t + \frac{k}{\omega^2+k^2}e^{kt}\cos\omega t + c$$

Hence

$$e^{kt}\,u = k\left(\frac{80e^{kt}}{k} + 10\left(\frac{\omega}{\omega^2+k^2}e^{kt}\sin\omega t + \frac{k}{\omega^2+k^2}e^{kt}\cos\omega t + c\right)\right)$$

so that

$$u = 80 + \frac{10k\omega}{\omega^2+k^2}\sin\omega t + \frac{10k^2}{\omega^2+k^2}\cos\omega t + 10\,k\,c\,e^{-kt}$$

What can we say about this? Well, note that as $t \to \infty$ the last term, tends to zero. Therefore we can say that, once enough time has elapsed, the indoor temperature will be well approximated by

$$u = 80 + \frac{10k\omega}{\omega^2+k^2}\sin\omega t + \frac{10k^2}{\omega^2+k^2}\cos\omega t$$

Recall that $\omega = \pi/12$. Let $k = 0.2$ (recall k is the insulation parameter). Then

$$u = 80 + 4.82\sin\frac{\pi t}{12} + 3.69\cos\frac{\pi t}{12}$$

For reasons which will become clear, let us write the last two terms in the form $R\cos(\pi t/12 - \alpha)$. Now

$$R\cos\left(\frac{\pi t}{12} - \alpha\right) = R\cos\frac{\pi t}{12}\cos\alpha + R\sin\frac{\pi t}{12}\sin\alpha$$

By comparison, $R\cos\alpha = 3.69$ and $R\sin\alpha = 4.82$. Hence $R = 6.07$ and $\alpha = 0.92$ radians. Thus, we can now write the expression for the indoor temperature u in the form

$$u = 80 + 6.07\cos\left(\frac{\pi t}{12} - 0.92\right)$$

or

$$u = 80 + 6.07\cos\frac{\pi}{12}(t - 3.51)$$

Compare this expression with the corresponding expression for the outdoor temperature $A(t)$, namely

$$A(t) = 80 + 10\cos\frac{\pi t}{12}$$

We may make two observations:

- the indoor temperature has lower variability, always being within 6.07^{o}F of 80^{o}F
- indoor temperature lags behind outdoor temperature by 3.51 hours. For example, if it is hottest outside at 4pm, it will be hottest inside at 7.30pm.

3.7 Practical Example 3

This concerns a holding tank, and is possibly the kind of problem that might arise in the chemical engineering industry.

There is a large tank, and an inflow pipe at the top through which polluted water can enter the tank. At the bottom of the tank there is an outflow pipe (with a tap) through which the polluted water leaves the tank. It is assumed that the contents of the tank are well mixed throughout.

Initially the tank has 360 gallons of water containing 2 pounds of pollutant per gallon.

At time $t = 0$, polluted water starts entering at a rate of 80 gallons per hour and contains 3 pounds of pollutant per gallon. Simultaneously a tap is opened at the bottom and the water flows out at a rate of 40 gallons per hour. Let $P(t)$ be the amount of pollutant (in lbs) in the tank at time t. We shall find an expression for $P(t)$. Note that it not immediately obvious what the answer will be since the water that starts coming in is dirtier than the water that was already in the tank, so that thereafter the well stirred tank will contain water of an intermediate level of dirtiness ranging between 2 and 3 pounds of pollutant per gallon, and there is the added complication of the water of unknown dirtiness going out at the bottom.

However, methods of differential equations can help us to solve this problem. Since dP/dt is the rate of change of the amount of pollutant, we can say that

$$\frac{dP}{dt} = \text{rate in} - \text{rate out} \tag{3.4}$$

The rate at which pollutant comes in is easy to calculate and is given by $80 \times 3 = 240$ lbs per hour.

The rate out is more tricky. Water enters at 80 gallons per hour and leaves at 40 gallons per hour. So it builds up at a rate of 40 gallons per hour. The amount of water in the tank at time t is therefore $360 + 40t$ gallons, 360 being the initial amount.

The amount of pollutant per gallon in the tank at time t is therefore

$$\frac{P(t)}{360 + 40t}$$

The pollutant therefore goes out at a rate

$$\underbrace{40}_{\text{gallons per hour going out}} \quad \underbrace{\frac{P(t)}{360 + 40t}}_{\text{pollutant per gallon}}$$

which equals

$$\frac{P}{9 + t}$$

Therefore, the differential equation (3.4) becomes

$$\frac{dP}{dt} = 240 - \frac{P}{9+t}$$

or

$$\frac{dP}{dt} + \frac{P}{9+t} = 240$$

which can be solved using the integrating factor method. The integrating factor is

$$\text{I.F.} = e^{\int \frac{1}{9+t}\,dt} = e^{\ln(9+t)} = 9+t$$

The differential equation becomes

$$(9+t)\frac{dP}{dt} + P = 240(9+t)$$

or

$$\frac{d}{dt}((9+t)P) = 240(9+t) = 2160 + 240t$$

Integrating,

$$(9+t)P = 2160t + 120t^2 + A$$

so that

$$P = \frac{2160t + 120t^2 + A}{9+t}$$

Now, when $t = 0$, $P = 360 \times 2 = 720$ lbs. Thus $720 = A/9$ so that $A = 6480$. Hence the amount of pollutant in the tank at time t is given by

$$P = \frac{2160t + 120t^2 + 6480}{9+t}$$

Also of interest is the amount of pollutant *per gallon* (i.e. concentration) in the tank at time t. This is $P(t)$ divided by $360 + 40t$. It is easily seen that the resulting quantity ranges between 2 (at time $t = 0$) to 3 (as $t \to \infty$) which is what we would intuitively expect.

4 Other methods for 1st order ODEs

We shall briefly discuss two other methods for first order ODEs. One of these is a transformation that can reduce a certain type of first order ODE to separable form. Also we shall look at Bernoulli's differential equation, which can be reduced by a transformation to an equation to which the integrating factor method applies.

4.1 Reducing to separable form

Differential equations of the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \tag{4.5}$$

i.e., the right hand side depends on y and x only through the single combination y/x, can be made separable by a change of variables. Let

$$u = \frac{y}{x}$$

Then $y = xu$ so that, by the product rule for differentiation,

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

Therefore, the original differential equation becomes

$$x\frac{du}{dx} + u = g(u)$$

which can be written as

$$\frac{1}{g(u) - u}\,du = \frac{1}{x}\,dx$$

Then, we can integrate both sides. Obviously, to proceed further we need to look at a specific example:

4.2 Example

Solve

$$2xy\frac{dy}{dx} - y^2 + x^2 = 0 \tag{4.6}$$

Solution. This equation does not immediately have the form of (4.5) but it can be seen to have that form after some rearranging. Indeed, (4.6) can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)}$$

and the right hand side of this is a function of y/x only, so that our method will work.

Let $u = y/x$ so that $y = xu$ and

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

Then the differential equation (4.6) becomes

$$2x(xu)\left(x\frac{du}{dx} + u\right) - x^2u^2 + x^2 = 0$$

Cancelling x^2, and then expanding out and rearranging,

$$2ux\frac{du}{dx} = -(1+u^2)$$

or

$$\frac{2u}{1+u^2}\,du = -\frac{1}{x}\,dx$$

Integrating

$$\ln(1+u^2) = -\ln x + c$$

so that

$$1+u^2 = e^{-\ln x+c} = e^{-\ln x}e^c = Ae^{-\ln x} = Ae^{\ln(x^{-1})} = \frac{A}{x}$$

Therefore

$$u^2 = \frac{A}{x} - 1$$

But $u = y/x$ so

$$\frac{y^2}{x^2} = \frac{A}{x} - 1$$

Hence the solution of the differential equation is

$$y^2 = Ax - x^2$$

4.3 Bernoulli's differential equation

Sometimes it is possible to take a differential equation and transform it into one to which the integrating factor method can be applied. The best known example of such a differential equation is the Bernoulli equation:

$$\frac{dy}{dx} + R(x)y = S(x)y^n \tag{4.7}$$

The integrating factor method cannot be applied to this equation as it stands, because the right hand side would need to depend on x only (recall that the differential equation has to have the structure of (3.2)). But we can always introduce a new variable u defined by

$$u = y^{1-n}$$

and transform (4.7) into an equation involving u and x, rather than y and x.

Now $u = y^{1-n}$ so that, differentiating,

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

giving

$$\frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx}$$

Therefore, (4.7) becomes

$$\frac{y^n}{1-n}\frac{du}{dx} + R(x)y = S(x)y^n$$

i.e.

$$\frac{1}{1-n}\frac{du}{dx} + R(x)y^{1-n} = S(x)$$

or

$$\frac{du}{dx} + (1-n)R(x)u = (1-n)S(x) \tag{4.8}$$

and the integrating factor method can certainly be applied to this.

4.4 Example

Solve the differential equation

$$x\frac{dy}{dx} + y = y^2x^2\ln x$$

Solution. The differential equation can be written as

$$\frac{dy}{dx} + \frac{1}{x}y = y^2 x\ln x$$

which has the structure of (4.7) with $R(x) = 1/x$, $S(x) = x\ln x$ and $n = 2$.

So, in this case, the variable u becomes

$$u = y^{1-n} = \frac{1}{y}$$

and the equation for u (equation (4.8)) becomes

$$\frac{du}{dx} - \frac{1}{x}u = -x\ln x$$

We shall solve this by the integrating factor method. The integrating factor is

$$\text{I.F.} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x}$$

Multiplying through by the integrating factor,

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -\ln x$$

or

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = -\ln x$$

Hence

$$\begin{aligned}\frac{1}{x}u &= -\int \ln x\,dx \\ &= -\left[x\ln x - \int \frac{1}{x}x\,dx\right] \\ &= -x\ln x + x + c\end{aligned}$$

Hence

$$u = -x^2 \ln x + x^2 + cx$$

But $u = 1/y$. Thus, the solution of the differential equation is

$$y = \frac{1}{-x^2 \ln x + x^2 + cx}$$

5 Second order differential equations

In this section we shall treat second order linear differential equations with constant coefficients.

5.1 Homogeneous equations

Homogeneous means the RHS= 0. We study differential equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \tag{5.9}$$

where a, b, c are all **constants**. Let us make two important observations:

(1) if y_1 and y_2 are solutions then so is $Ay_1 + By_2$ for any constants A and B;

(2) $y = e^{mx}$ is a solution whenever m is a root of the quadratic equation

$$am^2 + bm + c = 0.$$

The proof of (1) is very trivial, you just check that $y = Ay_1 + By_2$ satisfies (5.9) when both y_1 and y_2 do.

To see (2), substitute $y = e^{mx}$ into the differential equation (5.9). It becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

or, cancelling e^{mx},

$$am^2 + bm + c = 0.$$

The above quadratic equation is called the **characteristic equation** or **auxiliary equation.** It has to be solved for m. The method for solving the differential equation (5.9) may be summarised thus:

5.2 Method for solving the homogeneous equation (5.9)

To solve

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where a, b, c are all constants, find the characteristic equation

$$am^2 + bm + c = 0$$

Then there are three cases:

(i) Real and distinct roots m_1 and m_2. The solution to the differential equation is

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

(ii) Only one real root m. Solution in this case is

$$y = (Ax + B)e^{mx}$$

(iii) Complex roots. Write them as $p \pm iq$. Solution is

$$y = e^{px}(A \cos qx + B \sin qx)$$

5.3 Comments

- The method is completely useless unless a, b, c are all constants. Equations with coefficients that depend on x are treated in certain 2nd year modules.
- In the repeated root case (ii), one solution is e^{mx} and it is easily checked that $x\,e^{mx}$ is also a solution in this case. The expression $(Ax + B)e^{mx}$ arises by combining these two solutions as in observation (1) of Section 5.1.
- In the complex roots case, here's where the solution comes from. The roots are $p + iq$ and $p - iq$ and so, by observations (1) and (2) of Section 5.1, the solution is

$$\begin{aligned} y &= Ce^{(p+iq)x} + De^{(p-iq)x} \\ &= e^{px}(Ce^{iqx} + De^{-iqx}) \\ &= e^{px}(C \cos qx + Ci \sin qx + D \cos qx - Di \sin qx) \end{aligned}$$

  i.e.

$$y = e^{px}(A \cos qx + B \sin qx)$$

  where $A = C + D$ and $B = i(C - D)$.

5.4 Example

Solve the differential equation

$$\frac{d^2y}{dx^2} - y = 0$$

Solution. The characteristic equation is $m^2 - 1 = 0$ and has roots 1 and -1. These are real and distinct so case (i) applies and the solution is

$$y = Ae^{x} + Be^{-x}$$

5.5 Example

Solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

subject to $y(0) = 4$ and $y'(0) = -5$.

Solution. The characteristic equation is $m^2 + m - 2 = 0$, i.e. $(m+2)(m-1) = 0$ so that $m = -2$ or 1. Again, this is case (i). The general solution is

$$y = Ae^{-2x} + Be^{x}$$

Now, $y(0) = 4$ so

$$4 = A + B$$

Also, $y' = -2Ae^{-2x} + Be^{x}$, so that, since $y'(0) = -5$,

$$-5 = -2A + B$$

Solving these two equations simultaneously gives $A = 3$ and $B = 1$. Hence the solution is

$$y = 3e^{-2x} + e^{x}$$

5.6 Example

Solve

$$y'' + 8y' + 16y = 0$$

Solution. The characteristic equation is $m^2 + 8m + 16 = 0$ which has just one real root $m = -4$. This is case (ii). The general solution is

$$y = (Ax + B)e^{-4x}$$

5.7 Example

Solve

$$y'' + 2y' + 10y = 0$$

Solution. The characteristic equation is $m^2 + 2m + 10 = 0$. The roots are

$$m = \frac{1}{2}\left(-2 \pm \sqrt{-36}\right) = \frac{1}{2}(-2 \pm 6i) = -1 \pm 3i$$

These are complex roots so we are in case (iii). The roots are of the form $p \pm iq$ with $p = -1$ and $q = 3$, giving the general solution to be

$$y = e^{-x}(A\cos 3x + B\sin 3x)$$

5.8 Example

Solve

$$y'' + 4y = 0 \qquad \text{subject to } y(0) = -3 \text{ and } y(\pi/4) = 0$$

This is called a *boundary value problem* since y is prescribed at two different values of x, 0 and $\pi/4$. A problem in which $y(0)$ and $y'(0)$ are both prescribed would be called an *initial value problem.*

Solution. The characteristic equation is $m^2 + 4 = 0$ and has roots $m = \pm 2i$ which is case (iii). The roots are of the from $p \pm iq$ where, in this case, $p = 0$ and $q = 2$. The general expression for the solution is

$$y = e^{px}(A \cos qx + B \sin qx)$$

which in this particular case gives

$$y = A \cos 2x + B \sin 2x$$

Now $y(0) = -3$ so $A = -3$. Also, $y(\pi/4) = 0$ so $B = 0$. Hence

$$y = -3 \cos 2x$$

5.9 Physical application: damped pendulum

The differential equation for a damped pendulum is

$$lM \frac{d^2\theta}{dt^2} + kl \frac{d\theta}{dt} + Mg \sin\theta = 0 \tag{5.10}$$

where M is the mass of the bob, l is the length of the pendulum, g is acceleration due to gravity and k is representative of the amount of damping present. Let us write down what we would intuitively expect:

- We would expect that if there is a very large amount of damping (e.g. if the pivot is oiled with a very thick grease) then the pendulum would go straight in to its downward equilibrium position without swinging back and forth. In other words, if k is large then $\theta(t) \to 0$ monotonically as $t \to \infty$.

- We would expect that if there is only a small amount of damping (k small) then the pendulum would still come to rest but would swing back and forth as it does so.

Although the differential equation (5.10) includes all possible motions of the pendulum (including, for example, motions in which it is kicked very hard and describes several complete revolutions before coming to rest), it can only be solved by our methods in this course if θ is small. In this case, we use the approximation

$$\sin\theta \approx \theta$$

so that the differential equation becomes

$$lM\frac{d^2\theta}{dt^2} + kl\frac{d\theta}{dt} + Mg\theta = 0$$

The characteristic equation is

$$lMm^2 + klm + Mg = 0$$

and has real distinct roots if $k^2l^2 > 4lM^2g$, one real repeated root if $k^2l^2 = 4lM^2g$ and complex roots if $k^2l^2 < 4lM^2g$. Let us concentrate on one of these cases only. We shall assume that

$$k^2l^2 < 4lM^2g$$

so that we are in the complex roots case. The roots are then

$$\begin{aligned} m &= \frac{1}{2lM}\left(-kl \pm \sqrt{k^2l^2 - 4lM^2g}\right) \\ &= \frac{1}{2lM}\left(-kl \pm i\sqrt{4lM^2g - k^2l^2}\right) \\ &= p \pm iq \end{aligned}$$

where

$$p = -\frac{k}{2M} \qquad \text{and} \qquad q = \frac{\sqrt{4lM^2g - k^2l^2}}{2lM}$$

The solution is

$$\theta(t) = e^{pt}(A\cos qt + B\sin qt)$$

i.e.

$$\theta(t) = e^{-\frac{kt}{2M}}\left\{A\cos\left(\frac{\sqrt{4lM^2g - k^2l^2}}{2lM}\,t\right) + B\sin\left(\frac{\sqrt{4lM^2g - k^2l^2}}{2lM}\,t\right)\right\}$$

From the above expression (recall it assumes that $k^2l^2 < 4lM^2g$, i.e. the damping parameter k is small) we see that the pendulum comes to rest (because of the negative exponential) but swings from side to side as it does so (because of the oscillatory sine and cosine terms).

If $k^2l^2 > 4lM^2g$ (large damping) the solution will instead be of the form

$$\theta(t) = Ae^{m_1t} + Be^{m_2t}$$

and the pendulum goes straight into the vertical equilibrium position without oscillating.

5.10 Inhomogeneous equations

Inhomogeneous equations are equations where the RHS is non-zero, i.e. differential equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

To solve such an equation (provided a, b, c are constants), proceed as follows:

(i) find the **complementary solution**, i.e. the solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

and call it y_c;

(ii) find a **particular solution** of the original differential equation and call it y_p. The appropriate guess for the particular solution depends on what is in the RHS (see below);

(iii) general solution is $y = y_c + y_p$.

Why does this work? We just have to verify that $y = y_c + y_p$ satisfies the original differential equation. Just substitute it in:

$$\begin{aligned} & a\frac{d^2}{dx^2}(y_c + y_p) + b\frac{d}{dx}(y_c + y_p) + c(y_c + y_p) \\ = & \underbrace{\left(a\frac{d^2y_c}{dx^2} + b\frac{dy_c}{dx} + cy_c\right)}_{=0} + \underbrace{\left(a\frac{d^2y_p}{dx^2} + b\frac{dy_p}{dx} + cy_p\right)}_{=f(x)} \\ = & f(x) \end{aligned}$$

Hence $y = y_c + y_p$ satisfies the original differential equation.

The complementary solution y_c will contain two undetermined constants. These will also be in the final solution unless the problem comes with initial or boundary data which would be used to find their values.

5.11 Particular solution table

The following table tells you what to guess for the particular solution y_p, based on what is in the RHS of the differential equation you are trying to solve.

RHS containing:	Try:	Doesn't work?
polynomial	general polynomial of same degree	
ke^{px}	Ce^{px}	Try $C\,x\,e^{px}$ or $C\,x^2\,e^{px}$
$k\cos\omega x$ or $k\sin\omega x$	$C\cos\omega x + D\sin\omega x$	$C\,x\,\cos\omega x + D\,x\,\sin\omega x$

5.12 Example

Solve the differential equation

$$\frac{d^2y}{dx^2} + y = x^2$$

Solution. First, we find the complementary solution (C.S.), i.e. we solve

$$\frac{d^2y}{dx^2} + y = 0$$

The characteristic equation of this is $m^2 + 1 = 0$ which has roots $m = \pm i$. The complementary solution is therefore

$$y_c = A\cos x + B\sin x$$

Next, we find a particular solution y_p. The RHS of the original differential equation is x^2, which is a polynomial. The table says to try a general polynomial of the same degree so we try

$$y_p = ax^2 + bx + c$$

[**N.B.** it is *not* enough just to have ax^2]. Substituting this in to the original differential equation gives

$$2a + ax^2 + bx + c = x^2$$

Comparing coefficients of x^2 and x and comparing constant terms gives $a = 1$, $b = 0$ and $2a + c = 0$ so that $c = -2$. Hence the particular solution is

$$y_p = x^2 - 2$$

and the general solution is given by $y = y_c + y_p$, i.e.

$$y = A\cos x + B\sin x + x^2 - 2$$

5.13 Example

Solve

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 10e^{-2t}$$

subject to $y(0) = \frac{5}{3}$ and $y'(0) = \frac{5}{3}$.

Solution. For the C.S. we solve

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0$$

The characteristic equation is $m^2 - 4m + 3 = 0$, i.e. $(m-3)(m-1) = 0$ so that $m = 3$ and 1. This is case (i) and the C.S. is

$$y_c = Ae^{3t} + Be^{t}$$

For the particular solution (P.S.), the RHS is $10e^{-2t}$ so, from the table, the appropriate guess is $y_p = Ce^{-2t}$. Substituting this into the original differential equation gives

$$4Ce^{-2t} + 8Ce^{-2t} + 3Ce^{-2t} = 10e^{-2t}$$

Hence $15C = 10$ so that $C = 2/3$ and $y_p = \frac{2}{3}e^{-2t}$. The general solution is $y = y_c + y_p$, i.e.

$$y = Ae^{3t} + Be^{t} + \frac{2}{3}e^{-2t}$$

Now $y(0) = 5/3$ so $5/3 = A + B + 2/3$, i.e.

$$A + B = 1$$

Also $y'(0) = 5/3$ so $5/3 = 3A + B - 4/3$, i.e.

$$3A + B = 3$$

Solving these simultaneously gives $A = 1$ and $B = 0$. Therefore, the final answer is

$$y = e^{3t} + \frac{2}{3}e^{-2t}$$

5.14 Example

Solve

$$y'' - 2y' + 5y = \sin 3x$$

Solution. For the C.S. we solve

$$y'' - 2y' + 5y = 0$$

The characteristic equation is $m^2 - 2m + 5 = 0$ which has roots

$$m = \frac{1}{2}(2 \pm \sqrt{-16}) = \frac{1}{2}(2 \pm 4i) = 1 \pm 2i$$

The complementary solution is therefore

$$y_c = e^x(A\cos 2x + B\sin 2x)$$

For the P.S., the right hand side of the differential equation is $\sin 3x$, so the table says that we should try

$$y_p = C\sin 3x + D\cos 3x$$

Substituting into the original differential equation gives

$$\begin{gathered} -9C\sin 3x - 9D\cos 3x - 2(3C\cos 3x - 3D\sin 3x) \\ + 5(C\sin 3x + D\cos 3x) = \sin 3x \end{gathered}$$

Comparing $\sin 3x$ terms:

$$-4C + 6D = 1$$

Comparing $\cos 3x$ terms:

$$-4D - 6C = 0$$

Solving simultaneously gives $C = -\frac{1}{13}$ and $D = \frac{3}{26}$. Hence the particular solution is $y_p = -\frac{1}{13}\sin 3x + \frac{3}{26}\cos 3x$ and the general solution is given by $y = y_c + y_p$, i.e.

$$y = e^x(A\cos 2x + B\sin 2x) - \frac{1}{13}\sin 3x + \frac{3}{26}\cos 3x$$

5.15 Example (less straightforward)

Solve

$$\frac{d^2y}{dt^2} - 4y = e^{2t}$$

Solution. The complementary solution is $y_c = Ae^{2t} + Be^{-2t}$.

For the particular solution, the right hand side of the differential equation is e^{2t} so we are initially let to try $y_p = Ce^{2t}$. Substituting this in gives

$$4Ce^{2t} - 4Ce^{2t} = e^{2t}$$

i.e.

$$0 = e^{2t}$$

This is obviously impossible, and we deduce from this that our guess $y_p = Ce^{2t}$ has failed.

The particular solution table in Section 5.11 says that we should now try

$$y_p = C\,t\,e^{2t}$$

Differentiating this gives, by the product rule,

$$\frac{dy_p}{dt} = 2Cte^{2t} + Ce^{2t}$$

and, differentiating again,

$$\frac{d^2y_p}{dt^2} = 4Cte^{2t} + 4Ce^{2t}$$

Therefore, the differential equation becomes

$$\underbrace{4Cte^{2t} + 4Ce^{2t}}_{=d^2y_p/dt^2} - 4\underbrace{Cte^{2t}}_{=y_p} = e^{2t}$$

i.e.

$$4Ce^{2t} = e^{2t}$$

so that $C = \frac{1}{4}$. Therefore, the particular solution is

$$y_p = \frac{1}{4}\,t\,e^{2t}$$

and the solution of the differential equation is $y = y_c + y_p$, i.e.

$$y = Ae^{2t} + Be^{-2t} + \frac{1}{4}\,t\,e^{2t}$$

5.16 Remarks on the above example

In the above example we tried $y_p = Ce^{2t}$ and it didn't work so we tried $y_p = C\,t\,e^{2t}$. If this had also failed the next possibility would have been $y_p = C\,t^2\,e^{2t}$.

We could have predicted the failure of $y_p = Ce^{2t}$ without actually trying it. The rule is this: the particular solution *cannot* be something that is already in the complementary solution. In the above example we originally tried $y_p = Ce^{2t}$. But the complementary solution already had an e^{2t} term.

5.17 Another less straightforward example

Solve the differential equation

$$\frac{d^2y}{dt^2} + y = \sin t$$

Solution. For the complementary solution we solve the LHS= 0 and obtain $y_c = A\cos t + B\sin t$.

Now let's find the particular solution. The right hand side of the differential equation is $\sin t$ and the particular solution table in Section 5.11 suggests $y_p = C\sin t + D\cos t$. But this will not work because the complementary solution y_c already has $\sin t$ and $\cos t$ terms.

The table indicates that we should instead try

$$y_p = C\,t\,\sin t + D\,t\,\cos t$$

Differentiating,

$$\frac{dy_p}{dt} = Ct\cos t + C\sin t - Dt\sin t + D\cos t$$

Differentiating again,

$$\frac{d^2y_p}{dt^2} = -Ct\sin t + 2C\cos t - Dt\cos t - 2D\sin t$$

The differential equation becomes

$$\underbrace{-Ct\sin t + 2C\cos t - Dt\cos t - 2D\sin t}_{=d^2y_p/dt^2} + \underbrace{Ct\sin t + Dt\cos t}_{=y_p} = \sin t$$

The $t\sin t$ and $t\cos t$ terms cancel and therefore

$$2C\cos t - 2D\sin t = \sin t$$

Comparing $\cos t$ and $\sin t$ terms gives $C = 0$ and $D = -\frac{1}{2}$. Therefore the particular solution is

$$y_p = -\frac{1}{2}\,t\,\cos t$$

and the solution of the differential equation is

$$y = A\cos t + B\sin t - \frac{1}{2}\,t\,\cos t$$

6 Resonance

Resonance means that something is being forced with a frequency (i.e. cycles per second) that is the same as the frequency with which it is trying to oscillate at anyway. Examples of resonance phenomena include:

- In 1850 a bridge was destroyed in France by soldiers marching across it in unified step. Two hundred soldiers were killed. A bridge is not a rigid structure. A simple pendulum has a tendency to swing back and forth with a certain frequency that is determined by its length and the acceleration due to gravity. A bridge is similar, but there will be more than one frequency (two such frequencies are associated with the tendency to swing from side to side, and the tendency to bob up and down). Now, soldiers marching in synchronised step are all simultaneously pounding on the bridge at set intervals of time, which amounts to a powerful periodic *forcing*. Should the frequency of this coincide with one of the bridge's natural frequencies of oscillation then the bridge is in danger of developing oscillations that get out of control. This is exactly what happened. Nowadays when soldiers cross a bridge they are instructed to break step (i.e. to walk normally).

- In 1981 in Kansas City a hotel balcony, on which people were dancing, collapsed. A hotel balcony is not a rigid structure and the cause of the disaster was similar to the bridge disaster in France described above.

- Many early aircraft crashed due to a problem called *flutter* which is a type of resonance phenomenon.

- Cars have a certain natural frequency of bouncing, determined by the suspension. But if a car is driven over evenly spaced bumps there is also an enforced bouncing and thus resonance may occur. We shall study this example in detail and calculate the velocity with which the car must be driven to obtain resonance.

Now, of course, all this raises the question of what do we mean by **natural frequency**. It is really a self-explanatory term, meaning the frequency with which something tries to oscillate at *naturally*. Simple systems (like the simple pendulum) have only one natural frequency, complicated ones (like suspension bridges) may have hundreds. We shall illustrate the concept with three simple examples.

6.1 Natural frequency: the simple pendulum

The pendulum equation, for small oscillations and no damping, is

$$lM\frac{d^2\theta}{dt^2} + Mg\theta = 0$$

where M is the bob's mass, or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

The characteristic equation of this is $m^2 + g/l = 0$ with roots $m = \pm i\sqrt{g/l}$. Therefore the solution is

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{l}}\,t\right) + B\sin\left(\sqrt{\frac{g}{l}}\,t\right)$$

where A and B are determined from the initial displacement and initial velocity but are of no concern here.

Recall that the period of $\cos \omega t$ and $\sin \omega t$ is $2\pi/\omega$ so that the period of the pendulum is $2\pi/(\sqrt{g/l})$ or

$$2\pi\sqrt{\frac{l}{g}}$$

The *frequency* is given by

$$\text{frequency} = \frac{1}{\text{period}}$$

Thus the (natural) frequency of the simple pendulum is

$$\frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

cycles per unit time. We call this the *natural frequency* because it is the frequency the pendulum naturally oscillates at.

Some people leave out the factor $\frac{1}{2\pi}$ and say the natural frequency is $\sqrt{g/l}$. We shall do this ourselves in what follows.

6.2 Natural frequency: another simple illustration

Imagine a disk of radius a that rolls without slipping on a horizontal surface. The centre of the disk is connected by a spring to a wall. There will obviously be some equilibrium position such that, if the disk is placed there, it will not move. If the disk is rolled slightly from that equilibrium position and released it will roll back and forth and adopt a periodic motion. Letting k be the spring constant and m the mass of the disk, then by using Hooke's Law

$$\text{tension} = (\text{spring constant})(\text{extension})$$

it can be deduced that, if x is the displacement of the centre of the disk from the equilibrium position thereof, then x is the solution of

$$\frac{d^2x}{dt^2} + \frac{2k}{3m}x = 0$$

which is similar to the pendulum equation. You can derive this quite easily if you know about moments of inertia (recall that the moment of inertia of a disk of mass m and radius a is $\frac{1}{2}ma^2$). You will also need the parallel axes theorem.

Anyway, the solution of the above differential equation is

$$x(t) = A\cos\left(\sqrt{\frac{2k}{3m}}\,t\right) + B\sin\left(\sqrt{\frac{2k}{3m}}\,t\right)$$

and therefore the natural frequency of the system is

$$\sqrt{\frac{2k}{3m}}$$

6.3 Natural frequency: a buoy in water

A cylindrical buoy of mass M and radius r floats in water of density ρ. It can remain at rest with a certain portion of it below the water surface and the remainder above. If it is depressed from this equilibrium position and then released it will bob up and down. We want to know the frequency with which this happens.

Let the submerged portion of the buoy extend down to depth d. By the principle of Archimedes, the upthrust force on the buoy is equal to the weight of the displaced water, which equals its volume times density times g, i.e. the upthrust is $\pi r^2 d\rho g$. The downward force on the buoy is its weight, which is Mg. Therefore, when in equilibrium,

$$Mg = \pi r^2 d\rho g$$

Now, if the buoy is pressed down and then released, then at a typical instant in the subsequent motion the submerged portion extends to depth $d + y$ (i.e. y is displacement from equilibrium). The expression for the buoyancy force becomes $\pi r^2(d+y)\rho g$ while the weight is still Mg, so that the net upwards force is

$$\pi r^2(d + y)\rho g - Mg$$

which, since $Mg = \pi r^2 d\rho g$, simplifies to

$$\pi r^2 y\rho g$$

But force equals mass times acceleration. The acceleration is $-d^2y/dt^2$ (minus, because y is measured downwards). Hence

$$\pi r^2 y\rho g = -M\frac{d^2y}{dt^2}$$

or

$$\frac{d^2y}{dt^2} + \frac{\pi r^2\rho g}{M}y = 0$$

The frequency with which the buoy bobs up and down is therefore

$$r\sqrt{\frac{\pi\rho g}{M}}$$

6.4 Forcing and resonance

Many problems involving *forcing* give rise to a differential equation of the form

$$m\frac{d^2y}{dt^2} + ky = f(t)$$

where $f(t)$ represents the forcing. Forcing something means applying an external input that doesn't depend on the state of the system or care about it. For example, if you drive your car over regularly spaced bumps (mathematically it is the same as the car being stationary and the road moving underneath it) then it is the bumps that are doing the forcing.

We can solve such differential equations using the methods of this course if $f(t)$ is suitable. Let's solve

$$m\frac{d^2y}{dt^2} + ky = F_0 \cos \omega t$$

where F_0 is a constant. The forcing is represented by $F_0 \cos \omega t$ and has frequency ω.

The complementary solution is easily seen to be

$$y_c = A \cos \omega_0 t + B \sin \omega_0 t \qquad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

For the particular solution, let's try

$$y_p = C \cos \omega t + D \sin \omega t$$

Substituting this in:

$$m(-C\omega^2 \cos \omega t - D\omega^2 \sin \omega t) + k(C \cos \omega t + D \sin \omega t) = F_0 \cos \omega t$$

i.e.

$$C(k - m\omega^2) \cos \omega t + D(k - m\omega^2) \sin \omega t = F_0 \cos \omega t$$

Comparing $\cos \omega t$ and $\sin \omega t$ terms gives $D = 0$ and

$$C = \frac{F_0}{k - m\omega^2} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

The solution of the differential equation is therefore

$$y = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

This solution has two frequencies associated with it, the *natural* frequency ω_0 and the *forcing* frequency ω and the solution breaks down when the two are equal, i.e. $\omega_0 = \omega$. The case when $\omega_0 = \omega$ is the *resonance* case; the above solution breaks down and we must find another solution appropriate to this case.

When $\omega = \omega_0$ the particular solution table indicates that we should try

$$y_p = C\,t\,\cos \omega_0 t + D\,t\,\sin \omega_0 t$$

The calculations involved are similar to those of Example 5.17 and we find that

$$C = 0, \qquad D = \frac{F_0}{2\omega_0 m}$$

The solution in the case of resonance is therefore

$$y = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{2\omega_0 m}\,t\,\sin \omega_0 t$$

6.5 Resonance in a car

A car is being driven over evenly spaced bumps. For simplicity, we assume that the car has only one wheel (or two at the ends of the same axle) and that, although the suspension is fine, the shock absorber is broken or disconnected so that there is no mechanism for damping out oscillations. We shall assume that the road is described by the equation

$$y = a \cos \frac{2\pi s}{L}$$

The reason for writing it like this, is that L can be interpreted as the distance between successive bumps.

We want to find the velocity v with which the car must be driven so that resonance occurs. Letting k be the spring constant for the cars suspension, m the mass of the car, x the vertical displacement of the car and v the car's speed, then it can be shown that $x(t)$ satisfies the differential equation

$$m\frac{d^2x}{dt^2} + kx = ka \cos \frac{2\pi vt}{L}$$

The complementary solution is

$$x_c(t) = A \cos\left(\sqrt{\frac{k}{m}}\, t\right) + B \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

It is not necessary to do any heavy analysis to determine the resonance velocity. Resonance occurs when the "nice" particular solution fails to work. The "nice" particular solution would be

$$x_p(t) = C \cos \frac{2\pi vt}{L} + D \sin \frac{2\pi vt}{L}$$

It fails when the complementary solution already contains such terms, i.e. when

$$\frac{2\pi v}{L} = \sqrt{\frac{k}{m}}$$

or

$$v = \frac{L}{2\pi}\sqrt{\frac{k}{m}}$$

This is the velocity with which the car must be driven, to make resonance occur.