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FORMALLY GRADIENT REACTION-DIFFUSION SYSTEMS IN $\mathbb{R}^n$ HAVE ZERO SPATIO-TEMPORAL TOPOLOGICAL ENTROPY.

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Abstract. We prove that the spatio-temporal topological entropy (= the topological entropy per unit volume) is equal to zero for formally gradient reaction-diffusion systems in $\mathbb{R}^n$. This result generalizes the well-known fact that gradient ODEs have zero topological entropy.

1. **Introduction.** We consider the following spatially homogeneous reaction-diffusion system in $\mathbb{R}^n$:

$$\partial_t u = a\Delta_x u - (\vec{L}, \nabla_x)u - \lambda_0 u - f(u), \quad x \in \mathbb{R}^n, \quad u\big|_{t=0} = u_0, \tag{1}$$

where $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to x, a is a given diffusion matrix which has positive symmetric part $(a + a^* > 0)$, $\vec{L} \in \mathbb{R}^n$ is a given constant vector,

$$(\vec{L}, \nabla_x)u := \sum_{l=1}^{n} L_i \partial_{x_i} u,$$

$\lambda_0 > 0$ is a given constant and $f(u)$ is a given nonlinear interaction function which is assumed to satisfy the following conditions:

$$\begin{cases} 1. \ f \in C^2(\mathbb{R}^k, \mathbb{R}^k), \\ 2. \ f(v).v \geq -C, \quad f'(v) \geq -K, \quad \forall v \in \mathbb{R}^k, \\ 3. \ |f(v)| \leq C(1 + |v|^p), \quad p < p_{max} := 1 + \frac{4}{n-4} \ (\text{if } n > 4), \end{cases} \tag{2}$$

where $w.v$ stands for the standard inner product of the vectors v and w of $\mathbb{R}^k$.

It is well-known (see [3], [4], [19] and the references therein) that, in many cases, the behavior of the solutions of (1) can be described in terms of global attractors of the corresponding semigroups. In particular, in case of equations (1) in *bounded* domains $\Omega \subset \mathbb{R}^n$, the associated attractor $\mathcal{A}$ usually has finite Hausdorff and fractal

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dimensions and, consequently (since the semigroup $\{S_t, t \ge 0\}$ generated by equation (1) is Lipschitz continuous in the appropriate phase space), the topological entropy of the action of the semigroup S_t on the attractor $\mathcal{A}$ is also finite:

$$h_{top}(\mathcal{A}, S_t) < \infty, \tag{3}$$

exactly as in case of classical ODEs (see [3, 4], [10] and [19] for the details). Moreover, if equation (1) has a gradient form:

$$\vec{L} = 0, \quad a = a^* > 0, \quad f(v) = \nabla_v F(v), \quad F \in C^3(\mathbb{R}^k, \mathbb{R}), \tag{4}$$

then it possesses a global Liapunov function

$$L(u) := \frac{1}{2}\int_\Omega a\nabla_x u.\nabla_x u + 2F(u) + \lambda_0 u.u\,dx \tag{5}$$

and, consequently, topological entropy (3) is equal to zero.

The longtime behavior of solutions of (1) in *unbounded* domains was studied (under various assumptions on the diffusion matrix a and the nonlinear interaction function f) by many authors, see [1],[2], [5, 6, 7, 8, 9], [12, 13, 14], [17], [20, 21, 22, 23] and the references therein. In particular, it is proved in [22, 23] that, under assumptions (2), equation (1) generates a dissipative semigroup $\{S_t, t \ge 0\}$ in the phase space $\Phi := L^\infty(\mathbb{R}^n)$:

$$S_t : \Phi \to \Phi, \quad S_t u_0 := u(t), \quad \text{where } u(t) \text{ solves (1)}. \tag{6}$$

Moroeover, this semigroup possesses a (locally compact) global attractor $\mathcal{A}$ in the phase space Φ (by definition, this attractor is bounded in $L^\infty(\mathbb{R}^n)$, compact in the local topology of $\Phi_{loc} := L^\infty_{loc}(\mathbb{R}^n)$ and attracts the images of any bounded subset of Φ also in a local topology of Φ_{loc}, which is natural for the attractors theory in unbounded domains, see [7, 8] and [13, 14]). It is also shown there that the attractor $\mathcal{A}$ is bounded in $C^3_b(\mathbb{R}^n)$ and is compact in $C^3_{loc}(\mathbb{R}^n)$:

$$\|\mathcal{A}\|_{C^3_b(\mathbb{R}^n)} \le C < \infty, \quad \mathcal{A} \subset\subset C^3_{loc}(\mathbb{R}^n). \tag{7}$$

We however note that, in contrast to the case of bounded domains, the attractor $\mathcal{A}$ of equation (1) usually has infinite Hausdorff and fractal dimensions, see [2], [5, 6, 7, 8] and [20, 21, 22, 23, 15] (see also [1, 2], [7] and [12] for some particular cases of equation (1) in unbounded domains where the dimension of the attractor remains finite). Consequently, there are no reasons to expect that the topological entropy of semigroup (6) on the attractor is always finite. Moreover, as shown in [23], this value is indeed infinite for many interesting (from the applications point of view) examples of equations (1).

In order to introduce the finite quantitative characteristics of the dynamics on the attractor, we first note that, due to the spatial homogeneity of equation (1), a group of spatial translations $\{T_h, h \in \mathbb{R}^n\}$ acts on the attractor $\mathcal{A}$:

$$T_h\mathcal{A} = \mathcal{A}, \quad (T_h u_0)(x) := u_0(x+h), \quad \forall h \in \mathbb{R}^n \text{ and } u_0 \in \mathcal{A}. \tag{8}$$

Moreover, this group, obviously, commutes with temporal dynamics (6) and, consequently, we may define an action of the extened $(n+1)$-parametrical semigroup $\{\mathbb{S}_{(t,h)}, t \in \mathbb{R}_+, h \in \mathbb{R}^n\}$ as follows:

$$\mathbb{S}_{(t,h)} := S_t \circ T_h, \quad \mathbb{S}_{(t,h)}\mathcal{A} = \mathcal{A}, \quad \forall t \in \mathbb{R}_+ \text{ and } \ h \in \mathbb{R}^n. \tag{9}$$

Following [23], it is natural to describe the spatio-temporal complexity of the attractor $\mathcal{A}$ in terms of the dynamical properties of semigroup (9) (which is interpreted as a dynamical system with multidimensional 'time' $(t,h) \in \mathbb{R}_+ \times \mathbb{R}^n$ acting on the attractor). In particular, it is natural to consider the topological entropy $h_{top}(\mathcal{A}, \mathbb{S}_{(t,h)})$ of the action of semigroup (9) on the attractor (= spatio-temporal topological entropy of the attractor $\mathcal{A}$, see §2 for the rigorous definition). Moreover, as proved in [23], this value is always finite for the attractors of equations (1) in $\mathbb{R}^n$:

$$h_{top}(\mathcal{A}, \mathbb{S}_{(t,h)}) \le C < \infty \tag{10}$$

(in contrast to usual temporal topological entropy (5) which is usually infinite in the case of unbounded domains, see also [15] for some examples of equations of the form (1), for which this value is strictly positive). We also note that quantity (10) coincides with the so-called topological entropy per unit volume which has been earlier introduced in [6] for studying some particular cases of equations (1).

The aim of the present article is to prove that quantity (10) equals zero identically for the case of formally gradient systems of the form (1). To be more precise, the main result of the paper is the following theorem.

Theorem 1. *Let the nonlinear interaction function f satisfy conditons (2) and, in addition, the following assumptions hold:*

$$a = a^* > 0, \quad \lambda_0 > 0, \quad f(u) = \nabla_u F(u), \quad F \in C^3(\mathbb{R}^k, \mathbb{R}). \tag{11}$$

Then, the spatio-temporal topological entropy of the attractor $\mathcal{A}$ associated with equation (1) equals zero identically:

$$h_{top}(\mathcal{A}, \mathbb{S}_{(t,h)}) = 0. \tag{12}$$

We note that, in contrast to the case of bounded domains, functional (5) is equal to infinity, for generic $u_0 \in \mathcal{A}$, and, consequently, we do not have the global Liapunov function on the attractor. Instead, our proof of identity (12) is based on the fact that (under the assumptions of the theorem) equation (1) generates a dynamical system (DS) on the space of all spatially invariant probabilistic Borel measures and this DS possesses a global Liapunov function (which was established in a slightly different form in [9] and [17]). This fact allows to prove that the measure-theoretical analogue of entropy (10) equals zero for any invariant Borel measure μ and verify (12) using the multidimensional analogue of the variational principle, see [16] and [18].

To conclude, we note that the usual temporal topological entropy $h_{top}(\mathcal{A}, S_t)$ is not necessarily equal to zero under the assumptions of Theorem 1. Indeed, let us consider the following one dimensional scalar Chafee-Infante equation perturbed by sufficiently large convective term:

$$\partial_t u = \partial_x^2 u - L\partial_x u + u - u^3, \quad x \in \mathbb{R}^1, \tag{13}$$

where $L > 2$. Then, it is known that the temporal dynamics generated by (13) on the attractor can be desribed in terms of the embeddings of Bernoulli shifts dynamics with the continuous set of symbols $\omega \in [0, 1]$, see [23] for the details. Consequently, quantity (3) equals infinity for this case. On the other hand, equation (13) satisfies the assumptions of Theorem 1 and, therefore, the spatio-temporal entropy of the attractor is equal to zero:

$$h_{top}(\mathcal{A}, S_t) = \infty, \quad \text{but} \quad h_{top}(\mathcal{A}, \mathbb{S}_{(t,h)}) = 0. \tag{14}$$

We also note that the concrete form of assumptions (2) is used only in order to verify that equation (1) generates a dissipative semigroup (6) in the phase space $\Phi := L^\infty(\mathbb{R}^n)$. If this fact is a priori known, for some equation of the form (1), then we obtain identity (12) without assumptions (2) on the nonlinear term f.

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2. **Definitions and the proof of the main theorem.** In this section, we recall the definition of the spatio-temporal topological entropy and give the proof of Theorem 1 which was formulated in the introduction. We first introduce the Kolmogorovs entropy of a compact set in a metric space.

Definition 1. Let K be a compact set in a metric space (M, d_M). Then, due to the Hausdorff criterium, for every positive $\varepsilon > 0$, the set K can be covered by a finite number of ε-balls in M. Let $N_\varepsilon(K, M)$ be the minimal number of such balls. Then, by definition, the Kolmogorov's ε-entropy of K in M is the following number:

$$\mathbb{H}_\varepsilon(K, M) = \mathbb{H}_\varepsilon(K, d_M) := \log_2 N_\varepsilon(K, M) \tag{15}$$

(see [11] for the details).

Now we are ready to give the definition of $h_{top}(\mathcal{A}, \mathbb{S}_{(t,h)})$.

Definition 2. Let $d(u, v)$ be an arbitrary metric which generates the local topology of $L^\infty_{loc}(\mathbb{R}^n)$ on the attractor. For instance, let

$$d(u, v) := \|u - v\|_{L^\infty_{e^{-|x|}}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \{e^{-|x|}|u(x) - v(x)|\}. \tag{16}$$

Then, by definition, $(\mathcal{A}, d)$ is a compact metric space. Let us define, for every $R \in \mathbb{R}_+$, a new metric d_R on $\mathcal{A}$ as follows:

$$d_R(u, v) := \sup_{(t,h) \in [0,R]^{n+1}} d(\mathbb{S}_{(t,h)}u, \mathbb{S}_{(t,h)}v). \tag{17}$$

Then, obviously, $(\mathcal{A}, d_R)$ is also a compact metric space and, consequently, the Kolmohgorov's ε-entropy $\mathbb{H}_\varepsilon(\mathcal{A}, d_R)$ is well defined, for every $\varepsilon > 0$ and $R \in \mathbb{R}_+$. By definition, the spatio-temporal topological entropy of the attractor $\mathcal{A}$ is the following number:

$$h_{top}(\mathcal{A}, \mathbb{S}_{(t,h)}) := \lim_{\varepsilon \to 0} \limsup_{R \to \infty} \frac{1}{R^{n+1}} \mathbb{H}_\varepsilon(\mathcal{A}, d_R) \tag{18}$$

(see [10] for the details).

It is well-known that quantity (18) is independent of the choice of the metric d on the attractor. Moreover, the following proposition (which is proved in [22, 23]) gives more convenient formula for its computation in terms of the Kolmogorov's ε-entropy.

Proposition 1. *Quantity (18) can be computed as follows:*

$$h_{top}(\mathcal{A}, \mathbb{S}_{(t,h)}) = \lim_{\varepsilon \to 0} \lim_{R \to \infty} \frac{1}{R^{n+1}} \mathbb{H}_\varepsilon(\mathcal{K}, L^\infty([0, R]^{n+1}), \tag{19}$$

where $\mathcal{K} \subset L^\infty(\mathbb{R}^{n+1})$ *is the set of all solutions of (1) which are defined for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$.

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. We first note that the vector $\vec{L}$ in equation (1) can be transformed into the following form: $\vec{L} := (L, 0, \cdots, 0)$ by the appropriate rotation of the x-coordinate. Futhermore, using the following travelling wave change of the independent variables: $(t, x) \to (t', x')$ $(t := t' - Lx_1,\ x = x')$, we transform equation (1) into an analogous equation, for $\vec{L} \equiv 0$. Since the spatio-temporal topological entropy is obviously preserved under this change of variables then, it is sufficient to prove the theorem for the case $\vec{L} = 0$ only.

Let us consider the space $\mathbb{M}(\mathcal{A})$ of all Borel probability measures on $\mathcal{A}$ which are invariant with respect to the spatial translations $\{T_h, h \in \mathbb{R}^n\}$. This space is not empty since the attractor $\mathcal{A}$ is compact in the local topology of Φ_{loc}. Equation (1) generates a semigroup on the space $\mathbb{M}(\mathcal{A})$ by the following standard expression:

$$S_t^* : \mathbb{M} \to \mathbb{M}, \quad (S_t^* \mu)(B) := \mu\left(S_t^{-1} B\right). \tag{20}$$

The proof of identity (12) is based on the following lemma.

Lemma 1. *Let the assumptions of Theorem 1 hold and $\vec{L} = 0$. Then, DS (20) possesses the following global Liapunov function:*

$$\mathcal{L}(\mu) = \int_{u_0 \in \mathcal{A}} [a\nabla_x u_0(x_0).\nabla_x u_0(x_0) + \lambda_0 u_0(x_0).u_0(x_0) + 2F(u_0(x_0))]\mu(du). \tag{21}$$

In particular $\mathcal{L}(\mu)$ is independent of $x_0 \in \mathbb{R}^n$.

Proof. Since the diffusion matrix a is selfadjoint then (as proved in [22, 23]) the operator $S_t : \mathcal{A} \to \mathcal{A}$ is a homeomorphism (in the $L^\infty_{loc}(\mathbb{R}^n)$-topology), for every $t \geq 0$. Using now the smoothing property $S_t : L^\infty_{e^{-|x|}}(\mathbb{R}^n) \to C^3_{e^{-|x|}}(\mathbb{R}^n)$, we may verify in a standard way (see [23] for the details) that the local topologies induced on the attractor $\mathcal{A}$ by the embeddings $\mathcal{A} \subset L^\infty_{loc}(R^n)$ and $\mathcal{A} \subset C^3_{loc}(\mathbb{R}^n)$ coincide. Therefore, expression (21) has a sense and finite (we recall that the measure μ is assumed to be Borel). Moreover, since $\mu \in \mathbb{M}(\mathcal{A})$ is invariant with respect to the spatial translations, then expression (21) is independent of $x_0 \in \mathbb{R}^n$. Let us now prove that (21) is a Liapunov function. Indeed, let $\mu \in \mathbb{M}(\mathcal{A})$. Then

$$\mathcal{L}(t) := \mathcal{L}(S_t^* \mu) = \int_{u_0 \in \mathcal{A}} [a\nabla_x u(t,x_0).\nabla_x u(t,x_0) + \\ + \lambda_0 u(t,x_0).u(t,x_0) + 2F(u(t,x_0))]\mu(du_0), \tag{22}$$

where $u(t,x_0) := (S_t u_0)(x_0)$. Differentiating expression (22) with respect to t and using equation (1), we have

$$\frac{d}{dt}\mathcal{L}(t) = -2\int_{u_0 \in \mathcal{A}} [\partial_t u(t,x_0).\partial_t u(t,x_0)]\mu(du_0) + \\ + 2\sum_{i=1}^n \int_{u_0 \in \mathcal{A}} \partial_{x_i}[a\partial_{x_i} u(t,x_0).\partial_t u(t,x_0)]\mu(du_0). \tag{23}$$

Let us verify that the second term in the right-hand side of (23) equals zero identically. Indeed, since the measure μ is invariant with respect to the spatial translations, then

$$\int_{u_0 \in \mathcal{A}} \partial_{x_i}[a\partial_{x_i} u(t,x_0).\partial_t u(t,x_0)]\mu(du_0) = \\ \lim_{h\to 0}\frac{1}{h}\int_{u_0 \in \mathcal{A}} [a\partial_{x_i} u(t,x_0+he_i).\partial_t u(t,x_0+he_i) - a\partial_{x_i} u(t,x_0).\partial_t u(t,x_0)]\mu(du_0) \\ = \lim_{h\to 0}\frac{1}{h}\left\{\int_{u_0 \in \mathcal{A}} a\partial_{x_i} u(t,x_0).\partial_t u(t,x_0)\mu(du_0) - \right. \\ \left. \int_{u_0 \in \mathcal{A}} a\partial_{x_i} u(t,x_0).\partial_t u(t,x_0)\mu(du_0)\right\} = 0. \tag{24}$$

Integrating (23) with respect to t, we have

$$\mathcal{L}(S^*_{t_2}\mu) - \mathcal{L}(S^*_{t_1}\mu) = -2\int_{t_1}^{t_2}\int_{u_0\in\mathcal{A}} |\partial_t u(t,x_0)|^2\mu(du_0) \le 0. \tag{25}$$

Thus, (21) is a nonincreasing function along the trajectories of (20). We now assume that

$$\mathcal{L}(S^*_{t_1}\mu) = \mathcal{L}(S^*_{t_2}\mu), \tag{26}$$

for some $\mu \in \mathbb{M}(\mathcal{A})$ and $t_2 > t_1$. Let us prove that (26) implies that that the support of μ is a subset of the set $\mathcal{R} \subset \mathcal{A}$ which consists of all equilibria of equation (1):

$$\operatorname{supp}\mu \subset \mathcal{R}. \tag{27}$$

Indeed, it follows from the spatial invariance of the measure μ and expression (25) that

$$\int_{t_1}^{t_2}\int_{u_0\in\mathcal{A}} |\partial_t u(t,x)|^2\mu(du_0)\,dt = 0, \tag{28}$$

for every $x \in \mathbb{R}^n$. Thus, for μ-almost every $u_0 \in \mathcal{A}$, we have

$$\int_{t_1}^{t_2} |\partial_t u(t,x)|^2\,dt = 0, \quad \text{for every} \quad x \in \mathbb{Q}^n. \tag{29}$$

Since the function $t \to \partial_t u(t,x)$ is continuous (we recall that $\mathcal{A} \subset C^3_b(\Omega)$ and, consequently, $\partial_t u \in C_b(\mathbb{R}, C^1_b(\mathbb{R}^n))$, for every trajectory $u(t)$ belonging to the attractor), then (29) implies that, for μ-almost every $u_0 \in \mathcal{A}$, we have $\partial_t u(t,x) \equiv 0$, for $t \in [t_1,t_2]$ and every $x \in \mathbb{Q}^n$. Since $x \to \partial_t u(t,x)$ is also continuous, then we finally derive that $\partial_t u(t,x) \equiv 0$, for all $(t,x) \in [t_1,t_2]\times\mathbb{R}^n$ and, consequently, $u_0 \in \mathcal{R}$. Thus, we have proved that

$$\mu(\mathcal{R}) = 1 \tag{30}$$

which implies embedding (27). There only remains to note that every measure which satisfies (30) is, obviously, an equilibrium of DS (20) and Lemma 1 is proved. □

We are now ready to finish the proof of Theorem 1. Indeed, let μ be an arbitrary Borel probability measure on $\mathcal{A}$ which is invariant with respect to the extended semigroup $\mathbb{S}_{(t,h)}$. Then, it is obviously an equilibrium of DS (20) and, consequently (due to Lemma 1), this measure satisfies condition (27). Therefore, the measure-theoretical entropy $h_\mu(\mathcal{A},\mathbb{S}_{(t,h)})$ (see e.g. [10] and [16] for the rigorous definition) of the action of $\mathbb{S}_{(t,h)}$ on the attractor $\mathcal{A}$ with respect to this measure equals zero:

$$h_\mu(\mathcal{A},\mathbb{S}_{(t,h)}) = 0. \tag{31}$$

Since identity (31) is valid for every $\mathbb{S}_{(t,h)}$-invariant measure μ, then, according to the variational principle

$$h_{top}(\mathcal{A},\mathbb{S}_{(t,h)}) = \sup_\mu h_\mu(\mathcal{A},\mathbb{S}_{(t,h)}) = 0 \tag{32}$$

(see e.g. [10], [16] and [18]) and Theorem 1 is proved. □

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