WEAK SPATIALLY NONDECAYING SOLUTIONS OF 3D NAVIER-STOKES EQUATIONS IN CYLINDRICAL DOMAINS

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ABSTRACT. The weighted energy theory for the Navier-Stokes equations in 3D cylindrical domains is developed. Based on this theory, the existence of a weak solution belonging to the uniformly local phase space (without any spatial decaying assumptions), its dissipativity and existence of the so-called trajectory attractor are verified. In particular, this phase space contains the 3D Poiseuille flows.

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1. INTRODUCTION

It is well-known that the Navier-Stokes system

$$
\begin{cases} \partial_t u + (u, \nabla_x)u = \nu\Delta_x u - \nabla_x p + g, \\ \operatorname{div} u = 0, \quad u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \end{cases} \tag{1.1}
$$

in a *bounded* 2D domain $\Omega \subset\subset \mathbb{R}^2$ is well-posed and generates a dissipative semi-group $S(t)$ in the appropriate phase space (of square integrable divergent-free vector fields). It is also known that in the case of bounded 3D domains, we have only the global existence of weak solutions (without uniqueness) and local in time existence of strong solutions (with uniqueness), see [6], [27], [28] and references therein. These results are strongly based

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on the so-called energy estimate. In order to obtain this energy estimate, one multiplies equation (1.1) by u, integrate over Ω and uses the fact that the nonlinear term disappears:

$$((u,\nabla_x),u) := \int_{x\in\Omega}(u(x),\nabla_x)u(x).u(x)\,dx \equiv 0, \tag{1.2}$$

for every divergent-free vector field with Dirichlet boundary conditions.

The situation becomes much more difficult when the domain Ω is unbounded. Moreover, although there exists a highly developed theory of dissipative PDEs in unbounded domains (mainly based on the so-called weighted energy estimates, see [7]-[12], [20]-[21], [32]-[35] and references therein), during the long time, it was not clear how to apply it to the concrete Navier-Stokes problem in unbounded domains, due to several principal obstacles.

Indeed, in contrast to bounded domains, in the unbounded ones, the space of square integrable (divergent-free) vector fields is not a convenient phase space, since the assumption $u \in L^2(\Omega)$ imposes too restrictive *decay* conditions on $u(x)$ as $x \to \infty$. So, under this choice of the phase space, many classical hydrodynamical objects, like Poiseuille flows, Couette-Taylor flows, Kolmogorov flows etc. are automatically out of the consideration. Thus, following the general theory mentioned above, it is reasonable to replace the assumption $u \in L^2(\Omega)$ by more relevant one: $u \in L^2_b(\Omega)$ where the uniformly local Sobolev spaces $W^{l,p}_b(\Omega)$ are defined via the following standard expression:

$$W^{l,p}_b(\Omega) := \{u \in D'(\Omega),\quad \|u\|_{W^{l,p}_b(\Omega)} := \sup_{x_0\in\Omega}\|u\|_{W^{l,p}(\Omega\cap B^1_{x_0})} < \infty\}.$$

Here $B^1_{x_0}$ denotes the ball of radius one of $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ and $W^{l,p}$ means the classical Sobolev space, see Section 1 for details. But here arises the main difficulty: how to obtain a priori estimates for the solution $u(t)$ in the uniformly local spaces?

Indeed, since $u(t)$ is not square integrable any more, we cannot multiply (1.1) simply by u and use identity (1.2) (the integrals do not have sense). So, following the general strategy, we need to multiply it by ϕu where $\phi = \phi(x)$ is an appropriate *weight* function. But in that case the nonlinear term does not vanish and produce the additional cubic term like $\phi' u^3$. We note that this cubic term *is not sign-defined* and the rest terms in the energy equality are at most quadratic with respect to u, so it was not clear how to control this cubic term in order to produce reasonable a priori estimate.

Another obstacle is related with the fact that ϕu is not *divergent free*, so the pressure p does not disappear in the weighted energy equality and one should be also able to control the term $(\phi' p, u)$. Of course, this problem is closely related with finding the reasonable extension of the Helmholtz projector (to divergent free vector fields) to uniformly local spaces.

The above mentioned difficulties stimulated the developing of the alternative methods to handle the Navier-Stokes equations in unbounded domains. In particular, in the 2D case, very helpful is the so-called vorticity equation

$$\partial_t\omega - \Delta_x\omega + (u,\nabla_x)\omega = \partial_{x_2}g_1 - \partial_{x_1}g_2 \tag{1.3}$$

where $\omega := \partial_{x_2}u_1 - \partial_{x_1}u_2$. Indeed, if Ω does not contain boundary, e.g. $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{S}^1 \times \mathbb{R}$ where $\mathbb{S}^1$ is a circle (like in the Kolmogorov problem), the maximum principle applied to (1.3) allows to obtain global a priori estimate for the vorticity ω which, together with the accurate analysis of the explicit formulas for the Helmholtz projectors, allow to obtain the global in time a priori estimates for the solution $u(t)$ and, thus, to prove the global solvability of the Navier-Stokes equation in the uniformly local phase spaces, see [2] and [14]. Unfortunately, a priori estimate for vorticity obtained from the maximum principle grows linearly in time, so all of the further estimates will also growing in time (to the best of our knowledge, for the case $\Omega = \mathbb{R}^2$, it gives double exponential ($\sim e^{Ce^{Ct}}$) growth rate and polynomial ($\sim t^3$) growth rate for $\Omega = \mathbb{S}^1 \times \mathbb{R}$). The other essential drawback is that this method seems to be non-applicable to the problems with boundary, e.g. in the case where Ω is a cylindrical domain and does not work in 3D case.

Another attractive possibility to avoid direct weighted energy estimates is to use the bifurcation analysis. Indeed, in the situation where the basic steady state of the Navier-Stokes problem is slightly above the instability threshold, the solutions remaining close to that steady state can be described in terms of the so-called *modulation* equations which are essentially simpler than the initial Navier-Stokes problem (usually it is Ginzburg-Landau or Swift-Hohenberg equations), see [1], [15]-[17], [19] and references therein. Since the well-posedness and dissipativity of these modulation equations is well-understood, the standard perturbation methods allow sometimes to obtain global in time estimates for solutions of the initial Navier-Stokes problem starting from the small neighborhood of the basic steady state. In particular, the global existence and dissipativity of such solutions for the 3D Couette-Taylor flow is obtained in [23] and "almost global solvability" (on the exponentially long with respect to perturbation parameter time interval) for the case of Poiseuille flow can be found in [24].

It is worth to emphasize that, in the case where the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ possesses the Friedrich's inequality

$$\|u\|^2_{L^2(\Omega)} \le \lambda_1 \|\nabla_x u\|^2_{L^2(\Omega)}, \quad u \in W^{1,2}_0(\Omega) \tag{1.4}$$

with *positive* λ_1 and under the restrictive assumption that u *is square integrable*, all of the above mentioned obstacles disappear and Navier-Stokes problem (1.1) possesses a standard (unweighted) energy theory similar to the case of bounded domains, see [5], [28]. We also mention the survey paper [3] on existence of spatially decaying solutions of the Navier-Stokes problem in various domains (not necessarily satisfying (0.4)), see also [13] and [30].

However, the above mentioned obstacles for applying the general weighted energy theory to Navier-Stokes equations in unbounded domains have been recently overcome in [37] for the case of 2D cylindrical domains. This allowed to verify the global existence, uniqueness and dissipativity of the 2D Navier-Stokes equations in the classes of spatially non-decaying solutions. Moreover, this result embeds the 2D Navier-Stokes problem in a strip into a general scheme of investigating dissipative PDEs in unbounded domains mentioned above, including the study of the dimension and Kolmogorov's entropy of attractors, topological entropies, spatial and temporal chaos, etc., see [36].

The main aim of this paper is to extend (of course, up to uniqueness and further regularity) this result to the case of 3D cylindrical domains. Although the general strategy of the paper is similar to [37], there are several essential differences and complications in comparison to the 2D case. Namely, in the 3D case we do not have a scalar stream function and, consequently, we cannot reduce the study of the Helmholtz projector and

Stokes operator in weighted spaces to simple model problems for the Laplace and bi-Laplace equations and should use the theory of general elliptic problems.

Next, due to the lack of uniqueness for the 3D Navier-Stokes equations, we cannot directly apply the methods of [37] to them, but should first consider the regularizing Leray approximations to the Navier-Stokes equations, prove the existence of spatially non-decaying solutions for that equations and, after that, obtain the required solution by passing to the limit.

Finally, again due to the lack of uniqueness, we cannot construct a usual *global* attractor for the problem considered and should use the so-called *trajectory* approach, see [8, 25, 31, 9] and references therein.

The paper is organized as follows. We recall in Sections 2 and 3 some basic facts on the theory of weighted spaces and the regularity of elliptic boundary value problems in these spaces which will be systematically used throughout of the paper.

Section 4 is devoted to study the Helmholtz projector Π and stationary Stokes equations in weighted and uniformly local Sobolev spaces. The results of this section are somehow close to [4] and [5] (and are, factually, inspired by these papers).

In Section 5, we study the auxiliary linear non-divergent free problem

$$-\partial_t v = \Delta_x v + \nabla_x q, \quad \Pi v\big|_{t=T} = 0, \quad \operatorname{div} v = \phi' u, \quad v\big|_{\partial\Omega} = 0 \tag{1.5}$$

where $\phi(x)$ is the appropriate weight function and $u(t)$ is a solution of the Navier-Stokes problem. This auxiliary problem is necessary in order to overcome the obstacle related with the appearance of the term containing pressure in the weighted energy equality. Roughly speaking, we will multiply equation (1.1) by the function $\phi u(t) - v(t)$ where v solves (1.5). Then, since $\operatorname{div}(\phi u - v) = 0$ the pressure term disappears (and the derivative of our weights will be small, so the corrector v will be also small and do not produce any essential difficulties in its estimating, see Sections 5 and 6 for the details).

We note that it is not clear how to overcome this obstacle in more simple way. Indeed, the "most natural" multiplication by $\Pi(\phi u)$ does not work since $\Pi(\phi u)$ has nonzero trace at the boundary which leads to additional uncontrollable boundary terms under the integration by parts in $(\Delta_x u, \Pi(\phi u))$. Another possibility is to construct a new "projector" Q to divergent free vector fields which preserves the boundary conditions and multiply the equation by $Q(\phi u)$. This, however, leads to essential difficulties with the term $(\partial_t u, Q(\phi u))$ which should be a complete time derivative from something. We also note that the multiplication of the equation by the combination of $\phi\partial_t u$ and $\phi\Pi\Delta_x u$ (as in [4] and [5]) is useless for us, since it works *only* if the unweighted L^2-norm of $\Delta_x u$ is a priori known.

In Section 6, we verify the basic (uniform with respect to $\alpha \to 0$) a priori estimate and prove the global existence of solutions of the following Leray-Navier-Stokes problem:

$$\begin{cases} \partial_t u + (\Pi w, \nabla_x) u + c\partial_{x_1} u = \Delta_x u - \nabla_x p + g, \\ w - \alpha\Delta_x w = u, \\ \operatorname{div} u = 0, \quad \mathbb{S} u_1 = c, \\ u\big|_{\partial\Omega} = w\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \end{cases} \tag{1.6}$$

where Π is a Helmholtz projector to the divergent-free vector fields, $\alpha > 0$ is a small parameter, $\mathbb{S}$ is the averaging operator with respect to the cross section $x' := (x_2, x_3)$:

$$\mathbb{S}v := \frac{1}{|\omega|}\int_{x'\in\Omega} v(x')\,dx'$$

and c is a given constant.

The additional projector Π is necessary since, in contrast to the spatially periodic case, w is no more divergent free and we will not have zero integral analogous to (1.2) without this projector. The term $c\partial_{x_1}u$ appears in order to have the classical Navier-Stokes problem as $\alpha = 0$ (since, due to our choice of projector Π, the mean flux of Πw equals zero and, consequently, $\Pi u = u - (c, 0, 0)$.

In order to obtain the required estimate, we use, following [37], the special weights

$$\theta_{\varepsilon,x_0}(x) := (1 + \varepsilon^2|x - x_0|^2)^{1/2} \tag{1.7}$$

with very small ε which factually depends on the solution u considered. Then, the careful analysis of the obtained weighted energy inequality allows us to obtain the globally in time bounded a priori estimate of the L^2_b-norm of $u(t)$. Based on this a priori estimate, we then establish the existence of such solution. In a fact, we first consider the case of zero flux $c = 0$ (see Theorem 6.5) and, after that reduce the general case to that particular one using the trick with the auxiliary "energy stable" equilibrium (see Theorem 6.6).

The uniqueness of such solutions is verified in Section 7 (see Theorem 7.1). Moreover, we also verify here the uniform with respect to $\alpha \to 0$ *dissipative* estimate for that solutions and verify the existence of global attractors $\mathcal{A}_\alpha$ for the approximating problems (1.6).

Finally, in Section 8, we obtain the existence of a dissipative weak solution for the classical Navier-Stokes problem by passing to the limit $\alpha \to 0$. Moreover, the appropriate *trajectory* attractor $\mathcal{A}_{tr}$ for the Navier-stokes problem is also constructed here. Finally, using the proper scaling , we obtain the following estimate for the size of attractor in L^2_b-norm in terms of the kinematic viscosity ν:

$$\|\mathcal{A}_{tr}\|_{L^\infty(\mathbb{R}_+, L^2_b(\Omega))} \le C\nu^{-3}(|c|^3\nu + \|g\|^2_{L^2_b(\Omega)} + \nu^4) \tag{1.8}$$

where the constant C is independent of ν, c and g. We recall that in bounded domains (in square integrable case), the best known estimate is the following one:

$$\|\mathcal{A}_{tr}\|_{L^\infty(\mathbb{R}_+, L^2(\Omega))} \le C\nu^{-1}\|g\|_{L^2(\Omega)}. \tag{1.9}$$

We see that, although estimate (1.8) is "worse" than (1.9), but it remains polynomial as $\nu \to 0$ (with a reasonable degree 3). Thus, our method is not "extremely rough" and can be used in order to obtain reasonable *quantitative* bounds for the solutions.

2. Functional spaces

In this section, we briefly recall the definitions and basic properties of weight functions and weighted functional spaces which will be systematically used throughout of the paper (see also [11], [33] for more details). We start with the class of admissible weight functions.

Definition 2.1. A function $\phi \in C_{loc}(\mathbb{R}^n)$ is a weight function of exponential growth rate $\mu > 0$ if the following inequalities hold:

$$\phi(x + y) \le C_\phi \phi(x) e^{\mu|y|}, \quad \phi(x) > 0, \tag{2.1}$$

for all $x, y \in \mathbb{R}^n$.

The following proposition collects the evident properties of that weights.

Proposition 2.2. *1. Let ϕ be a weight function with exponential growth rate μ. Then, for every $\varepsilon > \mu$, ϕ is a weight function of exponential growth rate ε (with the same constant C_ϕ).*

2. Let ϕ and ψ be weight functions of exponential growth rate μ. Then the functions $\Psi_1 = \phi(x)\psi(x)$ and $\Psi_2 = \phi(x)/\psi(x)$ are weight functions of exponential growth rate 2μ with the constant $C_{\Psi_i} \le C_\phi C_\psi$.

3. Let ϕ be a weight function of exponential growth rate μ and let $\psi \in C_{loc}(\mathbb{R}^n)$ satisfies

$$C_1\phi(x) \le \psi(x) \le C_2\phi(x), \quad x \in \mathbb{R}^n. \tag{2.2}$$

Then ψ is also a weight function of exponential growth rate μ and $C_\psi \le C_1^{-1}C_2C_\phi$.

4. Let $\varepsilon > 0$ and $\phi(x)$ be a weight function of exponential growth rate μ. Then the function $\phi_\varepsilon(x) := \phi(\varepsilon x)$ is of exponential growth rate $\varepsilon\mu$ and with $C_{\phi_\varepsilon} = C_\phi$.

All of the assertions of the proposition are simple corollaries of estimate (2.1).

The natural example of such weights is the following one:

$$\phi_{\mu,x_0}(x) := e^{-\mu|x-x_0|}, \quad x_0 \in \mathbb{R}^n, \quad \mu \in \mathbb{R}. \tag{2.3}$$

Obviously, they are of exponential growth rate $|\mu|$ and the constant $C_{\phi_{\mu,x_0}} = 1$ (independent of $x_0 \in \mathbb{R}^n$). However, these weights are non-smooth at $x = x_0$. In order to overcome this drawback, it is natural to use the following equivalent weights:

$$\varphi_{\mu,x_0}(x) := e^{-\mu\sqrt{1+|x-x_0|^2}}, \quad x_0 \in \mathbb{R}^n. \tag{2.4}$$

Indeed, since $|x| \le \sqrt{x^2+1} \le |x| + 1$, then these weights satisfy

$$e^{-|\mu|}\phi_{\mu,x_0}(x) \le \varphi_{\mu,x_0}(x) \le e^{|\mu|}\phi_{\mu,x_0}(x), \quad x \in \mathbb{R}^n \tag{2.5}$$

and, consequently, φ_{μ,x_0} are also weight functions of exponential growth rate μ (with $C_{\varphi_{\mu,x_0}} = e^{2|\mu|}$). Moreover, in contrast to (2.3) these weights are smooth and satisfy, for $\mu \le 1$ the additional obvious inequality

$$|D_x^k\varphi_{\mu,x_0}(x)| \le C_k|\mu|\varphi_{\mu,x_0}(x), \quad x \in \mathbb{R}^n \tag{2.6}$$

where $k \in \mathbb{N}$, D_x^k denotes a collection of all x-derivatives of order k and the constant C_k is independent of x and μ. This inequality is crucial for obtaining the regularity estimates in weighted spaces (see [11]–[12], [32]–[35] and Section 3 below).

Another important class of weight functions is the so-called polynomial ones:

$$\theta_{x_0}^m(x) := (1 + |x - x_0|^2)^{-m/2}, \quad m \in \mathbb{R}. \tag{2.7}$$

It is not difficult to verify that these weights are of exponential growth rate μ for every $\mu > 0$ with the constant $C_{\theta_{m,x_0}}$ depending on μ and m, but independent of $x_0 \in \Omega$.

We now introduce a class of weighted Sobolev spaces in a regular unbounded domain Ω associated with weights introduced above. Since we factually need below only the case where $\Omega := \mathbb{R} \times \omega$ is a cylinder which obviously have regular boundary, in order to avoid the technicalities, we do not formulate precise assumptions on the boundary $\partial\Omega$ (which can be found e.g. in [11] or [12]).

Definition 2.3. Let Ω be a regular domain and let ϕ be a weight function of exponential growth rate. Then, for every $1 \le p \le \infty$, we set

$$L^p_\phi(\Omega) := \{u \in L^p_{loc}(\Omega),\ \ \|u\|^p_{L^p_\phi} := \int_\Omega \phi(x)^p |u(x)|^p\, dx < \infty\} \tag{2.8}$$

and

$$L^p_{b,\phi}(\Omega) := \{u \in L^p_{loc}(\Omega),\ \ \|u\|_{L^p_{b,\phi}} := \sup_{x_0\in\Omega} (\phi(x_0)\|u\|_{L^p(\Omega\cap B^1_{x_0})}) < \infty\}. \tag{2.9}$$

Here and below $B^r_{x_0}$ denotes an r-ball of $\mathbb{R}^n$ centered at x_0 and we write L^p_b instead of $L^p_{b,1}$.

Moreover, for every $l \in \mathbb{N}$, we define the weighted Sobolev spaces $W^{l,p}_\phi(\Omega)$ and $W^{l,p}_{b,\phi}(\Omega)$ as spaces of distributions whose derivatives up to order l belong to $L^p_\phi(\Omega)$ and $L^p_{b,\phi}(\Omega)$ respectively.

Furthermore, the weighted Sobolev spaces $W^{l,p}_\phi(\partial\Omega)$ and $W^{l,p}_{b,\phi}(\partial\Omega)$ on the boundary $\partial\Omega$ can be defined analogously only the integral over Ω (resp. supremum in (2.9)) in (2.8) should be naturally replaced by the integral (resp. supremum) over the boundary $\partial\Omega$, see [11], [12].

Remark 2.4. In the sequel, we will also use the functions $u(t)$ with values in the weighted Sobolev spaces defined above. In slight abuse the notations, we denote by $L^p_b(\mathbb{R}, W^{l,p}_b)$ the space, generated by the following norm:

$$\|u\|_{L^p_b(\mathbb{R},W^{l,p}_b)} := \sup_{x_0\in\Omega}\sup_{T\in\mathbb{R}} \|u\|_{L^p([T,T+1],W^{l,p}(\Omega\cap B^1_{x_0}))}. \tag{2.10}$$

The following proposition collects some useful facts on the spaces introduced before.

Proposition 2.5. *Let Ω be a regular domain and ϕ be a weight of exponential growth rate μ. Then,*

1) For every $r > 0$ and every $u \in L^p_\phi(\Omega)$, $1 \le p < \infty$,

$$C_r^{-1}\|u\|_{L^p_\phi(\Omega)} \le \left(\int_{x_0\in\Omega} \phi^p(x_0)\|u\|^p_{L^p(\Omega\cap B^r_{x_0})}\, dx_0\right)^{1/p} \le C_r\|u\|_{L^p_\phi(\Omega)} \tag{2.11}$$

where the constant C_r depends on r, μ and on the constant C_ϕ from (2.1), but is independent of ϕ and of the concrete choice of the weight ϕ.

2) For every $\alpha > \mu$, every $q \in [1,\infty]$ and every $u \in L^1_\phi(\Omega)$, we have

$$\left(\int_{x_0\in\Omega} \phi(x_0)^q \left(\int_{x\in\Omega} e^{-\alpha|x-x_0|}|u(x)|\, dx\right)^q dx_0\right)^{1/q} \le C_\alpha\|u\|_{L^1_\phi(\Omega)} \tag{2.12}$$

where the constant C_α depends on α, μ and on the constant C_ϕ, but is independent of u and of the concrete choice of ϕ and q.

3) For every $\alpha > \mu$ and every $u \in L^p_{b,\phi}(\Omega)$, we have

$$C_\alpha^{-1}\|u\|^p_{L^p_{b,\phi}(\Omega)} \le \sup_{x_0\in\Omega}\{\phi(x_0)^p \int_{x\in\Omega} e^{-\alpha p|x-x_0|}|u(x)|^p\, dx\} \le C_\alpha\|u\|^p_{L^p_{b,\phi}(\Omega)} \tag{2.13}$$

where the constant C_α depends on α, μ and on the constant C_ϕ, but is independent of u and of the concrete choice of ϕ.

The proof of that estimates is given in [11] (see also [12], [30]).

Remark 2.6. As we will see below, estimate (2.11) allows to reduce the proofs of embedding and interpolation theorems for weighted Sobolev spaces to the classical unweighted case in a bounded domain. Estimates (2.12) and (2.13) allow, in turns, to obtain the elliptic regularity in weighted spaces with *arbitrary* weights of exponential growth rate if the analogous result for the *special* weights $e^{-\alpha|x-x_0|}$ (or which is the same, for the equivalent smooth weights (2.4)) is known, see Section 3. Moreover, these estimates allow to control the dependence of the constants in embedding, interpolation and regularity theorems on the concrete choice of the weights which is crucial for our study of the non-decaying solutions of NS equations.

We need now to introduce also the weighted Sobolev spaces with fractional derivatives. To this end, we first recall that in the unweighted case the space $W^{l+s,p}(\Omega)$ for $s\in(0,1)$ and $l\in\mathbb{Z}_+$ is usually defined via

$$\|u\|^p_{W^{l+s,p}(\Omega)} := \|u\|^p_{W^{l,p}(\Omega)} + \int_{x\in\Omega}\int_{y\in\Omega}\frac{|D^l_xu(x)-D^l_xu(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \tag{2.14}$$

and, for negative l, the space $W^{l,p}(\Omega)$ is defined as a conjugate space to $W^{-l,q}_0(\Omega)$ where $1/p+1/q=1$, see [18], [29]. Then, estimate (2.11) justifies the following definition.

Definition 2.7. Let Ω be a regular domain and ϕ be a weight function of exponential growth rate. For every $1<p\le\infty$ and every $l\in\mathbb{R}$, we define the space $W^{l,p}_\phi(\Omega)$ as a subspace of distributions for which the following norm is finite:

$$\|u\|^p_{W^{l,p}_\phi(\Omega)} := \int_{x_0\in\Omega}\phi(x_0)^p\|u\|^p_{W^{l,p}(\Omega\cap B^r_{x_0})}\,dx_0 \tag{2.15}$$

where r is some positive number (it is not difficult to verify that, this space is independent of r). Analogously the norm in $W^{l,p}_{b,\phi}$ is defined via

$$\|u\|^p_{W^{l,p}_{b,\phi}(\Omega)} := \sup_{x_0\in\Omega}\{\phi(x_0)^p\|u\|^p_{W^{l,p}(\Omega\cap B^r_{x_0})}\}, \tag{2.16}$$

for simplicity, we fix below $r=1$ in definitions (2.15) and (2.16) of the weighted norms.

Indeed, according to (2.11), we see that, for $l\in\mathbb{Z}_+$ the spaces thus defined coincide with the spaces from Definition 2.1. Moreover, it is not difficult to verify, using the explicit formula (2.14) that in the unweighted case $\phi=1$, the norm (2.15) is equivalent to (2.14).

The following proposition describes the weighted negative Sobolev spaces in terms of conjugate spaces.

Proposition 2.8. *Let Ω be a regular domain and let ϕ be a weight function of exponential growth rate μ. Then, for every $l>0$, and every $1<p,q<\infty$ with $1/p+1/q=1$,*

$$W^{-l,p}_\phi(\Omega) = [W^{l,q}_{0,\phi^{-1}}(\Omega)]^* \tag{2.17}$$

where $W^{l,q}_{0,\phi}(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in the $W^{l,q}_\phi$-norm and $*$ means the conjugate space (with respect to the standard inner product in $L^2(\Omega)$). Moreover,

$$C_1\|u\|_{W^{-l,p}_\phi(\Omega)} \le \|u\|_{[W^{l,q}_{0,\phi^{-1}}(\Omega)]^*} \le C_2\|u\|_{W^{-l,p}_\phi(\Omega)} \tag{2.18}$$

where the constants C_1 and C_2 depend on μ, l, p and C_ϕ, but are independent of the concrete choice of u and C_ϕ.

Proof. In order to avoid the technicalities, we give below the proof of (2.18) only for the case of a cylindrical domain $\Omega := \mathbb{R}\times\omega$ where ω is a smooth bounded domain of $\mathbb{R}^{n-1}$ (only that case will be used in the sequel) although the slightly modified proof works for a general regular domain. In that particular case, we can restrict ourselves to consider only one dimensional weights $\phi\in C_{loc}(\mathbb{R})$. Indeed, since ω is bounded, (2.1) implies that

$$C_1\phi(s,\xi_0)\le\phi(s,\xi)\le C_2\phi(s,\xi_0),\quad s\in\mathbb{R},\quad \xi\in\omega \tag{2.19}$$

where $\xi_0\in\omega$ is some fixed point and, consequently, the weight $\phi(s,\xi)$ is equivalent to $\phi_{\xi_0}(s):=\phi(s,\xi_0)$. Moreover, it is more convenient to use, instead of balls $B^r_{x_0}$ the finite cylinders $\Omega_s := (s,s+1)\times\omega$, i.e. to define the norm in $W^{l,p}_\phi(\Omega)$ via

$$\|u\|^p_{W^{l,p}_\phi(\Omega)}=\int_{s\in\mathbb{R}}\phi(s)^p\|u\|^p_{W^{l,p}(\Omega_s)}\,ds \tag{2.20}$$

(since the norms (2.15) are equivalent for different r and ω is bounded then (2.15) and (2.20) are also equivalent).

We first verify the right inequality of (2.18). To this end, we introduce a partition of unity $\{\psi_y\}_{y\in\mathbb{R}}\in C_0^\infty(\mathbb{R})$ such that

$$\begin{cases}1.\ \ \operatorname{supp}\psi_y\subset(y,y+1),\\ 2.\ \ \int_{y\in\mathbb{R}}\psi_y(s)\,dy\equiv1,\\ 3.\ \ |D^k_s\psi_y(s)|\le C_k,\end{cases} \tag{2.21}$$

where the constant C_k is independent of $s\in\mathbb{R}$ (obviously such partition of unity exists and can be chosen in a smooth way with respect to $y\in\mathbb{R}$).

Let now $u\in[W^{l,q}_{0,\phi^{-1}}(\Omega)]^*$ be a functional over $W^{l,q}_{0,\phi^{-1}}(\Omega)$ and let v be an arbitrary test function from that space. Then, using (2.21) and Hölder inequality, we have

$$\begin{aligned}|\langle u,v\rangle|&\le\int_{y\in\mathbb{R}}|\langle u,\psi_y v\rangle|\,dy\le\int_{y\in\mathbb{R}}\|u\|_{W^{-l,p}(\Omega_y)}\|\psi_y v\|_{W^{l,q}(\Omega_y)}\,dy\le\\ &\le C\int_{y\in\mathbb{R}}\phi(y)\|u\|_{W^{-l,p}(\Omega_y)}\cdot\phi(y)^{-1}\|v\|_{W^{l,q}(\Omega_y)}\,dy\le C\|u\|_{W^{-l,p}_\phi(\Omega)}\|v\|_{W^{l,q}_{\phi^{-1}}(\Omega)}\end{aligned} \tag{2.22}$$

which, together with the definition of the norm in a conjugate space gives the right-hand side of inequality (2.18).

Let us now verify the left-hand side of that inequality. Indeed, let $u\in W^{-l,p}_\phi(\Omega)$. We fix a family of functions $v_y\in W^{l,q}_0(\Omega_y)$, such that

$$\langle u,v_y\rangle=\|u\|_{W^{-l,p}(\Omega_y)}\|v_y\|_{W^{l,q}(\Omega_y)} \tag{2.23}$$

and normalize these functions as follows:

$$\|v_y\|_{W^{l,q}(\Omega_y)}=\phi(y)^p\|u\|^{p-1}_{W^{-l,p}(\Omega_y)}. \tag{2.24}$$

Since the spaces $W^{l,q}(\Omega_y)$ are uniformly convex, these family are uniquely defined and, moreover, continuous with respect to $y\in\mathbb{R}$.

Let us define also the function $v(x)$ as follows

$$v(x) := \int_{y\in\mathbb{R}} v_y(x)\,dy. \tag{2.25}$$

We claim that $v \in W^{l,q}_{0,\phi^{-1}}(\Omega)$. Indeed, since $v_y \in W^{l,q}_0(\Omega_y)$, it can be naturally continued by zero to the function $v_y \in W^{l,q}_0(\Omega)$ with $\operatorname{supp} v_y \subset \Omega_y$. Thus, the integral (2.25) is well posed and defines a function $v \in W^{l,q}_{loc}(\overline{\Omega})$ vanishing at the boundary $\partial\Omega$. So, we only need to estimate the $W^{l,q}_{\phi^{-1}}(\Omega)$-norm of it.

Using now that $\|v_y\|_{W^{l,q}(\Omega_s)} = 0$ if $|s-y| \ge 1$, we have

$$\begin{aligned}\|v\|_{W^{l,q}(\Omega_s)} \le \int_{|s-y|\le 1} \|v_y\|_{W^{l,q}(\Omega_y)}\,dy = \int_{|s-y|\le 1} \phi(y)^p \|u\|^{p-1}_{W^{-l,p}(\Omega_y)}\,dy \le \\ \le C\phi(s)^p \int_{|s-y|\le 1} \|u\|^{p-1}_{W^{-l,p}(\Omega_y)}\,dy \le C_1\phi(s)^p \int_{y\in\mathbb{R}} e^{-\alpha|s-y|}\|u\|^{p-1}_{W^{-l,p}(\Omega_y)}\,dy\end{aligned} \tag{2.26}$$

where the constant $\alpha > 2p\mu/q$ can be arbitrary (here we have implicitly used (2.1) in order to estimate $\phi(y)$ via $\phi(s)$). Taking the q-th power from the both sides of that relation, applying the Hölder inequality and using that $q(p-1) = p$, we arrive at

$$\phi(s)^{-q}\|v\|^q_{W^{l,q}(\Omega_s)} \le C\phi(s)^p \int_{y\in\mathbb{R}} e^{\alpha q|s-y|/2}\|u\|^p_{W^{-l,p}(\Omega_y)}\,dy.$$

Integrating this relation over $s \in \mathbb{R}$ and using (2.12), we finally infer

$$\|v\|^q_{W^{l,q}_{\phi^{-1}}(\Omega)} \le C_2\|u\|^p_{W^{-l,p}_\phi(\Omega)}. \tag{2.27}$$

We are now ready to finish the proof of the proposition. Indeed, due to (2.23)–(2.25), we have

$$\langle u, v\rangle = \int_{y\in\Omega} \|u\|_{W^{-l,p}(\Omega_y)}\|v_y\|_{W^{l,q}(\Omega_y)}\,dy = \|u\|^p_{W^{-l,p}_\phi(\Omega)}$$

and, consequently, due to (2.27),

$$\|u\|_{[W^{l,q}_{0,\phi^{-1}}(\Omega)]^*} \ge \frac{\langle u, v\rangle}{\|v\|_{W^{l,q}_{\phi^{-1}}(\Omega)}} \ge C\|u\|^{p(1-1/q)}_{W^{-l,p}_\phi(\Omega)}. \tag{2.28}$$

Since $p(1-1/q) = 1$, then (2.28) implies the left-hand side of inequality (2.18). Proposition 2.8 is proved. □

Remark 2.9. Proposition 2.8 shows, in particular, that in the case $\phi = 1$, the spaces $W^{l,p}(\Omega)$ introduced in Definition 2.7, coincide with the standard Sobolev spaces for any $l \in \mathbb{R}$. Moreover, arguing analogously to the proof of Proposition 2.8, one can verify the interpolation representation of the weighted spaces $W^{l+\alpha,p}_\phi(\Omega)$ with fractional derivatives ($l \in \mathbb{Z}$, $\alpha \in (0,1)$)

$$W^{l+\alpha,p}_\phi(\Omega) = \left(W^{l,p}_\phi(\Omega), W^{l+1,p}_\phi(\Omega)\right)_{\alpha,p} \tag{2.29}$$

in a complete analogy with the unweighted case, see e.g. [29].

For the convenience of the reader, we show below how to obtain the weighted analogues of interpolation and embedding inequalities.

Proposition 2.10. *Let Ω be a regular domain, ϕ_1 and ϕ_2 be two weight functions of exponential growth rate μ, $0 \le l_1, l_2 < \infty$, $1 < p_1, p_2 < \infty$. Let also $\theta \in [0,1]$ be arbitrary and*

$$l := \theta l_1 + (1-\theta) l_2, \quad \frac{1}{p} := \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \phi := \phi_1^{\theta} \cdot \phi_2^{1-\theta}$$

Then, $W^{l,p}_{\phi}(\Omega) \subset W^{l_1,p_1}_{\phi_1}(\Omega) \cap W^{l_2,p_2}_{\phi_2}(\Omega)$ and the following estimate holds:

$$\|u\|_{W^{l,p}_{\phi}} \le C \|u\|^{\theta}_{W^{l_1,p_1}_{\phi_1}} \cdot \|u\|^{1-\theta}_{W^{l_2,p_2}_{\phi_2}}$$

where the constant C depends on l_i, p_i, μ, C_{ϕ_i} and on some regularity constant of the domain Ω, but is independent of the concrete choice of the weights ϕ_i and of the form of the domain Ω. Moreover, the analogous estimate holds for the spaces $W^{l,p}_{b,\phi}$ as well.

Proof. As in the proof of Proposition 2.8, we restrict ourselves to consider only the case of a cylindrical domain $\Omega := \mathbb{R} \times \omega$, one dimensional weights and the equivalent norms (2.20). Moreover, we will consider below only the case of spaces $W^{l,p}_{\phi}$ (the spaces $W^{l,p}_{b,\phi}$ can be considered analogously).

Indeed, according to the standard unweighted interpolation inequality for domains Ω_s, we have

$$\|u\|^p_{W^{l,p}(\Omega_s)} \le c_1 \|u\|^{p\theta}_{W^{l_1,p_1}(\Omega_s)} \|u\|^{p(1-\theta)}_{W^{l_2,p_2}(\Omega_s)}$$

where the constant C_1 is independent of s, see [29]. Multiplying this inequality by the weight $\phi^p(s)$, integrating over $s \in \mathbb{R}$ and using (2.11), we get

$$\|u\|^p_{W^{l,p}_{\phi}(\Omega)} \le C_2 \int_{s\in\mathbb{R}} \left(\phi_1 \|u\|_{W^{l_1,p_1}(\Omega_s)}\right)^{p\theta} \left(\phi_2 \|u\|_{W^{l_2,p_2}(\Omega_s)}\right)^{p(1-\theta)} ds.$$

Applying the Hölder inequality with exponents $\frac{p_1}{p\theta}$ and $\frac{p_2}{p(1-\theta)}$ to the right-hand side of this inequality and using estimate (2.11) once more, we deduce the required weighted interpolation inequality and finish the proof of the proposition. ☐

The next proposition gives the weighted analogue of embedding and trace inequalities.

Proposition 2.11. *Let Ω be a regular domain and ϕ be a weight function of exponential growth rate μ. Then*

1) For every $1 < p_1 \le p_2 < \infty$ and every $0 \le l_2 \le l_1$ satisfying

$$\frac{1}{p_2} - \frac{l_2}{n} \ge \frac{1}{p_1} - \frac{l_1}{n}, \tag{2.30}$$

there is a continuous embedding $W^{l_1,p_1}_{\phi}(\Omega) \subset W^{l_2,p_2}(\Omega)$ and the norm of the embedding operator depends on l_i, p_i, μ and C_ϕ, but is independent of the concrete form of the weight function ϕ. If the inequality (2.30) is strict, then we can take also $p_2 = \infty$.

2) For every $m \in \mathbb{Z}_+$, $1 < p < \infty$ and $l > m + 1/p$ the trace operator $\Pi^m_{\partial\Omega}$

$$\Pi^m_{\Omega} u := (u\big|_{\partial\Omega}, \partial_n u\big|_{\partial\Omega}, \cdots, \partial^m_n u\big|_{\partial\Omega}) \tag{2.31}$$

(where $\partial_n u$ denotes the normal derivative of the function u at the boundary $\partial\Omega$) maps $W^{l,p}_{\phi}(\Omega)$ to $\otimes^m_{k=0} W^{l-k-1/p,p}_{\phi}(\partial\Omega)$ and there exists the associated extension operator $[\Pi^m_{\partial\Omega}]^{-1}$

(right inverse to $\Pi^m_{\partial\Omega}$) and the norms of that operators depend on l, m, p, μ and C_ϕ, but are independent of the concrete choice of the weight ϕ.

Furthermore, the above results hold also for the family of spaces $W^{l,p}_{b,\phi}(\Omega)$.

Proof. As before, we restrict ourselves to consider only the case of a cylindrical domain $\Omega := \mathbb{R}\times\omega$, one dimensional weights and the equivalent norms (2.20). Moreover, we will consider below only the case of spaces $W^{l,p}_\phi$ (the spaces $W^{l,p}_{b,\phi}$ can be considered analogously).

Indeed, let $u \in W^{l_1,p_1}_\phi(\Omega)$. Then, according to the classical Sobolev embedding theorem (see [29]), we have

$$\|u\|_{W^{l_2,p_2}(\Omega_s)} \le C\|u\|_{W^{l_1,p_1}(\Omega_s)} \tag{2.32}$$

where the constant C is independent of s. Taking the power p_2 from the both sides of that inequality, we transform it to the following form (for simplicity, we consider only the case $p_2 < \infty$)

$$\|u\|^{p_2}_{W^{l_2,p_2}(\Omega_s)} \le C^{p_2}\|u\|^{p_2}_{W^{l_1,p_1}(\Omega_s)} \le C_1\left(\int_{s\in\mathbb{R}} e^{-\alpha p_1|s-y|}\|u\|^{p_1}_{W^{l_1,p_1}(\Omega_y)}\,dy\right)^{p_2/p_1}$$

where $\alpha > \mu$ is arbitrary and the constant C_1 is independent of u. Multiplying this relation by $\phi(s)^{p_2}$ integrating by $s \in \mathbb{R}$ and using inequality (2.12), we infer

$$\|u\|^{p_2}_{W^{l_2,p_2}_\phi(\Omega)} \le C_2\|u\|^{p_2}_{W^{l_1,p_1}_\phi(\Omega)}$$

which proves the first part of the proposition.

Let us verify the second assertion of the proposition. Indeed, the existence and boundedness of the trace operator $\Pi^m_{\partial\Omega}$ can be verified based on the analogous property for domains Ω_s exactly as before (so we rest it to the reader). Thus, we only need to construct the extension operator $[\Pi^m_{\partial\Omega}]^{-1}$. Indeed, let $U := \{u_k\}_{k=0}^m \in \otimes_{k=0}^m W^{l-k-1/p,p}_\phi(\partial\Omega)$ be arbitrary. Using now the partition of unity (2.21), we construct the family $U_s := \psi_s U = \{\psi_s u_k\}_{k=0}^m$. Then, since all of that functions vanish at the origins of the cylinder Ω_s, there exists an extension operator $[\Pi^m_{\partial\Omega_s}]^{-1}$ for bounded domain Ω_s which maps U_s to $W^{l,p}(\Omega_s)$ and its norm is independent of U and s, see [29]. The required extension operator $[\Pi^m_{\partial\Omega}]^{-1}$ can be now constructed as follows:

$$[\Pi^m_{\partial\Omega}]^{-1}U := \int_{s\in\mathbb{R}}[\Pi^m_{\partial\Omega_s}]^{-1}U_s\,ds. \tag{2.33}$$

Indeed, the fact that this operator is well defined and the required uniform (with respect to ϕ) estimate for its norm as the map from $\otimes_{k=0}^m W^{l-k-1/p,p}_\phi(\partial\Omega)$ to $W^{l,p}_\phi(\Omega)$ can be verified exactly as estimate (2.27) for the function (2.25) from the proof of Proposition 2.8. Proposition 2.11 is proved. □

Our next task is formulate some trace theorems for classes of less smooth functions which are closely related with the theory of NS equations. To this end, we need the following definition.

Definition 2.12. Let Ω be a regular domain of $\mathbb{R}^n$, ϕ be a weight function of exponential growth rate μ and $1 < p < \infty$. Let us define the space $E^p_\phi(\Omega)$ of vector-valued functions $u := (u^1, \cdots, u^n) \in [D'(\Omega)]^n$ by the following norm:

$$\|u\|^p_{E^p_\phi(\Omega)} := \|u\|^p_{[L^p_\phi(\Omega)]^n} + \|\operatorname{div} u\|^p_{L^p_\phi(\Omega)}. \tag{2.34}$$

The spaces $E^p_{b,\phi}(\Omega)$ are defined analogously. Moreover, for every sufficiently smooth vector-valued function $u := (u^1, \cdots, u^n)$, we denote by $l_n u := (\vec{u}, \vec{n})\big|_{\partial\Omega}$ the normal component of that function at the boundary.

Proposition 2.13. *Let Ω be a regular domain and ϕ be a weight function of exponential growth rate μ. Then the operator $l_n : E^p_\phi(\Omega) \to W^{-1/p,p}_\phi(\partial\Omega)$ is well-defined and*

$$\|l_n u\|_{W^{-1/p,p}_\phi(\partial\Omega)} \le C\|u\|_{E^p_\phi(\Omega)} \tag{2.35}$$

where the constant C depends on μ and C_ϕ, but is independent of the concrete choice of the weight function ϕ. Moreover, the analogous result holds also for the spaces $E^p_{b,\phi}(\Omega)$.

Proof. As before, we verify estimate (2.35) only for the cylindrical domains. Indeed, let u and v_s be smooth functions in Ω_s. Then, due to Green's formula

$$(l_n u, v)_{\partial\Omega_s} := (\operatorname{div} u, v)_{\Omega_s} + (u, \nabla_x v)_{\Omega_s}. \tag{2.36}$$

As usual, we see that the right-hand side of (2.36) is well-defined for all $u \in E^p(\Omega_s)$ and $v \in W^{1,q}(\Omega_s)$ where $1/p + 1/q = 1$. Moreover, due to the classical trace theorems, there exists an extension operator $[\Pi_s]^{-1} : W^{1-1/q,q}(\partial\Omega_s) \to W^{1,q}(\Omega_s)$ whose norm is, obviously independent of s. Thus, (2.36) shows that the functional $l_n u$ is well-defined and satisfies

$$\|l_n u\|_{W^{-1/p,p}(\partial\Omega_s)} = \|l_n u\|_{[W^{1-1/q,q}(\partial\Omega_s)]^*} \le C\|u\|_{E^p(\Omega_s)}. \tag{2.37}$$

Multiplying this relation by $\phi(s)^p$ and integrating over $s \in \mathbb{R}$, we deduce (2.35) and finish the proof of the proposition. Here we have implicitly used that

$$\|l_n u\|_{W^{-1/p,p}((s,s+1)\times\partial\omega)} \le \|l_n u\|_{W^{-1/p,p}(\partial\Omega_s)}.$$

The estimate for $E^p_{b,\phi}(\Omega)$ can be obtained analogously using the supremum instead of integral over $s \in \mathbb{R}$. ☐

As we have already mentioned, estimates of Proposition 2.5 allow to reduce the proofs of elliptic regularity in arbitrary weighted spaces to the particular case of special weights (2.4). The following evident proposition will be useful in order to reduce the case of that special weights to the classical unweighted case $\phi = 1$.

Proposition 2.14. *Let Ω be a regular domain and let $\mathbb{T}_{\mu,x_0}$ be a multiplication operator by the weight $\varphi_{\mu,x_0}(x)$ (i.e. $(\mathbb{T}_{\mu,x_0}u)(x) := \varphi_{\mu,x_0}(x)u(x)$). Then, for every $l \in \mathbb{R}$ and $1 \le p \le \infty$, this operator realizes an isomorphism between the spaces $W^{l,p}_{\varphi_{\mu,x_0}}(\Omega)$ and $W^{l,p}(\Omega)$. Moreover,*

$$C^{-1}\|u\|_{W^{l,p}_{\varphi_{\mu,x_0}}(\Omega)} \le \|\mathbb{T}_{\mu,x_0}u\|_{W^{l,p}(\Omega)} \le C\|u\|_{W^{l,p}_{\varphi_{\mu,x_0}}(\Omega)} \tag{2.38}$$

where the constant C depends on l, p and μ, but is independent of u and $x_0 \in \mathbb{R}^n$.

Indeed, this estimate is an immediate corollary of inequalities (2.6) and Definition 2.7 of the corresponding weighted spaces.

We conclude by formulating some useful results on the weighted and local topologies on bounded sets of $W^{l,p}_b(\Omega)$.

Proposition 2.15. *Let Ω be a bounded domain $l \in \mathbb{R}$ and $p \in [1,\infty]$ and let $\mathbb{B}$ be a bounded subset of $W^{l,p}_b(\Omega)$. Then, for every weight function ϕ of exponential growth rate μ satisfying*

$$\|\phi\|_{L^p(\mathbb{R}^n)} < \infty, \tag{2.39}$$

the set $\mathbb{B}$ belongs to $W^{l,p}_\phi(\Omega)$ and the topology, generated on $\mathbb{B}$ by this embedding is independent of the weight ϕ and coincides with the local topology on $\mathbb{B}$ generated by embedding to $W^{l,p}_{loc}(\overline{\Omega})$.

Proof. Indeed, due to (2.39), we have

$$\|u\|^p_{W^{l,p}_\phi(\Omega)} = \int_{x_0\in\Omega} \phi^p(x_0)\|u\|^p_{W^{l,p}(\Omega\cap B^1_{x_0})}\,dx_0 \le \|\phi\|^p_{L^p(\mathbb{R}^n)}\|u\|^p_{W^{l,p}_b(\Omega)}$$

which shows that $W^{l,p}_b(\Omega) \subset W^{l,p}_\phi(\Omega)$. Let us now the sequence $u_n \to u$ in $W^{l,p}_{loc}(\overline{\Omega})$. This means that, for every $x_0 \in \Omega$ and every $R \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \|u_n - u\|_{W^{l,p}(\Omega\cap B^R_{x_0})} = 0. \tag{2.40}$$

Let also $u_n, u \in \mathbb{B}$ and ϕ be an integrable (in the sense of (2.39)) weight. Then, since the set $\mathbb{B}$ is assumed to be bounded in $W^{l,p}_b(\Omega)$,

$$\lim_{R\to\infty} \|u_n\|_{W^{l,p}_\phi(\Omega\setminus B^R_0)} = 0 \tag{2.41}$$

uniformly with respect to $n \in \mathbb{N}$. Assertions (2.40) and (2.41) imply in a standard way that $u_n \to u$ in $W^{l,p}_\phi(\Omega)$. Since the embedding $W^{l,p}_\phi(\Omega) \subset W^{l,p}_{loc}(\overline{\Omega})$ is obvious, then Proposition 2.15 is proved. $\square$

3. Elliptic regularity in weighted spaces

In this Section, we recall some standard elliptic regularity results in weighted Sobolev spaces which are necessary to deals with the Navier-Stokes equations in unbounded domains. For simplicity, we restrict ourselves to consider only the case of a 3D cylinder $\Omega := \mathbb{R} \times \omega$, ω is a bounded smooth domain of $\mathbb{R}^2$ ($x := (x_1, x_2, x_3) \in \Omega$, $x_1 \in \mathbb{R}$, $x' := (x_2, x_3) \in \omega$) although some of the results of this section remain true for general regular domains, see [11]-[12], [32]-[35] for details. We start with the weighted regularity estimate for the Laplacian with Dirichlet boundary conditions.

Proposition 3.1. *Let us consider the following Dirichlet problem in a cylinder Ω:*

$$\Delta_x u = h, \quad u\big|_{\partial\Omega} = 0. \tag{3.1}$$

Then, for every $1 < p < \infty$ and $l = -1, 0, 1$, there exists positive $\mu_0 = \mu_0(p)$ such that, for every weight function ϕ with sufficiently small exponential growth rate μ ($\mu \le \mu_0$)

and every $h \in W^{l,p}_{\phi}(\Omega)$, equation (2.1) possesses a unique solution $u \in W^{l+2,p}_{\phi}(\Omega)$ and the following estimate holds:

$$\|u\|_{W^{l+2,p}_{\phi}(\Omega)} \le C\|h\|_{W^{l,p}_{\phi}(\Omega)} \tag{3.2}$$

where the constant C depends on C_ϕ, but is independent of the concrete choice of the weight ϕ. Moreover, the analogous estimate holds also for the spaces $W^{l,p}_{b,\phi}(\Omega)$.

Proof. We restrict ourselves to verify a priori estimate (3.2) only (the existence and uniqueness of a solution can be then verified in a standard way, see e.g. [11], [12]).

As we have already mentioned, due to estimates (2.12) and (2.13), it is sufficient to verify estimate (3.2) only for the special class of weights $\varphi_{\mu_0,x_0}(x)$ introduced in (2.4). Indeed, if we have estimate (3.2) for such weights with the constant C independent of x_0, then we obviously have the following estimate:

$$\|u\|^p_{W^{l+2,p}(\Omega_s)} \le C_{\mu_0}\|u\|^p_{W^{l+2,p}_{\varphi_{\mu_0,s}}(\Omega)} \le C_1\|h\|^p_{W^{l,p}_{\varphi_{\mu_0,s}}(\Omega)} \le \\ \le C_2\int_{y\in\mathbb{R}} e^{-p\mu_0|s-y|}\|h\|^p_{W^{l,p}(\Omega_y)}\,dy \tag{3.3}$$

where the constant C_2 is also independent of $s \in \mathbb{R}$. Multiplying now estimate (3.3) by $\phi(s)^p$ (where ϕ is a weight function with exponential growth rate $\mu < \mu_0$), integrating over $s \in \mathbb{R}$ and using estimate (2.12), we infer the required estimate (2.2). Analogously, estimate (3.2) for the spaces $W^{l,p}_{b,\phi}$ can be obtained by multiplication (3.3) by $\phi(s)^p$, taking the supremum over $s \in \mathbb{R}$ and using estimate (2.13).

Thus, it only remains to verify (3.2) for the special weights $\varphi_{\mu_0,s}$ with a sufficiently small positive μ_0 and every $s \in \mathbb{R}$. In turns, due to Proposition 2.14 and estimates (2.6), the case of special weights $\varphi_{\mu_0,s}$ can be easily reduced to the unweighted case $\phi \equiv 1$. Indeed, the function $u \in W^{l+2,p}_{\varphi_{\mu_0}}(\Omega)$ solves (3.3) if and only if the function $v := \varphi_{\mu_0,s}u \in W^{l+2,p}(\Omega)$ solves the following perturbed version of problem (3.2):

$$\Delta_x v = \varphi_{\mu_0,s}h - \varphi_{\mu_0,s}\varphi''_{-\mu_0,s}v - 2\phi'_{-\mu_0,s}\phi_{\mu_0,s}\partial_{x_1}v := \mathbb{T}_{\mu_0,s}h + h_{\mu_0}(v), \quad v\big|_{\partial\Omega} = 0. \tag{3.4}$$

We recall that, due to (2.6),

$$\|h_{\mu_0}(v)\|_{W^{l,p}(\Omega)} \le C\mu_0\|v\|_{W^{l+2,p}(\Omega)} \tag{3.5}$$

where the constant C is independent of s and μ_0. Thus, if estimate (3.2) for $\phi \equiv 1$ is known, then applying it to equation (3.4) and using (3.5), we infer

$$\|\mathbb{T}_{\mu_0,s}u\|_{W^{l+2,p}(\Omega)} \le C(\|\mathbb{T}_{\mu_0,s}h\|_{W^{l,p}(\Omega)} + \mu_0\|v\|_{W^{l+2,p}(\Omega)})$$

with the constant C independent of μ_0 and s. Fixing now μ_0 to be small enough that $C\mu_0 < 1/2$, we deduce from the last estimate that

$$\|v\|_{W^{l+2,p}(\Omega)} \le 2C\|\mathbb{T}_{-\mu_0,s}h\|_{W^{l,p}(\Omega)} \tag{3.6}$$

which together with Proposition 2.14 imply estimate (3.2) for special weights $\varphi_{\mu_0,s}$.

Thus, we have reduced the verifying of the regularity estimate (3.2) in weighted spaces to the unweighted case $\phi \equiv 1$. It only remains to note that (3.2) with $\phi \equiv 1$ is a classical L^p-regularity estimate for the solutions of the Laplace operator, see e.g. [18], [29]. Proposition 3.1 is proved. ∎

Remark 3.2. Surely, regularity estimate (3.2) holds not only for $l = -1, 0, 1$, but we will need it in the sequel only for that values of l. We also note that estimate (3.2) holds for the unweighted space since the spectrum of the Laplacian in a cylinder with Dirichlet boundary conditions is strictly negative.

Next proposition which gives some uniform estimate for the singular perturbed Laplace equation, will be useful for approximating the 3D Navier-Stokes problem.

Proposition 3.3. *Let $\alpha > 0$ be small, ϕ be a weight function with exponential growth rate μ and let u solve the equation:*

$$u - \alpha\Delta_x u = h, \quad u\big|_{\partial\Omega} = 0 \tag{3.7}$$

for some $h \in L^p_\phi(\Omega)$. Then, the following estimate holds:

$$\alpha\|u\|_{W^{2,p}_\phi(\Omega)} + \|u\|_{L^p_\phi(\Omega)} \le C\|h\|_{L^p_\phi(\Omega)} \tag{3.8}$$

where the constant C depends only on μ and C_ϕ and is independent of α and the concrete form of the weight ϕ. Moreover, the analogous result holds also for the spaces $W^{l,p}_{b,\phi}$.

Proof. Indeed, after the scaling $\bar x := \alpha^{-1/2}x$, equation (3.7) reads

$$\bar u - \Delta_{\bar x}\bar u = \bar h, \quad \bar u\big|_{\partial\bar\Omega} = 0, \quad \bar\Omega := \alpha^{-1/2}\Omega$$

and the weight ϕ should be now replaced by $\bar\phi(\bar s) := \phi(\alpha^{1/2}\bar x)$. It is clear that the regularity constant of the domain $\bar\Omega$ is at least not worse than for Ω (if α is small enough) and the weight $\bar\phi$ will be of exponential growth rate $\alpha^{1/2}\mu \le \mu$ with $C_{\bar\phi} = C_\phi$. By this reason, estimate (3.2) of Proposition 3.1 will hold for the scaled equation uniformly with respect to α, i.e.,

$$\|\bar u\|_{W^{2,p}_{\bar\phi}(\bar\Omega)} \le C\|\bar h\|_{L^p_{\bar\phi}(\Omega)},$$

see eg,[12, 30] for details. Returning back to variable x, we obtain the desired estimate (3.8) and finish the proof of the proposition. ☐

Remark 3.4. The analogue of Proposition 3.3 for more regular external forces $h \in W^{l,p}_\phi(\Omega)$ with $l > 0$ is not true, since boundary layer terms may appear. Indeed, in the simplest 1D case:

$$y(x) - \alpha^2 y''(x) = 1, \quad x \in [0,1], \quad y(0) = y(1) = 0,$$

the external force belongs to C^∞ and the associated solution

$$y(x) = 1 - \frac{\sinh(\alpha^{-1}x)}{\sinh(\alpha^{-1})} - \frac{\sinh\alpha^{-1}(1-x)}{\sinh(\alpha^{-1})}$$

is a typical boundary layer solution which does not uniformly bounded in any C^β with $\beta > 0$ as $\alpha \to 0$.

We are now going to consider the Neumann-type boundary value problems for the Laplacian in a cylinder Ω. The main difficulty here is the fact that, in contrast to the Dirichlet problems considered above, the Neumann problem for the Laplacian has an essential spectrum at $\lambda = 0$, which makes the situation much more delicate. We however start with the regularized Neumann-type problem where the spectrum remains strictly negative.

Proposition 3.5. *Let Ω be a cylinder and let us consider the following boundary value problem in Ω:*

$$\Delta_x u - u = 0, \quad \partial_n u\big|_{\partial\Omega} = h_0, \tag{3.9}$$

Then, for every $1 < p < \infty$ and $l = 0, 1, 2$, there exists $\mu_0 = \mu_0(p)$ such that, for every weight function of sufficiently small exponential growth rate μ ($\mu \le \mu_0$) and every $h_0 \in W^{l-1/p,p}_\phi(\partial\Omega)$ problem (3.9) has a unique solution $u \in W^{l+1,p}_\phi(\Omega)$ and the following estimate holds:

$$\|u\|_{W^{l+1,p}_\phi(\Omega)} \le C\|h_0\|_{W^{l-1/p,p}_\phi(\partial\Omega)} \tag{3.10}$$

where the constant C depends on C_ϕ, but is independent of the concrete choice of weight function ϕ. Moreover, the analogous result holds for the spaces $W^{l,p}_{b,\phi}$ as well.

Proof. Indeed, in the case $l = 1, 2$ estimate (3.10) can be verified exactly as in Propositions 3.1 and 3.3 (by reducing to the homogeneous and unweighted case), so we rest it to the reader. In the case $l = 0$ the situation is slightly more delicate since we do not formulate the extension theorem for the space $W^{-1/p,p}_\phi(\partial\Omega)$ in Proposition 2.11 and, consequently, we need to work with the non-homogeneous boundary value problem. Nevertheless, the reduction to the unweighted case based on introducing the function $v := \varphi_{\mu_0,s} u$ works in this case as well. Indeed, this function obviously satisfies

$$\Delta_x v - v = h_{\mu_0}(v), \quad \partial_n v\big|_{\partial\Omega} := \mathbb{T}_{-\mu_0,s} h_0 \tag{3.11}$$

and

$$\|h_{\mu_0}(v)\|_{L^p(\Omega)} \le C\mu_0\|v\|_{W^{1,p}(\Omega)} \tag{3.12}$$

Thus, we can split the solution v of (3.11) as follows: $v = v_1 + v_2$ where v_1 solves the homogeneous problem

$$\Delta_x v_1 - v_1 = h_{\mu_0}(v), \quad \partial_n v_1\big|_{\partial\Omega} = 0 \tag{3.13}$$

and the remainder v_2 solves the analogue of (3.9) with h_0 replaced by $\mathbb{T}_{-\mu_0,s}h_0$. We see also that the right-hand side of (3.11) belongs to $L^p(\Omega)$ and, consequently, due to the classical L^p-regularity, we have

$$\|v_1\|_{W^{2,p}(\Omega)} \le C\|h_{\mu_0}(v)\|_{L^p(\Omega)} \le C_1\mu_0\|v\|_{W^{1,p}(\Omega)}. \tag{3.14}$$

If we assume now that estimate (3.10) for the unweighted case $\phi = 1$ and $l = 0$ is known, then, due to (3.14), we infer

$$\|v\|_{W^{1,p}(\Omega)} \le \|v_1\|_{W^{1,p}(\Omega)} + \|v_2\|_{W^{1,p}(\Omega)} \le C\|\mathbb{T}_{\mu_0,s}h_0\|_{W^{-1/p,p}(\partial\Omega)} + C\mu_0\|v\|_{W^{1,p}(\Omega)}$$

which implies the estimate

$$\|v\|_{W^{1,p}(\Omega)} \le 2C\|\mathbb{T}_{\mu_0,s}h_0\|_{W^{-1/p,p}(\partial\Omega)} \tag{3.15}$$

if μ_0 is small. Thus, the case of general weight naturally reduces to the case of $\phi \equiv 1$ for $l = 0$ as well. It remains to recall that, for $\phi \equiv 1$, estimate (3.10) is a classical L^p-regularity result for the Laplacian, see [29]. Proposition 3.5 is proved. $\square$

In order to treat the case of Neumann problem without the regularizing term $-u$, we need to introduce the following averaging operator with respect to the variable x' ($(x_1,x')\in\mathbb{R}\times\omega:=\Omega$):

$$(\mathbb{S}u)(x_1):=\frac{1}{|\omega|}\int_{s\in\omega}u(x_1,s)\,ds. \tag{3.16}$$

The next proposition gives the solvability of the Neumann problem for some natural closed subspace of the the space of external forces h.

Proposition 3.6. *Let Ω be a cylinder and let us consider the following boundary value problem in Ω:*

$$\Delta_x u=h,\quad \partial_n u\big|_{\partial\Omega}=0. \tag{3.17}$$

Then, for every $1<p<\infty$ and $l=0,1,2$, there exists $\mu_0=\mu_0(p)$ such that, for every weight function of a sufficiently small exponential growth rate μ ($\mu\le\mu_0$) and every $h\in W^{l,p}_\phi(\Omega)$ satisfying

$$\mathbb{S}h\equiv 0,$$

problem (3.17) has a unique solution $u\in W^{l+2,p}_\phi(\Omega)$, $\mathbb{S}u\equiv 0$ and the following estimate holds:

$$\|u\|_{W^{l+2,p}_\phi(\Omega)}\le C\|h\|_{W^{l,p}_\phi(\Omega)} \tag{3.18}$$

where the constant C depends on C_ϕ, but is independent of the concrete choice of weight function ϕ. Moreover, the analogous result holds for the spaces $W^{l,p}_{b,\phi}$ as well.

Proof. We first note that the operator $\mathbb{S}$ commutes with the multiplication operator $\mathbb{T}_{\mu_0}$ and with the x_1-derivatives ∂_{x_1}. Thus, arguing exactly as before, we can reduce the proof of (3.18) to the unweighted case $\phi\equiv 1$. So, we will prove below (3.18) for the case $\phi\equiv 1$ only.

To this end, we first consider the case $p=2$. In that case we can multiply equation (3.17) by u and obtain, after the integration by parts that

$$\|\nabla_x u\|^2_{L^2(\Omega)}\le\|h\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \tag{3.19}$$

Since we have assumed additionally that $\mathbb{S}u\equiv 0$ then, we have the Friedrich's inequality

$$\|u\|_{W^{1,2}(\Omega)}\le C\|\nabla_x u\|_{L^2(\Omega)} \tag{3.20}$$

which together with (3.19) implies that

$$\|u\|_{W^{1,2}(\Omega)}\le C\|h\|_{L^2(\Omega)}. \tag{3.21}$$

In order to prove estimate (3.18) for $p=2$ and $\phi\equiv 1$, we now use the following standard interior regularity estimate:

$$\|u\|^2_{W^{l+2,2}(\Omega_s)}\le C(\|u\|^2_{W^{1,2}(\Omega_{s-1}\cup\Omega_s\cup\Omega_{s+1})}+\|h\|^2_{W^{l,2}(\Omega_s)})\le \\ \le C_1\int_{y\in\Omega}e^{-\alpha|s-y|}(\|u\|^2_{W^{1,2}(\Omega_y)}+\|h\|^2_{W^{l,2}(\Omega_y)})\,dy. \tag{3.22}$$

Integrating this estimate over $s\in\mathbb{R}$ and using (2.12) and (3.21), we infer the unweighted estimate (3.18) for $p=2$. Thus, due to the trick with the multiplication operator $\mathbb{T}_{\mu_0,s}$, estimate (3.18) is verified for $p=2$ and all weights with sufficiently small exponential growth rate. Moreover, we have also the analogue of estimate (3.18) with $p = 2$ for the spaces $W^{l,p}_{b,\phi}(\Omega)$.

Let us now consider the case $p \neq 2$. We first consider the case $p > 2$ and will prove estimate (3.18) for the spaces $W^{l,p}_b(\Omega)$. Indeed, since $W^{l,p}_b(\Omega) \subset W^{l,2}_b(\Omega)$, then we already have the estimate

$$\|u\|_{W^{1,2}_b(\Omega)} \le C\|h\|_{L^2_b(\Omega)} \le C_1\|h\|_{L^p_b(\Omega)}. \tag{3.23}$$

Using now the interior regularity estimate

$$\|u\|_{W^{l+2,p}(\Omega_s)} \le C(\|u\|_{W^{1,2}(\Omega_{s-1}\cup\Omega_s\cup\Omega_{s+1})} + \|h\|_{W^{l,p}(\Omega_s)}) \le \\ \le C_1 \sup_{y\in\mathbb{R}}\{e^{-\alpha|s-y|}(\|u\|_{W^{1,2}(\Omega_y)} + \|h\|_{W^{l,p}(\Omega_y)})\},$$

taking a supremum over $s \in \mathbb{R}$ from the both parts of that inequality and using (2.3) and (3.23), we finally infer

$$\|u\|_{W^{l+2,p}_b(\Omega)} \le C\|h\|_{W^{l,p}_b(\Omega)}. \tag{3.24}$$

Let now $1 < p < 2$. Then, we split the solution u of (3.17) as follows: $u = u_1 + u_2$ where u_1 solves problem

$$\Delta_x u_1 - u_1 = h, \quad \partial_n u_1\big|_{\partial\Omega} = 0 \tag{3.25}$$

and the remainder u_2 solves

$$\Delta_x u_2 = -u_1, \quad \partial_n u_2\big|_{\partial\Omega} = 0. \tag{3.26}$$

We first note that, due to the L^p-regularity (see Proposition 3.5), for equation (3.25), we have

$$\|u_1\|_{W^{l+2,p}_b(\Omega)} \le C\|h\|_{W^{l,p}_b(\Omega)}. \tag{3.27}$$

Moreover, applying the operator $\mathbb{S}$ to both sides of equation (3.25) and using that $\mathbb{S}h \equiv 0$, we have

$$(\mathbb{S}u_1)'' - \mathbb{S}u_1 \equiv 0 \ \text{ and, consequently, } \ \mathbb{S}u_1 \equiv 0. \tag{3.28}$$

Furthermore, due to the embedding theorem (see Proposition 2.11), we have

$$\|u_1\|_{W^{l,2}(\Omega)} \le C\|u_1\|_{W^{l+2,p}_b(\Omega)}, \tag{3.29}$$

for every $1 < p < 2$. Thus, we can apply estimate (3.23) for equation (3.26) which together with (3.27) gives estimate (3.24) for $1 < p < 2$ as well.

Thus, estimate (3.24) is verified for all $1 < p < \infty$. Then, due to the above described trick with the multiplication operator $\mathbb{T}_{\mu_0,s}$, we deduce estimate (3.18) for the spaces $W^{l+2,p}_{b,\phi}(\Omega)$ for all weight functions of sufficiently small exponential growth rate.

So, it only remains to obtain it for the spaces $W^{l,p}_{\phi}(\Omega)$. To this end, we note that (3.18) for the spaces $W^{l,p}_{b,\varphi_{\mu_0,s}}(\Omega)$ implies, in particular, that

$$\|u\|^p_{W^{l+2,p}(\Omega_s)} \le C\sup_{y\in\mathbb{R}}\{e^{-\mu_0 p|s-y|}\|h\|^p_{W^{l,p}(\Omega_y)}\} \le C_1\int_{y\in\Omega} e^{-\mu_0 p|s-y|}\|h\|^p_{W^{l,p}(\Omega)}\, dy. \tag{3.30}$$

Multiplying (3.30) by $\phi(s)^p$, integrating over $s \in \mathbb{R}$ and using (2.12), we deduce finally estimate (3.18) and finish the proof of Proposition 3.6. ☐

Remark 3.7. As we see from the proof of Proposition 3.6, the weighted regularity estimates can be deduced not only from the unweighted estimates in $W^{l,p}(\Omega)$, but also from their analogs in the spaces $W^{l,p}_b(\Omega)$. The last scale of spaces is sometimes (e.g., in the proof of Proposition 3.6) more convenient, since, in contrast to spaces $L^p(\Omega)$, the spaces $L^p_b(\Omega)$ have usual (for bounded domains) embedding properties ($L^{p_1}_b(\Omega) \subset L^{p_2}_b(\Omega)$, for $p_1 \ge p_2$).

We now note that assumption $\mathbb{S}h \equiv 0$ in Proposition 3.6 is essential for the weighted estimate (3.18). Indeed, in general case $\mathbb{S}h \ne 0$, for the quantity $\mathbb{S}u = (\mathbb{S}u)(x_1)$ we have the following equation:

$$(\mathbb{S}u)(x_1)'' = (\mathbb{S}h)(x_1), \quad x_1 \in \mathbb{R} \tag{3.31}$$

whose solution $\mathbb{S}u$, obviously, does not possess any weighted regularity estimates for general h. Fortunately, for problems arising in the weighted regularity theory for the Helmholtz operator, the function $\mathbb{S}h$ has a special structure which allows to take one primitive of it remaining in weighted Sobolev classes. To be more precise, the following proposition holds.

Proposition 3.8. *Let Ω be a cylinder and let us consider the following Neumann boundary value problem in Ω:*

$$\Delta_x u = 0, \quad \partial_n u\big|_{\partial\Omega} = l_n g \tag{3.32}$$

where $g \in [L^p(\Omega)]^2$ is a divergent free vector field

$$\operatorname{div} g \equiv 0. \tag{3.33}$$

Then, for every $1 < p < \infty$ and $l = 0, 1, 2$, there exists $\mu_0 = \mu_0(p)$ such that, for every weight function of a sufficiently small exponential growth rate μ ($\mu \le \mu_0$) and every $g \in W^{l,p}_\phi(\Omega)$ satisfying (3.33), problem (3.32) has a unique solution (up to adding a constant) satisfying $\nabla_x u \in W^{l,p}_\phi(\Omega)$, and

$$(\mathbb{S}u)(x_1)' = (\mathbb{S}g_1)(x_1), \quad x_1 \in \mathbb{R} \tag{3.34}$$

and the following estimate holds:

$$\|\nabla_x u\|_{W^{l,p}_\phi(\Omega)} \le C\|g\|_{W^{l,p}_\phi(\Omega)} \tag{3.35}$$

where the constant C depends on C_ϕ, but is independent of the concrete choice of weight function ϕ. Moreover, the analogous result holds for the spaces $W^{l,p}_{b,\phi}$ as well.

Proof. For simplicity, we deduce below only a priori estimate (3.35). The existence and uniqueness of a solution can be verified in a standard way (see also [4]).

We first define an auxiliary function v as a solution of the following problem:

$$\Delta_x v - v = 0, \quad \partial_n v\big|_{\partial\Omega} = l_n g. \tag{3.36}$$

Then, due to Propositions 3.5 and 2.13, we have

$$\|v\|_{W^{l+1,p}_\phi(\Omega)} \le C\|l_n g\|_{W^{l-1/p,p}_\phi(\partial\Omega)} \le C_2\|g\|_{W^{l,p}_\phi(\Omega)}. \tag{3.37}$$

Moreover, applying the x'-averaging operator $\mathbb{S}$ to equation (3.36), we have

$$(\mathbb{S}v)(x_1)'' - (\mathbb{S}v)(x_1) = -\frac{1}{|\omega|}\int_{s\in\partial\omega}(\vec{n}, ng(x_1,s))\,ds, \quad x_1\in\mathbb{R}. \tag{3.38}$$

Furthermore, since the vector field g is divergence free, we have

$$\frac{1}{|\omega|}\int_{s\in\partial\omega}(\vec{n}, g(x_1,s))\,ds = (\mathbb{S}[\partial_{x_2}g_2 + \partial_{x_3}g_3])(x_1) = -(\mathbb{S}g_1)(x_1)'$$

and, consequently,

$$(\mathbb{S}v)(x_1)'' - (\mathbb{S}v)(x_1) = (\mathbb{S}g_1)(x_1)'. \tag{3.39}$$

Let us consider now the remainder $w := u - v$ which obviously satisfies the following equation:

$$\Delta_x w = -v, \quad \partial_n w\big|_{\partial\Omega} = 0. \tag{3.40}$$

Then, according to Proposition 3.6, the function $\bar{w} := w - \mathbb{S}w$ satisfies the following estimate:

$$\|\bar{w}\|_{W^{l+1,p}_\phi(\Omega)} \le C\|\bar{v}\|_{W^{l,p}_\phi(\Omega)} \le C_1\|g\|_{W^{l,p}_\phi(\Omega)}. \tag{3.41}$$

So, it only remains to consider the equation for $\mathbb{S}w$, i.e.

$$(\mathbb{S}w)(x_1)'' = -(\mathbb{S}v)(x_1)$$

which together with (3.39) gives

$$(\mathbb{S}u)(x_1)'' = (\mathbb{S}g_1)(x_1)'. \tag{3.42}$$

This relation shows that we can indeed to take one primitive and satisfy condition (3.34). It only remains to note that the function $(\mathbb{S}u)(x_1)$ is independent of x' and, consequently,

$$\nabla_x u = \nabla_x\bar{u} + ((\mathbb{S}u)', 0, 0). \tag{3.43}$$

Thus, estimates (3.37), (3.41) together with the obvious fact that

$$\|\mathbb{S}g\|_{W^{l,p}_\phi(\mathbb{R})} \le C\|g\|_{W^{l,p}_\phi(\Omega)} \tag{3.44}$$

implies (3.35) and finishes the proof of Proposition 3.8. □

4. The Helmholtz projector and stationary Stokes problem

In this Section, we discuss the weighted analogue of the classical Helmholtz decomposition of the space $[L^2(\Omega)]^2$ to divergent free and gradient vector fields which is necessary for excluding the pressure from Navier- Stokes equations. To this end, we first need to define the corresponding spaces of divergent free vector fields.

Definition 4.1. Let Ω be a cylinder. Then, for every $l \ge 0$, $1 < p < \infty$ and every weight function ϕ of exponential growth rate, we define the following space of divergent free vector fields:

$$\mathcal{H}^{l,p}_\phi(\Omega) := \{v \in [W^{l,p}_\phi(\Omega)]^3, \ \ \operatorname{div} v \equiv 0, \ \ l_n v\big|_{\partial\Omega} = 0, \ \ \mathbb{S}v_1 \equiv 0\} \tag{4.1}$$

which is considered as a closed subspace of $W^{l,p}_\phi(\Omega)$ and endowed by the norm induced by this embedding. Here the normal component $l_n v$ of the trace on the boundary is well-defined due to Proposition 2.13 and the x'-averaging operator $\mathbb{S}$ is defined by (3.16). The spaces $\mathcal{H}^{l,p}_{b,\phi}(\Omega)$ can be defined analogously. Moreover, for simplicity, we will write below $\mathcal{H}^p_\phi(\Omega)$ and $\mathcal{H}^p_{b,\phi}(\Omega)$ instead of $\mathcal{H}^{0,p}_\phi(\Omega)$ and $\mathcal{H}^{0,p}_{b,\phi}(\Omega)$ respectively.

We also define the space $\mathcal{V}^p_\phi(\Omega)$ as follows:

$$\mathcal{V}^p_\phi(\Omega) := \{v \in \mathcal{H}^{1,p}_\phi(\Omega), \quad v\big|_{\partial\Omega} = 0\}$$

and the analogous space $\mathcal{V}^p_{b,\phi}(\Omega)$.

The following natural proposition clarifies the additional conditions $l_n v\big|_{\partial\Omega} = 0$ and $\mathbb{S}v_1 \equiv 0$ in formula (4.1).

Proposition 4.2. *Let Ω be a cylinder and ϕ be a weight function of exponential growth rate μ and $1 < p < \infty$. Then the space $\mathcal{H}^p(\Omega)$ coincides with the closure of all divergent free vector fields $v \in [\mathcal{D}(\Omega)]^3$ in the topology of $[L^p_\phi(\Omega)]^3$:*

$$\mathcal{H}^p_\phi(\Omega) = \big[v \in [\mathcal{D}(\Omega)]^3, \quad \operatorname{div} v = 0\big]_{[L^p_\phi(\Omega)]^2} \tag{4.2}$$

where $[\cdot]_V$ denotes the closure in the topology of the space V.

Proof. Indeed, let v be a divergent free vector field from $[D(\Omega)]^3$. Then, obviously, $l_n v\big|_{\partial\Omega} = 0$. Moreover, integrating the relation $\partial_{x_1}v_1 = -\partial_{x_2}v_2 - \partial_{x_3}v_3$, we infer that $\mathbb{S}v_1 \equiv const = 0$ (since v_1 has a finite support). Since all these properties preserve under the closure (see Proposition 2.13), then the right-hand side of (4.2) is a subset of the left one.

Thus, it only remains to approximate every function from $u \in \mathcal{H}^p_\phi(\Omega)$ by divergent free vector fields belonging to $[\mathcal{D}(\Omega)]^3$. Moreover, since for the case of bounded domains the assertion is well-known (see eg, [27]), it is sufficient to approximate u by the functions $u_n \in \mathcal{H}^p_\phi(\Omega)$ with *bounded* support. In order to do so, we introduce a family of cut-off functions θ_n such that $\theta_n(s) \in [0,1]$, $\theta_n(s) = 1$, for $s \in [-n,n]$, $\theta_n(s) = 0$, for $s \notin [-n-1, n+1]$ and $\phi'_n(s)$ is uniformly bounded with respect to s and n.

Let us now consider the vector-field $\tilde u^n(x) := \theta_n(x_1)u(x)$. Then, obviously, $\tilde u^n \to u$ in $[L^p_\phi(\Omega)]^3$, the support of $\tilde u^n$ is bounded and is contained in the sub-domain $\Omega_{[-n-1,n+1]} := [-n-1,n+1]\times\omega$ and has zero trace of the normal component on the boundary and zero mean flux. The only problem is that vector field is not divergent free:

$$\operatorname{div} \tilde u^n = \phi'_n u_1 := h^n(x) = h^n(x)\chi_{\Omega_{-n-1}}(x) + h^n(x)\chi_{\Omega_n}(x) := h^n_+(x) + h^n_-(x)$$

(here we have implicitly used that $\operatorname{supp}\theta'_n \subset [-n-1,-n]\cup[n,n+1]$). Moreover, since u_1 has a zero mean, we conclude that

$$\int_{\Omega_{-n-1}} h^n_-(x)\,dx = \int_{\Omega_n} h^n_+(x)\,dx = 0.$$

Thus, there exist vector fields $u^n_- \in [W^{1,2}_0(\Omega_{-n-1})]^3$ and $u^n_+ \in [W^{1,2}_0(\Omega)]^3$ such that

$$\operatorname{div} u^n_\pm = h^n_\pm, \quad \|u^n_-\|_{W^{1,p}(\Omega_{-n-1})} \le C\|u_1\|_{L^p(\Omega_{-n-1})}, \quad \|u^n_+\|_{W^{1,p}(\Omega_n)} \le C\|u_1\|_{L^p(\Omega_n)} \tag{4.3}$$

where the constant C is independent of n, see [27]. Extending now vector fields $u^n_\pm$ by zero outside of $\Omega_{-n-1}\cup\Omega_n$, we obtain the vector fields $u^n_\pm$ defined already in the whole cylinder Ω and satisfying (4.3) (here we have used zero boundary conditions). Finally, estimates (4.3) show that $u_\pm^n \in L^p_\phi(\Omega)$ (and even $W^{1,p}_\phi(\Omega)$) and tend to zero as $n \to \infty$ in that spaces. Defining finally

$$u^n := \tilde u^n - u^n_+ - u^n_-, \tag{4.4}$$

we obtain the desired converging sequence of divergent free vector fields with finite supports an finish the proof of the proposition. □

Remark 4.3. Arguing analogously, one can also verify the description for $\mathcal{V}^p_\phi(\Omega)$:

$$\mathcal{V}^p_\phi(\Omega) = \big[v \in [\mathcal{D}(\Omega)]^3,\quad \operatorname{div} v = 0\big]_{[W^{1,p}_\phi(\Omega)]^2} \tag{4.5}$$

As usual, we define the operator $\Pi : [L^2(\Omega)]^3 \to \mathcal{H}^2(\Omega)$ as an orthoprojector to the divergent free vector fields. Then, as known (see e.g. [27] or [28]), every vector field $u \in [L^2(\Omega)]^3$ can be split in a unique way in a sum of a divergent free vector field $v \in \mathcal{H}^2(\Omega)$ and a potential one $\nabla_x p \in [L^2(\Omega)]^3$ for the appropriate $p \in H^1_{loc}(\overline{\Omega})$:

$$u = v + \nabla_x p,\quad \operatorname{div} v = 0,\quad v := \Pi u. \tag{4.6}$$

The next theorem shows that the analogous splitting holds in weighted spaces as well.

Theorem 4.4. *Let Ω be a cylinder and let Π be the orthoprojector defined above. Then, for every $1 < p < \infty$ and $l = 0, 1, 2$, there exists a sufficiently small positive μ_0 such that, for every weight function with exponential growth rate $\mu \le \mu_0$, this projector can be uniquely extended by continuity to a bounded operator from $[W^{l,p}_\phi(\Omega)]^3$ to $\mathcal{H}^{l,p}_\phi(\Omega)$ and the following estimate holds:*

$$\|\Pi u\|_{\mathcal{H}^{l,p}_\phi(\Omega)} \le C\|u\|_{[W^{l,p}_\phi(\Omega)]^3} \tag{4.7}$$

where the constant C depends only on p, l and C_ϕ, but is independent of the concrete choice of the weight ϕ. Thus, for every $u \in [W^{l,p}_\phi(\Omega)]^3$ there exists a unique decomposition in the form of (4.6) with $v \in \mathcal{H}^{l,p}_\phi(\Omega)$ and $p \in W^{l+1,p}_{loc}(\overline{\Omega})$. In this formula $v = \Pi u$. Moreover, the analogous result holds also for the spaces $W^{l,p}_{b,\phi}$.

Proof. Let $u \in [W^{l,p}_\phi(\Omega)]^3$ and let us construct the pressure p in decomposition (4.6). Indeed, taking formally a divergence from the both parts of (4.6), we get

$$\Delta_x p = \operatorname{div} u \tag{4.8}$$

and using that $l_n v\big|_{\partial\Omega} = 0$, we infer the boundary condition for p:

$$\partial_n p\big|_{\partial\Omega} = l_n u\big|_{\partial\Omega}. \tag{4.9}$$

We however note that the right-hand side of (4.9) is ill-posed for general $u \in [L^p(\Omega)]^3$. In order to overcome this difficulty (following [4]), we introduce an auxiliary function p_1 which solves

$$\Delta_x p_1 = \operatorname{div} u,\quad p_1\big|_{\partial\Omega} = 0 \tag{4.10}$$

and then, the remainder $\bar p := p - p_1$ solves

$$\Delta_x \bar p = 0,\quad \partial_n \bar p\big|_{\partial\Omega} = l_n(u - \nabla_x p)\big|_{\partial\Omega}. \tag{4.11}$$

We now note that $\operatorname{div}(u - \nabla_x p_1) = 0$ and, consequently, due to Proposition 2.13, the trace $l_n(u - \nabla_x p_1)$ on the boundary is well-defined and we can apply Proposition 3.8 which gives a unique solvability (up to a constant) of (4.11) and estimate (3.35) for the gradient of $\bar{p}$. It remains to note that, condition (3.34) now reads

$$\partial_{x_1}\mathbb{S}\bar{p} = \mathbb{S}u_1 - \partial_{x_1}\mathbb{S}p_1 \ \text{ and, thus } \ \mathbb{S}\partial_{x_1}p = \mathbb{S}u_1$$

which shows that p is indeed correctly defined ($\mathbb{S}v_1 = \mathbb{S}u_1 - \mathbb{S}\partial_{x_1}p = 0$, $\operatorname{div} v = 0$ and $l_n v = 0$). From estimate (3.35) (for $\nabla_x\bar{p}$) and estimate (3.2) for $\nabla_x p_1$, we immediately deduce the analog of estimate (4.7) for $\nabla_x p$. Since $\Pi u := v = u - \nabla_x p$ this gives also estimate (4.7) for Π. Theorem 4.4 is proved. □

Corollary 4.5. *Let the assumptions of Theorem 4.4 hold and let $v \in \mathcal{H}^p_\phi(\Omega)$. Then, for every potential vector field $w = \nabla_x p$ such that $w \in [L^q_{\phi^{-1}}(\Omega)]^3$, we have*

$$(v, w)_{[L^2(\Omega)]^3} = 0. \tag{4.12}$$

Indeed, according to Proposition 4.2, the function v can be approximated (in the metric of $L^p_\phi(\Omega)$) by a sequence of smooth divergent free vector fields with a compact support. Since for such vector fields (4.12) is obvious, then passing to the limit, we obtain (4.12) for all $v \in \mathcal{H}^p_\phi(\Omega)$.

The next proposition gives the estimate for the weighted norms of the commutator of Π and the multiplication operator $\mathbb{T}_{\mu,x_0}$ introduced in Proposition 2.14.

Proposition 4.6. *Let Ω be a cylinder, $1 < p < \infty$, $l = 0, 1, 2$ and $\mathbb{T}_{\mu,x_0}$ is a multiplication by the special weight $\varphi_{\mu,x_0}(x_1)$. Then, there exists $\mu_0 = \mu_0(p) > 0$ such that, for every weight function of exponential growth rate $\varepsilon \le \mu_0$, every $\mu \le \mu_0$ and every $x_0 \in \mathbb{R}$, we have*

$$\|(\mathbb{T}_{\mu,x_0}\circ\Pi - \Pi\circ\mathbb{T}_{\mu,x_0})u\|_{W^{l+1,p}_{\phi(\varphi_{\mu,x_0})^{-1}}(\Omega)} \le C\mu\|u\|_{W^{l,p}_\phi(\Omega)} \tag{4.13}$$

where the constant C depends on C_ϕ, but is independent of μ, u, x_0 and on the concrete choice of the weight ϕ. Moreover, the analogous result holds for the spaces $W^{l,p}_{b,\phi}(\Omega)$ as well.

Proof. Indeed, let $u \in W^{l,p}_\phi(\Omega)$ be arbitrary and let

$$u = v + \nabla_x p, \quad \varphi u = v_\varphi + \nabla_x p_\varphi \tag{4.14}$$

be decompositions (4.6) for functions u and φu respectively (which exist due to Theorem 4.4) and $\varphi := \varphi_{\mu,x_0}$. Then,

$$\Delta_x p_\varphi = \operatorname{div}(\varphi u), \quad \Delta_x p = \operatorname{div} u, \quad \partial_n p = l_n u, \quad \partial_n p_\varphi = \varphi l_n u.$$

Let now $P_\varphi := p_\varphi - \varphi p$. Then, this function solves:

$$\Delta_x P_\varphi = \varphi' u_1 - 2\varphi'\partial_{x_1}p - \varphi'' p, \quad \partial_n P_\varphi = 0. \tag{4.15}$$

However, from Proposition 3.6, we obtain the weighted estimate of $\bar{P}_\varphi := P_\varphi - \mathbb{S}P_\varphi$ only. Moreover, from Theorem 4.4, we are able to extract the weighted estimate only for $\bar{p} := p - \mathbb{S}p$. To overcome this difficulty, we recall that, for proving (4.13), we only need that

$$\|v_\varphi - \varphi v\|_{W^{l+1,p}_{\phi\varphi^{-1}}(\Omega)} = \|\nabla_x p_\varphi - \varphi\nabla_x p\|_{W^{l+1,p}_{\phi\varphi^{-1}}(\Omega)} \le C\mu\|u\|_{W^{l,p}_\phi(\Omega)} \tag{4.16}$$

We claim that, for estimating this quantity, it is sufficient to have the proper estimates for $\bar{P}_\varphi$ and $\bar{p}$ only. Indeed,

$$\nabla_x p_\varphi - \varphi\nabla_x p = \begin{pmatrix} \partial_{x_1}\bar{P}_\varphi + \varphi'\bar{p} - \partial_{x_1}\mathbb{S}P_\varphi - \varphi'\mathbb{S} \\ \nabla_{x'}\bar{P}_\phi p \end{pmatrix}.$$

Furthermore, since the mean flux of v and v_φ equal zero, we infer from (4.14) that

$$\partial_{x_1}\mathbb{S}p_\varphi = \varphi\mathbb{S}u_1,\ \partial_{x_1}\mathbb{S}p = \mathbb{S}u_1 \tag{4.17}$$

and, consequently,

$$\partial_{x_1}\mathbb{S}P_\varphi - \varphi'\mathbb{S}p = \partial_{x_1}(\mathbb{S}p_\varphi - \varphi\mathbb{S}p) + \varphi'\mathbb{S}p = \partial_{x_1}\mathbb{S}p_\varphi - \varphi\partial_{x_1}\mathbb{S}p = \varphi\mathbb{S}u_1 - \varphi\mathbb{S}u_1 = 0.$$

Thus, we really need only the estimates for $\bar{P}_\varphi$ and $\bar{p}$ in order to finish the proof of the proposition.

We also recall that, exactly as in the proof of Theorem 4.4, we have the following estimate

$$\|\bar{p}\|_{W^{l+1,p}_\phi(\Omega)} \le C\|u\|_{W^{l,p}_\phi(\Omega)} \tag{4.18}$$

which, together with estimate (2.6) for φ' give the required estimate for $\varphi'\bar{p}$. So, we only need to estimate $\nabla_x\bar{P}_\varphi$. To this end, applying the operator $(\mathrm{Id} - \mathbb{S})$ to equation (4.15), we get

$$\Delta_x\bar{P}_\varphi = H_u(x) := \varphi'\bar{u}_1 - 2\varphi'\partial_{x_1}\bar{p} - \varphi''\bar{p},\quad \partial_n\bar{P}_\varphi = 0. \tag{4.19}$$

Using now estimate (4.18) together with inequality (2.6) for weights φ, we conclude that

$$\|H_u\|_{W^{l,p}_{\varphi^{-1}\phi}(\Omega)} \le C\mu\|u\|_{W^{l,p}_\phi(\Omega)}$$

where C is independent of μ. Applying now the result of Proposition 3.6 to equation (4.19) ($\mathbb{S}\bar{P}_\varphi = 0!$), we finally arrive at

$$\|\bar{P}_\varphi\|_{W^{l+2,p}_{\varphi^{-1}\phi}(\Omega)} \le C\mu\|u\|_{W^{l,p}_\phi(\Omega)}$$

which finishes the proof of Proposition 4.6 for the spaces $W^{l,p}_\phi$. The case of spaces $W^{l,p}_{b,\phi}$ is completely analogous and Proposition 4.6 is proved. □

We start now to study the Stokes operator $A := \Pi\Delta_x$ in weighted spaces. To this end, we need to define the spaces of distributions associated with divergent free vector fields. the corresponding functional spaces.

Definition 4.7. Let Ω be a cylinder and let $\mathcal{D}_{\mathrm{div}}(\Omega)$ be the space of all smooth divergent free vector fields in Ω with compact support. As usual, we denote by $\mathcal{D}'_{\mathrm{div}}(\Omega)$ the space of all linear continuous functionals on $\mathcal{D}_{\mathrm{div}}(\Omega)$. We denote also by $\mathcal{H}^{-1,p}(\Omega_s) \subset \mathcal{D}'_{\mathrm{div}}(\Omega_s)$ the conjugate space to $\mathcal{V}^q(\Omega_s)$ with the standard norm.

Finally, for every weight function ϕ of exponential growth rate μ, we define the spaces $\mathcal{H}^{-1}_\phi(\Omega)$ and $\mathcal{H}^{-1}_{b,\phi}(\Omega)$ as subspaces of $\mathcal{D}'_{\mathrm{div}}(\Omega)$ with the following finite norms:

$$\|u\|^p_{\mathcal{H}^{-1,p}_\phi(\Omega)} := \int_{s\in\Omega}\phi(s)^p\|u\|^p_{\mathcal{H}^{-1,p}(\Omega_s)}\,ds < \infty,$$
$$\|u\|_{\mathcal{H}^{-1,p}_{b,\phi}(\Omega)} := \sup_{s\in\mathbb{R}}\{\phi(s)\|u\|_{\mathcal{H}^{-1,p}(\Omega_s)}\} < \infty.$$

Arguing exactly as in Proposition 2.8, one can show that

$$\mathcal{H}_\phi^{-1,p}(\Omega) = [\mathcal{V}^q_{\phi^{-1}}(\Omega)]^*. \tag{4.20}$$

We however note that the spaces $\mathcal{H}_\phi^{-1,p}(\Omega)$ *are not* the subspaces of usual distributions $\mathcal{D}'(\Omega)$ and, in a fact, larger than the corresponding spaces $[W_\phi^{-1,p}(\Omega)]^2$ of distributions. Nevertheless, there is a natural map of $[W_\phi^{-1,p}(\Omega)]^2$ to $\mathcal{H}_\phi^{-1,p}(\Omega)$ (which is usually considered as an extension of the projector Π to the negative Sobolev spaces and is also denoted by Π)

$$\langle \Pi u, v\rangle_{\operatorname{div}} := \langle u, v\rangle\,, \quad \operatorname{div} v = 0 \tag{4.21}$$

where in the left-hand side we have the pairing in $\mathcal{D}'_{\operatorname{div}}(\Omega)\times\mathcal{D}_{\operatorname{div}}(\Omega)$ and in the right-hand side the standard pairing in distributional sense is written.

Thus, the Stokes operator $A = \Pi\Delta_x$ can be naturally extended to the operator from $\mathcal{V}^p_\phi(\Omega)$ to $\mathcal{H}_\phi^{-1,p}(\Omega)$ (and, analogously, in the spaces $\mathcal{V}^p_{b,\phi}(\Omega)$).

We are going to study the linear Stokes equation in a cylinder Ω:

$$\Delta_x u + \nabla_x p = g, \quad u\big|_{\partial\Omega} = 0, \ \operatorname{div} u = 0. \tag{4.22}$$

Or, in the equivalent operator form: $Au = \Pi g$. We start with recalling the standard regularity result for the unweighted case

Proposition 4.8. *For any $g \in W^{l,p}(\Omega)$, $l \ge -1$ is integer and $1 < p < \infty$, there exists a unique solution (u,p) (up to adding a constant to p; in the sense of distributions), such that $u \in \mathcal{V}^p(\Omega)\cap\mathcal{H}^{l+2,p}(\Omega)$, $\bar p := p - \mathbb{S}p \in W^{l+1,p}(\Omega)$, such that*

$$\|u\|_{W^{l+2,p}(\Omega)} + \|\nabla_x p\|_{W^{l,p}(\Omega)} + \|\bar p\|_{W^{l+1,p}(\Omega)} \le C\|g\|_{W^{l,p}(\Omega)} \tag{4.23}$$

where the constant C is independent on u.

Indeed, at least for bounded domains, this assertion is well-known (see, eg, [27], Proposition I.2.3) even in the case of non-homogeneous Stokes problem. For the Hilbert case $p = 2$, $l = -1$, the result follows immediately from the energy estimate and for other lth and pth, it can be easily verified by reducing the problem to the case of bounded domains via the standard localization technique, see also [3] and references therein.

We, however, note that, in contrast to the case of bounded domains, we are now able to control the L^p-norms pressure function $\bar p := p - \mathbb{S}p$ and $\partial_{x_1}\mathbb{S}p$ and the mean pressure $\mathbb{S}p$ may even grow as $|x_1| \to \infty$.

Our aim now is to obtain the weighted analogue of Proposition 4.8. To this end, it is more convenient to consider more general non-homogeneous analog of that equation:

$$\begin{cases} \Delta_x u + \nabla_x p = g, \quad \mathbb{S}u_1 \equiv 0, \ u\big|_{\partial\Omega} = 0,\\ \operatorname{div} u = h, \end{cases} \tag{4.24}$$

for some function h satisfying

$$\mathbb{S}h \equiv 0. \tag{4.25}$$

Theorem 4.9. *For every integer $l \ge -1$ and every $1 < p < \infty$, there exists $\mu_0 = \mu_0(p,\omega) > 0$ such that for every weight function ϕ of exponential growth rate $\mu \le \mu_0$, every $g \in W^{l,p}_\phi(\Omega)$ and every $h \in W^{l+1,p}_\phi(\Omega)$ satisfying zero flux condition (4.25), problem*

(4.24) *possesses a unique solution* (u,p) *such that* $u \in W_\phi^{l+2,p}(\Omega)$, $\bar p \in W_\phi^{l+1,p}(\Omega)$ *and the following estimate holds:*

$$\|u\|_{W_\phi^{l+2,p}(\Omega)} + \|\nabla_x p\|_{W_\phi^{l,p}(\Omega)} + \|\bar p\|_{W_\phi^{l+1,p}(\Omega)} \le C(\|g\|_{W_\phi^{l,p}(\Omega)} + \|h\|_{W_\phi^{l+1,p}(\Omega)}) \tag{4.26}$$

where the constant C *depends on* C_ϕ, *but is independent of the concrete form of* ϕ *and* g *and* h. *Moreover, the analogous result holds also for the spaces* $W_{b,\phi}^{l,p}$.

Proof. Let us first consider the unweighted non-homogeneous case $\phi = 1$ and $h \neq 0$. In the case $l = -1$, it can be easily reduced to the homogeneous case $h = 0$ by substructing a function $\tilde u \in W_0^{1,p}(\Omega)$ satisfying $\operatorname{div} \tilde u = h$. In order to construct this function, it is sufficient to solve the problem

$$\operatorname{div} u_s = h, \quad u\big|_{\partial\Omega_s} = 0, \quad x \in \Omega_s.$$

Since $\mathbb{S}h \equiv 0$, the mean value of h in Ω_s is also equal zero and, consequently, this problem has indeed a solution u_s which satisfies

$$\|u_s\|_{W^{1,p}(\Omega_s)} \le C\|h\|_{L^p(\Omega_s)}$$

with C independent of s. The required function $\tilde u$ can be then defined as follows:

$$\tilde u(x) := \tilde u_n(x), \quad x \in \Omega_n.$$

Thus, the assertion of the theorem is verified in the case $l = -1$ and $\phi = 1$. Using that estimate and the localization technique, one can verify the estimate for every integer $l \ge -1$.

Let us now consider the weighted case $\phi \neq 0$. As usual, it is sufficient to verify estimate (4.26) for the special exponential weights $\varphi_{\mu,x_0}(x_1)$ only (a general result will follow then from representations (2.11) and (2.13)). We also restrict ourselves by verifying only a priori estimate (4.26) (the existence of a solution will follow then in a standard way eg, by approximation g and h by functions with finite support and passing to the limit).

Let (u,p) be a desired solution of problem (4.24) and let $\varphi := \varphi_{\mu,x_0}(x_1)$ for some $x_0 \in \mathbb{R}$. We set

$$u_\varphi := \varphi u, \quad p_\varphi := \varphi p - \varphi' \int_0^{x_1} (\mathbb{S}p)(y)\, dy.$$

In this new variables problem (4.24) reads

$$\begin{cases} \Delta_x u_\varphi + \nabla_x p_\varphi = g_\varphi := \varphi g + 2\varphi' \partial_{x_1} u_1 + \varphi'' u_1 + \varphi' \bar p \vec e_1, \\ \operatorname{div} u_\varphi = h_\varphi := \varphi h + \varphi' u_1 \end{cases} \tag{4.27}$$

where $\vec e_1 := (1,0,0)$. We see that again $\mathbb{S}h_\varphi \equiv 0$ (since $\mathbb{S}u_1 \equiv 0$). Moreover, using inequalities (2.6) and obvious fact that $\bar p_\varphi = \varphi \bar p$, we establish that

$$\begin{aligned} &\|g_\varphi\|_{W^{l,p}(\Omega)} + \|h_\varphi\|_{W^{l+1,p}(\Omega)} \le \\ &\qquad \le C\left(\|g\|_{W_\varphi^{l,p}(\Omega)} + \|h\|_{W_\varphi^{l+1,p}(\Omega)}\right) + C\mu\left(\|u_\varphi\|_{W^{l+2,p}(\Omega)} + \|\bar p_\varphi\|_{W^{l+1,p}(\Omega)}\right) \end{aligned} \tag{4.28}$$

where the constant C is independent of μ (here we have implicitly used also Proposition 2.14). Applying now already proved non-weighted estimate (4.26) to equation (4.27), we will have

$$\|u_\varphi\|_{W^{l+2,p}(\Omega)} + \|\nabla_x p_\varphi\|_{W^{l,p}(\Omega)} + \|\bar p_\varphi\|_{W^{l+1,p}(\Omega)} \le C(\|g_\varphi\|_{W^{l,p}(\Omega)} + \|h_\varphi\|_{W^{l+1,p}(\Omega)}).$$

Combining now this estimate with (4.28) and fixing $\mu = \mu_0$ being small enough, we arrive at

$$\|u_\varphi\|_{W^{l+2,p}(\Omega)} + \|\nabla_x p_\varphi\|_{W^{l,p}_\phi(\Omega)} + \|\bar p_\varphi\|_{W^{l+1,p}(\Omega)} \le C(\|g\|_{W^{l,p}_\varphi(\Omega)} + \|h\|_{W^{l+1,p}_\varphi(\Omega)})$$

which together with Proposition 2.14 gives (4.26) and finishes the proof of the theorem. □

We conclude the section by formulating several useful corollaries of the proved theorem.

Corollary 4.10. *Let Ω be a cylinder and let $A := \Pi\Delta_x$. Then, for every $1 < p < \infty$ and $l = 0, -1$ there exists positive $\mu_0 = \mu_0(p)$ such that, for every weight function of a sufficiently small exponential growth rate ($\mu \le \mu_0$) operator A realizes an isomorphism between spaces $\mathcal{V}^p_\phi(\Omega) \cap \mathcal{H}^{l+2,p}_\phi(\Omega)$ and $\mathcal{H}^{l,p}_\phi(\Omega)$ and the following estimate holds:*

$$C^{-1}\|u\|_{\mathcal{H}^{l+2,p}_\phi(\Omega)} \le \|\Pi\Delta_x u\|_{\mathcal{H}^{l,p}_\phi(\Omega)} \le C\|u\|_{\mathcal{H}^{l+2,p}_\phi(\Omega)} \tag{4.29}$$

where the constant C depends on C_ϕ, but is independent of the concrete choice of the weight function ϕ. Moreover, the analogous result holds for the spaces $\mathcal{H}^{l,p}_{b,\phi}(\Omega)$ as well.

Proof. Indeed, estimates in the right-hand side of (4.29) follow from Theorem 4.4 (in the case $l = 0$) and from the definition of Π for negative Sobolev spaces (if $l = -1$). The left-hand side of (4.29) follows immediately from Theorem 4.9 with $h \equiv 0$ by applying the projector Π to both sides of equation (4.22) (in the case $l = -1$ one needs to use also the obvious fact that, for every $g \in \mathcal{H}^{-1,p}_\phi(\Omega)$ there exists an extension $\tilde g \in W^{-1,2}_\phi(\Omega)$ such that $\Pi\tilde g = g$). □

Corollary 4.11. *Let the assumptions of Corollary 4.10 hold and let $p = 2$. Then, for every weight function with a sufficiently small growth rate μ, we have*

$$C^{-1}(\phi\Delta_x u, \phi\Delta_x u) \le (\phi\Pi\Delta_x u, \phi\Pi\Delta_x u) \le C(\phi\Delta_x u, \phi\Delta_x u) \tag{4.30}$$

where $(\cdot,\cdot)$ denotes the standard inner product in $[L^2(\Omega)]^3$ and the constant C is independent of the concrete choice of the weight ϕ and $u \in \mathcal{V}^2_\phi(\Omega) \cap \mathcal{H}^{2,2}_\phi(\Omega)$.

Indeed, estimate (4.30) is an immediate corollary of (4.29) with $p = 2$ and the following elliptic regularity estimate for the Laplacian in Ω with Dirichlet boundary conditions:

$$C^{-1}\|u\|_{W^{2,2}_\phi(\Omega)} \le \|\Delta_x u\|_{L^2_\phi(\Omega)} \le C\|u\|_{W^{2,2}_\phi(\Omega)}, \tag{4.31}$$

see Proposition 3.1.

5. An auxiliary non-stationary Stokes problem

In this section, we study the following non-stationary linear Stokes problem in a cylindrical domain Ω:

$$\begin{cases} \partial_t w = \Delta_x w - \nabla_x q, \\ \operatorname{div} w = h(t), \quad \mathbb{S}w_1 \equiv 0, \\ w\big|_{\partial\Omega} = 0, \quad \Pi w\big|_{t=0} = 0, \end{cases} \tag{5.1}$$

where $h(t) = h(t,x)$ is a given function satisfying

$$\mathbb{S}h(t)(x_1) \equiv 0, \quad t \in [0,T], \quad x_1 \in \mathbb{R}. \tag{5.2}$$

This auxiliary problem will be essentially used in the next section in order to obtain the weighted energy estimates for weak solutions of the nonlinear Navier-Stokes system.

The following theorem gives a priori estimates and the solvability result for problem (5.1).

Theorem 5.1. *There exists a positive* μ_0 *such that, for every weight function* ϕ *of sufficiently small exponential growth rate* μ $(\mu \le \mu_0)$ *and every*

$$h \in L^2([0,T], W^{1,2}_\phi(\Omega)) \cap C([0,T], L^2_\phi(\Omega)) \tag{5.3}$$

for which (5.2) *is satisfied problem* (5.1) *possesses a unique solution* w *from the class*

$$w \in L^2([0,T], W^{2,2}_\phi(\Omega)) \cap C([0,T], W^{1,2}_\phi(\Omega)), \\ \partial_t \Pi w \in L^2([0,T], L^2_\phi(\Omega)), \quad q \in \mathcal{D}'([0,T]\times\Omega) \tag{5.4}$$

and satisfying the following estimates:

$$\int_0^T e^{-\alpha|t-s|}(\|\partial_t \Pi w(s)\|^2_{L^2_\phi(\Omega)} + \|w(s)\|^2_{W^{2,2}_\phi(\Omega)})\,ds \le \\ \le C\int_0^T e^{-\alpha|t-s|}\|h(s)\|^2_{W^{1,2}_\phi(\Omega)}\,ds, \\ \|w(t)\|^2_{W^{1,2}_\phi(\Omega)} \le C\left(\|h(t)\|^2_{L^2_\phi(\Omega)} + \int_0^T e^{-\alpha|t-s|}\|h(s)\|^2_{W^{1,2}_\phi(\Omega)}\,ds\right) \tag{5.5}$$

where α *is a sufficiently small positive constant depending only on* μ_0 *and the constant* C *depends on* C_ϕ*, but is independent of the concrete choice of the weight* ϕ*.*

Proof. In order to solve (5.1), we are going to reduce it to the divergent free case. To this end, for every $t \in [0,T]$, we introduce $v(t) = K_{h(t)}$ as a solution of the following stationary Stokes problem:

$$\Delta_x v - \nabla_x r = 0, \quad \operatorname{div} v = h(t), \quad v\big|_{\partial\Omega} = 0. \tag{5.6}$$

Indeed, due to Theorem 4.9, there exists positive μ_0 such that, for every weight function of sufficiently small exponential growth rate μ $(\mu \le \mu_0)$ and every $h \in W^{l,2}_\phi(\Omega)$, $l = 0,1$ satisfying (5.2), equation (5.6) possesses a unique solution $v \in W^{l+1,2}_\phi(\Omega)$, $\mathbb{S}v_1 \equiv 0$, so the operator $K_{h(t)}$ is well-defined. Moreover, the following estimate holds:

$$\|v\|_{W^{l+1,2}_\phi(\Omega)} \le C\|h\|_{W^{l,2}_\phi(\Omega)} \tag{5.7}$$

where the constant C depends on C_ϕ, but is independent of the concrete choice of the weight ϕ.

Let us introduce now a new dependent variable $\bar w(t) := w(t) - v(t)$. This function obviously satisfies the following equation:

$$\partial_t(\bar w + v) = \Delta_x \bar w - \nabla_x \bar q, \quad \operatorname{div} \bar w = 0, \ \bar w\big|_{\partial\Omega} = 0, \ \bar w\big|_{t=0} = -\Pi v\big|_{t=0}. \tag{5.8}$$

Applying the projector Π to both parts of (5.8), we infer

$$\partial_t(\bar w+\Pi v)=\Pi\Delta_x\bar w,\quad \operatorname{div}\bar w=0,\quad \bar w\big|_{\partial\Omega}=0,\ \bar w\big|_{t=0}=-\Pi v\big|_{t=0}. \tag{5.9}$$

In order to obtain a priori estimate for solutions of (5.9), we multiply it by the expression

$$\varphi_{2\mu,x_0}(x_1)(\partial_{x_2}^2+\partial_{x_3}^2)(\bar w+\Pi v)+\partial_{x_1}[\varphi_{2\mu,x_0}(x_1)\partial_{x_1}(\bar w+\Pi v)]$$

where $x_0\in\mathbb{R}$ is arbitrary, $\mu>0$ is small enough and the weight φ is defined by (2.4). Then, we get

$$\begin{aligned}1/2\partial_t(\varphi_{2\mu,x_0},|\nabla_x(\bar w+\Pi v)|^2)+(\varphi_{2\mu,x_0}\Pi\Delta_x\bar w,\Delta_x\bar w)=\\ =-(\varphi'_{2\mu,x_0}\Pi\Delta_x\bar w,\partial_{x_1}\bar w)-(\varphi_{2\mu,x_0}\Pi\Delta_x\bar w,\Delta_x\Pi v)-(\varphi'_{2\mu,x_0}\Pi\Delta_x\bar w,\partial_{x_1}\Pi v).\end{aligned} \tag{5.10}$$

We estimate the second term in the left-hand side of (5.10) using estimates (4.13), (4.7) and (4.30) in the following way:

$$\begin{aligned}(\varphi_{2\mu,x_0}\Pi\Delta_x\bar w,\Delta_x\bar w)=(\varphi_{2\mu,x_0},|\Pi\Delta_x\bar w|^2)-\\ -(\Pi\Delta_x\bar w,(\varphi_{2\mu,x_0}\circ\Pi-\Pi\circ\varphi_{2\mu,x_0})\Delta_x\bar w)\ge C(\varphi_{2\mu,x_0},|\Delta_x\bar w|^2)-\\ -C_1(\varphi_{-2\mu,x_0},|(\varphi_{2\mu,x_0}\circ\Pi-\Pi\circ\varphi_{2\mu,x_0})\Delta_x\bar w|^2)\ge(C_2-C_3\mu)\|\Delta_x\bar w\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)}\end{aligned} \tag{5.11}$$

where the constants C_i are independent of μ and x_0. Fixing now μ to be small enough, estimating the right-hand side of (5.10) by Hölder inequality and using (4.7) and (4.29), we have

$$\begin{aligned}\partial_t(\|\nabla_x(\bar w+\Pi v)\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)})+\\ +\alpha(\|\Delta_x\bar w\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)}+\|\nabla_x(\bar w+\Pi v)\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)})\le C\|v\|^2_{W^{2,2}_{\varphi_{\mu.x_0}}(\Omega)}\end{aligned} \tag{5.12}$$

where the positive constants α and C are independent of $x_0\in\mathbb{R}$ (here we have also implicitly used that $\|\nabla_x(\bar w+\Pi v)\|_{L^2_{\varphi_{\mu,x_0}}(\Omega)}\le C(\|\nabla_x\bar w\|_{L^2_{\varphi_{\mu,x_0}}(\Omega)}+\|v\|_{W^{2,2}_{\varphi_{\mu,x_0}}(\Omega)}))$

Applying the Gronwall inequality to (5.12) and using estimate (5.7) with $l=1$ (for every fixed t), we arrive at

$$\begin{aligned}\|\nabla_x(\bar w(t)+\Pi v(t))\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)}+\int_0^t e^{-\alpha(t-s)}\|\bar w(s)\|^2_{W^{2,2}_{\varphi_{\mu,x_0}}(\Omega)}\,ds\le\\ \le C\int_0^t e^{-\alpha(t-s)}\|h(s)\|^2_{W^{1,2}_{\varphi_{\mu,x_0}}(\Omega)}\,ds\end{aligned} \tag{5.13}$$

(here we have used also that $\bar w(0)+\Pi v(0)=\Pi u(0)=0$). Moreover, since the constant C in (5.13) is independent of $x_0\in\mathbb{R}$, then, multiplying (5.13) by $\phi^2(x_0)$, integrating over $x_0\in\mathbb{R}$ and using (2.12), we obtain (exactly as in Section 3) the analogue of estimate (5.13) not only for the special weights φ_{μ,x_0}, but also for arbitrary weight ϕ of exponential growth rate $\varepsilon<\mu$.

In order to deduce a priori estimate (5.5) from (5.13), it only remains to recall that $w=\bar w+v$ and (due to (5.7) with $l=0$)

$$\|w(t)\|_{W^{1,2}_\phi(\Omega)}\le C(\|\nabla_x(\bar w(t)+\Pi v(t))\|_{L^2_\phi(\Omega)}+\|h(t)\|_{L^2_\phi(\Omega)}).$$

Indeed, this estimate together with (5.13) gives the required estimate for the $W^{1,2}_\phi$-norm of $w(t)$, estimate for the $W^{2,2}_\phi$-norm of w is also an immediate corollary of (5.13) and (5.7) with $l = 1$. Finally, the required estimate for $\partial_t \Pi w = \partial_t(\bar w + \Pi v)$ can be now obtained from equation (5.9). Thus, a priori estimate (5.5) is proved.

We also note that,

$$(\mathrm{Id} - \Pi)\partial_t w(t) = \partial_t (Id - \Pi) K_{h(t)} = (Id - \Pi) K_{\partial_t h(t)}, \tag{5.14}$$

and we see that, in contrast to the divergence free component of $\partial_t w$ its potential component *does not* belong to $L^2_\phi(\Omega)$ for general external forces h, but if, in addition, we have $\partial_t h \in L^2_\phi(\Omega)$, then (5.14) show that $\partial_t u$ will be also in $L^2_\phi(\Omega)$ and equation (5.1) can be naturally understood as an equality in $L^2([0,T], L^2_\phi(\Omega))$.

The above observation gives a natural way to construct the required solution $w(t)$ of (5.1) based on the obtained a priori estimate. Indeed, let us approximate the external force $h \in C([0,T], L^2_\phi(\Omega)) \cap L^2([0,T], W^{1,2}_\phi(\Omega))$ by a sequence of smooth (with respect to t and x) functions h^n having the compact support in x_1 and satisfying (5.2). Having such h^n, we construct the associated functions $v^n \in C^1([0,T], W^{2,2}(\Omega))$ by Theorem 4.9. Then, the associated equation (5.8) for $\bar w^n$ will be the standard non-stationary Stokes equation with the external forces $\partial_t v(t)$ belonging to the unweighted space $C([0,T], W^{2,2}(\Omega))$.

It is well-known that, for such external forces the non-stationary Stokes equation possesses a unique solution $\bar w^n \in W^{1,2}([0,T], L^2(\Omega)) \cap L^2([0,T], W^{2,2}(\Omega))$, see e.g. [4] or [5]. Thus, the approximating sequence of solutions w^n is constructed. We also note that, since $w^n(t)$ belongs to $L^2(\Omega)$ and divergent free, one has

$$\mathbb{S}\bar w^n_1 \equiv 0 \ \text{ and, consequently } \ \mathbb{S} w^n_1 \equiv 0. \tag{5.15}$$

Moreover, since h^n have compact support in x_1, then a priori estimate (5.5) holds for w^n uniformly with respect to $n \to \infty$. Passing now to the limit $n \to \infty$ and using (5.15) we construct the required solution $w(t)$. Theorem 5.1 is proved. □

Remark 5.2. Condition $\mathbb{S} w_1 \equiv 0$ is essential for the uniqueness part of Theorem 5.1. As we will see below, for every function $c(t) \in C_b(R)$, equation (5.1) possesses a solution w satisfying $\mathbb{S} w_1(t) \equiv c(t)$.

We conclude this section by preparing some technical tools for obtaining the energy estimates for the nonlinear Navier-Stokes equation in a cylindrical domain. To this end, we need to introduce some more functional spaces.

Definition 5.3. Let Ω be a cylinder and let $\mathbb{W}_b([0,T] \times \Omega)$ consists of vector fields $u \in L^2_b([0,T], \mathcal{V}^2_b(\Omega))$ (see Remark 2.4) such that the t-derivative $\partial_t u$ belongs to $\mathcal{D}'_{\mathrm{div}}(\Omega)$ a.e. and satisfies

$$\partial_t u \in L^2_b([0,T], \mathcal{H}^{-1,2}_b(\Omega)). \tag{5.16}$$

Let us consider also an arbitrary weight function θ of a sufficiently small exponential growth rate μ and a smooth nonnegative function ϕ satisfying the following assumptions:

$$|\phi'(s)| + \phi(s) \le C\theta(s), \quad s \in \mathbb{R}, \qquad \int_{s\in\mathbb{R}} \theta^2(s)\, ds < \infty. \tag{5.17}$$

In order to obtain the weighted energy estimates for the solution $u \in \mathbb{W}_b([0,T]\times\Omega)$ of the Navier-Stokes equation in $L^2_\phi(\Omega)$ (which contains $L^2_b(\Omega)$ due to the integrability assumption on ϕ), it would be natural to multiply it by the function $\phi^2 u$ and integrate over Ω, but, unfortunately, this function is no more divergent free and, consequently, this way does not allow to exclude the pressure. Instead of that, we will multiply, following [37]) it by the function $\phi^2 u - v$ where $v(t) := (\mathbb{P}_\phi u)(t)$ is the appropriate corrector which makes this multiplier divergent free. To this end, the function $v(t)$ should satisfy

$$\operatorname{div} v(t) \equiv h_u(t) := 2\phi\phi' u_1(t) \tag{5.18}$$

(here we have used that $\operatorname{div} u = 0$). Due to the integrability assumption on ϕ, the function $h \in L^2([0,T], W^{1,2}_{\theta^{-1}}(\Omega))$ and, moreover, since $\mathbb{S}u_1 \equiv 0$, we have $\mathbb{S}h \equiv 0$ and (5.2) is satisfied.

Furthermore, it is convenient for us to fix the corrector $v(t) := (\mathbb{P}_\phi u)(t)$ as a solution of the following auxiliary non-stationary Stokes problem in Ω:

$$-\partial_t v = \Delta_x v - \nabla_x q,\quad \operatorname{div} v(t) = h_u(t),\quad v\big|_{\partial\Omega} = 0,\quad \Pi v\big|_{t=T} = 0. \tag{5.19}$$

This equation, obviously, can be reduced to (5.1) by the time change $t \to T - t$. Thus, Theorem 5.1 and estimate (5.5) holds for this equation as well. The following theorem justifies our choice of the corrector $\mathbb{P}_\phi$ and gives the main technical tool for the weighted energy estimates of the Navier-Stokes equations.

Theorem 5.4. *Let Ω be a cylinder and let ϕ be a smooth nonnegative function, satisfying* (5.17) *for some square integrable weight θ of sufficiently small exponential growth rate μ. Then,*

$$\mathbb{W}_b([0,T]\times\Omega) \subset C([0,T], L^2_\theta(\Omega)). \tag{5.20}$$

Let also $\mathbb{P}_\phi$ be defined as the solving operator for problem (5.19)*. Then, the following equality holds:*

$$\frac{d}{dt}\big[1/2(\phi^2 u(t), u(t)) - (u(t), (\mathbb{P}_\phi u)(t))\big] + (\nabla_x u(t), \nabla_x(\phi^2 u(t))) = \\ = (\partial_t u(t) - \Pi\Delta_x u(t), \phi^2 u - (\mathbb{P}_\phi u)(t)) \tag{5.21}$$

which means that the function $1/2(\phi^2 u, u) - (u, \mathbb{P}_\phi u)$ is absolutely continuous as a scalar function on $[0,T]$ and (5.21) *holds almost everywhere.*

Proof. We give below only the formal derivation of (5.21) which can be justified in a standard way (see [37]; the detailed proof of embedding (5.20) also can be found there).

Indeed, since $\partial_t \Pi v + \Pi\Delta_x v \equiv 0$ and $\operatorname{div} u = \operatorname{div}(\phi^2 u - v) = 0$, integrating by parts, we have

$$(\partial_t u - \Pi\Delta_x u, \phi^2 u - v) = (\partial_t u - \Delta_x u, \phi^2 u - v) = \\ = \partial_t[1/2(\phi^2 u, u) - (u, v)] + (\nabla_x u, \nabla_x(\phi^2 u)) + (u, \partial_t v + \Delta_x v) = \\ = \partial_t[1/2(\phi^2 u, u) - (u, v)] + (\nabla_x u, \nabla_x(\phi^2 u)). \tag{5.22}$$

Theorem 5.4 is proved. ☐

6. Leray approximations to Navier-Stokes equations

The aim of that section is to verify the existence of spatially non-decaying solutions for the following Leray-Navier-Stokes equation in a cylindrical unbounded domain Ω:

$$\begin{cases} \partial_t u + (\Pi w, \nabla_x)u + \mathbb{S}u_1\partial_{x_1}u = \Delta_x u - \nabla_x p + g, \\ w - \alpha\Delta_x w = u, \quad w\big|_{\partial\Omega} = u\big|_{\partial\Omega} = 0, \quad \operatorname{div} u = 0, \\ u\big|_{t=0} = u_0 \end{cases} \tag{6.1}$$

where $\alpha > 0$ is a small regularizing parameter which is now fixed. In order to obtain the unique solvability, we endow the problem by the additional mean flux assumption

$$\mathbb{S}u_1(t) \equiv c \tag{6.2}$$

where c is a given constant which plays the role of a "boundary" condition at $x_1 = \pm\infty$.

The additional term $\mathbb{S}u_1\partial_{x_1}u = c\partial_{x_1}u$ is related with the fact that, in the case $\alpha = 0$, on the one hand, we should have the classical Navier-Stokes problem and, on the other hand, $w(t) = u(t)$ and $\Pi w(t) = u(t) - (c, 0, 0)$.

For simplicity we start our consideration with the case of zero flux

$$\mathbb{S}u_1(t) \equiv 0 \tag{6.3}$$

and the case of general flux c will be considered at the end of this section. We assume also that

$$g \in L^2_b(\mathbb{R}_+, L^2_b(\Omega)), \quad u_0 \in \mathcal{H}^2_b(\Omega) \tag{6.4}$$

and the solution u belongs to

$$u \in \mathbb{W}_b([0,T] \times \Omega) \tag{6.5}$$

(see Definition 5.3) and satisfies equation (6.1) in the sense of distributions $\mathcal{D}'_{\text{div}}(\Omega)$ over the divergent free vector fields.

Remark 6.1. Due to Theorem 5.4, $u \in L^\infty([0,T], \mathcal{H}^2_b(\Omega)) \cap C([0,T], \mathcal{H}^2_\phi(\Omega))$ for every square integrable weight function of exponential growth rate, so the initial condition $u\big|_{t=0} = u_0$ is well-defined. Moreover, since $u \in L^\infty([0,T], L^2_b(\Omega)) \cap L^2_b([0,T], \mathcal{V}^2_b(\Omega))$ then, due to Proposition 3.3 and Theorem 4.4 together with the embedding $W^{2,2} \subset L^\infty$, we conclude that we have

$$\Pi w \in L^\infty([0,T] \times \Omega) \tag{6.6}$$

Then, the inertial term $(\Pi w, \nabla_x u)$ satisfies

$$(\Pi w, \nabla_x)u \in L^2_b([0,T] \times \Omega) \subset L^2_b([0,T], \mathcal{H}^{-1,2}_b(\Omega)). \tag{6.7}$$

Thus,

$$\Pi[(\Pi w, \nabla_x)u] \in L^2_b([0,T], \mathcal{H}^{-1,2}_b(\Omega)) \tag{6.8}$$

(where Π is the projector on the divergent free vector fields introduced in Section 4). Applying this projector to equation (6.1), we obtain

$$\partial_t u = \Pi\Delta_x u - \Pi[(\Pi w, \nabla_x)u] + \Pi g \tag{6.9}$$

which shows that, indeed, the derivative $\partial_t u$ should belong to the space

$$\partial_t u \in L^2_b([0,T],\mathcal{H}_b^{-1,2}(\Omega)) \tag{6.10}$$

(see Corollary 4.10 for the term $\Pi\Delta_x u$). This shows that the definition of a solution u in the form (5.5) is not contradictory and equation (6.1) can be understood as equality (6.9) in the space (6.10). We also note that zero flux assumption (6.3) is now incorporated into the definition of the space $\mathbb{W}_b([0,T]\times\Omega)$.

We now introduce a special family of polynomial weight functions $\theta_\varepsilon(s) = \theta_{\varepsilon,x_0}(s)$ by the following expression:

$$\theta_{\varepsilon,x_0}(s) := \left(1+\varepsilon^2|s-x_0|^2\right)^{-1/2}, \quad \varepsilon>0, \quad s,x_0\in\mathbb{R}. \tag{6.11}$$

Obviously these functions are weight functions of exponential growth rate μ, for every $\mu>0$ with the constant C_{θ_ε} depending on μ, but is independent of $x_0\in\Omega$ and $\varepsilon\in[0,1]$. This means that all of the weighted estimates formulated in previous sections will hold for weights (6.11) with the constants independent of $\varepsilon\to 0$ which is crucial for our method. Moreover, these weights satisfy also the following improved version of (5.17):

$$|\phi'_{\varepsilon,x_0}(s)| \le \varepsilon[\phi_{\varepsilon,x_0}(s)]^2, \quad \|\phi_\varepsilon\|_{L^2(\mathbb{R}^n)}<\infty. \tag{6.12}$$

Thus, Theorem 5.4 holds for these weights as well.

The next proposition gives basic *uniform* with respect to α a priori estimate for the solutions of (6.1).

Proposition 6.2. *Let the above assumptions hold and let $u\in\mathbb{W}_b([0,T]\times\Omega)$ be a solution of the Leray-Navier-Stokes problem* (6.1). *Then, the following estimate holds:*

$$\sup_{s\in[0,T]}\{e^{-\beta|t-s|}\|u(s)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\} + (C_1 - C_2\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\times \\ \times\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds \le \\ \le C_3e^{-\beta t}\|u(0)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C_3\int_0^T e^{-\beta|t-s|}\|g(s)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\,ds \tag{6.13}$$

where the positive constants β and C_i, $i=1,2,3$ are independent of small $\alpha>0$, u, u_0, g, $\varepsilon\to 0$, T and x_0 (we recall that we write for brevity θ_ε instead of θ_{ε,x_0}).

Proof. Indeed, let u be a solution of (6.9) belonging to the above class. Then, due to Theorem 5.4, we have the following identity:

$$\frac{d}{dt}[1/2(\theta_\varepsilon^2u(t),u(t)) - (u(t),v(t))] + (\nabla_x u(t),\nabla_x(\theta^2_\varepsilon u(t))) = \\ = -(\theta^2_\varepsilon u(t)-v(t),(\Pi w(t),\nabla_x)u(t)-g(t))) \tag{6.14}$$

where $v := \mathbb{P}_{\theta_\varepsilon} u$ solves the auxiliary problem (5.19). Using (6.12) and the inequality $\|u\|_{L^2_{\theta_\varepsilon}(\Omega)} \le C\|\nabla_x u\|_{L^2_{\theta_\varepsilon}(\Omega)}$, we transform (6.14) as follows:

$$\frac{d}{dt}R_u(t) + \beta R_u(t) + 1/2\|u(t)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} \le |(\theta_\varepsilon^2 u(t), (\Pi w(t), \nabla_x u(t))|+ \\ + |(v(t), (\Pi w(t), \nabla_x) u(t))| + C\|g(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C\|v\|^2_{L^2_{[\theta_\varepsilon]^{-1}}(\Omega)} := H_u(t) \tag{6.15}$$

where $R_u(t) := 1/2\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} - (u(t), v(t))$. Applying now the Gronwall inequality to (6.15), we infer

$$R_u(t) + \int_0^t e^{-\beta(t-s)}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\, ds \le Ce^{-\beta t}R_u(0) + C\int_0^t e^{-\alpha(t-s)}H_u(s)\, ds. \tag{6.16}$$

We now need to estimate the auxiliary function $v(t)$. To this end, we note that, due to (6.11), the function $h_u(t) := 2\theta_\varepsilon\theta_\varepsilon' u(t)$ satisfies

$$\|h_u(t)\|_{W^{l,2}_{[\theta_\varepsilon]^{-2}}(\Omega)} \le C\varepsilon\|u(t)\|_{W^{l,2}_{\theta_\varepsilon}(\Omega)} \tag{6.17}$$

where the constant C is independent of $\varepsilon \to 0$. Applying now Theorem 5.1 to the auxiliary equation (5.19), we deduce the following estimate:

$$\begin{gathered} \|v(t)\|^2_{W^{1,2}_{[\theta_\varepsilon]^{-2}}(\Omega)} \le C\varepsilon^2\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C\varepsilon^2\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\, ds, \\ \int_0^T e^{-\beta|t-s|}\|v(s)\|^2_{W^{2,2}_{[\theta_\varepsilon]^{-2}}(\Omega)}\, ds \le C\varepsilon^2\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\, ds \end{gathered} \tag{6.18}$$

where $\beta > 0$ is small enough and the constant C are independent of α and $\varepsilon \to 0$. Inserting these estimates into (6.16) after simple transformations, we get

$$\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + \int_0^t e^{-\beta(t-s)}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\, ds \le Ce^{-\beta t}\|u_0\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + \\ + C\varepsilon^2\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\, ds + \int_0^T e^{-\beta|t-s|}H_u(s)\, ds. \tag{6.19}$$

This estimate, in turns implies in a standard way that, for sufficiently small $\varepsilon > 0$,

$$\sup_{s\in[0,T]}\{e^{-\beta|t-s|}\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\} + C'\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\, ds \le \\ \le Ce^{-\beta t}\|u_0\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C\int_0^T e^{-\beta|t-s|}\|g(s)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\, ds + \\ + C\int_0^T e^{-\beta|t-s|}|(\theta_\varepsilon^2 u(s), (u(s), \nabla_x)u(s))|\, ds + \\ + C\int_0^T e^{-\beta|t-s|}|(v(s), (u(s), \nabla_x)u(s))|\, ds := I_{u_0} + I_g + I_1 + I_2. \tag{6.20}$$

Indeed, in order to obtain the estimate for the first term in the left-hand side of (6.20), it is sufficient to multiply (6.19) by $e^{-\gamma|t_1-t|}$, where $\gamma < \beta$, take the supremum over $t \in [0, T]$ and use Proposition 2.5. Analogously, in order to obtain the estimate for the second term, we only need to integrate over $t\in[0,T]$ instead of taking the supremum (rigorously speaking, we obtain (6.20) for some new exponent γ which is less than β (say, $\gamma=\beta/2$), but, in order to simplify the notations, we denote this new exponent by β as well).

Thus, in order to finish the proof of Proposition 6.2, we only need to estimate the integrals I_1 and I_2 in the right-hand side of (6.20) uniformly with respect to $\alpha\to0$. To this end, we will use the uniform (with respect to α) estimate

$$\|\Pi w(t)\|_{L^2_{\theta_\varepsilon}(\Omega)}\le C\|u(t)\|_{L^2_{\theta_\varepsilon}(\Omega)} \tag{6.21}$$

which is an immediate corollary of Proposition 3.3 and Theorem 4.4.

Indeed, for the term I_1, integrating by parts in $(\theta_\varepsilon^2u,(\Pi w,\nabla_x)u)$ and using that $\operatorname{div}u=0$, (6.21) and inequality (6.12), we have

$$\begin{aligned}|(\theta_\varepsilon^2u,(\Pi w,\nabla_x)u)|=|(\theta_\varepsilon\theta_\varepsilon'(\Pi w)_1,|u|^2)|\le C\varepsilon([\theta_\varepsilon]^3|\Pi w|,|u|^2)\le\\ \le C_1\varepsilon\|\Pi w\|_{L^2_{\theta_\varepsilon}(\Omega)}\|u\|^2_{L^4_{\theta_\varepsilon}(\Omega)}\le C_2\varepsilon\|u\|_{L^2_{\theta_\varepsilon}(\Omega)}\|u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\end{aligned} \tag{6.22}$$

where the constant C_2 is independent of ε and α (here we have implicitly used also the embedding $W^{1,2}_{\theta_\varepsilon}(\Omega)\subset L^4_{\theta_\varepsilon}(\Omega)$ where the embedding constant is independent of ε, see Proposition 2.11).

Inserting this estimate into the expression for I_1, we arrive at

$$I_1\le C_3\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))}\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds. \tag{6.23}$$

Let us now estimate the integral I_2. To this end we will use the following embedding estimate of Proposition 2.11:

$$\|v\|_{L^\infty_{[\theta_\varepsilon]^{-2}}(\Omega)}\le C\|v\|_{W^{2,2}_{[\theta_\varepsilon]^{-2}}(\Omega)}$$

where again the constant C is independent of ε. Thus, we can estimate the term I_2 as follows:

$$\begin{aligned}I_2\le C\int_0^T e^{-\beta|t-s|}\|\Pi w(s)\|_{L^2_{\theta_\varepsilon}(\Omega)}\|\nabla_x u(s)\|_{L^2_{\theta_\varepsilon}(\Omega)}\|v(s)\|_{W^{2,2}_{[\theta_\varepsilon]^{-2}}(\Omega)}\,ds\le\\ \le C\|\Pi w(s)\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))}\int_0^T e^{-\beta|t-s|}(\varepsilon\|u(t)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}+\varepsilon^{-1}\|v(s)\|^2_{W^{2,2}_{[\theta_\varepsilon]^{-2}}(\Omega)})\,ds.\end{aligned} \tag{6.24}$$

Using now (6.18) and (6.21), we finally arrive at

$$I_2\le C_3\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))}\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds. \tag{6.25}$$

Inserting estimates (6.23) and (6.25) into the right-hand side of (6.20), we obtain (6.13) and finish the proof of Proposition 6.2. □

In order to deduce the existence of a solution $u\in\mathbb{W}_b([0,T]\times\Omega)$ of problem (6.1) from a priori estimate (6.13), we need the following simple proposition.

Proposition 6.3. *Let $w\in L^2_b(\Omega)$ and let the weight $\theta_\varepsilon=\theta_{\varepsilon,x_0}$ be the weight function defined by* (6.11). *Then, the following estimate holds:*

$$\|w\|_{L^2_{\theta_\varepsilon}(\Omega)}\le C\varepsilon^{-1/2}\|w\|_{L^2_b(\Omega)} \tag{6.26}$$

where the constant C is independent of $\varepsilon \to 0$ and $x_0 \in \mathbb{R}$.

Proof. Indeed, according to (2.11), we have

$$\|w\|^2_{L^2_{\theta_\varepsilon}(\Omega)} \le C\int_{s\in\mathbb{R}} \theta_\varepsilon(s)^2\|w\|^2_{L^2(\Omega_s)}\,ds \le C\|w\|^2_{L^2_b(\Omega)}\int_{s\in\mathbb{R}}(1+\varepsilon^2|s-x_0|^2)^{-1}\,ds =$$
$$= C\|w\|^2_{L^2_b(\Omega)}\varepsilon^{-1}\int_{s\in\mathbb{R}}(1+|s|^2)^{-1}\,ds = C_1\varepsilon^{-1}\|w\|^2_{L^2_b(\Omega)}$$

and Proposition 6.3 is proved. □

Proposition 6.3 allows to simplify basic a priori estimate (6.13) as follows.

Corollary 6.4. *Let the assumptions of Proposition 6.2 hold and let $u \in \mathbb{W}_b([0,T]\times\Omega)$ be a solution of* (6.1). *Then, the following estimate holds:*

$$\|u\|^2_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} + (C_1 - C_2\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\|u\|^2_{L^2_b([0,T],W^{1,2}_{\theta_\varepsilon}(\Omega))} \le$$
$$\le C_3\varepsilon^{-1}(\|u(0)\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T],L^2_b(\Omega))}) \tag{6.27}$$

where the positive constants α and C_i, $i = 1,2,3$ are independent of u, u_0, g, $\varepsilon \to 0$, T and x_0 (we recall that we write for brevity θ_ε instead of θ_{ε,x_0}).

Indeed, in order to deduce (6.27) from (6.13), it is sufficient to use (6.26), take the supremum over $t \in [0,T]$ and use (2.13).

We are now ready to prove the existence of a bounded solution of the Leray-Navier-Stokes problem (6.1).

Theorem 6.5. *Let the above assumptions hold. Then, problem* (6.1) *possesses at least one solution $u \in \mathbb{W}_b([0,T]\times\Omega)$ which satisfies the following estimate:*

$$\|u\|_{L^\infty([0,T],L^2_b(\Omega))\cap L^2_b([0,T],W^{1,2}_b(\Omega))} \le C(1 + \|u_0\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T]\times\Omega)}) \tag{6.28}$$

where the constant C is independent of small $\alpha > 0$ T, g and u_0.

Proof. The idea of the proof is based on the following observation: let

$$K_{u_0,g} := (1 + \|u_0\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T]\times\Omega)})^{1/2}. \tag{6.29}$$

Then, a priori estimate (6.27) gives the following conditional result: let the solution u a priori satisfy

$$\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} \le \frac{C_1}{2C_2\varepsilon}. \tag{6.30}$$

Then, we necessarily have

$$\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega)} + C_1/2\|u\|_{L^2_b([0,T],W^{1,2}_b(\Omega))} \le C_3^{1/2}\varepsilon^{-1/2}K_{u_0,g}. \tag{6.31}$$

Let us now fix $\varepsilon \ll 1$ in such way that

$$C_3^{1/2}\varepsilon^{-1/2}K_{g,u_0} < \frac{C_1}{2C_2\varepsilon} \tag{6.32}$$

or which is the same

$$\varepsilon \sim [K_{u_0,g}]^{-2}. \tag{6.33}$$

In this case estimates (6.30) and (6.31) allow to deduce estimate of the form (6.28) using the standard continuation by parameter arguments. Indeed, let u^s, $s \in [0,1]$ be a continuous curve of solutions of (6.1) such that

$$K_{u_0^s,g^s} \le K_{u_0^1,g^1} \tag{6.34}$$

and estimate (6.31) is satisfied for $s = 0$. Then, it is satisfied for $s = 1$ as well, since, due to (6.32), we cannot achieve the bound (6.30) before crossing the bound (6.31) and, consequently, the continuity arguments show that (6.31) holds for every $s \in [0,1]$.

Let us now proceed in more rigorous way. To this end, we first prove estimate (6.28) for the square integrable case:

$$u_0 \in \mathcal{H}^2(\Omega), \quad g \in L^2([0,T], L^2(\Omega)). \tag{6.35}$$

In this case, the Leray-Navier-Stokes problem has a unique square integrable solution u:

$$u \in C([0,T], L^2(\Omega)) \cap L^2([0,T], W^{1,2}(\Omega)). \tag{6.36}$$

which can be obtained exactly as in the case of bounded domains. Moreover, this solution depends continuously (in the metric of (6.36)) on the initial data u_0 and external forces g (this can be verified in a standard way, since the above α-regularization makes the inertial term subordinated to the linear part of the equation, see e.g. [4], [5], [28]).

Thus, the solutions u^s, $s \in [0,1]$ associated with the initial data $u_0^s := su_0$, $g^s := sg$ generate a continuous curve in the space (6.36) and, evidently, (6.31) is satisfied for $u^0 \equiv 0$. Therefore, due to the above continuity arguments, we have estimate (6.31) for $s = 1$ as well. Taking into account (6.33), we can rewrite it in the following way:

$$\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon,x_0}(\Omega))\cap L^2_b([0,T],W^{1,2}_{\theta_\varepsilon,x_0}(\Omega))} \le C[K_{u_0,g}]^2 \tag{6.37}$$

where the constant C is independent of $x_0 \in \mathbb{R}$. Using now the obvious estimate

$$\|v\|_{W^{l,2}_b(\Omega)} \le C \sup_{x_0\in\mathbb{R}} \|v\|_{W^{l,2}_{\theta_\varepsilon,x_0}(\Omega)}, \quad l = 0,1$$

where C is independent of $\varepsilon \ll 1$, we deduce the required estimate (6.28).

Thus, the assertion of the theorem is verified in the square integrable case (6.35). Let us now consider the general case of u_0 and g satisfying only assumption (6.4). To this end, we approximate the data u_0 and g by a sequence of square integrable ones u_0^n and g^n satisfying (6.35). Moreover, we assume that

$$\|u_0^n\|_{\mathcal{H}^2_b(\Omega)} + \|g^n\|_{L^2_b([0,T]\times\Omega)} \le C \tag{6.38}$$

where C is independent of n and that

$$u_0^n \to u_0 \ \text{ in } \ L^2_{loc}(\overline{\Omega}), \quad g^n \to g \ \text{ in } \ L^2_{loc}([0,T]\times\overline{\Omega}). \tag{6.39}$$

Then, due to already proved part of estimate (6.28), the associated solution u^n of the Navier-Stokes equation (belonging to the class (6.36)) satisfies

$$\|u^n\|_{L^\infty([0,T],L^2_b(\Omega))} + \|u^n\|_{L^2_b([0,T],W^{1,2}_b(\Omega))} \le C_1 \tag{6.40}$$

where C_1 is also independent of n. Moreover, from equation (6.9), we infer also that

$$\|\partial_t u^n\|_{L^2_b([0,T],\mathcal{H}^{-1,2}_b(\Omega))} \le C. \tag{6.41}$$

Thus, passing to the subsequence if necessary, we can assume without loss of generality that the sequence u^n converge weakly to some $u \in \mathbb{W}_b([0,T]\times\Omega)$ in the local topology, i.e., for every square integrable weight ϕ satisfying (5.17), we have

$$u^n \to u \ \text{ weakly in } \ \mathbb{W}_\phi([0,T]\times\Omega). \tag{6.42}$$

Moreover, due to the embedding $\mathbb{W}_\phi([0,T]\times\Omega) \subset C([0,T], L^2_\phi(\Omega))$ (see Theorem 5.4), the limit function u satisfies the initial condition $u(0) = u_0$.

Thus, we only need to verify that the constructed function u satisfy equation (6.1) (or which is the same, equation (6.9)) in the sense of distributions, i.e., we need to verify that, for every $U \in C_0^\infty((0,T)\times\Omega)$ with $\operatorname{div} U = 0$, we have

$$-\langle u, \partial_t U\rangle = \langle u, \Delta_x U\rangle - \langle(\Pi w, \nabla_x)u, U\rangle + \langle g, U\rangle\,. \tag{6.43}$$

(the passing to the limit in the linear equation $w^n - \alpha\Delta_x w^n = u^n$ is obvious) Indeed, since u^n solves the Leray-Navier-Stokes equations, we have

$$-\langle u^n, \partial_t U\rangle = \langle u^n, \Delta_x U\rangle - \langle(\Pi w^n, \nabla_x)u^n, U\rangle + \langle g^n, U\rangle\,. \tag{6.44}$$

Moreover, passing to the limit $n\to\infty$ in all linear terms of (6.44) is evident and we only need to pass to the limit in the inertial term $(\Pi w^n, \nabla_x)u^n$. To this end, it is sufficient to verify that

$$u^n \to u \quad \text{strongly in the space} \quad L^2_{loc}([0,T]\times\overline{\Omega}), \tag{6.45}$$

Indeed, since $\nabla_x u^n \to \nabla_x u$ weakly in $L^2_{loc}([0,T]\times\overline{\Omega})$. Moreover, due to Proposition 3.1, Theorem 4.4 and convergence (6.45), we have the analogous strong convergence of Πw^N to Πw. This, in turns, implies the weak convergence $(\Pi w^n, \nabla_x)u^n \to (\Pi w, \nabla_x)u$ in $L^1_{loc}([0,T]\times\overline{\Omega})$.

In order to prove (6.45), we note that, due to (6.42), for every integrable weight ϕ we have

$$\partial_t u^n \to \partial_t u \ \text{ weakly in } \ L^2([0,T], \mathcal{H}^{-1,2}_{\phi^2}(\Omega)). \tag{6.46}$$

Furthermore, due to (6.42), we have also

$$u^n \to u \ \text{ weakly in } \ L^2([0,T], \mathcal{V}^2_\phi(\Omega)). \tag{6.47}$$

Since, we have the standard embeddings

$$\mathcal{V}^2_\phi(\Omega) \subset\subset \mathcal{H}^2_{\phi^2}(\Omega) \subset \mathcal{H}^{-1,2}_{\phi^2}(\Omega)$$

and the first embedding is compact, then, due to the compactness theorem (see e.g. [27]), we have the strong convergence $u^n \to u$ in $L^2([0,T], \mathcal{H}^2_{\phi^2}(\Omega))$. Thus, the convergence (6.45) is proved and Theorem 6.5 is also proved. □

We now return to the general case of nonzero flux $c \neq 0$ in (6.2). Then the Leray-Navier-Stokes equation (6.1) with $g = 0$ possesses the classical Poiseuille solution

$$\vec{v}_c(x) := c\Big(\frac{|\omega|}{\|v_c\|_{L^1(\omega)}} v_c(x'), 0, 0\Big) \tag{6.48}$$

where the scalar function $v_c = v_c(x')$ solves the Laplace equation in ω:

$$\Delta_{x'} v_p = 1, \quad v_p\big|_{\partial\omega} = 0. \tag{6.49}$$

Indeed, the associated vector field $\vec{w}_c$, obviously, has the following form:

$$\vec{w}_c(x) = (w_c(x'), 0, 0)$$

where $w_c(x')$ solves

$$w_c - \alpha\Delta_{x'} w_c = v_c, \quad w_c\big|_{\partial\omega} = 0. \tag{6.50}$$

In particular, the vector field $\vec{w}_c$ is divergent free and, consequently,

$$\Pi\vec{w}_c = \vec{w}_c - (\mathbb{S}w_c, 0, 0) = (\bar{w}_c, 0, 0). \tag{6.51}$$

Using (6.51), one can immediately verify that $\vec{v}_c$ solves the Leray-Navier-Stokes problem (6.1), (6.2) with $g \equiv 0$.

Therefore, the difference $u - \vec{v}_c$ has zero flux and, consequently, it is natural to define a weak solution of (6.1) as a function $u \in \vec{v}_c + \mathbb{W}_b([0, T] \times \Omega)$ which satisfies (6.1) in the sense of distributions over the divergent free vector fields. Moreover, the assumption on u_0 should be also naturally replaced by

$$u_0 \in \vec{v}_c + \mathcal{H}^2_b(\Omega).$$

The next theorem is an analogue of Theorem 6.5 for the case of nonzero flux.

Theorem 6.6. *Let the above assumptions hold. Then, for every $c \in \mathbb{R}$, $u_0 \in \vec{v}_c + \mathcal{H}^2_b(\Omega)$ and $g \in L^2_b([0, T] \times \Omega)$, the Navier-Stokes problem (6.1), (6.2) possesses at least one weak solution $u \in v_c + \mathbb{W}_b([0, T] \times \Omega)$ which satisfies the following estimate:*

$$\|u\|_{L^\infty([0,T],L^2_b(\Omega))\cap L^2_b([0,T],W^{1,2}_b(\Omega))} \le C(1 + |c|^3 + \|u_0\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T]\times\Omega)}) \tag{6.52}$$

where the constant C is independent of α, T, u_0, g and c.

Proof. We want to reduce the general case to the particular case of zero flux considered above. The most natural way to do so is to make the variable change $\bar{u} := u - \vec{v}_c$ where the $\vec{v}_c$ is the Poiseuille flow, but this scheme does not work, since the Poiseuille flow can be unstable. Instead of this, we construct below some special solution of the stationary Navier-Stokes problem (6.1), (6.2) of the form $V_c(x) := (V_c(x'), 0, 0)$, $V_c\big|_{\partial\omega} = 0$ (with the appropriate nonzero external force g_c) and introduce a new unknown $\bar{u} := u - V_c$. Then, this function belongs to $\mathbb{W}_b([0, T] \times \Omega)$ and solves

$$\begin{cases} \partial_t\bar{u} + (\Pi\bar{w}, \nabla_x)\bar{u} = \Delta_x\bar{u} + L_{V_c}\bar{u} - \nabla_x p + g - g_c,\\ \bar{w} - \alpha\Delta_x\bar{w} = \bar{u}, \quad \bar{w}\big|_{\partial\omega} = 0,\\ \operatorname{div}\bar{u} = 0, \quad \bar{u}\big|_{\partial\Omega} = 0, \quad \mathbb{S}\bar{u}_1 \equiv 0,\\ \bar{u}\big|_{t=0} = \bar{u}_0 := u_0 - V_c. \end{cases} \tag{6.53}$$

which differs from (6.1) by the presence of the additional linear operator L_{V_c}

$$L_{V_c}z := (\Pi W_c, \nabla_x)z + (\Pi w, \nabla_x)V_c - c\partial_{x_1}z, \ w - \alpha\Delta_x w = z \tag{6.54}$$

The next Lemma specifies the choice of the special function V_c.

Lemma 6.7. *Let $c \in \mathbb{R}$ be arbitrary. Then, there exist a vector field $V_c(x) = (V_c(x'), 0, 0)$, $V_c\big|_{\partial\omega} = 0$ such that*

$$(L_{V_c}z, z) \le 1/2\|z\|^2_{W^{1,2}(\Omega)}, \quad \forall w \in W^{1,2}_0(\Omega) \tag{6.55}$$

and

$$\|V_c\|_{C(\omega)} \le \kappa|c|, \quad \|\nabla_{x'}V_c'\|_{L^2(\omega)} \le \kappa(|c|^{3/2}+|c|), \tag{6.56}$$

where the constant κ is independent of c and α and $g_c = -\Delta_{x'}V_c$.

Proof. Let $\delta > 0$ be small and $\omega_\delta := \{x' \in \omega,\ \operatorname{dist}(x', \partial\omega) < \delta\}$. Then, as known ω_δ will be smooth sub-domain of ω if δ is small enough.

We seek for the the required function $V_c(x') \in W_0^{1,2}(\omega)$ in the following form:

$$V_c(z) = \begin{cases} \lambda, & z \in \omega\backslash\omega_\delta, \\ \lambda\delta^{-1}\operatorname{dist}(z,\partial\omega), & z \in \omega_\delta \end{cases} \tag{6.57}$$

where $\delta \ll 1$ is small positive constant and λ is some parameter. Obviously, in order to satisfy the flux condition, we need

$$|\omega|c = \int_\omega V_c(z)\,dz = \lambda(|\omega| - |\omega_\delta| + \delta^2/2|\partial\omega|). \tag{6.58}$$

We will fix below $\delta \sim |c|^{-1}$. Then, formula (6.58) shows that $\lambda = c + o(c^{-1})$.

So, we now need to fix δ in such way that (6.55) would be satisfied. Indeed, let $w \in [W_0^{1,2}(\Omega)]^2$. Then, direct calculation gives

$$(L_{V_c}z, z) = ((\Pi w)_2\partial_{x_2}V_c + (\Pi)_3\partial_{x_3}V_c, z_1) = \lambda\delta^{-1}(\Pi w.\vec{n}, z_1)_{L^2(\mathbb{R}\times\omega_\delta)} \tag{6.59}$$

where $\vec{n}(x') := \nabla_{x'}\operatorname{dist}(x',\partial\omega)$. Since $\vec{n}\big|_{\partial\omega}$ coincides with normal vector to $\partial\omega$, we have that the function $Z := \Pi w.\vec{n}$ has zero trace at $\partial\omega$:

$$Z\big|_{\partial\omega} = 0$$

(here we have implicitly used that $l_n u = 0$ for every $u \in \mathcal{H}^2(\Omega)$). Thus, (6.59) can be rewritten in the form:

$$|(L_{V_c}z, z)| \le \lambda\delta^{-1}\int_{\mathbb{R}}\int_{\omega_\delta} |Z(x_1,x')|^2 + |z(x_1,x')|^2\,dx'\,dx_1. \tag{6.60}$$

Using now the standard estimate

$$\int_{\omega_\delta} |u(x')|^2\,dx' \le C\delta^2\int_{\omega_\delta}|\nabla_{x'}u(x)|^2\,dx'$$

which holds for every $u \in W^{1,2}(\omega_\delta)$ such that $u\big|_{\partial\omega} = 0$, we transform (6.60) as follows:

$$|(L_{V_c}w, w)| \le C\lambda\delta(\|Z\|^2_{W^{1,2}(\Omega)} + \|z\|^2_{W_0^{1,2}(\Omega)}) \tag{6.61}$$

where C is independent of λ and δ.

Furthermore, according to Theorem 4.4, we have

$$\|Z\|_{W^{1,2}(\Omega)} \le C\|w\|_{W^{1,2}(\Omega)}.$$

We now recall that w solves the Laplace equation

$$w - \alpha\Delta_x w = z, \quad z\big|_{\partial\Omega} = 0. \tag{6.62}$$

Therefore, multiplying (6.62) by $\Delta_x w$ and using that $u\big|_{\partial\Omega} = 0$, we infer

$$\|w\|_{W^{1,2}(\Omega)} \le C\|z\|_{W^{1,2}(\Omega)}.$$

where the constant C is independent of α. Thus, finally,

$$|(L_{V_c}z,z)|\le C'\lambda\delta\|z\|^2_{W^{1,2}_0(\Omega)}$$

with C' independent of α, λ and δ. So, it only remains to fix δ in such way that $C'\lambda\delta\le 1/2$. Since λ should be close to c, this gives the estimate for the desired δ:

$$\delta\sim C|c|^{-1} \tag{6.63}$$

for some constant C independent of c and λ. It is also not difficult to verify that the function V_c thus defined satisfies also inequalities (6.56). Lemma 6.7 is proved. □

We are now ready to finish the proof of Theorem 6.6. This proof repeats with minor changing the proof of Theorem 6.5 for the case of zero flux. The only difference is that we now have the additional linear term $L_{V_c}\bar u$ in equation (6.53) which is not essential due to estimate (6.55).

Indeed, proving the analogue of basic a priori estimate (6.13), we will only have the additional terms

$$(L_{V_c}\bar u,\theta_\varepsilon^2\bar u)-(L_{V_c}\bar u,v)-(\nabla_{x'}V_c,\nabla_{x'}(\theta_\varepsilon^2\bar u-v)) \tag{6.64}$$

in the right-hand side of (6.15). In order to estimate the first term in (6.64), we will use the following commutation relation:

$$|(\theta_\varepsilon L_{V_c}\bar u-L_{V_c}(\theta_\varepsilon\bar u),\theta_\varepsilon\bar u)|\le\kappa|c|\varepsilon\|\bar u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} \tag{6.65}$$

for some κ independent of α, ε and c. Indeed, let us start from the most complicated second term in the expression (6.54) for L_{V_c}. Indeed, let w_θ solve the equation

$$w_\theta-\alpha\Delta_x w_\theta=\theta_\varepsilon\bar u,\quad w_\theta\big|_{\partial\Omega}=0.$$

Then, the difference $\theta_\varepsilon w-w_\theta$ solves

$$(\theta_\varepsilon w-w_\theta)-\alpha\Delta_x(\theta_\varepsilon-w_\varepsilon)=H_\varepsilon:=-2\alpha\theta_\varepsilon'\partial_{x_1}w-\alpha\theta_\varepsilon''w \tag{6.66}$$

Using now estimate (6.12) for derivatives of θ_ε and Proposition 3.3 for estimating w, we infer

$$\|H_\varepsilon\|_{W^{1,2}(\Omega)}\le\kappa\varepsilon\|\bar u\|^2_{L^2(\Omega)}$$

where κ is independent of α and ε. Moreover, since $H_u\big|_{\partial\Omega}=0$, multiplying equation (6.66) by $\Delta_x(\theta_\varepsilon-w_\theta)$, we can write

$$\|\theta_\varepsilon w-w_\theta\|_{W^{1,2}(\Omega)}\le C\|H_\varepsilon\|_{W^{1,2}(\Omega)}\le\kappa_1\varepsilon\|\bar u\|_{L^2_{\theta_\varepsilon}(\Omega)}.$$

Furthermore, using this estimate together with Theorem 4.4 and the analog of Proposition 4.6 for weights θ_ε, we will have

$$\|\theta_\varepsilon\Pi w-\Pi w_\theta\|_{W^{1,2}(\Omega)}\le\kappa_2\varepsilon\|\bar u\|_{L^2_{\theta_\varepsilon}(\Omega)}. \tag{6.67}$$

Thus, for the commutator of the second term in the expression (6.54), we get

$$|((\theta_\varepsilon\Pi w-\Pi w_\theta,\nabla_x V_c,\theta_\varepsilon\bar u)|\le 2\|V_c\|_{L^\infty}\|\|\theta_\varepsilon\Pi w-\Pi w_\theta\|_{W^{1,2}(\Omega)}\|\bar u\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\le\kappa|c|\varepsilon\|\bar u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}$$

where we have integrated by parts in order to avoid the terms $\nabla_x V_c$ and use estimate (6.56). Thus, estimate (6.66) is verified for the second term in expression (6.54). The one for the third term is obvious and, for the first term, it is sufficient to note that

$$\|\Pi W_c\|_{L^\infty(\omega)} \le \kappa|c|$$

(due to (6.56) and the maximum principle applied to the equation for W_c. Thus, estimate (6.66) is proved.

Using estimate (6.66) and Lemma 6.7, we estimate the first additional term in (6.64) as follows:

$$(6.68)\quad |(L_{V_c}\bar u, \theta_\varepsilon^2\bar u)| \le |(L_{V_c}(\theta_\varepsilon\bar u), \theta_\varepsilon\bar u)| + \kappa|c|\varepsilon\|\bar u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} \le \\ \le 1/2\|\nabla_x(\theta_\varepsilon\bar u)\|^2_{L^2(\Omega)} + \kappa|c|\varepsilon\|u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}$$

where the constant κ is independent of $\bar u$, c, ε and α.

The second additional term of (6.64) can be then estimated in the following way:

$$(6.69)\quad |L_{V_c}\bar u, v)| \le |((\Pi w, \nabla_x)V_c, v)| + |((\Pi W_c, \nabla_x)\bar u, v)| + |c(\partial_{x_1}u, v)| \le \\ \le 2\|V_c\|_{L^\infty}\|\Pi w\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\|v\|_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)} + \\ + \|\Pi W_c\|_{L^\infty}\|\bar u\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\|v\|_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)} + |c|\|\bar u\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\|v\|_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)} \le \\ \le \kappa(|c|+1)(\varepsilon\|\bar u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} + \varepsilon^{-1}\|v\|^2_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)}.$$

where the constant κ is independent of ε, c, α, u and v.

Finally, the third additional term of (6.64) can be estimates using (6.56) and Hölder inequality:

$$(6.70)\quad |(\nabla x' V_c, \nabla_{x'}(\theta_\varepsilon^2\bar u - v))| \le C_\beta\|\nabla_{x'}V_c\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + \beta(\|\bar u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} + \|v\|^2_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)}) \le \\ \le \kappa_\beta(c^3+1)\varepsilon^{-1} + \beta(\|\bar u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} + \|v\|^2_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)})$$

where $\beta > 0$ is arbitrary and the constant κ_β depends on δ, but is independent of c, ε, α, u and v.

Estimates (6.68)–(6.70) show that, under the additional assumption

$$(6.71)\qquad |c|\varepsilon \le \kappa$$

where $\kappa > 0$ is a sufficiently small number independent of c, α and ε (we recall that, due to (6.18), $v \sim \varepsilon[\theta_\varepsilon]^2\bar u$), we can repeat word by word the proof of (6.13) and obtain the following analogue of (6.27):

$$(6.72)\quad \|\bar u\|^2_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} + (C_1 - C_2\varepsilon\|\bar u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\|\bar u\|^2_{L^2_b([0,T],W^{1,2}_{\theta_\varepsilon}(\Omega))} \le \\ \le C_3\varepsilon^{-1}(1 + |c|^3 + \|\bar u(0)\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T],L^2_b(\Omega))})$$

where the positive constants and C_i, $i = 1,2,3$ are independent of u, u_0, g, α $\varepsilon \to 0$, T, c and x_0.

Furthermore, arguing exactly as in the proof of estimate (6.28), we deduce a priori estimate (6.52) (see (6.29)–(6.33)). The existence of a solution can be than verified exactly as in the case of zero flux c. Theorem 6.6 is proved. $\square$

Remark 6.8. Arguing analogously, it is not difficult to verify the existence of a solution of more general Leray-Navier-Stokes problem with the *non-autonomous* flux

$$\mathbb{S}u_1(t) \equiv c(t) \tag{6.73}$$

where $c \in C^1([0,T])$ is an arbitrary given function. Moreover, the assumption on the external force g is also can be relaxed till

$$g \in L^2_b([0,T], \mathcal{H}_b^{-1,2}(\Omega)). \tag{6.74}$$

Furthermore, the weighted theory developed in this section allows to consider not only bounded with respect to $x_1 \to \infty$ solutions, but also slowly growing solutions of the NS equation (growing not faster than $|x_1|^{1/2-\delta}$, $\delta > 0$ is arbitrary). We however will not use these facts in the sequel and, by this reason, do not give their rigorous proofs here.

7. Leray-Navier-Stokes equations: uniqueness and dissipativity

In this section, we prove the uniqueness of the spatially non-decaying solution of the Leray approximations and verify the dissipativity of that system in $\mathcal{H}^2_b(\Omega)$. We start with the uniqueness which is now almost obvious (due to the regularization of the inertial term).

Theorem 7.1. *Let the assumptions of Theorem 6.6 hold. Then, there exists positive μ such that, for every two solutions $u_1, u_2 \in \vec{v}_c + \mathbb{W}_b([0,T] \times \Omega)$ of problem (6.1) and every weight function ϕ of sufficiently small exponential growth rate ε ($\varepsilon \le \mu$), the following estimate holds:*

$$\|u_1(t) - u_2(t)\|_{L^2_\phi(\Omega)} \le Ce^{Kt}\|u_1(0) - u_2(0)\|_{L^2_\phi(\Omega)}, \tag{7.1}$$

where the constants K and C depend on the L^2_b-norms of $u_1(0)$ and $u_2(0)$, g, $\alpha > 0$ and constant C_ϕ, but are independent of the concrete choice of u_1, u_2 and ϕ.

In particular, the energy solution of the Leray-Navier-Stokes is unique. Moreover, the analogous estimate holds for the spaces $L^2_{b,\phi}$ as well.

Proof. The proof of the theorem is based on the following solvability result for the linear Stokes problem:

$$\partial_t v - \Delta_x v + \nabla_x q = h(t), \ \ v\big|_{\partial\Omega} = 0, \ \operatorname{div} v = 0, \ \mathbb{S}v_1 = 0, \ v\big|_{t=0} = v_0. \tag{7.2}$$

Lemma 7.2. *Let Ω be a cylindrical domain. Then, there exist $\mu_0 > 0$ such that, for every weight function ϕ of exponential growth rate $\mu \le \mu_0$, every $v_0 \in \mathcal{H}^2_\phi(\Omega)$ and every $h \in L^2_{loc}(\mathbb{R}_+, \mathcal{H}_\phi^{-1,2}(\Omega))$, equation (7.2) possesses a unique weighted energy solution v and the following estimate holds:*

$$\|v(T)\|^2_{L^2_\phi(\Omega)} + \int_0^T e^{-\beta(T-s)}\|v(s)\|^2_{W^{1,2}_\phi(\Omega)}\,ds \le \\ \le Ce^{-\beta T}\|v(0)\|^2_{L^2_\phi(\Omega)} + C\int_0^T e^{-\beta(T-s)}\|h(s)\|^2_{\mathcal{H}_\phi^{-1,2}(\Omega)}\,ds \tag{7.3}$$

where the positive constants C and β depend on Ω and C_ϕ, but are independent of v and h.

Indeed, a priori estimate (7.3) can be verified based on the energy identity of Theorem 5.4 analogously to the derivation of (6.19) (so, we rest it to the reader). The existence of a solution can be obtained then exactly as in Theorem 6.5. Thus, Lemma 7.2 is proved.

We are now ready to finish the proof of the theorem. Indeed, let u_1 and u_2 be two solutions of equation (6.1) belonging to $\vec{v}_c + \mathbb{W}_b(\mathbb{R}\times\Omega)$. Then, the difference $v := u_1 - u_2$ belongs to $\mathbb{W}_b(\mathbb{R}\times\Omega)$ and solves the following equation (7.2) with

$$h(t) := -(\Pi w_1, \nabla_x)v - (\Pi w, \nabla_x)u_2 - c\partial_{x_1} v \tag{7.4}$$

with

$$w_i - \alpha\Delta_x w_i = u_i, \ i = 1, 2, \quad w := w_1 - w_2.$$

Let us now estimate function h in the norm of $W^{-1,2}_{\varphi_\varepsilon}(\Omega)$ with $\varphi_\varepsilon = \varphi_{\varepsilon,x_0}$. Indeed, integrating by parts and using Hölder inequality and embedding $W^{1,2} \subset L^6$, we infer:

$$|((\Pi w_1, Nx)v, \varphi_\varepsilon^2 W)| \le |\nabla_x \Pi w_1\|_{W^{1,3}_b(\Omega)}\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}\|W\|_{W^{1,2}_{\varphi_\varepsilon}(\Omega)} + \\ + \|\Pi w_1\|_{L^\infty(\Omega)}\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}\|Z\|_{W^{1,2}_{\varphi_\varepsilon}(\Omega)}$$

which holds for every $Z \in W^{1,2}(\Omega)$ and, consequently,

$$\|(\Pi w_1, \nabla_x)v\|_{W^{-1,2}_{\varphi_\varepsilon}(\Omega)} \le (\|\Pi w_1\|_{W^{1,3}_b(\Omega)} + \|\Pi w_1\|_{L^\infty(\Omega)})\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}$$

Moreover, since u_1 is bounded in $L^\infty(\mathbb{R}, L^2_b(\Omega))$ (we recall that $u_1 \in \vec{v}_c + \mathbb{W}(\mathbb{R}_+)$, see Theorem 5.4) then w_1 is bounded in $L^\infty(\mathbb{R}_+, W^{2,2}_b(\Omega))$ and, consequently, due to Theorem 4.4, one has

$$\|\Pi w_1\|_{W^{1,3}_b(\Omega)} + \|\Pi w\|_{L^\infty(\Omega)} \le C\|u_1\|_{L^2_b(\Omega)} \le C_1 \tag{7.5}$$

where C_1 now depends on α. Thus,

$$\|(\Pi w_1, \nabla_x)v\|_{W^{-1,2}_{\varphi_\varepsilon}(\Omega)} \le C_1\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}. \tag{7.6}$$

The second term in (7.4) can be estimated analogously, the only difference that one should use the weighted analogue of (7.5) and the unweighted one for u_2. This gives us estimate (7.6) for the second term. Finally, the analogous estimate for the third term is immediate and we arrive at

$$\|h(t)\|_{W^{-1,2}_{\varphi_\varepsilon}(\Omega)} \le C\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}.$$

Using now this estimate together with (7.3), we will have

$$\|v(T)\|^2_{L^2_{\varphi_\varepsilon}(\Omega)} \le C\|v(0)\|^2_{L^2_{\varphi_\varepsilon}(\Omega)} + C\int_0^T e^{-\beta(T-s)}\|v(s)\|^2_{L^2_{\varphi_\varepsilon}(\Omega)}\,ds$$

and the Gronwall inequality applied to this estimate finishes the proof of the theorem. $\square$

We now recall the uniformly-local analogue of smoothing property for the solutions of Leray-Navier-Stokes equations.

Theorem 7.3. *Let the assumptions of Theorem 6.6 be satisfied and let* $u \in v_c + \mathbb{W}_b([0,T]\times\Omega)$ *be a weak solution of* (6.1) *constructed in this theorem. Then,*

$$t^{1/2}u(t) \in L^\infty([0,T], W^{1,2}_b(\Omega)) \cap L^2_b([0,T], W^{2,2}_b(\Omega)) \tag{7.7}$$

and the following estimate holds:

$$t\|v(t)\|^2_{W^{1,2}_b(\Omega)} \le Q(\|u_0\|_{L^2_b(\Omega)} + \|g\|_{L^2_b([0,T]\times\Omega)}),\quad t\in[0,1] \tag{7.8}$$

where the monotonic function Q *depends on* α*, but is independent of* u*.*

The proof of this theorem is more or less standard and is based on multiplication of equation (6.1) by the expression

$$t\Pi\left(\partial_{x_1}(\varphi^2_\varepsilon\partial_{x_1}u) + \varphi^2_\varepsilon\partial^2_{x_2}u + \varphi^2_\varepsilon\partial^2_{x_3}u\right), \tag{7.9}$$

see the proof of Theorem 5.1. So, we rest the rigorous proof of this result to the reader.

Our next aim is to verify that the Leray-Navier-Stokes problem (6.1) generates a dissipative dynamical system in the corresponding phase space and obtain a *dissipative* estimate for the solutions of that problem which will be *uniform* with respect to $\alpha\to 0$. This dissipative estimate will be used in the next section in order to verify the dissipativity of the classical Navier-Stokes problem with $\alpha = 0$.

For simplicity, we restrict ourselves to consider the autonomous case only:

$$g(t) \equiv g \in [L^2_b(\Omega)]^2. \tag{7.10}$$

Then, due to Theorems 6.6 and 7.1, the Leray-Navier-Stokes problem (6.1) generates (for all $\alpha > 0$) semi-groups $S_\alpha(t) = S_{c,\alpha}(t)$ in the phase spaces

$$\Phi_b := \Phi_b(c) = \vec{v}_c + \mathcal{H}^2_b(\Omega) \tag{7.11}$$

via the standard expression

$$S_\alpha(t)u_0 := u(t),\quad S_\alpha(t_1+t_2) = S_\alpha(t_1)\circ S_\alpha(t_2),\quad t_1,t_2\ge 0. \tag{7.12}$$

The following theorem, which gives a dissipative estimate for the solutions of the Leray-Navier-Stokes problem, can be considered as the main result of the section.

Theorem 7.4. *Let the assumptions of Theorem 6.6 holds and, in addition,* (7.10) *be satisfied. Then, there exist positive constants* β *and* K *and a monotonic function* Q *such that, for every weak energy solution* $u(t)$ *of the Leray-Navier-Stokes problem* (6.1)–(6.2)*, the following uniform estimate holds:*

$$\|u(t)\|_{L^2_b(\Omega)} \le Q(\|u(0)\|_{L^2_b(\Omega)} + \|g\|_{L^2_b(\Omega)})e^{-\beta t} + K(1+|c|^3 + \|g\|^2_{L^2_b(\Omega)}) \tag{7.13}$$

(we emphasize that the constant K *in* (7.13) *is independent of* α*,* t*,* $\|u(0)\|_{L^2_b(\Omega)}$ *and the flux* $c = \mathbb{S}u_1(0)$*).*

Proof. In order to verify (7.13), it is sufficient to prove that the ball

$$\mathcal{B} := \{u_0 \in [L^2_b(\Omega)]^3,\ \ \|u_0\|_{L^2_b(\Omega)} \le K(1+|c|^3+\|g\|^2_{L^2_b(\Omega)})\} \tag{7.14}$$

is an absorbing set for Leray-Navier-Stokes problem (6.1), i.e., that, for every bounded subset $B\subset\Phi$ there exists time $T = T(\|B\|_\Phi, \|g\|_{L^2_b(\Omega)})$ such that

$$S(t)B \subset \mathcal{B},\quad \forall t\ge T. \tag{7.15}$$

Moreover, for simplicity, we restrict ourselves to consider only the case of zero flux $c = 0$. The general case can be reduced to this particular one exactly as in Theorem 6.6.

The proof of embedding (7.15) requires a little more detailed analysis of basic a priori estimate (6.13) which we rewrite in following more convenient way:

$$\begin{aligned}(7.16)\quad \|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + (C_1 - C_2\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\times \\ \times\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds \le \\ \le C_3^2(e^{-\beta t}\|u(0)\|_{L^2_{\theta_\varepsilon}(\Omega)} + \|g\|_{L^2_{\theta_\varepsilon}(\Omega)})^2\end{aligned}$$

where the positive constants β and C_i, $i = 1, 2, 3$ are independent of α, u, u_0, g, $\varepsilon \to 0$, T and x_0 (in order to deduce (7.16) from (6.13), it is sufficient to take $s = t$ in the left-hand side of it).

Lemma 7.5. *Let the assumptions of Theorem 6.5 holds and let, in addition, the initial data $u(0)$ for problem* (6.1) *satisfy*

$$(7.17)\qquad C_1 - 2C_2C_3\varepsilon(\|u(0)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}) \ge 0$$

where all of the constants are the same as in (7.16). *Then the associated energy solutions $u(t)$ of the Leray-Navier-Stokes problem (with zero flux $c = 0$) satisfies*

$$(7.18)\qquad \|u(t)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} \le C_3(\|u(0)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}e^{-\beta t} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}),$$

for all $t \ge 0$.

Proof. Indeed, estimate (7.16) implies (7.18) under the additional assumption that

$$(7.19)\qquad C_1 - C_2\varepsilon\|u\|_{L^\infty(\mathbb{R}_+,L^2_{\theta_{\varepsilon,x_0}}(\Omega))} \ge 0.$$

On the other hand, (7.18) gives

$$(7.20)\qquad \|u\|_{L^\infty(\mathbb{R}_+,L^2_{\theta_{\varepsilon,x_0}}(\Omega))} \le C_3(\|u(0)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}).$$

which *formally* implies (7.19). Thus, using the continuity arguments (analogously to the proof of Theorem 6.5), we can verify that (7.18) really holds if the initial data satisfies (7.17) and Lemma 7.5 is proved. □

We now note that, although (7.18) looks like a dissipative estimate (in the phase space $L^2_{\theta_{\varepsilon,x_0}}(\Omega)$), it is not sufficient to finish immediately the proof of the theorem, since the exponent $\varepsilon > 0$ in it *depends on* $\|u(0)\|_{L^2_b(\Omega)}$ (through assumption (7.17)), namely,

$$(7.21)\qquad \varepsilon \le \varepsilon_0 := C(\|u(0)\|_{L^2_b(\Omega)} + \|g\|_{L^2_b(\Omega)} + 1)^{-2}$$

for some positive C, see the proof of Theorem 6.5.

Thus, we need to be able to increase the exponent ε as $t \to \infty$ which is guaranteed by the following lemma.

Lemma 7.6. *Let the above assumptions hold. Then, for every bounded subset $B \subset \Phi$, there exists time $T = T(\|B\|, \|g\|)$ such that, for every $x_0 \in \mathbb{R}$, we have*

$$(7.22)\qquad C_1 - 2C_2C_3\varepsilon(\|u(T)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}) \ge 0$$

with $\varepsilon \ge \bar\varepsilon := L(1+\|g\|_{L^2_b(\Omega)})^{-2}$ *(where the constant* L *is independent of* α*,* u_0 *and* g*), if the initial data* $u(0)\in B$*.*

Proof. We will prove the lemma by the iteration procedure. Indeed, let $T_0=0$ and $\varepsilon=\varepsilon_0$ is given by (7.21). Then, estimate (7.22) is satisfied with $\varepsilon=\varepsilon_0$ and $T=T_0$. Let us assume that (7.22) is already proved for some $T_k>0$ and $\varepsilon_k:=2^k\varepsilon_0<\bar\varepsilon$. Then, we only need to prove that there exists $T_{k+1}>T_k$ such that (7.22) is satisfied with $\varepsilon=\varepsilon_{k+1}:=2\varepsilon_k$ and $T=T_{k+1}$. To this end, we note that

$$\theta_{2\varepsilon,x_0}(x):=(1+4\varepsilon^2|x-x_0|^2)^{-1/2}\le 2(1+\varepsilon^2|x-x_0|^2)^{1/2}=2\theta_{\varepsilon,x_0}(x)$$

and, consequently,

$$\|v\|_{L^2_{\theta_{2\varepsilon,x_0}}(\Omega)}\le 2\|v\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}. \tag{7.23}$$

Let us now fix $T_{k+1}>T_k$ in such way that

$$\|u(T_{k+1})\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)}\le 2C_3\|g\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)}, \tag{7.24}$$

for all $u(t)$ such that $u(0)\in B$ (it is possible to do due to our assumptions on ε_k and "dissipative" estimate (7.18)). Estimates (7.23) and (7.24) together with (6.26) give

$$\begin{aligned}\varepsilon_{k+1}(\|u(T_{k+1})\|_{L^2_{\theta_{\varepsilon_{k+1},x_0}}(\Omega)}+\|g\|_{L^2_{\theta_{\varepsilon_{k+1},x_0}}(\Omega)})\le\\ \le 4\varepsilon_k(\|u(T_{k+1})\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)}+\|g\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)})\le 4(2C_3+1)\varepsilon_k\|g\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)}\le\\ \le 4(2C_3+1)C\varepsilon_k^{1/2}\|g\|_{L^2_b(\Omega)}\le\\ \le 4(2C_3+1)CL^{1/2}\|g\|_{L^2_b(\Omega)}(1+\|g\|_{L^2_b(\Omega)})^{-1}\le 4C(2C_3+1)L^{1/2}.\end{aligned}$$

Thus, if the constant L is small enough to satisfy

$$C_1-8C_2CC_3(2C_3+1)L^{1/2}\ge 0,$$

then estimate (7.22) is satisfied with $T=T_{k+1}$ and $\varepsilon=\varepsilon_{k+1}=2\varepsilon_k$. Thus, the iteration finishes the proof of the lemma. □

It is not difficult now to finish the proof of the theorem. Indeed, due to Lemma 7.6 and estimate (7.18), there exists $T=T(\|B\|,\|g\|)$ such that

$$\|u(t)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}\le 2C_3\|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)},\quad t\ge T \tag{7.25}$$

holds with $\varepsilon\ge\bar\varepsilon:=L(1+\|g\|_{L^2_b(\Omega)})^{-2}$ and uniformly with respect to $x_0\in\mathbb{R}$. Taking now supremum over $x_0\in\mathbb{R}$ from the both sides of inequality (7.25) and using again (6.26), we arrive at

$$\|u(t)\|_{L^2_b(\Omega)}\le 2C_3CL^{-1/2}\|g\|_{L^2_b(\Omega)}(1+\|g\|_{L^2_b(\Omega)}),\quad t\ge T \tag{7.26}$$

which shows that the ball (7.14) is really the absorbing set if $K\ge 2C_3CL^{-1/2}$. Theorem 7.4 is proved. □

Remark 7.7. It is worth to note that the intermediate estimate (7.25) gives slightly more information on the solutions than the final one (7.26). Indeed, assume that $c = 0$ and g is square integrable $g \in [L^2(\Omega)]^3$. Then, instead of (6.26), we will have $\|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} \le C\|g\|_{L^2(\Omega)}$ with the constant C *independent* of ε. Thus, instead of (7.26), we will have better estimate

$$\|u(t)\|_{L^2_b(\Omega)} \le 2C_3C\|g\|_{L^2(\Omega)}, \quad t \ge T$$

for the radius of the absorbing set (which grows now *linearly* with respect to g in contrast to the quadratic growth rate in general case).

We are now in a position to prove the existence of a global attractor for semi-groups (7.12) associated with the Leray-Navier-Stokes equation. We however note that, in contrast to the dissipative systems in bounded domains, in unbounded ones the global attractor is usually *not compact* in the initial phase space (Φ_b in our case). That is the reason why one need to use the following weaker definition of a global attractor (following [6], [11], [20]).

Definition 7.8. A set $\mathcal{A} = \mathcal{A}_\alpha \subset \Phi_b$ is a *locally compact* (global) attractor for the semi-group $S_\alpha(t) : \Phi_b \to \Phi_b$ if the following assumptions are satisfied:

1) The set $\mathcal{A}$ is bounded in Φ_b and compact in $\Phi_{loc} := \vec{v}_c + \mathcal{H}^2_{loc}(\overline{\Omega})$ (i.e., the restriction $\mathcal{A}\big|_{\Omega_1}$ of the attractor $\mathcal{A}$ to any bounded sub-domain Ω_1 of Ω is compact in $L^2(\Omega_1)$.

2) The set $\mathcal{A}$ is strictly invariant: $S_\alpha(t)\mathcal{A} = \mathcal{A}$.

3) The set $\mathcal{A}$ is a attracting set for the semi-group $S_\alpha(t)$, i.e., for every neighborhood $\mathcal{O}(\mathcal{A})$ (in the local topology of the space Φ_{loc}) and every bounded (in Φ_b) subset B, there exists time $T = T(\mathcal{O}, B)$ such that

$$S_\alpha(t)B \subset \mathcal{O}(\mathcal{A}), \tag{7.27}$$

for all $t \ge T$.

Corollary 7.9. *Under the assumptions of Theorem 7.4, semi-group* (7.12) *associated with the Leray-Navier-Stokes problem* (6.1)–(6.2) *possesses locally compact attractor* $\mathcal{A}_\alpha = \mathcal{A}^c_\alpha$ *which is bounded in* $\vec{v}_c + \mathcal{V}^2_b(\Omega)$. *Moreover, the following uniform estimate holds:*

$$\|\mathcal{A}_\alpha\|_{L^2_b(\Omega)} \le K(1 + |c|^3 + \|g\|^2_{L^2_b(\Omega)}) \tag{7.28}$$

where the constant K is independent of α, c and g.

Proof. As usual, in order to verify the attractor's existence, we need to check the standard conditions, namely, the existence of a compact absorbing set and the continuity, see eg, [6].

Indeed, due to Theorem 7.4, semi-group (7.11) possesses an absorbing set $\mathcal{B} \subset \Phi$ which is, however, not compact in the space Φ_{loc}. But, due Theorem 7.3, the set $S_\alpha(1)\mathcal{B}$ is bounded in $\vec{v}_c + \mathcal{V}^2_b(\Omega)$ and, consequently, is compact in Φ_{loc}. Thus, a compact absorbing set $\mathcal{B}_1 := S_\alpha(1)\mathcal{B}$ for semi-group (7.11) is constructed. Moreover, due to Theorem 7.1, the operators $S_\alpha(t) : \mathcal{B}_1 \to \Phi$ are continuous (in the topology of Φ_{loc}) for every fixed $t > 0$. Thus, due to the standard attractor existence theorem, semi-group $S_\alpha(t)$ possesses a global attractor $\mathcal{A}_\alpha \subset \mathcal{B}_1 \cap \mathcal{B}$. Estimate (7.28) is now an immediate corollary of Theorem 7.4. Corollary 7.9 is proved. □

8. The classical Navier-Stokes problem

In this concluding section, we construct (by passing to the limit $\alpha \to 0$) a dissipative weak solution of the classical 3D Navier-Stokes equation in a cylindrical domain Ω:

$$\partial_t u + (u, \nabla_x)u = \Delta_x u - \nabla_x p + g, \quad \operatorname{div} u = 0, \ \mathbb{S}u_1 = c, \ u\big|_{t=0} = u_0. \tag{8.1}$$

The following theorem can be considered as the main result of the paper.

Theorem 8.1. *For every $c \in \mathbb{R}$, $g \in L^2_b(\Omega)$ and every $u_0 \in \vec{v}_c + \mathcal{H}^2_b(\Omega)$, there exists at least one weak solution $u = u(t)$*

$$u \in \Psi_b := L^\infty(\mathbb{R}_+, \mathcal{H}^2_b(\Omega)) \cap L^2_b(\mathbb{R}_+, \mathcal{V}^2_b(\Omega)) \tag{8.2}$$

satisfying the Navier-Stokes equation (8.1) *in the sense of distributions over the divergence free vector fields which satisfies the following dissipative estimate:*

$$\|u(t)\|_{L^2_b(\Omega)} + \|u\|_{L^2_b([t,t+1],W^{1,2}_b(\Omega))} \le Q(\|u_0\|_{L^2_b(\Omega)})e^{-\beta t} + K(1 + |c|^3 + \|g\|^2_{L^2_b(\Omega)}) \tag{8.3}$$

where the monotone function Q and positive constants β and K are independent of c, g and u_0.

Proof. Let $u_\alpha(t)$, $\alpha > 0$, $\alpha \to 0$, be the unique solutions of the Leray approximations (6.1) to the Navier-Stokes equations (with fixed c, g and u_0) constructed above. Then, according to Theorems 6.6 and 8.1, these functions satisfy estimate (8.3) *uniformly* with respect to $\alpha \to 0$. In particular, u_α are uniformly bounded in Ψ_b. Thus, we can extract a subsequence $u_n := u_{\alpha_n}$ which converges to some function $u \in \Psi_b$ in the following sense:

$$u_n \to u \text{ weakly star in } L^\infty_{loc}(\mathbb{R}_+, L^2_{loc}(\Omega)) \ \text{ and weakly in } L^2_{loc}(\mathbb{R}_+, W^{1,2}_{loc}(\Omega)), \tag{8.4}$$

see eg, [22]. Moreover, passing to the weak limit in estimates (8.3) for u_n, we see that the limit function u also satisfies this estimate. So, we only need to verify that u solves the limit Navier-Stokes equation. As usual, to this end, we need *strong* convergence $u_n \to u$ in the appropriate space and this, in turns requires to control $\partial_t u$ in some negative Sobolev space. Let us obtain such control.

For simplicity, we will consider below the case $c = 0$ (the general case can be easily reduced to this particular one by the variable change $\bar{u} := u - \vec{v}_c$, see the proof of Theorem 6.6). Applying the projector Π to equation (6.1), we will have

$$\partial_t u_n = \Pi\Delta_x u_n - \Pi(\Pi w_n, \nabla_x)u_n + \Pi g \tag{8.5}$$

where w_n solves

$$w_n - \alpha_n \Delta_x w_n = u_n, \quad w_n\big|_{\partial\Omega} = 0. \tag{8.6}$$

We see that the first term in the right-hand side of (8.5) belongs to $L^2_b(\mathbb{R}_+, \mathcal{H}^{-1,2}_b(\Omega))$ and is uniformly bounded in it since u_n are bounded in Ψ_b. In order to estimate the second term, we note that, due to the standard interpolation theorem, we have the following embedding:

$$\Psi_b \subset L^{10/3}_b(\mathbb{R}_+ \times \Omega),$$

see eg, [29]. Consequently, u_n is uniformly bounded in $L^{10/3}_b(\mathbb{R}_+ \times \Omega)$ and Proposition 3.3 then gives that w_n are uniformly bounded in $L^{10/3}_b$.

In turns, Theorem 4.4 now guarantees that Πw_n are uniformly bounded in $L^{10/3}_b(\mathbb{R}_+\times\Omega)$. Therefore, due to Hölder inequality $(\Pi w_n,\nabla_x)u_n$ is uniformly bounded in $L^{5/4}_b(\mathbb{R}_+\times\Omega)$ (since $\frac{4}{5}=\frac{3}{10}+\frac{1}{2}$). Thus, applying Theorem 4.4 again, we infer that the second term in (8.5) belongs to $L^{5/4}_b(\mathbb{R}_+\times\Omega)$. Finally, since $L^{5/4}\subset W^{-1,2}$, we have obtained the following inequality:

$$\|\partial_t u_n\|_{L^{5/4}_b(\mathbb{R}_+,\mathcal{H}^{-1,2}_b(\Omega))}\le C\|u_n\|_{\Psi_b}\le C_1 \tag{8.7}$$

where the constants C and C_1 is independent of n.

Arguing now as in the proof of Theorem 6.6, we conclude from this that

$$u_n\to u \ \text{ strongly in } \ L^2_{loc}(\mathbb{R}_+\times\Omega). \tag{8.8}$$

We claim now that (8.8) implies the analogous strong convergence

$$w_n\to u \ \text{ strongly in } \ L^2_{loc}(\mathbb{R}_+\times\Omega). \tag{8.9}$$

Indeed, let us split w_n as follows: $w_n=w^0_n+w^1_n$ where

$$w^0_n-\alpha_n\Delta_x w^0_n=u_n-u,\quad w^1_n-\alpha_n\Delta_x w^1_n=u.$$

Then, due to (8.8) and Proposition 3.3, the functions w^0_n converge strongly to zero as $n\to\infty$ in the space $L^2_{loc}(\mathbb{R}_+\times\Omega)$. So, we only need to study functions w^1_n. To this end, we note that, multiplying the equations for w^1_n by $\varphi^2_{\varepsilon,x_0}\Delta_x w^1_n$, using that $u\in L^2_b(W^{1,2}_b(\Omega))$, $u\big|_{\partial\Omega}=0$ and arguing in a standard way, one can easily check that

$$\alpha_n\|\Delta_x w^1_n\|^2_{L^2_b(\mathbb{R}_+\times\Omega)}\le C\|u\|_{L^2_b(\mathbb{R}_+,W^{1,2}_b(\Omega))}\le C_1$$

and, consequently, $\alpha_n\Delta_x w^1_n$ tends to zero in $L^2_b(\mathbb{R}_+\times\Omega)$. Thus, $w^1_n\to u$ strongly even in $L^2_b(\mathbb{R}_+\times\Omega)$ and convergence (8.9) is verified.

We are now ready to finish the proof of the theorem. Indeed, according to Theorem 4.4 and convergence (8.9), $\Pi w_n\to\Pi u=u$ *strongly* in $L^2_{loc}(\mathbb{R}_+\times\Omega)$. Since $\nabla_x u_n\to\nabla_x u$ weakly in $L^2_{loc}(\mathbb{R}_+\times\Omega)$, we conclude from here that

$$(\Pi w_n,\nabla_x)u_n\to(u,\nabla_x)u \ \text{ weakly in } \ L^1_{loc}(\mathbb{R}_+\times\Omega).$$

As usual, the passing to the limit (in the sense of distributions) in linear terms of the equations for u_n is obvious and we have proved that u solves indeed the classical Navier-Stokes problem (8.1). Theorem 8.1 is proved. □

Remark 8.2. As usual, it is not difficult to verify that the space

$$\Theta_b:=\{u\in\Psi_b,\ \partial_t u\in L^{5/4}(\mathbb{R}_+,\mathcal{H}^{-1,2}_b(\Omega))$$

is compactly embedded, eg, in $C([0,T],\mathcal{H}^{-1,2}_\phi(\Omega))$ for every $T>0$ and every square integrable weight ϕ. Thus, the solution u constructed above satisfies indeed the initial condition $u(0)=u_0$.

Our next aim is to construct an attractor for the classical Navier-Stokes equation in a cylindrical domain. However, in contrast to the previous section, the uniqueness of a solution u is still out of reach of the theory (even in the case of bounded domains) and, consequently, the limit semi-group $S_0(t)$ can be defined only as as a semi-group of multi-valued maps. In order to overcome this difficulty, we will use the so-called trajectory approach which allows to restore the uniqueness by changing the phase space of the problem and to construct a global attractor for the so-called trajectory dynamical system related with the problem considered, see [8, 25, 31, 9] for the details.

We start with constructing this trajectory phase space and trajectory semi-group for the Navier-Stokes problem (8.1).

Definition 8.3. Let $K_{tr} = K_{tr}(c)$ be a set of all weak solutions $u \in \Psi_b$ of equation (8.1) (for all initial data $u_0 \in \vec{v}_c + \mathcal{H}_b^{-1,2}(\Omega)$) which satisfy, additionally, the following dissipative estimate:

$$\|u(t)\|_{L^2_b(\Omega)} + \|u\|_{L^2_b([t,t+1],W^{1,2}_b(\Omega)} \le C_u e^{-\beta t} + K(|c|^3 + 1 + \|g\|^2_{L^2_b(\Omega)}) \tag{8.10}$$

where positive constants K and β are the same as in Theorem 8.1 and C_u is an arbitrary constant depending on u. Then, due to this theorem, K_{tr} is not empty. Moreover, since our equation is autonomous, and estimate (8.10) is translation invariant, the semi-group of temporal translations will act on K_{tr}:

$$T(t) : K_{tr} \to K_{tr}, \quad (T(t)u)(s) := u(t+s), \quad t, s \ge 0. \tag{8.11}$$

We refer to the translation semi-group $(T(t), K_{tr})$ acting on the trajectory phase space K_{tr} as a trajectory dynamical system associated with equation (8.1).

Finally, we endow the trajectory phase space K_{tr} by the topology, induced by the embedding

$$K_{tr} \subset \Psi_{loc} := [L^\infty_{loc}(\mathbb{R}_+, L^2_{loc}(\Omega))]^{w^*} \cap [L^2_{loc}(\mathbb{R}_+, W^{1,2}_{loc}(\Omega))]^w \tag{8.12}$$

where symbols w^* and w mean weak-star an weak topology respectively. We recall that a sequence u_n converges to u in Θ_{oc} if for every $T > 0$ and every $N > 0$, the restrictions of this sequence to the domain $t \in [0, T]$, $x \in \Omega_{-N,N} := [-N, N] \times \omega$ converge weakly in the space $L^2([0, T], W^{1,2}(\Omega_N))$ and weakly-star in $L^\infty([0, T], L^2(\Omega_N))$.

Remark 8.4. We note that

(i) If we assume that the uniqueness theorem holds, then the solution operator $S : u_0 \to K_{tr}$ generates a one-to-one map between the usual phase space $\Phi_b = \vec{v}_c + \mathcal{H}^2_b(\Omega)$ and trajectory phase space K_{tr}. Moreover, the translation semi-group $T(t)$ on K_{tr} will be conjugated with the usual semi-group $S_0(t)$ ($S_0(t)u_0 := u(t)$) by this map:

$$T(t) = S \circ S_0(t) \circ S^{-1}, \quad S^{-1}u := u(0). \tag{8.13}$$

Thus, in the case of uniqueness, the trajectory dynamical system $(T(t), K_{tr})$ is formally equivalent to the classical one $(S_0(t), \Phi_b)$ and, without uniqueness, can be considered as a natural generalization of it which allows to avoid the usage of rather unfriendly theory of multi-valued maps.

(ii) We have to include some form of *dissipative* estimate into the definition of a solution belonging to K_{tr} since it is still not known whether or not there exist other "pathological" weak solutions $u \in \Psi_b$ which do not satisfy energy inequalities and which are, possibly, non-dissipative. By including estimate (8.10) into the definition of a solution, we automatically exclude such solutions. We also emphasize that estimate (8.10) is slightly weaker than estimate (8.3) obtained in the proof of Theorem (8.1), namely, we have an arbitrary constant C_u instead of $Q(\|u(0)\|_{L^2_b(\Omega)})$. This is related with the fact that estimate (8.3) is not translation-invariant (because, we have proved it only on the time interval $[0, t]$,

but not for the interval $[\tau, t+\tau]$, the weak convergence $u_n(\tau)$ to $u(\tau)$ obtained in the proof of Theorem 8.1 is not sufficient to pass to the limit in $Q(\|u_n(\tau)\|_{L^2_b(\Omega)})$. By this reason, we cannot use the dissipative estimate with $C_u = Q(\|u(0)\|_{L^2_b})$ for defining K_{tr} (otherwise, the translation semi-group may not act on it) and, following [9] use slightly different estimate (8.10) for which this translation invariance is immediate.

(iii) Rather unusual choice of topology on K_{tr} is motivated, on the one hand, by the necessity to have some kind of compactness/asymptotic compactness for the attractor's theory and, on the other hand, by the fact that no additional regularity and/or compactness is known for the 3D Navier-Stokes equations even in the case of bounded domains. So, we may speak only about *weak* attractors (ie, attractors in a weak topology of the phase space, where the proper bounded subsets are automatically precompact). In contrast to that, the choice of a *local* topology on the trajectory phase space K_{tr} is unavoidable for the trajectory approach, since even in the case of uniqueness and continuous dependence on the initial data, the solution map $S : \Phi_b \to K_{tr}$ will be a homeomorphism only under such choice of the topology on K_{tr}.

Our next task is to define properly the class of bounded sets in the trajectory phase space K_{tr} (which will be attracted by our trajectory attractor as $t \to \infty$). We first note that the most natural way to do so is to use the topology of the Banach space Ψ_b for defining bounded sets. However, this choice is incompatible with our dissipative estimate (8.10). Indeed, since we do not have the relation $C_u = Q(\|u(0)\|_{L^2_b(\Omega)})$ the constant C_u is, in general, may be not bounded on bounded subsets of Ψ_b, so, for such choice of bounded sets we are not able establish a dissipativity (=existence of a bounded absorbing set) which is crucial for the attractors theory.

This problem is overcome by using the abstract class of "bounded" sets (not related with any Banach or metric space), namely, a subset $B \subset K_{tr}$ is "bounded" if the constant C_u in the estimate (8.10) is uniformly bounded on B

$$C_u \le C_B < \infty, \quad \text{for all} \quad u \in K_{tr}.$$

Then, on the one side, this class of "bounded" sets obviously satisfies the property

$$\text{if } B \text{ "bounded" and } B_1 \subset B, \text{ then } B_1 \text{ is also "bounded"} \tag{8.14}$$

and, on the other hand, since for "reasonable" solutions (eg, constructed in Theorem 8.1), we expect that $C_u = Q(\|u(0)\|_{L^2_b(\Omega)})$, this definition is naturally related with bounded subsets of the classical phase space Φ_b.

We are now ready to introduce a concept of a trajectory attractor associated with the Navier-Stokes equation.

Definition 8.5. A set $\mathcal{A}_{tr} = \mathcal{A}_{tr}(c) \subset K_{tr}$ is a trajectory attractor of the Navier-Stokes system (8.1) (= a global attractor of the trajectory dynamical system $(T(s), K_{tr})$) if the following properties are satisfied:

(i) $\mathcal{A}_{tr}$ is "bounded" and compact in K_{tr} (in the topology of Ψ_{loc});

(ii) It is strictly invariant: $T(t)\mathcal{A} = \mathcal{A}$, $t \ge 0$;

(iii) It attracts the images of all "bounded" subsets of K_{tr} in the topology of Ψ_{loc}, ie, for every "bounded" subset $B \subset K_{tr}$ and every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of

Ψ_{loc}, there exists $T = T(B, \mathcal{A})$ such that

$$T(t)B \subset \mathcal{O}(\mathcal{A}), \quad \text{if} \quad t \geq T.$$

And, to formulate the existence theorem for that attractor.

Theorem 8.6. *Let the above assumptions hold. Then, for every $c \in \mathbb{R}$ and every $g \in L^2_b(\Omega)$, the Navier-Stokes problem* (8.1) *possesses a trajectory attractor $\mathcal{A}_{tr}(c)$ in the sense of Definition 8.5. Moreover, the following estimate holds:*

$$\|\mathcal{A}_{tr}(c)\|_{\Psi_b} \leq K(|c|^3 + 1 + \|g\|^2_{L^2_b(\Omega)}) \tag{8.15}$$

where the constant K is the same as in (8.3).

Proof. Indeed, according to the attractor existence theorem for abstract classes of "bounded" sets, see [26] (see also [9]), we need to verify that

1) There exists a "bounded", compact and metrisable absorbing set $\mathcal{B}$ for the semi-group $T(t)$ acting on K_{tr};

2) $T(t)$ is continuous on $\mathcal{B}$ for every fixed t.

The second condition is obvious, since $T(t)$ is continuous on the whole K_{tr} as a translation semi-group. Let us verify the first one.

Indeed, according to estimate (8.10), the set

$$\mathcal{B}_\varepsilon = \{u \in K_{tr}, \quad C_u \leq \varepsilon\} \tag{8.16}$$

are "bounded" absorbing sets for every $\varepsilon > 0$. Moreover, again due to estimate (8.3), the sets $\mathcal{B}_\varepsilon$ are bounded in Ψ_b and, therefore, precompact and metrisable in Ψ_{loc}, see [22]. So, we only need to check that $\mathcal{B}_\varepsilon$ are closed in K_{tr}. The fact that the limit point u solves again the Navier-Stokes equation can be verified exactly as in the proof of Theorem 8.1 (using the additional control of $\partial_t u$ provided by the equation and compactness arguments, see (8.7)). Finally, passing to the limit in estimates (8.10), wee see that the limit point u also should satisfy this estimate. Thus, $u \in K_{tr}$, $\mathcal{B}_\varepsilon$ is closed and Theorem 8.6 is proved. □

Remark 8.7. We note that:

(i) Using the fact that the attractor $\mathcal{A}_{tr}$ is factually bounded in a stronger space Θ_b, see Remark 8.2 and compactness arguments, one can easily verify that the weak attraction in Ψ_{loc} implies the following strong local attraction: for every "bounded" subset $B \subset K_{tr}$, every $T > 0$ and every $N \in \mathbb{R}_+$,

$$\lim_{t\to\infty} \operatorname{dist}_{C([0,T],\vec{v}_c+\mathcal{H}^{-\delta,2}(\Omega_{-N,N}))\cap L^2([0,T],W^{1-\delta,2}(\Omega_{-N,N}))} \left(T(t)B\big|_{[0,T]\times\Omega_{-N,N}}, \mathcal{A}_{tr}\big|_{[0,T]\times\Omega_{-N,N}}\right) = 0$$

where $\delta > 0$ is arbitrary.

(ii) One can define also a "global" attractor $\mathcal{A}^{gl}$ by projecting the trajectory attractor $\mathcal{A}_{tr}$ to the classical phase space Φ_b:

$$\mathcal{A}^{gl} := \mathcal{A}_{tr}\big|_{t=0}.$$

Then, as not difficult to show that the global attractors $\mathcal{A}_\alpha$ of Leray approximations tend as $\alpha \to 0$ to the limit attractor $\mathcal{A}^{gl}$ in the sense of upper semi-continuity in $[L^2_{loc}(\Omega)]^w$. Alternatively, one can lift global attractors $\mathcal{A}_\alpha$, $\alpha > 0$ to the equivalent trajectory attractors

$\mathcal{A}_{\alpha,tr}$ by the solution map. Then, one will have the upper semi-continuity of trajectory attractors $\mathcal{A}_{\alpha,tr}$ as $\alpha \to 0$ in the topology of Ψ_{loc}.

To conclude, we restore the physical parameters in the Navier-Stokes system (6.1), i.e. consider the problem

$$\partial_t u + (u, \nabla_x)u = \nu\Delta_x u - \nabla_x p + g, \quad \operatorname{div} u = 0 \tag{8.17}$$

in a cylindrical domain Ω and study the dependence of the size of attractor on ν.

Corollary 8.8. *The trajectory attractor $\mathcal{A}_{tr} = \mathcal{A}_{tr}(c, g, \nu)$ of problem (8.17) possesses the following estimate:*

$$\|\mathcal{A}_{tr}\|_{L^\infty(\mathbb{R}_+, L^2_b(\Omega))} \le C\nu^{-3}(|c|^3\nu + \|g\|^2_{L^2_b(\Omega)} + \nu^4) \tag{8.18}$$

where the constant C is independent of c, g and ν.

Indeed, the scaling $t' = \nu t$, $u' = \nu^{-1}u$ reduces equation (8.18) to equation (6.1)–(6.2) with $c' = \nu^{-1}c$ and $g' = \nu^{-2}g$. Since $\mathcal{A}' = \nu^{-1}\mathcal{A}$, then (7.28) implies (8.18).

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