EXPONENTIAL ATTRACTORS FOR THE CAHN-HILLIARD EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. We consider in this article the Cahn-Hilliard equation endowed with dynamic boundary conditions. By interpreting these boundary conditions as a parabolic equation on the boundary and by considering a regularized problem, we obtain, by the Leray-Schauder principle, the existence and uniqueness of solutions. We then construct a robust family of exponential attractors.

Introduction.

The Cahn-Hilliard equation

$$\partial_t \phi + \alpha\kappa\Delta_x^2\phi - \kappa\Delta_x f(\phi) = 0, \quad \alpha, \kappa > 0,$$

is very important in materials science. This equation describes important qualitative features of two-phase systems, in particular, the spinodal decomposition, i.e., a rapid separation of phases when the material is cooled down sufficiently. We refer the reader to [C] and [CH] for more details. Here, ϕ is the order parameter, α is related to the surface tension at the interface, κ is the mobility (we shall take

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$\alpha = \kappa \equiv 1$ in what follows) and f is the derivative of a double-well potential (a logarithmic potential; however, such a logarithmic potential is usually approximated by a polynomial of degree four, see [CH]).

This equation has been much studied and one has now satisfactory existence and uniqueness results, as well as results on the long time behavior of the solutions. We refer the reader to the review articles [El], [M], [NC2] and the references therein.

In all these studies, the Cahn-Hilliard equation is endowed with Neumann or periodic boundary conditions. Now, physicists have recently considered the study of phase separation in confined systems (see [FiMD1], [FiMD2] and [KEMRSBD]). In that case, one has to account for the interactions with the walls, which leads to additional terms in the free energy and then to the so-called dynamic boundary conditions (in the sense that the quantity $\partial_t \phi$ appears in the boundary conditions).

The Cahn-Hilliard equation, endowed with dynamic boundary conditions, has been studied in [PRZ], [RZ] and [WZ]. In particular, for a polynomial potential of degree four, the existence and uniqueness of solutions is proven in [RZ]. In order to overcome the new mathematical difficulties due to the boundary conditions, the authors introduce an approximate problem and use results on elliptic operators with highest-order derivatives in the boundary conditions (which still form elliptic boundary value problems in the sense of Hörmander). We can note that it is not clear, in [RZ], whether the solutions define a continuous semigroup in the phase space considered. This problem is overcome in [PRZ], where, based on the maximal L^p-regularity of the solutions, the authors prove that the solutions define a continuous semigroup in certain Sobolev spaces and then obtain the existence of the global attractor.

In this article, we consider a third approach in order to handle the problem. More precisely, we propose to treat the dynamic boundary conditions as a separate parabolic equation on the boundary. We then prove the existence and uniqueness of solutions by the Leray-Schauder principle. Actually, following [MZ], we regularize the problem by considering the viscous Cahn-Hilliard equation (see [NC1]), i.e., we add the term $-\epsilon \partial_t \Delta_x \phi$, $\epsilon \geq 0$, to the equation. Compared with the results of [PRZ] and [RZ], this approach allows us to consider potentials with arbitrary growths. Furthermore, we are also able to consider a nonlinear term in the dynamic boundary conditions. Finally, we are able to construct a robust (as $\epsilon \to 0$) family of exponential attractors $\mathcal{M}_\epsilon$ for the problem. As a consequence, we obtain the existence of the global attractors $\mathcal{A}_\epsilon$ with finite fractal dimension (we note that the finite dimensionality of the global attractor was not studied in [PRZ]).

We now introduce the problem and give the main assumptions for our study.

We consider in this article the following Cahn-Hilliard problem in a bounded smooth domain $\Omega \subset \mathbb{R}^3$:

$$(0.1) \qquad \partial_t \phi = \Delta_x \mu, \quad \partial_n \mu\big|_{\partial\Omega} = 0, \quad \mu = -\Delta_x \phi + \varepsilon \partial_t \phi + f(\phi), \quad \phi\big|_{t=0} = \phi_0,$$

where ϕ and μ are unknown functions (μ is the chemical potential), Δ_x is the Laplacian, $\varepsilon \geq 0$ is a small parameter and f is a given nonlinear interaction function.

Equation (0.1) is equipped with the following dynamic boundary conditions:

$$\partial_t\phi = \Delta_{\parallel}\phi - \lambda\phi - g(\phi) - \partial_n\phi, \quad x \in \partial\Omega, \tag{0.2}$$

where $\Delta_{\parallel}$ is the Laplace-Beltrami operator on the boundary $\partial\Omega$, g is another given nonlinear function and λ is some given positive constant.

We assume that the nonlinearities f and g belong to $C^2(\mathbb{R},\mathbb{R})$ and satisfy the following standard dissipativity assumptions:

$$\liminf_{|v|\to\infty} f'(v) > 0, \quad \liminf_{|v|\to\infty} g'(v) > 0. \tag{0.3}$$

We also recall that system (0.1)-(0.2) possesses a natural conservation law:

$$\langle\phi(t)\rangle \equiv \langle\phi_0\rangle = M_0, \tag{0.4}$$

where $\langle . \rangle$ denotes the average over Ω. Moreover, if the value of $\phi(t)$ is known for some $t = T$, then the value of $\mu(T)$ can be found by solving the following linear elliptic problem:

$$\mu(T) - \varepsilon\Delta_x\mu(T) = -\Delta_x\phi(T) + f(\phi(T)), \quad \partial_n\mu(T)\big|_{\partial\Omega} = 0. \tag{0.4$'$}$$

Thus, we only need to find the function $\phi(t)$. It is however more convenient to introduce one more unknown function, namely, $\psi(t) := \phi(t)\big|_{\partial\Omega}$, defined on the boundary $\partial\Omega$, and to rewrite the above system as follows:

$$\begin{cases} \partial_t\phi = \Delta_x\mu, \quad \partial_n\mu\big|_{\partial\Omega} = 0, \\ \mu = -\Delta_x\phi + \varepsilon\partial_t\phi + f(\phi), \quad \phi\big|_{t=0} = \phi_0, \\ \partial_t\psi = \Delta_{\parallel}\psi - \lambda\psi - g(\psi) - \partial_n\phi, \quad x \in \partial\Omega, \quad \psi\big|_{t=0} = \psi_0, \\ \phi\big|_{\partial\Omega} = \psi, \end{cases} \tag{0.5}$$

where the boundary condition (0.2) is now interpreted as an additional second-order parabolic equation on the boundary $\partial\Omega$.

Following [MZ], we introduce the following "natural" phase space for problem (0.5):

$$\mathbb{D}_\varepsilon := \{(\phi,\psi) \in H^2(\Omega)\times H^2(\partial\Omega), \quad \mu \in H^1(\Omega), \\ \varepsilon^{1/2}\mu \in H^2(\Omega), \quad \phi\big|_{\partial\Omega} = \psi, \quad \partial_n\mu\big|_{\partial\Omega} = 0\} \tag{0.6}$$

(where μ is computed from ϕ via (0.4$'$)), with the obvious norm

$$\|(\phi,\psi)\|^2_{\mathbb{D}_\varepsilon} := \|\phi\|^2_{H^2(\Omega)} + \|\mu\|^2_{H^1(\Omega)} + \varepsilon\|\mu\|^2_{H^2(\Omega)} + \|\psi\|^2_{H^2(\partial\Omega)}. \tag{0.7}$$

A solution of problem (0.5) is a pair of functions $(\phi(t), \psi(t))$ belonging to the space $L^\infty([0,T],\mathbb{D}_\varepsilon)$ with $\partial_t\phi \in L^2([0,T],H^1(\Omega))$ and $\partial_t\psi \in L^2([0,T],H^1(\partial\Omega))$ and which satisfies the equations in the sense of equalities in the spaces $L^2([0,T],L^2(\Omega))$ and $L^2([0,T],L^2(\partial\Omega))$.

Remark 0.1. We recall that, due to the embedding $H^2 \subset C$, we have $f(\phi) \in C([0,T]\times\Omega)$ and $g(\psi) \in C([0,T],L^2(\partial\Omega))$ and the nonlinearities in (0.5) are well-defined. Moreover, since $\partial_t\phi \in L^2([0,T],H^1(\Omega))$, then $\mu \in L^2([0,T],H^3(\Omega))$ and thus the boundary conditions are also well-defined.

We also need to introduce the weak energy spaces $\mathbb{L}_\varepsilon$ for problem (0.5) defined by the following norms:

$$\|(\phi,\psi)\|^2_{\mathbb{L}_\varepsilon} := \varepsilon\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{H^{-1}(\Omega)} + \|\psi\|^2_{L^2(\partial\Omega)}, \tag{0.8}$$

where $H^{-1}(\Omega) := [H^1(\Omega)]'$ endowed with the norm

$$\|v\|^2_{H^{-1}(\Omega)} = \|(-\Delta_x)_N^{-1/2}(v - \langle v\rangle)\|^2_{L^2(\Omega)} + \langle v\rangle^2,$$

$(-\Delta_x)_N^{-1} : L^2_0(\Omega) \to L^2_0(\Omega)$ being the inverse Laplacian with Neumann boundary conditions defined on the subspace of functions with null average. We note that, for $\varepsilon \neq 0$, the space $\mathbb{L}_\varepsilon$ coincides with $L^2(\Omega)\times L^2(\partial\Omega)$ and, for $\varepsilon = 0$, we have $\mathbb{L}_0 = H^{-1}(\Omega)\times L^2(\partial\Omega)$.

§1 A priori estimates.

The main task of this section is to derive uniform (with respect to $\varepsilon \to 0$) a priori estimates for the solutions of (0.5) in the phase space $\mathbb{D}_\varepsilon$. The existence of a solution will be verified in the next section.

We derive the required estimates in three steps. In a first step, we obtain a divergent estimate in $\mathbb{D}_\varepsilon$. In a second step, we derive dissipative estimates in the weak energy space $\mathbb{L}_\varepsilon$ and, in a third step, we prove some kind of a smoothing property which allows to eliminate this divergence and obtain *dissipative* estimates in $\mathbb{D}_\varepsilon$.

Theorem 1.1. *Let the nonlinearities f and g satisfy assumptions* (0.3) *and let a pair of functions $(\phi(t),\psi(t))$ be a solution of problem* (0.5). *Then, the following estimates hold:*

$$\|(\phi(t),\psi(t))\|^2_{\mathbb{D}_\varepsilon} + \int_0^t (\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\partial\Omega)})\,ds \le Q(\|(\phi(0),\psi(0))\|^2_{\mathbb{D}_\varepsilon} e^{Kt}), \tag{1.1}$$

where the positive constant K and the monotonic function Q depend on M_0 (introduced in (0.4)*), but are independent of ε.*

Proof. We differentiate (0.5) in t and set $(u(t), v(t), w(t)) := \partial_t(\phi(t), \mu(t), \psi(t))$. Then, we have

$$(1.2)\qquad \begin{cases} \partial_t u = \Delta_x v,\ \partial_n v\big|_{\partial\Omega} = 0, \\ v + \varepsilon\partial_t u = -\Delta_x u + f'(\phi)u, \quad u\big|_{\partial\Omega} = w, \\ \partial_t w = \Delta_{\|} w - \lambda w - g'(\psi)w - \partial_n u. \end{cases}$$

We then note that, due to (0.4), we have $\langle u\rangle = 0$. Consequently, multiplying the first equation of (1.2) scalarly by $(-\Delta_x)_N^{-1}u$, the second equation by u and the third one by ψ and taking the sum of the equations that we obtain, we have the following identity:

$$(1.3)\quad 1/2\partial_t(\|u(t)\|^2_{H^{-1}(\Omega)} + \varepsilon\|u(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{L^2(\partial\Omega)}) + \\ + \lambda\|w(t)\|^2_{L^2(\partial\Omega)} + \|\nabla_x u(t)\|^2_{L^2(\Omega)} + \|\nabla_{\|} w(t)\|^2_{L^2(\partial\Omega)} = \\ = (f'(\phi(t))u(t), u(t)) + (g'(\psi(t))w(t), w(t))_{\partial\Omega},$$

where $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\partial\Omega}$ denote the inner products in $L^2(\Omega)$ and $L^2(\partial\Omega)$ respectively and $\nabla_{\|}$ is the covariant gradient at $\partial\Omega$. We now note that, due to assumptions (0.3), $f'(\phi) \ge -K$ and $g'(\psi) \ge -K$ for some constant K. Consequently, applying Gronwall's inequality to (1.3) and using the interpolation inequality $\|u\|^2_{L^2(\Omega)} \le C\|u\|_{H^{-1}(\Omega)}\|\nabla_x u\|_{L^2(\Omega)}$, we have

$$(1.4)\quad \|u(t)\|^2_{H^{-1}(\Omega)} + \varepsilon\|u(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{L^2(\partial\Omega)} + \\ + \int_0^t (\|\nabla_x u(s)\|^2_{L^2(\Omega)} + \|\nabla_{\|} w(s)\|^2_{L^2(\partial\Omega)})\,ds \le \\ \le Ce^{K_1 t}(\|u(0)\|^2_{H^{-1}(\Omega)} + \varepsilon\|u(0)\|^2_{L^2(\Omega)} + \|w(0)\|^2_{L^2(\partial\Omega)}),$$

where the positive constants C and K_1 are independent of ε. We now recall that $u = \partial_t\phi = \Delta_x\mu$. Consequently, (1.4) can be rewritten as follows:

$$(1.5)\quad \|\nabla_x\mu(t)\|^2_{L^2(\Omega)} + \varepsilon\|\Delta_x\mu(t)\|^2_{L^2(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\partial\Omega)} \le \\ \le Ce^{K_1 t}(\|\nabla_x\mu(0)\|^2_{L^2(\Omega)} + \varepsilon\|\Delta_x\mu(0)\|^2_{L^2(\Omega)} + \|\partial_t\psi(0)\|^2_{L^2(\partial\Omega)}).$$

So, in order to obtain the μ-part of estimate (1.1), we only need to estimate the average of $\mu(t)$. To this end, averaging the expression for μ from the second equation of (0.5), we have

$$(1.6)\quad \langle\mu(t)\rangle = -\langle\Delta_x\phi(t)\rangle + \langle f(\phi(t))\rangle = -\langle\partial_n\phi(t)\rangle_{\partial\Omega} + \langle f(\phi(t))\rangle = \\ = \langle\lambda\psi(t) + g(\psi(t))\rangle_{\partial\Omega} - \langle\partial_t\psi(t)\rangle_{\partial\Omega} + \langle f(\phi(t))\rangle,$$

where we denote by $\langle\cdot\rangle_{\partial\Omega}$ the average value on the boundary and have used the third equation of (0.5) in order to find the average of $\partial_n\phi$. Thus, since the average of $\partial_t\psi$

is already estimated in (1.5), we only need to estimate the averages of ψ, $f(\phi)$ and $g(\psi)$. In order to do so, we introduce the functions $\bar\phi(t) := \phi(t) - \langle\phi(t)\rangle = \phi(t) - M_0$ and $\bar\psi(t) := \psi(t) - M_0$. Then, multiplying the expression for μ by $\bar\phi(t)$, we have, after standard transformations,

$$
\begin{aligned}
(1.7)\quad \|\nabla_x\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\parallel\psi(t)\|^2_{L^2(\partial\Omega)} + (f(\phi(t)),\bar\phi(t)) + (g(\psi(t)),\bar\psi(t))_{\partial\Omega}+\\
+\lambda(\psi(t),\psi(t)-M_0)_{\partial\Omega} =\\
= (\mu(t),\bar\phi(t)) - (\varepsilon\partial_t\phi(t),\bar\phi(t)) - (\partial_t\psi(t),\bar\psi(t))_{L^2(\partial\Omega)}.
\end{aligned}
$$

We note that the first term in the right-hand side of (1.7) can be estimated by $1/2\|\nabla_x\phi\|^2_{L^2} + C\|\nabla_x\mu(t)\|^2_{L^2}$ (since $\langle\bar\phi\rangle = 0$). Consequently, estimates (1.4), (1.5) and (1.7) give

$$
\begin{aligned}
(1.8)\quad \|\nabla_x\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\parallel\psi(t)\|^2_{L^2(\partial\Omega)}+\\
+2(f(\phi(t)),\bar\phi(t)) + 2(g(\psi(t)),\bar\psi(t))_{\partial\Omega} \le C(\|(\phi(0),\psi(0))\|^2_{\mathbb{D}_\varepsilon}+1)e^{K_1t}.
\end{aligned}
$$

It remains to note that assumptions (0.3) imply that

$$
(1.9)\qquad
\begin{aligned}
1/2|f(v)|(1+|v|) \le f(v)(v-M_0) + C_{f,M_0},\\
1/2|g(v)|(1+|v|) \le g(v)(v-M_0) + C_{g,M_0},\quad \forall v\in\mathbb{R},
\end{aligned}
$$

and, consequently, (1.8) yields

$$
\begin{aligned}
(1.10)\quad \|\psi(t)\|_{L^1(\partial\Omega)} + \|f(\phi(t))\|_{L^1(\Omega)} + \|g(\psi(t))\|_{L^1(\partial\Omega)} \le\\
\le C(1+\|(\phi(0),\psi(0))\|^2_{\mathbb{D}_\varepsilon})e^{K_1t}.
\end{aligned}
$$

Thus, the required average estimates for the terms ψ, $f(\phi)$ and $g(\psi)$ are obtained and (1.6) now implies that

$$
(1.11)\qquad |\langle\mu(t)\rangle| \le C(1+\|(\phi(0),\psi(0))\|^2_{\mathbb{D}_\varepsilon})e^{K_1t}.
$$

Consequently (taking into account (1.5) and (1.11)), the μ-part of estimate (1.1) is verified. In order to obtain the required estimates for the H^2-norms of ϕ and ψ and thus complete the proof of Theorem 1.1, we rewrite (for every fixed t) problem (0.5) as a second-order nonlinear elliptic boundary value problem:

$$
(1.12)\qquad
\begin{cases}
\Delta_x\phi - f(\phi) = h_1(t) := -\mu(t) + \varepsilon\partial_t\phi(t), \quad \phi\big|_{\partial\Omega} = \psi,\\
\Delta_\parallel\psi - \lambda\psi - g(\psi) - \partial_n\phi = h_2(t) := \partial_t\psi(t).
\end{cases}
$$

Indeed, according to (1.5), (1.8) and (1.11), we have

$$
(1.13)\qquad \|h_1(t)\|^2_{L^2(\Omega)} + \|h_2(t)\|^2_{L^2(\partial\Omega)} \le C(1+\|(\phi(0),\psi(0))\|^2_{\mathbb{D}_\varepsilon})e^{K_1t}.
$$

Applying now the maximum principle (see Appendix, Lemma A.2) to this problem, we deduce that

$$\|\phi(t)\|^2_{L^\infty(\Omega)} + \|\psi(t)\|^2_{L^\infty(\partial\Omega)} \le C_1(1 + \|(\phi(0), \psi(0))\|^2_{\mathbb{D}_\varepsilon})e^{K_1 t}. \tag{1.14}$$

Finally, having the L^∞-estimates (1.14), we can interpret the nonlinearities $f(\phi)$ and $g(\psi)$ as external forces as well and apply the H^2-regularity theorem (see Appendix, Lemma A.1) to the linear elliptic system that we obtain. This yields that

$$\|\phi(t)\|^2_{H^2(\Omega)} + \|\psi(t)\|^2_{H^2(\partial\Omega)} \le Q(\|(\phi(0), \psi(0))\|^2_{\mathbb{D}_\varepsilon} e^{K_1 t}), \tag{1.15}$$

for some monotonic function Q which depends on f, g and M_0, but is independent of ε, and Theorem 1.1 is proven.

In a next step, we deduce dissipative estimates for the solutions of (0.5) in the weak energy space introduced in (0.8).

Proposition 1.1. *Let the assumptions of Theorem 1.1 hold. Then, every solution* $(\phi(t), \psi(t))$ *of* (0.5) *satisfies the following estimates:*

$$\begin{aligned}(1.16)\quad &\varepsilon\|\phi(t)\|^2_{L^2(\Omega)} + \|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\partial\Omega)} + \\ &+ \int_t^{t+1} (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\partial\Omega)} + \|F(\phi(s))\|_{L^1(\Omega)} + \|G(\psi(s))\|_{L^1(\partial\Omega)})\, ds \le \\ &\le C(\varepsilon\|\phi(0)\|^2_{L^2(\Omega)} + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\partial\Omega)})e^{-\alpha t} + C_1,\end{aligned}$$

where $F(v) := \int_0^v f(u)du$, $G(v) := \int_0^v g(u)du$ *and the positive constants* C, C_1 *and* α *are independent of* ε *and* t.

Proof. Let $\bar\phi(t) := \phi(t) - M_0$ and $\bar\psi(t) := \psi(t) - M_0$ be the same as in the proof of Theorem 1.1. Then, recalling that $\mu = \langle\mu\rangle - (-\Delta_x)^{-1}_N \partial_t\phi$ and using the fact that $\langle\bar\phi\rangle = 0$, we derive from (1.7) that

$$\begin{aligned}(1.17)\quad &1/2\partial_t(\|\bar\phi(t)\|^2_{H^{-1}(\Omega)} + \varepsilon\|\bar\phi(t)\|^2_{L^2(\Omega)} + \|\bar\psi(t)\|^2_{L^2(\partial\Omega)}) + \\ &+ \|\nabla_x\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\parallel\psi(t)\|^2_{L^2(\partial\Omega)} + \\ &+ (f(\phi(t)), \bar\phi(t)) + (g(\psi(t)), \bar\psi(t))_{\partial\Omega} + \lambda(\psi(t), \psi(t) - M_0)_{\partial\Omega} = 0.\end{aligned}$$

Moreover, due to estimates (1.9), we obtain

$$\begin{aligned}(1.18)\quad &\partial_t(\|\bar\phi(t)\|^2_{H^{-1}(\Omega)} + \varepsilon\|\bar\phi(t)\|^2_{L^2(\Omega)} + \|\bar\psi(t)\|^2_{L^2(\partial\Omega)}) + \\ &+ 2\alpha(\|\bar\phi(t)\|^2_{H^1(\Omega)} + \|\bar\psi(t)\|^2_{H^1(\partial\Omega)}) + \|\nabla_x\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\parallel\psi(t)\|^2_{L^2(\partial\Omega)} + \\ &+ (|f(\phi(t))|, 1 + |\phi(t)|) + (|g(\psi(t))|, 1 + |\psi(t)|)_{\partial\Omega} \le C,\end{aligned}$$

for some positive constants C and α which are independent of ε (here, we have implicitly used the fact that $\|\phi\|_{H^1(\Omega)} \le C(\|\nabla_x\phi\|_{L^2(\Omega)} + \|\psi\|_{H^1(\partial\Omega)})$). Applying Gronwall's inequality to (1.18), we have

$$\tag{1.19} \begin{aligned} \varepsilon\|\phi(t)\|^2_{L^2(\Omega)} + \|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\partial\Omega)} + \\ + \int_t^{t+1} (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\partial\Omega)})\, ds + \\ + \int_t^{t+1} ((|f(\phi(s))|, 1 + |\phi(s)|) + (|g(\psi(s))|, 1 + |\psi(s)|)_{\partial\Omega})\, ds \le \\ \le C(\varepsilon\|\phi(0)\|^2_{L^2(\Omega)} + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\partial\Omega)})e^{-\alpha t} + C_1. \end{aligned}$$

In order to deduce (1.16) from (1.19), it remains to note that assumptions (0.3) imply that

$$\tag{1.20} |F(v)| \le |f(v)|(1+|v|) - C, \quad |G(v)| \le |g(v)|(1+|v|) - C,$$

for some positive C and for every $v \in \mathbb{R}$ (indeed, according to (0.3), the functions f and g are monotonic if $|v|$ is large enough). Proposition 1.1 is proven.

In a third step, we establish a smoothing-type property for the solutions of (0.5) as follows.

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold and let (ϕ, ψ) be a solution of problem* (0.5)*. Then, for every $T \in (0, 1)$, the following estimate holds:*

$$\tag{1.21} \|(\phi(t), \psi(t))\|^2_{\mathbb{D}_\varepsilon} \le Q_T\left(\varepsilon\|\phi(0)\|^2_{L^2(\Omega)} + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\partial\Omega)}\right), \ t \in [T, 1],$$

where the monotonic function Q_T depends obviously on $T > 0$, but is independent of t and ε.

Proof. We first note that, in order to verify (1.21), it is sufficient to verify only that there exists at least one time $T_0 \in [T/2, T]$ (which can depend on the solution $(\phi(t), \psi(t))$) such that

$$\tag{1.22} \begin{aligned} \|\partial_t\phi(T_0)\|^2_{H^{-1}(\Omega)} + \varepsilon\|\partial_t\phi(T_0)\|^2_{L^2(\Omega)} + \|\partial_t\psi(T_0)\|^2_{L^2(\partial\Omega)} \le \\ \le CT^{-1}(1 + \varepsilon\|\phi(0)\|^2_{L^2(\Omega)} + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\partial\Omega)}), \end{aligned}$$

for some constant C which is independent of ε and $(\phi(t), \psi(t))$. Indeed, if (1.22) is known, then, differentiating equation (0.5) with respect to t and arguing exactly as in the derivation of (1.4), we verify that the analogue of (1.22) holds for every $t \in [T_0, 1]$. Having this estimate, we can obtain the analogue of estimate (1.11) for $\langle\mu(t)\rangle$ (exactly as in (1.6)–(1.11)) and then derive the required H^2-estimates by applying the elliptic regularity theorem to problem (1.12) (see (1.13)–(1.15)). Thus, it only remains to verify (1.22). To this end, multiplying the first equation of (0.5) scalarly by $(-\Delta_x)^{-1}_N\partial_t\phi(t)$, using the fact that $\langle\partial_t\phi(t)\rangle=0$ and the expression for μ and integrating by parts, we have

$$(1.23)\quad \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)}=-(\mu(t),\partial_t\phi(t))=(\Delta_x\phi(t),\partial_t\phi(t))-\varepsilon\|\partial_t\phi(t)\|^2_{L^2(\Omega)}-(f(\phi(t)),\partial_t\phi(t))=$$
$$=-1/2\partial_t(\|\nabla_x\phi(t)\|^2_{L^2(\Omega)}+2(F(\phi(t)),1))-\varepsilon\|\partial_t\phi(t)\|^2_{L^2(\Omega)}+(\partial_n\phi(t),\partial_t\phi(t))_{\partial\Omega}.$$

Expressing $\partial_n\phi$ from the third equation of (0.5), inserting this expression into the last term of formula (1.23) and integrating by parts again, we find

$$(1.24)\quad \partial_t(\|\nabla_x\phi(t)\|^2_{L^2(\Omega)}+2(F(\phi(t)),1)+\|\nabla_\|\psi(t)\|^2_{L^2(\partial\Omega)}+$$
$$+2(\bar G(\psi(t)),1)_{\partial\Omega})+2\|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)}+2\varepsilon\|\partial_t\phi(t)\|^2_{L^2(\Omega)}+2\|\partial_t\psi(t)\|^2_{L^2(\partial\Omega)}=0,$$

where $\bar G(v):=G(v)+\frac{\lambda}{2}v^2$. Multiplying now (1.24) by t and integrating over $[0,t]$, $t\in[0,1]$, we have

$$(1.25)\quad t(\|\nabla_x\phi(t)\|^2_{L^2(\Omega)}+2(F(\phi(t)),1)+\|\nabla_\|\psi(t)\|^2_{L^2(\partial\Omega)}+2(\bar G(\psi(t)),1)_{\partial\Omega})+$$
$$+2\int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)}+\varepsilon\|\partial_t\phi(s)\|^2_{L^2(\Omega)}+\|\partial_t\psi(s)\|^2_{L^2(\partial\Omega)})\,ds=$$
$$=\int_0^t(\|\nabla_x\phi(s)\|^2_{L^2(\Omega)}+2(F(\phi(s)),1)+\|\nabla_\|\psi(s)\|^2_{L^2(\partial\Omega)}+2(\bar G(\psi(s)),1)_{\partial\Omega})\,ds.$$

Using finally inequality (1.16) in order to estimate the right-hand side of (1.25), we deduce that

$$(1.26)\quad \int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)}+\varepsilon\|\partial_t\phi(s)\|^2_{L^2(\Omega)}+\|\partial_t\psi(s)\|^2_{L^2(\partial\Omega)})\,ds\le$$
$$\le C(\varepsilon\|\phi(0)\|^2_{L^2(\Omega)}+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\partial\Omega)}),$$

where C is independent of ε and $(\phi(t),\psi(t))$. Estimates (1.22) follow immediately from (1.26) and Theorem 1.2 is proven.

We are now ready to obtain a *dissipative* estimate for the $\mathbb{D}_\varepsilon$-norm of the solution $(\phi(t),\psi(t))$.

Corollary 1.1. *Let the assumptions of Theorem 1.1 hold. Then, every solution* $(\phi(t),\psi(t))$ *of problem* (0.5) *satisfies the following estimate:*

$$(1.27)\quad \|(\phi(t),\psi(t))\|_{\mathbb{D}_\varepsilon}\le Q(\|(\phi(0),\psi(0))\|_{\mathbb{D}_\varepsilon})e^{-\alpha t}+C_*,$$

where the positive constants C_* *and* α *and the monotonic function* Q *are independent of* t, $u(t)$ *and* ε.

Indeed, in order to deduce (1.27), it is sufficient to use (1.1) for $t\le1$ and (1.16) and (1.21) for $t\ge1$.

In the sequel, we shall also need the uniform bounds on the solutions in $H^3(\Omega)\times H^3(\partial\Omega)$ which are formulated in the next theorem.

Theorem 1.3. *Let the assumptions of Theorem 1.1 hold and let $(\phi(t), \psi(t))$ be a solution of problem* (0.5)*. Then, $(\phi(t), \psi(t)) \in H^3(\Omega) \times H^3(\partial\Omega)$ for every $t > 0$ and the following estimates hold:*

$$\|\phi(t)\|_{H^3(\Omega)} + \|\psi(t)\|_{H^3(\partial\Omega)} \le t^{-1/2} Q\left(\|(\phi(0), \psi(0))\|_{\mathbb{D}_\varepsilon}\right), \ t \in (0, 1], \tag{1.28}$$

where the monotonic function Q is independent of t and ε.

Proof. We only give a formal derivation of (1.28), which can be easily justified in a standard way (using, e.g., t-regularizations of the solutions of (0.5) via $\tilde{\phi}(t) := \int_0^\infty K_\mu(t-s)\phi(s)\,ds$, where the kernel K_μ is smooth and satisfies $\operatorname{supp} K_\mu \subset [0, \mu]$, $\int_0^\infty K_\mu(z)\,dz = 1$, and then passing to the limit $\mu \to 0$). As in the proof of Theorem 1.1, we differentiate equations (0.5) with respect to t and set $(u(t), v(t), w(t)) := \partial_t(\phi(t), \mu(t), \psi(t))$. These functions satisfy equations (1.2). We multiply scalarly the first, second and third equations of (1.2) by $t(-\Delta_x)^{-1}\partial_t u(t)$, $t\partial_t u(t)$ and $t\partial_t w(t)$ respectively and take the sum of the equations that we obtain. Then, after standard transformations, we have

$$\begin{aligned} 1/2\partial_t(t\|\nabla_x u(t)\|^2_{L^2(\Omega)} + t\|\nabla_\| w(t)\|^2_{L^2(\partial\Omega)} + \lambda t\|w(t)\|^2_{L^2(\partial\Omega)}) + \\ + t\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \varepsilon t\|\partial_t u(t)\|^2_{L^2(\Omega)} + t\|\partial_t w(t)\|^2_{L^2(\partial\Omega)} = \\ = 1/2(\|\nabla_x u(t)\|^2_{L^2(\Omega)} + \|\nabla_\| w(t)\|^2_{L^2(\partial\Omega)} + \lambda\|w(t)\|^2_{L^2(\partial\Omega)}) + \\ + (f'(\phi(t))u(t), t\partial_t u(t)) + (g'(\psi(t))w(t), t\partial_t w(t))_{\partial\Omega}. \end{aligned} \tag{1.29}$$

We estimate the (most complicated) second term in the right-hand side of (1.29) as follows:

$$\begin{aligned} |(f'(\phi(t))u(t), t\partial_t u(t))| \le \\ \le Ct\|f'(\phi(t))u(t)\|_{H^1(\Omega)}\|\partial_t u(t)\|_{H^{-1}(\Omega)} \le \\ \le t\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + Q(\|\phi(t)\|_{H^2})\|\partial_t\phi(t)\|^2_{H^1(\Omega)}, \end{aligned} \tag{1.30}$$

and the last term in the right-hand side of (1.29) can be estimated analogously (the proof is however simpler, since the negative norms are not necessary). Integrating now (1.29) in t and using estimate (1.1), we deduce

$$t(\|\partial_t\phi(t)\|^2_{H^1(\Omega)} + \|\partial_t\psi(t)\|^2_{H^1(\partial\Omega)}) \le Q_1(\|(\phi(0), \psi(0))\|_{\mathbb{D}_\varepsilon}), \ t \in [0, 1], \tag{1.31}$$

for some monotonic function Q_1 which is independent of ε and t. Having obtained estimate (1.31), we can rewrite problem (0.5) as a linear elliptic boundary value problem:

$$\begin{cases} -\Delta_x\phi(t) = h_1(t) := \mu(t) + \varepsilon\partial_t\phi(t) - f(\phi(t)), \quad \phi(t)\big|_{\partial\Omega} = \psi(t), \\ -\Delta_\|\psi(t) + \lambda\psi(t) + \partial_n\phi(t) = h_2(t) := -\partial_t\psi(t) - g(\psi(t)). \end{cases} \tag{1.32}$$

Moreover, according to (1.1) and (1.31), we have

$$\|h_1(t)\|_{H^1(\Omega)} + \|h_2(t)\|_{H^1(\partial\Omega)} \le t^{-1/2} Q_2(\|(\phi(0),\psi(0))\|_{\mathbb{D}_\varepsilon}),\ \ t\in(0,1], \tag{1.33}$$

for some monotonic function Q_2. Applying now the H^1-H^3 regularity theorem to the linear elliptic problem (1.32) (see (A.8′)), we obtain (1.28) and finish the proof of Theorem 1.3.

We conclude this section by verifying the uniqueness of a solution of (0.5) and the Lipschitz continuity with respect to the initial data.

Proposition 1.2. *Let the assumptions of Theorem 1.1 hold and let the functions* $(\phi_1(t),\psi_1(t))$ *and* $(\phi_2(t),\psi_2(t))$ *be two solutions of* (0.5) *such that*

$$\langle \phi_1(0)\rangle = \langle\phi_2(0)\rangle\,. \tag{1.34}$$

Then, the following estimates hold:

$$\begin{aligned}\varepsilon\|\phi_1(t)-\phi_2(t)\|^2_{L^2(\Omega)} + \|\phi_1(t)-\phi_2(t)\|^2_{H^{-1}(\Omega)} + \|\psi_1(t)-\psi_2(t)\|^2_{L^2(\partial\Omega)} \le \\ \le Ce^{Kt}(\varepsilon\|\phi_1(0)-\phi_2(0)\|^2_{L^2(\Omega)} + \|\phi_1(0)-\phi_2(0)\|^2_{H^{-1}(\Omega)} + \|\psi_1(0)-\psi_2(0)\|^2_{L^2(\partial\Omega)}),\end{aligned} \tag{1.35}$$

where the positive constants C and K are independent of ε, t and the initial data for the solutions considered.

Proof. We set $(\tilde\phi,\tilde\mu,\tilde\psi) := (\phi_1-\phi_2,\mu_1-\mu_2,\psi_1-\psi_2)$. These functions satisfy the following problem:

$$\begin{cases}\partial_t\tilde\phi = \Delta_x\tilde\mu,\quad \partial_n\tilde\mu\big|_{\partial\Omega}=0,\\ \tilde\mu = -\Delta_x\tilde\phi + \varepsilon\partial_t\tilde\phi + [f(\phi_1)-f(\phi_2)],\\ \partial_t\tilde\psi = \Delta_\parallel\tilde\psi - \lambda\tilde\psi - [g(\psi_1)-g(\psi_2)] - \partial_n\tilde\phi,\quad x\in\partial\Omega,\\ \tilde\phi\big|_{\partial\Omega} = \tilde\psi.\end{cases} \tag{1.36}$$

We also note that $\left\langle\tilde\phi(t)\right\rangle = 0$ (thanks to assumption (1.34)). Consequently, multiplying the first equation of (1.36) by $(-\Delta_x)^{-1}_N\tilde\phi$ and arguing as in the proof of Theorem 1.2, we have

$$\begin{aligned}1/2\partial_t(\|\tilde\phi(t)\|^2_{H^{-1}(\Omega)} + \varepsilon\|\tilde\phi(t)\|^2_{L^2(\Omega)} + \|\tilde\psi(t)\|^2_{L^2(\partial\Omega)}) + \|\nabla_x\tilde\phi(t)\|^2_{L^2(\Omega)} + \\ + \lambda\|\tilde\psi(t)\|^2_{L^2(\partial\Omega)} + \\ + \|\nabla_\parallel\tilde\psi(t)\|^2_{L^2(\partial\Omega)} = -([f(\phi_1(t))-f(\phi_2(t))],\tilde\phi(t)) - ([g(\psi_1(t))-g(\psi_2(t))],\tilde\psi(t))_{\partial\Omega}.\end{aligned} \tag{1.37}$$

It remains to note that, due to assumptions (0.3), $f'(v)\ge -K$ and $g'(v)\ge -K$ for some positive constant K. Consequently, (1.37) implies

$$\begin{aligned}\partial_t(\|\tilde\phi(t)\|^2_{H^{-1}(\Omega)} + \varepsilon\|\tilde\phi(t)\|^2_{L^2(\Omega)} + \|\tilde\psi(t)\|^2_{L^2(\partial\Omega)}) + \\ + 2\|\nabla_x\tilde\phi(t)\|^2_{L^2(\Omega)} \le 2K(\|\tilde\phi(t)\|^2_{L^2(\Omega)} + \|\tilde\psi(t)\|^2_{L^2(\partial\Omega)}).\end{aligned} \tag{1.38}$$

Using now the interpolation inequality $\|\cdot\|^2_{L^2(\Omega)} \le C\|\cdot\|_{H^1(\Omega)}\cdot\|\cdot\|_{H^{-1}(\Omega)}$ in order to estimate the first term in the right-hand side of (1.38) and applying Gronwall's inequality, we find (1.35) and finish the proof of Proposition 1.2.

§2 Existence of solutions.

In this section, we establish the existence of a solution for problem (0.5). To this end, we first need to study the following linear nonhomogeneous problem, which corresponds to (0.5) with null boundary conditions:

$$\partial_t\phi = \Delta_x\mu, \quad \partial_n\mu\big|_{\partial\Omega} = 0, \quad \mu = -\Delta_x\phi + \varepsilon\partial_t\phi - h(t), \tag{2.1}$$

equipped with the null Dirichlet boundary condition $\phi\big|_{\partial\Omega} = 0$ (here, $h(t)$ corresponds to given external forces and satisfies $\langle h(t)\rangle \equiv 0$). It is however more convenient to rewrite this problem as a single equation for ϕ, using the operator $(-\Delta_x)_N^{-1}$, as follows:

$$(\varepsilon + (-\Delta_x)_N^{-1})\partial_t\phi = \Delta_x\phi - \langle\partial_n\phi\rangle_{\partial\Omega} + h(t), \quad \phi\big|_{\partial\Omega} = 0, \quad \phi\big|_{t=0} = \phi_0. \tag{2.2}$$

As usual, it is convenient to study equation (2.2) by using the anisotropic Sobolev spaces $W_p^{(1,2)}(\Omega_T)$, $1 < p < \infty$, $\Omega_T := [0,T] \times \Omega$ (which are, by definition, the spaces of functions whose t-derivative and x-derivatives up to the order 2 belong to $L^p(\Omega_T)$) associated with second-order parabolic operators. The next lemma gives the standard L^p-regularity estimate for equation (2.2).

Lemma 2.1. *We assume that $\varepsilon > 0$, $h \in L^p(\Omega_T)$, with $\langle h(t)\rangle \equiv 0$, and the initial datum $\phi_0 \in W_p^{2(1-1/p)}(\Omega)$ for some $2 \le p < \infty$. Then, problem (2.2) has a unique solution $u(t) \in W_p^{(1,2)}(\Omega_T)$ and the following estimate holds:*

$$\|\phi\|_{W_p^{(1,2)}(\Omega_T)} \le C(\|\phi_0\|_{W_p^{2(1-1/p)}(\Omega)} + \|h\|_{L^p(\Omega_T)}), \tag{2.3}$$

where the constant C depends on T and ε, but is independent of ϕ.

Proof. We first apply the operator $(\varepsilon + (-\Delta_x)_N^{-1})^{-1}$ to both sides of (2.2) and transform this equation as follows:

$$\varepsilon\partial_t\phi = \Delta_x\phi - \mathbb{K}\phi + \varepsilon(\varepsilon + (-\Delta_x)_N^{-1})^{-1}h(t), \quad \phi\big|_{\partial\Omega} = 0, \tag{2.4}$$

where

$$\mathbb{K}v := \langle\partial_n v\rangle_{\partial\Omega} + (\varepsilon + (-\Delta_x)_N^{-1})^{-1}(-\Delta_x)_N^{-1}\left(\Delta_x v - \langle\partial_n v\rangle_{\partial\Omega}\right). \tag{2.5}$$

We then recall that, according to the classical L^p-regularity theory for the Laplacian (see, e.g., [LU] or [Tri]), we have

$$\|\mathbb{K}v\|_{L^p(\Omega)} \le C_{s,p}\|v\|_{W_p^{1+1/p+s}(\Omega)}, \quad \|(\varepsilon + (-\Delta_x)_N^{-1})^{-1}v\|_{L^p(\Omega)} \le C_{s,p}\|v\|_{L^p(\Omega)}, \tag{2.6}$$

for all $1 < p < \infty$ and $s > 0$. Thus, the linear equation (2.4) is a compact perturbation of a heat equation and, consequently, the existence and uniqueness of a solution can be verified in a standard way (using, e.g., the Leray-Schauder fixed point theorem, see [Z]). So, we restrict ourselves to verifying estimate (2.3) only. To this end, we first multiply equation (2.4) scalarly by $\Delta_x\phi(t)$ and use the following estimate:

$$
\begin{aligned}
(2.7)\quad |(\mathbb{K}\phi, \Delta_x \phi)| \le C\|\mathbb{K}\phi\|^2_{L^2(\Omega)} + 1/4\|\Delta_x\phi\|^2_{L^2(\Omega)} \le \\
\le C_1\|\phi\|^2_{H^{2-\delta}(\Omega)} + 1/4\|\Delta_x\phi\|^2_{L^2(\Omega)} \le C_1\|\phi\|^2_{H^1(\Omega)} + 1/2\|\Delta_x\phi\|^2_{L^2(\Omega)},
\end{aligned}
$$

for some $0 < \delta < 1/2$ (here, we have used (2.6) with $p = 2$ and $s = -\delta$). Then, we have

$$
\varepsilon/2\partial_t\|\nabla_x\phi(t)\|^2_{L^2(\Omega)} \le C\|\nabla_x\phi(t)\|^2_{L^2(\Omega)} + C\|h(t)\|^2_{L^2(\Omega)}
$$

and, applying Gronwall's inequality to this relation, we infer

$$
(2.8)\qquad \|\phi(t)\|^2_{H^1(\Omega)} \le Ce^{Kt}(\|\phi_0\|^2_{H^1(\Omega)} + \|h\|^2_{L^2(\Omega_t)}).
$$

We are now ready to finish the derivation of estimate (2.3). To this end, we apply the classical L^p-regularity theorem for heat equations to (2.4), interpreting the term $\mathbb{K}v$ as external forces. Then, using the second estimate of (2.6) with $s = 0$, we have

$$
(2.9)\qquad \|\phi\|_{W^{(1,2)}_p(\Omega_T)} \le C(\|\mathbb{K}\phi\|_{L^p(\Omega_T)} + \|h\|_{L^p(\Omega_T)} + \|\phi_0\|_{W^{2(1-1/p)}_p(\Omega)}).
$$

We then estimate the first term in the right-hand side of (2.9) by using the first estimate of (2.6) with $s = -\delta$, a proper interpolation inequality and estimate (2.8):

$$
(2.10)\qquad \|\mathbb{K}\phi(t)\|_{L^p(\Omega)} \le C\|\phi(t)\|_{W^{2-\delta}_p(\Omega)} \le C_\nu\|\phi(t)\|_{H^1(\Omega)} + \nu\|\phi(t)\|_{W^2_p(\Omega)}.
$$

Fixing $\nu > 0$ small enough and inserting (2.10) into the right-hand side of (2.9), we infer (2.3) and finish the proof of Lemma 2.1.

The next corollary gives the analogue of estimate (2.3) for nonhomogeneous boundary conditions.

Corollary 2.1. *We assume that $\varepsilon > 0$ and that the function $\psi(t)$ belongs to the anisotropic Sobolev space $W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T)$ for some $2 \le p < \infty$ (here and below, $\partial\Omega_T := [0,T] \times \partial\Omega$). Then, the problem*

$$
(2.11)\qquad (\varepsilon + (-\Delta_x)^{-1}_N)\partial_t\phi = \Delta_x\phi - \langle\partial_n\phi\rangle_{\partial\Omega}, \quad \phi\big|_{\partial\Omega} = \psi, \quad \phi\big|_{t=0} = 0,
$$

possesses a unique solution $\phi \in W_p^{(1,p)}(\Omega_T)$ and the following estimate holds:

$$
(2.12)\qquad \|\phi\|_{W^{(1,2)}_p(\Omega_T)} \le C\|\psi\|_{W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T)},
$$

where the constant C depends on ε and T, but is independent of ψ. Moreover,

$$\int_0^t (\partial_n\phi(s),\psi(s))_{\partial\Omega}\,ds = \tag{2.13}$$

$$= 1/2\|\phi(t)\|^2_{H^{-1}(\Omega)} + \varepsilon/2\|\phi(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla_x\phi(s)\|^2_{L^2(\Omega)}\,ds \geq 0.$$

Proof. There exists a continuous linear extension operator

$$T_p : W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T) \to W_p^{(1,2)}(\Omega_T) \ \text{ such that } \ (T_p\psi)\big|_{\partial\Omega_T} = \psi \tag{2.14}$$

(see, e.g., [LSU]). Moreover, without loss of generality, we can also assume that

$$\langle (T_p\psi)(t)\rangle \equiv 0, \quad \forall t\geq 0, \quad \forall \psi \in W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T). \tag{2.15}$$

We now set $v := T_p\psi$. Then, this function satisfies equation (2.2) with $h(t) := (\varepsilon+(-\Delta_x)^{-1}_N)\partial_t v(t) - \Delta_x v(t) + \langle\partial_n v(t)\rangle_{\partial\Omega}$. Applying the L^p-regularity estimate (2.3) to this equation and using (2.14) and (2.15), we obtain estimate (2.12). In order to deduce (2.13), it remains to multiply equation (2.11) by $\phi(t)$ and to integrate in x and t. Corollary 2.1 is proven.

We are now ready to study the linear analogue of the complete system (0.5), which we rewrite, analogously to (2.3), in the following form:

$$\begin{cases} (\varepsilon + (-\Delta_x)^{-1}_N)\partial_t\phi = \Delta_x\phi - \langle\partial_n\phi\rangle_{\partial\Omega} + h_1(t), \quad \phi\big|_{t=0} = \phi_0, \\ \partial_t\psi = \Delta_{\|}\psi - \lambda\psi - \partial_n\phi + h_2(t), \quad \psi\big|_{t=0} = \psi_0, \\ \phi\big|_{\partial\Omega} = \psi. \end{cases} \tag{2.16}$$

The following lemma is the analogue of Lemma 2.1 for this problem.

Lemma 2.2. *We assume that $\varepsilon>0$, that the external forces h_1 and h_2 belong to $L^p(\Omega_T)$ and $L^p(\partial\Omega_T)$ respectively for some $2\leq p<\infty$ and that $\langle h_1(t)\rangle\equiv 0$. We also assume that the initial data ϕ_0 and ψ_0 belong to $W_p^{2(1-1/p)}(\Omega)$ and $W_p^{2(1-1/p)}(\partial\Omega)$ respectively. Then, problem (2.16) possesses a unique solution $(\phi(t),\psi(t))$ and the following estimates hold:*

$$\|\phi\|_{W_p^{(1,2)}(\Omega_T)} + \|\psi\|_{W_p^{(1,2)}(\partial\Omega_T)} \leq C(\|\phi_0\|_{W_p^{2(1-1/p)}(\Omega)} + \tag{2.17}$$

$$+ \|\psi_0\|_{W_p^{2(1-1/p)}(\partial\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\partial\Omega_T)}),$$

where the constant C depends on ε and T, but is independent of (ϕ,ψ) and (h_1,h_2).

Proof. We introduce the operator $\mathbb{T} : W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T) \to W_p^{(1,2)}(\Omega_T)$ as the unique solution of problem (2.11) and set $v(t) := (\mathbb{T}\psi)(t)$ and $\theta(t) := \phi(t) - v(t)$. Then, system (2.16) reads, in the new variables (θ,ψ),

$$\begin{cases} (\varepsilon + (-\Delta_x)^{-1}_N)\partial_t\theta = \Delta_x\theta - \langle\partial_n\theta\rangle_{\partial\Omega} + h_1(t), \quad \theta\big|_{t=0} = \phi_0, \quad \theta\big|_{\partial\Omega} = 0, \\ \partial_t\psi = \Delta_{\|}\psi - \lambda\psi - \partial_n v(t) + h_2(t) - \partial_n\theta(t), \\ \psi\big|_{t=0} = \psi_0. \end{cases} \tag{2.18}$$

Thus, the first equation of (2.18) is now independent of ψ and can be solved separately thanks to Lemma 2.1, which gives the existence and uniqueness of θ and the following estimate:

$$\|\theta\|_{W_p^{(1,2)}(\Omega_T)} \le C(\|\phi_0\|_{W_p^{2(1-1/p)}(\Omega)} + \|h_1\|_{L^p(\Omega_T)}). \tag{2.19}$$

So, it only remains to solve the second equation of (2.18), which has the following form:

$$\partial_t\psi = \Delta_{\|}\psi - \lambda\psi - \partial_n(\mathbb{T}\psi) + \tilde{h}(t),\ \psi\big|_{t=0} = \psi_0,\ \tilde{h}(t) := h_2(t) - \partial_n\theta(t). \tag{2.20}$$

We note that, due to (2.19), $\tilde{h} \in L^p(\partial\Omega_T)$. Moreover, according to (2.12) and to a proper trace theorem, we have

$$\|\partial_n(\mathbb{T}\psi)\|_{L^p(\partial\Omega_T)} \le C\|\psi\|_{W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T)}. \tag{2.21}$$

Consequently, equation (2.20) is again a compact perturbation of the heat equation (but now on the boundary $\partial\Omega$) and, therefore, the existence and uniqueness of a solution can be verified in a standard way (using, e.g., the Leray-Schauder principle). For the reader's convenience, we give below the derivation of the L^p-regularity estimate for the solutions of (2.20). Applying the classical L^p-regularity estimate for heat equations to (2.20) in which the nonlocal term is interpreted as external forces and using (2.21), we have

$$\|\psi\|_{W_p^{(1,2)}(\partial\Omega_T)} \le \\ \le C(\|\psi_0\|_{W_p^{2(1-1/p)}(\partial\Omega)} + \|\psi\|_{W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T)} + \|\tilde{h}\|_{L^p(\partial\Omega_T)}). \tag{2.22}$$

Using now the obvious interpolation inequality

$$\|\psi\|_{W_p^{(1-1/(2p),2-1/p)}(\partial\Omega_T)} \le C_\nu\|\psi\|_{L^2(\partial\Omega_T)} + \nu\|\psi\|_{W_p^{(1,2)}(\partial\Omega_T)}, \tag{2.23}$$

and fixing $\nu > 0$ small enough, we find

$$\|\psi\|_{W_p^{(1,2)}(\partial\Omega_T)} \le C_1(\|\psi_0\|_{W_p^{2(1-1/p)}(\partial\Omega)} + \|\psi\|_{L^2(\partial\Omega_T)} + \|\tilde{h}\|_{L^p(\partial\Omega_T)}). \tag{2.24}$$

So, we only need to estimate the L^2-norm of the solution $\psi(t)$. To this end, we multiply equation (2.20) scalarly in $L^2(\partial\Omega)$ by $\psi(t)$, integrate in t and use the positivity of the form (2.13). Then, we have

$$1/2\|\psi(t)\|^2_{L^2(\partial\Omega)} - 1/2\|\psi_0\|^2_{L^2(\partial\Omega)} \le \int_0^t (\tilde{h}(s), \psi(s))_{\partial\Omega}\, ds. \tag{2.25}$$

Applying Gronwall's inequality to this relation, we deduce the required estimate for the L^2-norm of ψ and finish the proof of Lemma 2.2.

We are now ready to verify the existence of a solution for the nonlinear problem (0.5).

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold. Then, for every $\varepsilon > 0$ and every initial datum belonging to $\mathbb{D}_\varepsilon$, problem* (0.5) *has a solution* $(\phi(t), \psi(t))$ *which satisfies all the estimates of Section 1.*

Proof. We verify the existence of a solution by interpreting problem (0.5) as a *nonlinear* compact perturbation of (2.16) and by using Lemma 2.2 and the Leray-Schauder principle. To this end, we consider the following homotopy of equation (0.5) to the linear one:

$$\begin{cases} (\varepsilon + (-\Delta_x)_N^{-1})\partial_t\phi = \Delta_x\phi - \langle\partial_n\phi\rangle_{\partial\Omega} - s[f(\phi) - \langle f(\phi)\rangle], \quad \phi\big|_{t=0} = \phi_0, \\ \partial_t\psi = \Delta_\parallel\psi - \lambda\psi - \partial_n\phi - sg(\psi), \quad \psi\big|_{t=0} = \psi_0, \\ \phi\big|_{\partial\Omega} = \psi. \end{cases} \tag{2.26}$$

Using now Lemma 2.2, we can rewrite (2.26) as follows:

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mathbb{M}_0 \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} + s\mathbb{M}_h \begin{pmatrix} \langle f(\phi)\rangle - f(\phi) \\ g(\psi) \end{pmatrix}, \tag{2.27}$$

where $\mathbb{M}_0 : (\phi_0, \psi_0) \mapsto (\phi, \psi)$ is the solving operator for problem (2.16) with zero external forces and $\mathbb{M}_h : (h_1, h_2) \mapsto (\phi, \psi)$ is the solving operator of the same problem, but with null initial data.

We solve equation (2.27) in the phase space

$$\Phi := W_p^{(1,2)}(\Omega_T) \times W_p^{(1,2)}(\partial\Omega_T), \quad 3 < p < 10/3 \tag{2.28}$$

(the assumption $p > 3$ is necessary in order to have the embedding $\Phi \subset C(\Omega_T) \times C(\partial\Omega_T)$ and the condition $p < 10/3$ is chosen in order to have the embedding $H^2(\Omega) \subset W_p^{2(1-1/p)}(\Omega)$, which guarantees that the operator $\mathbb{M}_0$ is well-defined for every initial datum belonging to $\mathbb{D}_\varepsilon$).

We now apply the Leray-Schauder fixed point theorem to equation (2.27). To this end, we first note that, due to the compact embedding $\Phi \subset C(\Omega_T) \times C(\partial\Omega_T)$ and estimate (2.17), the operator $(\phi, \psi) \mapsto \mathbb{M}_h(\langle f(\phi)\rangle - f(\phi),\ g(\psi))$ is a compact and continuous operator in Φ. Thus, the Leray-Schauder theory is indeed applicable and we only need to have uniform with respect to $s \in [0, 1]$ a priori estimates for the solutions of (2.27). Indeed, let (ϕ_s, ψ_s) be a solution of equation (2.27). Then, obviously, (ϕ_s, ψ_s) also solves system (2.26) and the triple (ϕ_s, μ_s, ψ_s), where $\mu_s = \mu_s(t)$ can be found as a solution of

$$\mu_s(t) - \varepsilon\Delta_x\mu_s(t) = -\Delta_x\phi_s(t) + f(\phi_s(t)), \quad \partial_n\mu_s(t)\big|_{\partial\Omega} = 0, \tag{2.29}$$

solves the initial system (0.5) (in which the nonlinearities f and g are replaced by sf and sg respectively). Moreover, it is not difficult to verify, arguing as in the proof of Theorem 1.1, that the a priori estimates (1.1) hold *uniformly* with respect to $s \in [0, 1]$. In particular, these estimates give

$$\|\phi_s\|_{L^\infty(\Omega_T)} + \|\psi_s\|_{L^\infty(\partial\Omega_T)} \le C_T, \tag{2.30}$$

where C_T is independent of s. Thus, we also have uniform L^∞-estimates on the terms $f(\phi_s)$ and $g(\psi_s)$ and, consequently, interpreting these terms as external forces in (2.26) and applying the L^p-regularity estimates (2.17), we obtain a *uniform* with respect to s estimate on (ϕ_s, ψ_s) in Φ. Therefore, according to the Leray-Schauder principle, equation (2.26) has a solution for every $s \in [0,1]$. It remains to note that, for $s = 1$, system (2.26) is equivalent to (0.5). Theorem 2.1 is proven.

It is not difficult to extend the existence result of Theorem 2.1 to the case $\varepsilon = 0$.

Corollary 2.1. *We assume that the assumptions of Theorem 1.1 hold and that $\varepsilon = 0$. Then, for every initial datum belonging to $\mathbb{D}_0$, problem* (0.5) *has a solution (in the sense defined in the introduction).*

Proof. Let the initial datum $(\phi_0, \psi_0) \in \mathbb{D}_0$. We recall that, in the case $\varepsilon = 0$, we have $\mu_0 = -\Delta_x \phi_0 + f(\phi_0)$ and, according to the definition of $\mathbb{D}_0$, this expression belongs to $H^1(\Omega)$. Using now this fact and equation (2.29) for μ in the case $\varepsilon > 0$, we can easily verify that the initial datum considered belongs to $\mathbb{D}_\varepsilon$ for every $\varepsilon > 0$ and the following *uniform* estimate holds:

$$\|(\phi_0, \psi_0)\|_{\mathbb{D}_\varepsilon} \le Q(\|(\phi_0, \psi_0)\|_{\mathbb{D}_0}), \quad \varepsilon \in [0,1], \tag{2.31}$$

where the function Q is independent of ε. Let $(\phi_\varepsilon(t), \psi_\varepsilon(t))$, $\varepsilon > 0$, be solutions of (0.5) whose existence is verified in Theorem 2.1. Then, according to (2.31) and Corollary 1.1, these solutions satisfy

$$\|(\phi_\varepsilon(t), \psi_\varepsilon(t))\|_{\mathbb{D}_\varepsilon} \le Q(\|(\phi_0, \psi_0)\|_{\mathbb{D}_0}) e^{-\alpha t} + C_*, \tag{2.32}$$

uniformly with respect to $\varepsilon \to 0$. Passing now in a standard way to the limit $\varepsilon \to 0$ in equations (0.5), we see that the limit triple $(\phi(t), \mu(t), \psi(t))$ satisfies equations (0.5) with $\varepsilon = 0$. Passing finally to the limit $\varepsilon \to 0$ in estimate (2.32), we verify that the solution (ϕ, ψ) satisfies (2.32) with $\varepsilon = 0$ and Corollary 2.1 is proven.

Thus, the existence of a solution of problem (0.5) with an initial datum belonging to $\mathbb{D}_\varepsilon$ is completely verified. We also recall that the uniqueness of this solution has been proven in Proposition 1.2. Thus, for every $\varepsilon \in [0,1]$, problem (0.5) generates a dissipative semigroup $S_t(\varepsilon)$ on the phase space $\mathbb{D}_\varepsilon$:

$$S_t(\varepsilon) : \mathbb{D}_\varepsilon \to \mathbb{D}_\varepsilon, \quad S_t(\varepsilon)(\phi_0, \psi_0) := (\phi(t), \psi(t)), \tag{2.33}$$

where $(\phi(t), \psi(t))$ is the unique solution of (0.5) with initial datum (ϕ_0, ψ_0). The rest of this section is devoted to the extension of this result to the weak energy spaces introduced in (0.8). Indeed, according to Proposition 1.2, we have the following uniform Lipschitz continuity in the $\mathbb{L}_\varepsilon$-norm:

$$\|S_t(\varepsilon)(\phi_1, \psi_1) - S_t(\varepsilon)(\phi_2, \psi_2)\|_{\mathbb{L}_\varepsilon} \le Ce^{Kt} \|(\phi_1 - \phi_2, \psi_1 - \psi_2)\|_{\mathbb{L}_\varepsilon} \tag{2.34}$$

for all $(\phi_i, \psi_i) \in \mathbb{D}_\varepsilon$, $i = 1, 2$, where the constants C and K *are independent* of the initial data. Consequently, the semigroup (2.33) can be extended in a unique way by continuity to a Lipschitz continuous semigroup acting on the whole space $\mathbb{L}_\varepsilon$ by the following standard expression:

$$S_t(\varepsilon)(\phi_0, \psi_0) := \mathbb{L}_\varepsilon - \lim_{n\to\infty} S_t(\varepsilon)(\phi_n, \psi_n), \tag{2.35}$$

where $(\phi_0, \psi_0) \in \mathbb{L}_\varepsilon$, $(\phi_n, \psi_n) \in \mathbb{D}_\varepsilon$ and $(\phi_n, \psi_n) \to (\phi_0, \psi_0)$ in the norm of $\mathbb{L}_\varepsilon$ (here, we have implicitly used the obvious fact that $\mathbb{D}_\varepsilon$ is dense in $\mathbb{L}_\varepsilon$). Moreover, since the solutions $(\phi_n(t), \psi_n(t))$ belong obviously to the space $C([0, T], \mathbb{L}_\varepsilon)$, then the limit function $(\phi(t), \psi(t)) := S_t(\varepsilon)(\phi_0, \psi_0)$ is also continuous with values in $\mathbb{L}_\varepsilon$. Furthermore, passing to the limit $n \to \infty$ in estimate (1.21) for the solutions $(\phi_n(t), \psi_n(t))$, we obtain the same estimate for the limit function $(\phi(t), \psi(t))$. Thus, for every $t > 0$, the extended semigroup $S_t(\varepsilon)$ maps $\mathbb{L}_\varepsilon$ into $\mathbb{D}_\varepsilon$:

$$S_t(\varepsilon) : \mathbb{L}_\varepsilon \to \mathbb{D}_\varepsilon, \ t > 0. \tag{2.36}$$

Finally, passing to the limit $n \to \infty$ in equations (0.5) and using (1.21), we can easily establish that, for every $t > 0$, the function $(\phi(t), \psi(t))$ also satisfies equations (0.5). Thus, we have proven the following result.

Theorem 2.2. *Let the assumptions of Theorem 1.1 hold. Then, for every initial datum* (ϕ_0, ψ_0) *belonging to* $\mathbb{L}_\varepsilon$, *problem* (0.5) *has a unique solution* $(\phi(t), \psi(t))$ *in the class* $C([0, T], \mathbb{L}_\varepsilon) \cap L^\infty_{loc}((0, T], \mathbb{D}_\varepsilon)$.

Indeed, this solution is given by the formula $(\phi(t), \psi(t)) := S_t(\varepsilon)(\phi_0, \psi_0)$, where the extension of $S_t(\varepsilon)$ to $\mathbb{L}_\varepsilon$ is defined by (2.35).

§3 Robust exponential attractors.

In this section, we construct a robust (as $\varepsilon \to 0$) family of exponential attractors for the semigroups (2.35) associated with problems (0.5). In order to do so, we first recall that system (0.5) possesses the conservation law (0.4) and, consequently, we cannot expect to have a dissipation in the whole phase space $\mathbb{L}_\varepsilon$ or $\mathbb{D}_\varepsilon$ (the constant C and the function Q in the dissipative estimate (1.27) depend on M_0) and we need to restrict ourselves to the hyperplanes (0.4), with a prescribed value of the average $\langle \phi(t) \rangle$. We note however that all the constants in the estimates obtained in Section 1 are uniform with respect to M_0 belonging to some bounded set. So, instead of fixing the prescribed value M_0 of the average (0.4), we will consider below the semigroups (2.35) in the phase spaces

$$\mathbb{L}_\varepsilon(M) := \mathbb{L}_\varepsilon \cap \{ |\langle \phi \rangle| \le M \} \quad \text{and} \quad \mathbb{D}_\varepsilon(M) := \mathbb{D}_\varepsilon \cap \{ |\langle \phi \rangle| \le M \}, \tag{3.1}$$

for every $M > 0$. The following theorem gives a robust family of exponential attractors for the semigroups (2.35) restricted to the spaces (3.1).

Theorem 3.1. *Let the assumptions of Theorem 1.1 hold and let $M > 0$ be arbitrary. Then, there exists a family of compact sets $\mathcal{M}_\varepsilon \subset\subset \mathbb{D}_\varepsilon$, $\varepsilon \in [0,1]$, which satisfy the following properties:*

1) Finite-dimensionality: the fractal dimension of $\mathcal{M}_\varepsilon$ (say, in $H^2(\Omega)\times H^2(\partial\Omega)$) is finite and uniformly bounded with respect to $\varepsilon \to 0$:

$$\dim_F(\mathcal{M}_\varepsilon, H^2(\Omega)\times H^2(\partial\Omega)) \le C, \tag{3.2}$$

where the constant C depends on M, but is independent of ε.

2) Invariance: $S_t(\varepsilon)\mathcal{M}_\varepsilon \subset \mathcal{M}_\varepsilon$, $t \ge 0$.

3) Uniform exponential attraction: there exist a positive constant α and a monotonic function Q (which are independent of ε) such that, for every bounded subset $B \subset \mathbb{L}_\varepsilon(M)$, we have

$$\operatorname{dist}_{H^2(\Omega)\times H^2(\partial\Omega)}(S_t(\varepsilon)B, \mathcal{M}_\varepsilon) \le Q(\|B\|_{\mathbb{L}_\varepsilon})e^{-\alpha t}, \quad t \ge 0, \tag{3.3}$$

where dist_V denotes the nonsymmetric Hausdorff semidistance between sets in the space V.

4) The sets $\mathcal{M}_\varepsilon$ tend to the limit set $\mathcal{M}_0$ as $\varepsilon \to 0$ in the following sense:

$$\operatorname{dist}^{symm}_{H^2(\Omega)\times H^2(\partial\Omega)}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C\varepsilon^\tau, \tag{3.4}$$

where $\operatorname{dist}^{symm}_V$ denotes the symmetric Hausdorff distance and the positive constants C and $\tau \in (0,1)$ depend on M, but are independent of ε.

Remark 3.1. Due to the parabolic nature of problem (0.5), we have a standard smoothing property for its solutions (see, e.g., (1.21) and (1.28)) and, consequently, the concrete choice of the space $H^2(\Omega)\times H^2(\partial\Omega)$ is not essential and can be replaced, e.g., by $\mathbb{D}_\varepsilon$, or even by $H^3(\Omega)\times H^3(\partial\Omega)$.

Proof of the theorem. As usual (see [EFNT], [EfMZ1], [EfMZ2] and [FGMZ]), we first construct the required exponential attractors for the semigroups (2.35) with discrete times and then extend the result to continuous times. To this end, we first introduce the following ball $\mathbb{B}$ with sufficiently large radius R in the space $H^3(\Omega)\times H^3(\partial\Omega)$:

$$\begin{aligned}\mathbb{B}_R = \mathbb{B}_R(M) := \{(\phi,\psi) \in H^3(\Omega)\times H^3(\partial\Omega),\\ \|(\phi,\psi)\|_{H^3(\Omega)\times H^3(\partial\Omega)} \le R,\quad |\langle\phi\rangle| \le M\}.\end{aligned} \tag{3.5}$$

Then, obviously, $\mathbb{B}_R \subset \mathbb{D}_\varepsilon(M)$ for all $\varepsilon \in [0,1]$ and

$$\|\mathbb{B}_R\|_{\mathbb{D}_\varepsilon} \le C_{R,M}, \tag{3.6}$$

where the constant $C_{R,M}$ depends on R and M, but is independent of ε. Moreover, due to the dissipative estimate (1.27) and the smoothing properties (1.21) and (1.28), there exist sufficiently large $\bar{R} = \bar{R}(M)$ and $T = T(M)$ which are independent of

ε such that the set $\mathbb{B} := \mathbb{B}_{\bar{R}}$ is an absorbing set for the semigroup $S_t(\varepsilon)$ acting on $\mathbb{L}_\varepsilon(M)$ (uniformly for all $\varepsilon \geq 0$) and

$$S_t(\varepsilon)\mathbb{B} \subset \mathbb{B}, \quad \varepsilon \in [0,1], \quad t \geq T. \tag{3.7}$$

So, it only remains to construct the required exponential attractors in the phase space $\mathbb{B}$. To this end, we first introduce the discrete semigroups $\mathcal{S}_n(\varepsilon) := S_{nT}(\varepsilon)$, $n \in \mathbb{N}$ (which, according to (3.7), act on the phase space $\mathbb{B}$) and construct the "discrete" exponential attractors $\mathcal{M}_\varepsilon^d$ by using the following exponential attractor's existence theorem proven in [EfMZ2].

Proposition 3.1. *Let the discrete semigroups $\{\mathcal{S}_n(\varepsilon), n \in \mathbb{N}\}$ acting on the space $\mathbb{B}$ (for every $\varepsilon \in [0,1]$) satisfy the following properties:*

1) Uniform smoothing property for the difference of two solutions:

$$\|\mathcal{S}_1(\varepsilon)(\phi_1,\psi_1) - \mathcal{S}_1(\varepsilon)(\phi_2,\psi_2)\|_{H^1(\Omega)\times H^1(\partial\Omega)} \leq K\|(\phi_1-\phi_2,\psi_1-\psi_2)\|_{L^2(\Omega)\times L^2(\partial\Omega)}, \quad (\phi_i,\psi_i) \in \mathbb{B}, \ i=1,2, \tag{3.8}$$

where the constant K is independent of ε.

2) Convergence as $\varepsilon \to 0$:

$$\|\mathcal{S}_n(\varepsilon)(\phi,\psi) - \mathcal{S}_n(0)(\phi,\psi)\|_{H^1(\Omega)\times H^1(\partial\Omega)} \leq C\varepsilon e^{Ln}, \quad (\phi,\psi) \in \mathbb{B}, \tag{3.9}$$

where the constants C and L are also independent of ε.

Then, there exists a robust family of exponential attractors $\mathcal{M}_\varepsilon^d$, $\varepsilon \in [0,1]$, which satisfy the analogues of properties 1)-4) of Theorem 3.1, namely:

1) Finite-dimensionality: the fractal dimension of $\mathcal{M}_\varepsilon^d$ (in $H^1(\Omega)\times H^1(\partial\Omega)$) is finite and uniformly bounded with respect to $\varepsilon \to 0$:

$$\dim_F(\mathcal{M}_\varepsilon, H^1(\Omega)\times H^1(\partial\Omega)) \leq C, \tag{3.10}$$

where the constant C is independent of ε.

2) Invariance: $\mathcal{S}_n(\varepsilon)\mathcal{M}_\varepsilon^d \subset \mathcal{M}_\varepsilon^d$, $n \in \mathbb{N}$.

3) Uniform exponential attraction: there exist positive constants α and C (which are independent of ε) such that

$$\operatorname{dist}_{H^1(\Omega)\times H^1(\partial\Omega)}(\mathcal{S}_n(\varepsilon)\mathbb{B}, \mathcal{M}_\varepsilon^d) \leq Ce^{-\alpha n}, \quad n \in \mathbb{N}. \tag{3.11}$$

4) The sets $\mathcal{M}_\varepsilon^d$ tend to the limit set $\mathcal{M}_0^d$ as $\varepsilon \to 0$ in the following sense:

$$\operatorname{dist}^{symm}_{H^1(\Omega)\times H^1(\partial\Omega)}(\mathcal{M}_\varepsilon^d, \mathcal{M}_0^d) \leq C_1\varepsilon^\tau, \tag{3.12}$$

where the positive constants C_1 and $\tau \in (0,1)$ are independent of ε.

Thus, in order to construct the family $\mathcal{M}_\varepsilon^d$ of discrete exponential attractors, we only need to verify estimates (3.8) and (3.9). We will do so in the next two lemmata.

Lemma 3.1. *Let the above assumptions hold. Then, for every* $(\phi_1(0), \psi_1(0))$ *and* $(\phi_2(0), \psi_2(0))$ *belonging to* $\mathbb{B}$, *the corresponding solutions of problem* (0.5) *satisfy the following estimates:*

$$\|\phi_1(T) - \phi_2(T)\|^2_{H^1(\Omega)} + \|\psi_1(T) - \psi_2(T)\|^2_{H^1(\partial\Omega)} \leq \\ \leq T^{-1}C_T\|(\phi_1(0) - \phi_2(0), \psi_1(0) - \psi_2(0))\|^2_{\mathbb{L}_\varepsilon}, \ T > 0, \tag{3.13}$$

where the constant C_T *depends on* T, *but is independent of* ε.

Proof. We first note that, due to estimates (1.27) and (3.6), we have

$$\|\phi_i(t)\|_{H^2(\Omega)} + \|\psi_i(t)\|_{H^2(\partial\Omega)} \leq C, \quad i = 1, 2, \quad t \geq 0, \tag{3.14}$$

where the constant C is independent of t and ε. Moreover, due to the embedding $H^2 \subset C$, we have analogous estimates for the L^∞-norms, which are necessary in order to handle the nonlinear terms in (0.5). In order to deduce (3.13), we first need to generalize (using the uniform estimates (3.14)) Proposition 1.2 to the case where condition (1.34) is not satisfied. Indeed, let $(\tilde\phi, \tilde\mu, \tilde\psi) := (\phi_1 - \phi_2, \mu_1 - \mu_2, \psi_1 - \psi_2)$. Then, these functions satisfy equations (1.36), but, in contrast to the proof of Proposition 1.2, we now have

$$\left\langle \tilde\phi(t) \right\rangle = \langle \phi_1(0)\rangle - \langle \phi_2(0)\rangle := M_{1,2} \neq 0. \tag{3.15}$$

Nevertheless, introducing the new functions $(\bar\phi(t), \bar\psi(t)) := (\tilde\phi(t) - M_{1,2}, \tilde\psi(t) - M_{1,2})$, multiplying the first equation of (1.36) by $(-\Delta_x)^{-1}_N\bar\phi(t)$ and arguing as in the proof of Propositions 1.1 and 1.2, we derive the following analogue of (1.37):

$$1/2\partial_t(\|\bar\phi(t)\|^2_{H^{-1}(\Omega)} + \varepsilon\|\bar\phi(t)\|^2_{L^2(\Omega)} + \|\bar\psi(t)\|^2_{L^2(\partial\Omega)}) + \|\nabla_x\bar\phi(t)\|^2_{L^2(\Omega)} + \\ + \lambda\|\bar\psi(t)\|^2_{L^2(\partial\Omega)} + \lambda M_{1,2}\left\langle \bar\psi(t)\right\rangle_{\partial\Omega} + \\ + \|\nabla_\parallel\bar\psi(t)\|^2_{L^2(\partial\Omega)} = -([f(\phi_1(t)) - f(\phi_2(t))], \bar\phi(t)) - ([g(\psi_1(t)) - g(\psi_2(t))], \bar\psi(t))_{\partial\Omega}. \tag{3.16}$$

In contrast to the proof of Proposition 1.2, we cannot estimate the nonlinear terms in (3.16) by using the facts that $f'(v) \geq -K$ and $g'(v) \geq -K$. Instead, we now have the uniform estimates (3.14) (and analogous estimates for the L^∞-norms). So, we can estimate these terms as follows:

$$-([f(\phi_1(t)) - f(\phi_2(t))], \bar\phi(t)) - ([g(\psi_1(t)) - g(\psi_2(t))], \bar\psi(t))_{\partial\Omega} \leq \\ \leq C(M^2_{1,2} + \|\bar\phi(t)\|^2_{L^2(\Omega)} + \|\bar\psi(t)\|^2_{L^2(\partial\Omega)}). \tag{3.17}$$

Inserting this estimate into the right-hand side of (3.16) and using Gronwall's inequality, we infer

$$\|(\bar\phi(t), \bar\psi(t))\|^2_{\mathbb{L}_\varepsilon} + \int_0^T (\|\nabla_x\tilde\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\parallel\tilde\psi(t)\|^2_{L^2(\partial\Omega)})\,dt \leq \\ \leq C_T(\|(\bar\phi(0), \bar\psi(0))\|^2_{\mathbb{L}_\varepsilon} + |M_{1,2}|^2) \leq C'_T\|(\tilde\phi(0), \tilde\psi(0))\|^2_{\mathbb{L}_\varepsilon}, \tag{3.18}$$

where the constants C_T and C'_T are independent of ε. We are now ready to verify estimate (3.13). To this end, we multiply the first equation of (1.36) by $t(-\Delta_x)^{-1}_N\partial_t\tilde\phi(t)$ and, arguing as in the derivation of (1.29), we have

$$
\begin{aligned}
(3.19)\quad & 1/2\partial_t(t\|\nabla_x\tilde\phi(t)\|^2_{L^2(\Omega)}+t\|\nabla_{\|}\tilde\psi(t)\|^2_{L^2(\partial\Omega)}+\lambda t\|\tilde\psi(t)\|^2_{L^2(\partial\Omega)})+\\
&\quad + t\|\partial_t\tilde\phi(t)\|^2_{H^{-1}(\Omega)}+\varepsilon t\|\partial_t\tilde\phi(t)\|^2_{L^2(\Omega)}+t\|\partial_t\tilde\psi(t)\|^2_{L^2(\partial\Omega)}=\\
&= 1/2(\|\nabla_x\tilde\phi(t)\|^2_{L^2(\Omega)}+\|\nabla_{\|}\tilde\psi(t)\|^2_{L^2(\partial\Omega)}+\lambda\|\tilde\psi(t)\|^2_{L^2(\partial\Omega)})+\\
&\quad +([f(\phi_1(t))-f(\phi_2(t))],t\partial_t\tilde\phi(t))+([g(\psi_1(t))-g(\psi_2(t))],t\partial_t\tilde\psi(t))_{\partial\Omega}.
\end{aligned}
$$

Estimating the last two terms in the right-hand side of (3.19) as follows:

$$
\begin{aligned}
(3.20)\quad & ([f(\phi_1(t))-f(\phi_2(t))],t\partial_t\tilde\phi(t))+([g(\psi_1(t))-g(\psi_2(t))],t\partial_t\tilde\psi(t))_{\partial\Omega}\le\\
&\le Ct(\|\tilde\phi(t)\|^2_{H^1(\Omega)}+\|\tilde\psi(t)\|^2_{L^2(\partial\Omega)})+1/2(t\|\partial_t\tilde\phi(t)\|^2_{H^{-1}(\Omega)}+t\|\partial_t\tilde\psi(t)\|^2_{L^2(\partial\Omega)})
\end{aligned}
$$

(where we have again used estimates (3.14)), integrating in t and using (3.18), we infer

$$
(3.21)\qquad T(\|\tilde\phi(T)\|^2_{H^1(\Omega)}+\|\tilde\psi(T)\|^2_{H^1(\partial\Omega)})\le C_T\|(\tilde\phi(0),\tilde\psi(0))\|^2_{\mathbb{L}_\varepsilon},
$$

which finishes the proof of the lemma.

Since, obviously, $\|(\tilde\phi,\tilde\psi)\|_{\mathbb{L}_\varepsilon}\le C\|(\tilde\phi,\tilde\psi)\|_{L^2(\Omega)\times L^2(\partial\Omega)}$, where C is independent of ε, then assumption (3.8) of Proposition 3.1 is verified.

Lemma 3.2. *Let $(\phi_\varepsilon(t),\psi_\varepsilon(t))$ and $(\phi_0(t),\psi_0(t))$ be two solutions of equations* (0.5) *with positive ε and with $\varepsilon=0$ respectively. We also assume that these solutions have the same initial datum, belonging to the set $\mathbb{B}$. Then, the following estimates hold:*

$$
(3.22)\qquad \|\phi_\varepsilon(t)-\phi_0(t)\|^2_{H^1(\Omega)}+\|\psi_\varepsilon(t)-\psi_0(t)\|^2_{H^1(\partial\Omega)}\le C\varepsilon^2e^{Kt},
$$

where the constants C and K are independent of ε and t.

Proof. We set $(\bar\phi,\bar\mu,\bar\psi):=(\phi_\varepsilon-\phi_0,\mu_\varepsilon-\mu_0,\psi_\varepsilon-\psi_0)$. Then, these functions satisfy the following system of equations:

$$
(3.23)\qquad \begin{cases}
\partial_t\bar\phi=\Delta_x\bar\mu,\quad \partial_n\bar\mu\big|_{\partial\Omega}=0,\\
\bar\mu=-\Delta_x\bar\phi+\varepsilon\partial_t\bar\phi+[f(\phi_\varepsilon)-f(\phi_0)]+\varepsilon\partial_t\phi_0,\\
\partial_t\bar\psi=\Delta_{\|}\bar\psi-\lambda\bar\psi-[g(\psi_\varepsilon)-g(\psi_0)]-\partial_n\bar\phi,\quad x\in\partial\Omega,\\
\bar\phi\big|_{\partial\Omega}=\bar\psi.
\end{cases}
$$

Multiplying now the first equation of (3.23) by $(-\Delta_x)^{-1}_N\partial_t\bar\phi$ and arguing exactly as in the derivation of (3.19), we obtain

$$
\begin{aligned}
(3.24)\quad & 1/2\partial_t(\|\nabla_x\bar\phi(t)\|^2_{L^2(\Omega)}+\|\nabla_{\|}\bar\psi(t)\|^2_{L^2(\partial\Omega)}+\lambda\|\bar\psi(t)\|^2_{L^2(\partial\Omega)})+\\
&\quad +\|\partial_t\bar\phi(t)\|^2_{H^{-1}(\Omega)}+\varepsilon\|\partial_t\bar\phi(t)\|^2_{L^2(\Omega)}+\|\partial_t\bar\psi(t)\|^2_{L^2(\partial\Omega)}=\\
&=([f(\phi_\varepsilon(t))-f(\phi_0(t))],\partial_t\bar\phi(t))+([g(\psi_\varepsilon(t))-g(\psi_0(t))],\partial_t\bar\psi(t))_{\partial\Omega}+\\
&\qquad +\varepsilon(\partial_t\phi_0(t),\partial_t\bar\phi(t)).
\end{aligned}
$$

We now recall that, according to (1.1) and (1.27), we have

$$\int_0^T \|\partial_t \phi_0(t)\|^2_{H^1(\Omega)}\, dt \le C(T+1), \tag{3.25}$$

where C is independent of ε. Estimating the nonlinear terms in the right-hand side of (3.24) by using (3.20), applying Gronwall's inequality and estimating the last term in the right-hand side of (3.24) by (3.25), we obtain estimate (3.22) and finish the proof of Lemma 3.2.

Thus, all the assumptions of Proposition 3.1 are verified and, consequently, the discrete semigroups $\{\mathcal{S}_n(\varepsilon), n\in\mathbb{N}\}$ possess a robust family of exponential attractors $\mathcal{M}_\varepsilon$ on $\mathbb{B}$ which satisfy properties 1)–4) of Proposition 3.1. Moreover, since $\mathbb{B}$ gives uniform absorbing sets for these semigroups in $\mathbb{L}_\varepsilon(M)$, then these attractors attract exponentially all the bounded subsets of $\mathbb{L}_\varepsilon(M)$. So, the required family of exponential attractors is constructed for discrete times. In order to extend this result to continuous times, we use the standard formula

$$\mathcal{M}_\varepsilon := \cup_{t\in[T,2T]} S_t(\varepsilon)\mathcal{M}^d_\varepsilon \tag{3.26}$$

(see, e.g., [EFNT] and [EfMZ1]). Then, since the semigroups $S_t(\varepsilon)$ are uniformly Lipschitz continuous on $[T,2T]\times\mathbb{B}$ in the norm of $\mathbb{L}_\varepsilon$ (the Lipschitz continuity with respect to the initial data was in fact verified in Proposition 1.2 and Lemma 3.1 and the Lipschitz continuity with respect to t follows from the fact that $(\partial_t\phi(t),\partial_t\psi(t))\in\mathbb{L}_\varepsilon$ if $(\phi(t),\psi(t))\in\mathbb{D}_\varepsilon$), arguing in a standard way (see, e.g., [EfMZ2]), we deduce that the family (3.26) of exponential attractors satisfies assumptions 1)–4) of Theorem 3.1, but with the $H^2(\Omega)\times H^2(\partial\Omega)$-norm replaced by the $\mathbb{L}_\varepsilon$-norm.

In order to extend the results obtained to the required $H^2(\Omega)\times H^2(\partial\Omega)$-norm, it remains to recall that, according to Theorems 1.2 and 1.3, the semigroups $S_t(\varepsilon)$ possess a uniform smoothing property from $\mathbb{L}_\varepsilon$ into $H^3(\Omega)\times H^3(\partial\Omega)$ and to use the obvious interpolation inequality:

$$\|(v,w)\|_{H^2(\Omega)\times H^2(\partial\Omega)} \le C\|(v,w)\|^{1/4}_{\mathbb{L}_\varepsilon}\cdot\|(v,w)\|^{3/4}_{H^3(\Omega)\times H^3(\partial\Omega)}, \tag{3.27}$$

where the constant C is also independent of ε. Thus, Theorem 3.1 is proven.

Appendix. The maximum principle and L^∞-bounds on the solutions for an auxiliary elliptic problem.

In this section, we formulate and prove several estimates which play a fundamental role for obtaining L^∞-bounds on the solutions of the initial Cahn-Hilliard problem. Since the existence of a solution has been verified in a more complicated situation in Section 2, we restrict ourselves only to the derivation of the corresponding a priori estimates.

We start with the study of the following linear problem:

$$\begin{cases} -\Delta_x u = h_1(x), \quad x \in \Omega, \quad u\big|_{\partial\Omega} = \phi, \\ -\Delta_{\|}\phi + \lambda\phi + \partial_n u = h_2(x), \ x \in \partial\Omega, \end{cases} \tag{A.1}$$

where λ is some positive constant. We start with the classical elliptic L^2-regularity estimates for this problem.

Lemma A.1. *Let the functions h_1 and h_2 belong to the spaces $L^2(\Omega)$ and $L^2(\partial\Omega)$ respectively. Then, the following estimates hold:*

$$\|u\|_{H^2(\Omega)} + \|\phi\|_{H^2(\partial\Omega)} \le C(\|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\partial\Omega)}), \tag{A.2}$$

for some positive constant C.

Proof. We first multiply the first equation of (A.1) scalarly in $L^2(\Omega)$ by u, integrate by parts and find the expression for the term $(\partial_n u, u)_{\partial\Omega} = (\partial_n u, \phi)_{\partial\Omega}$ from the second equation of (A.1). Then, after obvious transformations, we have

$$\|\nabla_x u\|^2_{L^2(\Omega)} + \|\nabla_{\|}\phi\|^2_{L^2(\partial\Omega)} + \lambda\|\phi\|^2_{L^2(\partial\Omega)} = (h_1, u) + (h_2, \phi)_{\partial\Omega} \tag{A.3}$$

and, consequently, using the fact that $\|u\|_{H^1(\Omega)} \le C(\|\nabla_x u\|_{L^2(\Omega)} + \|\phi\|_{H^1(\partial\Omega)})$, we have

$$\|u\|_{H^1(\Omega)} + \|\phi\|_{H^1(\partial\Omega)} \le C(\|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\partial\Omega)}), \tag{A.4}$$

for some positive constant C.

Applying now the H^2-regularity theorem to the first equation of (A.1) with Dirichlet boundary conditions, we have

$$\|u\|_{H^2(\Omega)} \le C(\|h_1\|_{L^2(\Omega)} + \|\phi\|_{H^{3/2}(\partial\Omega)}). \tag{A.5}$$

Analogously, applying this theorem to the second equation of (A.1), we deduce

$$\|\phi\|_{H^2(\partial\Omega)} \le C(\|h_2\|_{L^2(\partial\Omega)} + \|\partial_n u\|_{L^2(\partial\Omega)}). \tag{A.6}$$

Moreover, due to a proper interpolation inequality, we obtain, for every $0 < s < 1/2$,

$$\|\partial_n u\|_{H^s(\partial\Omega)} \le C_s\|u\|_{H^{3/2+s}(\Omega)} \le C_\nu\|u\|_{H^1(\Omega)} + \nu\|u\|_{H^2(\Omega)}, \tag{A.7}$$

where the positive constant ν can be arbitrarily small. Combining estimates (A.5)–(A.7), we have

$$\|u\|_{H^2(\Omega)} \le C_1(\|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\partial\Omega)}) + C'\nu\|u\|_{H^2(\Omega)} + C'_\nu\|u\|_{H^1(\Omega)}, \tag{A.8}$$

which, together with (A.4), gives the required estimate for the H^2-norm of u. The H^2-norm of ϕ can then be found from estimates (A.6) and (A.7). Lemma A.1 is proven.

Corollary A.1. *Let the assumptions of Lemma A.1 hold and let, in addition, the external forces h_1 and h_2 belong to $H^s(\Omega)$ and $H^s(\partial\Omega)$ respectively, where $s \geq 0$ is such that $s + 1/2$ does not belong to $\mathbb{N}$. Then, the solution (u, ϕ) of problem* (A.1) *belongs to the space $H^{s+2}(\Omega) \times H^{s+2}(\partial\Omega)$ and the following estimates hold:*

$$\|u\|_{H^{s+2}(\Omega)} + \|\phi\|_{H^{s+2}(\partial\Omega)} \leq C(\|h_1\|_{H^s(\Omega)} + \|h_2\|_{H^s(\partial\Omega)}). \tag{A.8$'$}$$

Indeed, estimate (A.8$'$) can be derived exactly as for (A.5–A.8), except that, instead of using the L^2-H^2 regularity estimate for linear elliptic equations, one needs to use its analogue for H^s-spaces.

We are now ready to formulate the main result of this section on L^∞-bounds on the solutions of the following *nonlinear* second-order elliptic problem:

$$\begin{cases} -\Delta_x u + f(u) = h_1(x), \quad u\big|_{t=0} = u_0, \quad u\big|_{\partial\Omega} = \phi, \\ -\Delta_\| \phi + \lambda\phi + \partial_n u + g(\phi) = h_2(x). \end{cases} \tag{A.9}$$

Lemma A.2. *Let the functions h_1 and h_2 be as in Lemma A.1 and the functions f and g satisfy assumptions* (0.3)*. Then, every solution of problem* (A.9) *satisfies the following estimates:*

$$\|u\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\partial\Omega)} \leq C_{f,g} + C(\|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\partial\Omega)}), \tag{A.10}$$

where the constants $C_{f,g}$ and C are independent of the solution (u, ϕ).

Proof. As usual, the L^∞-estimates (A.10) are based on the maximum principle. However, we cannot apply it *directly* to problem (A.9), since the external forces do not belong to L^∞. In order to overcome this difficulty, we introduce the function $(\bar{u}, \bar{\phi})$ solution of the simplified problem (A.9) with $f_1 = f_2 = 0$. Then, due to estimate (A.2) and the embedding $H^2 \subset C$,

$$\|\bar{u}\|_{L^\infty(\Omega)} + \|\bar{\phi}\|_{L^\infty(\partial\Omega)} \leq C(\|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\partial\Omega)}) := \bar{K}. \tag{A.11}$$

We also introduce the function $(\tilde{u}, \tilde{\phi}) := (u - \bar{u}, \phi - \bar{\phi})$ which obviously solves the following problem:

$$\begin{cases} -\Delta_x \tilde{u} + f(\tilde{u} + \bar{u}) = 0, \; \tilde{u}\big|_{\partial\Omega} = \tilde{\phi}, \\ -\Delta_\| \tilde{\phi} + \lambda\tilde{\phi} + \partial_n \tilde{u} + g(\tilde{\phi} + \bar{\phi}) = 0. \end{cases} \tag{A.12}$$

In contrast to (A.9), all the terms in equations (A.12) belong to L^∞ and we can apply the maximum principle. To this end, we fix the constant $C_{f,g} > 0$ such that

$$\begin{cases} \operatorname{sgn} v \cdot f(v) \geq 0, \; f'(v) \geq 0, \\ \operatorname{sgn} v \cdot g(v) \geq 0, \; g'(v) \geq 0, \end{cases} \tag{A.13}$$

for all $v \notin [-C_{f,g}, C_{f,g}]$ ($C_{f,g}$ exists thanks to conditions (0.3)), and consider the following test function which is independent of x:

$$U = \Phi := C_{f,g} + \bar{K}, \tag{A.14}$$

where the constant $\bar{K}$ is the same as in (A.11). Then, (A.13) and (A.11) imply that (U, Φ) is a supersolution of (A.12) and, consequently, the function $(\bar{U}, \bar{\Phi}) := (\tilde{u} - U, \tilde{\phi} - \Phi)$ satisfies the following differential inequalities:

$$\begin{cases} -\Delta_x \bar{U} + [f(\tilde{u} + \bar{u}) - f(U + \bar{u})] \le 0, \ \bar{U}\big|_{\partial\Omega} = \bar{\Phi}, \\ -\Delta_{\|} \bar{\Phi} + \lambda \bar{\Phi} + \partial_n \bar{U} + [g(\tilde{\phi} + \bar{\phi}) - g(\Phi + \bar{\phi})] \le 0. \end{cases} \tag{A.15}$$

We also recall that, due to (A.13),

$$[f(w + \bar{u}) - f(U + \bar{u})] \ge 0, \quad [g(w + \bar{u}) - g(\bar{\Phi} + \bar{\phi})] \ge 0, \tag{A.16}$$

for all $w \ge U = \Phi$. Multiplying now the first inequality of (A.15) scalarly in $L^2(\Omega)$ by $\bar{U}_+ := \max\{\bar{U}, 0\}$, integrating by parts and using (A.16), we have

$$\|\nabla_x \bar{U}_+\|^2_{L^2(\Omega)} + \|\nabla_{\|} \bar{\Phi}_+\|^2_{L^2(\partial\Omega)} + \lambda \|\bar{\Phi}_+\|^2_{L^2(\partial\Omega)} \le 0, \tag{A.17}$$

which immediately implies that $\bar{U}_+ = \bar{\Phi}_+ \equiv 0$ and, consequently,

$$(u(x), \phi(x)) \le (U + \bar{u}(x), \Phi + \bar{\phi}(x)), \tag{A.18}$$

which, together with (A.11) and (A.14), gives the required upper bound on the solution (u, ϕ). The lower bound can be obtained analogously by using the test function $(-U, -\Phi)$. Lemma A.2 is proven.

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