EXPONENTIAL ATTRACTORS FOR A SINGULARLY PERTURBED CAHN-HILLIARD SYSTEM

M.A. EFENDIEV [a], A. MIRANVILLE [b], S. ZELIK [c]

[a] Free University Berlin,
Department of Mathematics, Arnimallee 2-6,
D-14195 Berlin, Germany.
Email : efendiev@math.fu-berlin.de

[b] Université de Poitiers,
Laboratoire d'Applications des Mathématiques,
SP2MI, Téléport 2, Avenue Marie et Pierre Curie,
86962 Futuroscope Chasseneuil Cedex, France.
Email : miranv@mathlabo.univ-poitiers.fr

[c] Institute for Information Transmission Problems, Russian Academy of Sciences,
Bolshoi Karetnii 19, Moscow 101447, GSP-4, Russia.
Email : zelik@iitp.ru

Abstract. Our aim in this article is to give a construction of exponential attractors that are continuous under perturbations of the underlying semi-group. We note that the continuity is obtained without time shifts as it was the case in previous studies. Moreover, the explicit estimate for the symmetric distance between the perturbed and non-perturbed exponential attractors in terms of the perturbation parameter is also obtained. As an application, we prove the continuity of exponential attractors for a viscous Cahn-Hilliard system to an exponential attractor for the limit Cahn-Hilliard system.

Introduction

The study of the long time behavior of systems arising from physics and mechanics is a capital issue, as it is important, for practical purposes, to understand and predict the asymptotic behavior of the system.

For many parabolic and weakly damped wave equations, one can prove the existence of the finite dimensional (in the sense of the Hausdorff or the fractal dimension) global attractor, which is a compact invariant set which attracts uniformly the bounded sets of the phase space. Since it is the smallest set enjoying these properties, it is a suitable set

1991 *Mathematics Subject Classification.* 37L30, 35B40, 35B45.
Key words and phrases. Exponential attractors, continuity, viscous Cahn-Hilliard system.

for the study of the long time behavior of the system. We refer the reader to [BV], [H], [L], [R] and [T] for extensive reviews on this subject.

Now, the global attractor may present two major defaults for practical purposes. Indeed, the rate of attraction of the trajectories may be small and (consequently) it may be sensible to perturbations.

In order to overcome these difficulties, Foias, Sell and Temam proposed in [FoST] the notion of inertial manifold, which is a smooth finite dimensional hyperbolic (and thus robust) positively invariant manifold which contains the global attractor and attracts exponentially the trajectories. Unfortunately, all the known constructions of inertial manifolds are based on a restrictive condition, the so-called spectral gap condition. Consequently, the existence of inertial manifolds is not known for many physically important equations (e.g. the Navier-Stokes equations, even in two space dimensions). A non-existence result has even been obtained by Mallet-Paret and Sell for a reaction-diffusion equation in higher-space dimensions.

Thus, as an intermediate object between the two ideal objects that the global attractor and an inertial manifold are, Eden, Foias, Nicolaenko and Temam proposed in [EFNT] the notion of exponential attractor, which is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. So, compared with the global attractor, an exponential attractor is more robust under perturbations and numerical approximations (see [EFNT], [FGM] and [G] for discussions on this subject). Another motivation for the study of exponential attractors comes from the fact that the global attractor may be trivial (say, reduced to one point) and may thus fail to capture important transient behaviors. We note however that, contrarily to the global attractor, an exponential attractor is not necessarily unique, so that the actual/concrete choice of an exponential attractor is in a sense artificial.

Exponential attractors have been constructed for a large class of equations (see [BN], [EFNT], [EFK], [EfM], [EfMZ1], [EfMZ2], [FGM], [G], [M1], [M2], [MPR] and the references therein). The known constructions of exponential attractors (see for instance [BN], [EFNT], [EFK] and [M1]) make an essential use of orthogonal projectors with finite rank (in order to prove the so-called squeezing property) and are thus valid in Hilbert spaces only. Recently, Efendiev, Miranville and Zelik gave in [EfMZ1] (see also [EfMZ2]) a construction of exponential attractors that is no longer based on the squeezing property and that is thus valid in a Banach setting. So, exponential attractors are as general as global attractors.

Let us come back to the robustness of the global attractor. Generally, global attractors are only upper semi-continuous with respect to perturbations. The property of their lower semi-continuity is much more delicate and can be established only for some particular cases (see for instance [R]). One of them is the case where a semi-group has a global Liapunov function and all equilibria are hyperbolic. In this particular case the corresponding global attractor (so-called regular attractor) is exponential and is robust under the perturbation (i.e. it is upper and lower semi-continuous with respect to the perturbation, see [BV] or [H]). Moreover, if $\mathcal{A}_\varepsilon$ is the regular attractor of a perturbed system and $\mathcal{A}_0$ corresponds to the unperturbed one, then under the natural assumptions on perturbation one has

$$\operatorname{dist}_{symm}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \le C\varepsilon^\kappa$$

where $\kappa \in (0,1)$ and ε is a perturbation parameter (see [BV]).

As already mentioned, exponential attractors are more robust objects. In particular, one can prove the continuity of exponential attractors under perturbations in many cases (see [EFNT] for the continuity for classical Galerkin approximations and [FGM] and [G] for examples of (singular) pertubations of partial differential equations), even when this property is violated or is not known for the global attractor. However, in all these references, the continuity is obtained only up to a time shift. In this article, we give (in Section 3), conditions on the semigroup which ensure the continuity of exponential attractors without such time shifts. Moreover, we obtain analogous (to the case of regular attractors) estimate for the symmetric distance between the perturbed ($\mathcal{M}_\varepsilon$) and unperturbed ($\mathcal{M}_0$) exponential attractors

$$\operatorname{dist}_{symm}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C_1\varepsilon^{\kappa_1}$$

without the supposition that the system under the consideration possesses a global Liapunov function and all equilibria are hyperbolic. Note also, that in contrast to the case of regular attractors our approach allows in applications to compute constants C_1 and κ_1 in terms of the corresponding physical parameters.

As an example, we consider a viscous Cahn-Hilliard system in $\mathbb{R}^3$. The Cahn-Hilliard equation (see [C], [CH], [ElZ], [Gu], [NST] and [Gu]) is very important in materials science: it models important qualitative behaviors of two phase systems, namely the transport of atoms between unit cells. Several generalizations of the original equation have been proposed and studied (see for instance [BaB], [Bo], [CaMP], [ChD], [ElG], [ElK], [ElS], [Ey], [Ga], [Gu], [LiZ], [M2], [M3], [MPR] and [No]) and, among them, the viscous Cahn-Hilliard equation (see [ElK], [ElS] and [No]).

In [ElS], the authors proved the existence of the global attractor for a viscous Cahn-Hilliard equation. They also proved the upper semicontinuity of the global attractor to that of the limit Cahn-Hilliard equation. However, they proved the lower semicontinuity under the assumption that all the stationary solutions be hyperbolic only; this assumption was relaxed in [ElK], but only in one space dimension. We also note that all these results are obtained under growth restrictions on the potential (i.e. on the nonlinear term); in [EfGZ], the authors proved the existence of the finite dimensional global attractor without any growth condition on the potential for $\varepsilon \neq 0$ but the estimates for the dimension obtained there was not uniform with respect to $\varepsilon \to 0$; the case $\varepsilon = 0$ without growth restrictions has been considered in [LyZ] where the bondedness of the corresponding attractor in $H^4(\Omega)$ has been obtained.

In this article, we prove, in Section 4, the continuity of exponential attractors for the viscous Cahn-Hilliard system and derive the corresponding estimate for the symmetric distance. This result is obtained without any growth condition on the nonlinearity and thus necessitates several technical estimates that are obtained in Sections 1 and 2. Note also that the deriving of these estimates is based on a combination of techniques developed in [EfGZ], [LyZ], and [Ze].

§0 Setting of the problem.

We consider the following viscous Cahn-Hilliard system:

$$\begin{cases} \partial_t u = -\Delta_x(a\Delta_x u - \varepsilon\partial_t u - f(u) + \tilde{g}), \quad x \in \Omega, \\ u\big|_{\partial\Omega} = \Delta_x u\big|_{\partial\Omega} = 0, \\ u\big|_{t=0} = u_0, \end{cases} \tag{0.1}$$

where $\Omega \subset\subset \mathbb{R}^3$ is a bounded domain with a sufficiently smooth boundary, $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, f and $\tilde{g}$ are given functions, a is a given constant $k \times k$ matrix with a positive symmetric part $a + a^* > 0$ and $\varepsilon \geq 0$ is a small parameter.

It will be more convenient for us to rewrite equation (0.1) in the following form:

$$\begin{cases} \partial_t(\varepsilon + L)u = a\Delta_x u - f(u) + g, \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \end{cases} \tag{0.2}$$

where L denotes the operator $(-\Delta_x)^{-1}$ associated with Dirichlet boundary conditions and the functions g and $\tilde{g}$ satisfy the relation $g = \tilde{g} - V$, where V is solution of the following problem:

$$\Delta_x V = 0, V\big|_{\partial\Omega} = \tilde{g}\big|_{\partial\Omega}. \tag{0.3}$$

Furthermore, we assume that the nonlinear term $f(u)$ satisfies the following conditions:

$$\begin{cases} 1. \quad f \in C^2(\mathbb{R}^k, \mathbb{R}^k), \quad f(0) = 0, \\ 2. \quad f(u).u \geq -C, \\ 3. \quad f'(u) \geq -KId. \end{cases} \tag{0.4}$$

Here an below, we denote by $u.v$ the standard inner product in $\mathbb{R}^k$ and we assume that the external force g in (0.2) belongs to the space $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

A solution of problem (0.2) is a function $u(t)$ which belongs to the space $\Phi := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, for every fixed $t \geq 0$, and satisfies (0.2) in the sense of distributions. Consequently, we assume that the initial value u_0 belongs to Φ.

Remark 0.1. We note that, due to a classical Sobolev embedding theorem, $\Phi \subset C(\Omega)$. Consequently, the nonlinear term $f(u)$ in (0.2) is well defined. We also note that, for smooth solutions $u(t)$ $(u(t) \in W^{4,2}(\Omega))$, equations (0.1) and (0.2) are equivalent. Indeed, applying the invertible operator L to both sides of (0.1), we obtain (0.2). If $u(t)$ only belongs to Φ, equation (0.2) can be treated as a definition of solutions for equation (0.1). It is not difficult to verify however that the definition of a solution thus obtained is equivalent to the standard definition of the variational solution for equation (0.1) (see e.g. [BV]).

Remark 0.2. We note that we make no growth restriction on the potential f in (0.4). In particular, conditions (0.4) are satisfied for polynomials of arbitrary odd degree with strictly positive leading coefficients (such potentials are classical in the Cahn-Hilliard theory, see [C] and [CH]). Generally, one has to make restrictions on the growth of the potential (and also on the degree of the polynomial for a polynomial potential; see e.g. [ChD], [ElG], [ElS] and [NST]), in particular, in order to study the regularity of solutions and the finite dimensionality of attractors.

§1 Uniform a priori estimates. Existence and uniqueness of solutions.

In this Section, we derive several a priori estimates for the solutions of (0.1). We then verify the existence and the uniqueness of solutions and obtain some regularity results which will be essential for the sequel. We start with the following theorem.

Theorem 1.1. *Let the above assumptions hold and let $u(t)$ be a solution of problem* (0.2). *Then, the following estimate is valid:*

$$\|u(t)\|_\Phi \le Q(\|u(0)\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2}), \tag{1.1}$$

where $\alpha > 0$ and the monotonic function Q are independent of $\varepsilon_0 \ge \varepsilon \ge 0$ and where $\|.\|_{m,p}$ denotes the usual norm on $W^{m,p}(\Omega)$.

We give below the formal derivation of estimate (1.1) only (this estimate can be easily justified by using e.g. Galerkin approximations). To this end, we need the following lemmata.

Lemma 1.1. *Let $u(t)$ be a solution of equation* (0.1). *Then, the following estimate is valid uniformly with respect to $\varepsilon_0 \ge \varepsilon \ge 0$:*

$$\varepsilon\|u(t)\|_{0,2}^2 + \|u(t)\|_{-1,2}^2 + \int_t^{t+1} \|u(s)\|_{1,2}^2\, ds \le C\|u(0)\|_{0,2}^2 e^{-\alpha t} + C(1 + \|g\|_2^2), \tag{1.2}$$

for appropriate strictly positive constants C and α.

Proof. Multiplying equation (0.1) by $u(t)$ and integrating over Ω, we obtain, integrating by parts and using the assumptions $a + a^* > 0$ and $f(u).u \ge -C$

$$1/2\partial_t(\varepsilon\|u(t)\|_{0,2}^2 + \|u(t)\|_{-1,2}^2) + 2\alpha\|\nabla_x u(t)\|_{0,2}^2 \le C(1 + |(g, u(t))|), \tag{1.3}$$

for an appropriate constant $\alpha > 0$. Applying Holder inequality together with Friedrichs inequality to relation (1.3), we find

$$\partial_t(\varepsilon\|u(t)\|_{0,2}^2 + \|u(t)\|_{-1,2}^2) + \\ + \alpha_1\|u(t)\|_{1,2}^2 + \alpha'(\varepsilon\|u(t)\|_{0,2}^2 + \|u(t)\|_{-1,2}^2) \le C_1(1 + \|g\|_{0,2}^2), \tag{1.4}$$

for some strictly positive constants C_1, α_1 and α' which are independent of $\varepsilon \ge 0$. Applying now Gronwall inequality to estimate (1.4), we have (1.2) and Lemma 1.1 is proved.

Lemma 1.2. *Let $u(t)$ be a solution of* (0.2). *Then, the following estimate is valid uniformly with respect to $\varepsilon_0 \geq \varepsilon \geq 0$:*

$$(1.5)\quad \varepsilon\|u(t)\|_{1,2}^2 + \|u(t)\|_{0,2}^2 + \int_t^{t+1} \|u(s)\|_{2,2}^2\, ds \leq \\ \leq C(\varepsilon\|u(0)\|_{1,2}^2 + \|u(0)\|_{0,2}^2)e^{-\alpha t} + C(1 + \|g\|_{0,2}^2),$$

for appropriate strictly positive constants C and α.

Proof. Multiplying equation (0.2) by $\Delta_x u(t)$, integrating over Ω and using the fact that $a + a^* > 0$ and $f'(u) \geq -KId$, we find, after simple transformations

$$(1.6)\quad \partial_t(\varepsilon\|u(t)\|_{1,2}^2 + \|u(t)\|_{0,2}^2) + \alpha\|u(t)\|_{2,2}^2 + \alpha(\varepsilon\|u(t)\|_{1,2}^2 + \|u(t)\|_{0,2}^2) \leq \\ \leq C(1 + \|g\|_{0,2}^2 + \|u(t)\|_{1,2}^2),$$

(here, we have also used the $(W^{2,2}, L^2)$-elliptic regularity theorem for the Laplacian (see e.g. [ADN])).

Applying Gronwall inequality to relation (1.6) and estimating the t-integral of $\|u(t)\|_{1,2}^2$ which then appears in the right-hand side by estimate (1.2), we obtain (1.5) and Lemma 1.2 is proved.

Lemma 1.3. *Let the above assumptions hold and let $u(t)$ be a solution of* (0.2). *Then, there exists a time $T_0 = T_0(\|u_0\|_\Phi)$, $0 < T_0 < 1/2$, and a function Q (which are independent of ε) such that*

$$(1.7)\quad \|u(t)\|_\Phi \leq Q(\|u_0\|_\Phi) + C\|g\|_{2,2}, \quad t \leq T_0(\|u_0\|_\Phi).$$

Proof. Let us rewrite equation (0.2) (or (0.1)) in the following equivalent form:

$$(1.8)\quad \partial_t u = -(1 - \varepsilon\Delta_x)^{-1}\Delta_x(a\Delta_x u - f(u) + g), \quad u\big|_{\partial\Omega} = \Delta_x u\big|_{\partial\Omega} = 0.$$

Multiplying equation (1.8) by $\Delta_x^2 u(t)$ (formally; these formal calculations can be justified by Galerkin approximations), we obtain the following inequality:

$$(1.9)\quad 1/2\partial_t\|\Delta_x u(t)\|_{0,2}^2 + \alpha\|(1 - \varepsilon\Delta_x)^{-1/2}\Delta_x^2 u(t)\|_{0,2}^2 \leq \\ \leq |\Big((1 - \varepsilon\Delta_x)^{-1/2}\Delta_x f(u(t)), (1 - \varepsilon\Delta_x)^{-1/2}\Delta_x^2 u(t)\Big)| + \\ + |\Big((1 - \varepsilon\Delta_x)^{-1/2}\Delta_x g, (1 - \varepsilon\Delta_x)^{-1/2}\Delta_x^2 u(t)\Big)|.$$

Applying Holder inequality to the right-hand side of (1.9) and noting that

$$(1.10)\quad \|(1 - \varepsilon\Delta_x)^{-1/2}\|_{L^2 \to L^2} \leq C,$$

(where C is independent of ε), we obtain the estimate

$$\partial_t(\|\Delta_x u(t)\|^2_{0,2}) \le C_1\left(\|g\|^2_{2,2} + \|f(u(t))\|^2_{2,2}\right), \tag{1.11}$$

where C_1 is independent of ε.

We recall that $f \in C^2$ and that $W^{2,2} \subset C$. Consequently, there exists a monotonic function Q (depending on f) such that

$$\|f(u(t))\|^2_{2,2} \le Q(\|u(t)\|^2_{2,2}) \le Q_1(\|\Delta_x u(t)\|^2_{0,2}). \tag{1.12}$$

Thus, the function $y(t) := \|\Delta_x u(t)\|^2_{0,2}$ satisfies the inequality

$$y'(t) \le C_1(\|g\|^2_{2,2} + Q_1(y(t))). \tag{1.13}$$

Let $z(t)$ be a solution of the following equation:

$$z'(t) = C_1(\|g\|^2_{2,2} + Q_1(z(t))), \quad z(0) = y(0) = \|\Delta_x u(0)\|^2_{0,2}. \tag{1.14}$$

Then, due to the comparison principle

$$y(t) \le z(t), \tag{1.15}$$

for every $t \ge 0$ such that $z(t)$ is defined. The assertion of the lemma is then an immediate consequence of (1.14) and (1.15).

Lemma 1.4. *Let the above assumptions hold and let T_0 be the same as in Lemma 1.3. Then, the following estimate holds:*

$$\varepsilon\|\partial_t u(T_0)\|^2_{0,2} + \|\partial_t u(T_0)\|^2_{-1,2} \le Q(\|u_0\|_\Phi) + Q(\|g\|_{2,2}), \tag{1.16}$$

for some monotonic function Q which is independent of ε.

Proof. Let us multiply equation (0.2) by $a\partial_t u(t)$ and integrate over Ω. Then, we have, after simple transformations

$$\begin{aligned}\alpha(\varepsilon\|\partial_t u(t)\|^2_{0,2} + \|\partial_t u(t)\|^2_{-1,2}) + \partial_t(a\nabla_x u(t), a\nabla_x u(t)) + \\ + 2\partial_t(a^* g, u(t)) \le 2|(f(u(t)), a\partial_t u(t))| \le C\|f(u(t))\|^2_{1,2} + \alpha/2\|\partial_t u(t)\|^2_{-1,2},\end{aligned} \tag{1.17}$$

where the constants C and $\alpha > 0$ are independent of ε.

Integrating inequality (1.17) over $[0, T_0]$ and taking into account estimate (1.7) and the fact that $\Phi \subset C$, we obtain the following estimate:

$$\int_0^{T_0} (\varepsilon\|\partial_t u(s)\|^2_{0,2} + \|\partial_t u(s)\|^2_{-1,2})\, ds \le Q(\|u_0\|_\Phi) + Q(\|g\|_{2,2}). \tag{1.18}$$

In order to complete the proof of the lemma, we differentiate equation (0.2) with respect to t and set $\theta(t) := \partial_t u(t)$. This function obviously satisfies the equation:

$$
\begin{cases} \partial_t(\varepsilon + L)\theta = a\Delta_x\theta - f'(u(t))\theta, \\ \theta\big|_{\partial\Omega} = 0. \end{cases} \tag{1.19}
$$

Multiplying this equation by $t\theta(t)$, integrating over Ω and using the fact that $f'(u) \geq -KId$, we have

$$
\begin{aligned} \partial_t[t(\varepsilon\|\theta(t)\|_{0,2}^2 + \|\theta(t)\|_{-1,2}^2)] + \alpha t\|\theta(t)\|_{2,2}^2 \leq \\ \leq 2Kt\|\theta(t)\|_{0,2}^2 + 2t(g, \theta(t)) + (\varepsilon\|\partial_t u(t)\|_{0,2}^2 + \|\partial_t u(t)\|_{-1,2}^2). \end{aligned} \tag{1.20}
$$

In order to estimate the right-hand side of (1.20), we use the following interpolation inequality (see e.g. [LoM] and [Tr]):

$$
2K\|\theta(t)\|_{0,2}^2 \leq C\|\theta(t)\|_{1,2}\|\theta(t)\|_{-1,2} \leq \alpha/2\|\theta(t)\|_{1,2}^2 + C_2\|\theta(t)\|_{-1,2}^2. \tag{1.21}
$$

Estimating the right-hand side of (1.20) by Holder inequality and using (1.21), we obtain the estimate

$$
\partial_t[t(\varepsilon\|\partial_t u(t)\|_{0,2}^2 + \|\partial_t u(t)\|_{-1,2}^2)] \leq C(1+t)(\varepsilon\|\partial_t u(t)\|_{0,2}^2 + \|\partial_t u(t)\|_{-1,2}^2). \tag{1.22}
$$

Integrating (1.22) with respect to $t \in [0, T_0]$ and using (1.18), we finally find estimate (1.16) and Lemma 1.4 is proved.

Lemma 1.5. *Let $u(t)$ be a solution of equation* (0.2) *and let $t \geq T_0$, where T_0 is the same as in Lemmata 1.3-1.4. Then, the following estimate is valid uniformly with respect to $\varepsilon_0 \geq \varepsilon \geq 0$:*

$$
\begin{aligned} \varepsilon\|\partial_t u(t)\|_{0,2}^2 + \|\partial_t u(t)\|_{-1,2}^2 + \|u(t)\|_{2,2}^2 + \int_t^{t+1} \|\partial_t u(s)\|_{1,2}^2\, ds \leq \\ \leq e^{K_1 t}\left(Q(\|u(0)\|_{2,2}) + Q(\|g\|_{0,2})\right), \quad t \geq T_0, \end{aligned} \tag{1.23}
$$

where K_1 is some positive constant and Q is some monotonic function that are independent of ε.

Proof. We differentiate equation (0.2) with respect to t and set $\theta(t) := \partial_t u(t)$. This function obviously satisfies the equation:

$$
\begin{cases} \partial_t(\varepsilon + L)\theta = a\Delta_x\theta - f'(u(t))\theta, \\ \theta\big|_{\partial\Omega} = 0, \quad \theta\big|_{t=T_0} = \partial_t u(T_0). \end{cases} \tag{1.24}
$$

Multiplying equation (1.24) by $\theta(t)$, integrating over Ω and using the fact that $f'(u) \geq -KId$, we find

$$
1/2\partial_t(\varepsilon\|\theta(t)\|_{0,2}^2 + \|\theta(t)\|_{-1,2}^2) + \alpha\|\theta(t)\|_{1,2}^2 \leq K\|\theta(t)\|_{0,2}^2. \tag{1.25}
$$

Estimating the right-hand side of (1.25) by using the interpolation inequality (1.21) (as in (1.20)), we have the estimate

$$\partial_t(\varepsilon\|\theta(t)\|^2_{0,2}+\|\theta(t)\|^2_{-1,2})+\alpha'\|\theta(t)\|^2_{1,2}\le K_1(\varepsilon\|\theta(t)\|^2_{0,2}+\|\theta(t)\|^2_{-1,2}), \tag{1.26}$$

for appropriate strictly positive constants K_1 and α'. Applying Gronwall inequality to estimate (1.26), we obtain

$$\begin{aligned}\varepsilon\|\partial_t u(t)\|^2_{0,2}+\|\partial_t u(t)\|^2_{-1,2}+\int_t^{t+1}\|\partial_t u(s)\|^2_{1,2}\,ds\le\\ \le Ce^{2K_2t}(\varepsilon\|\partial_t u(T_0)\|^2_{0,2}+\|\partial_t u(T_0)\|^2_{-1,2}+\|g\|^2_{0,2}),\end{aligned} \tag{1.27}$$

where the constants K_2 and C are independent of ε.

Inserting estimate (1.16) into the right-hand side of (1.27), we find

$$\begin{aligned}\varepsilon\|\partial_t u(t)\|^2_{0,2}+\|\partial_t u(t)\|^2_{-1,2}+\int_t^{t+1}\|\partial_t u(s)\|^2_{1,2}\,ds\le\\ \le e^{2K_2t}\left(Q(\|u_0\|_\Phi)+Q(\|g\|_{2,2})\right),\end{aligned} \tag{1.28}$$

for appropriate positive constant K_2 and function Q which are independent of ε.

Having the uniform estimate (1.28), one can interpret the parabolic equation (0.2) as an elliptic boundary value problem

$$a\Delta_x u(t)-f(u(t))=h_u(t):=(\varepsilon+L)\partial_t u(t)-g,\quad u(t)\big|_{\partial\Omega}=0,\quad t\ge T_0, \tag{1.29}$$

for every fixed $t\ge 0$. Indeed, estimate (1.28) implies that

$$\|h_u(t)\|^2_{0,2}\le e^{K_2t}\left(Q(\|u(0)\|_{2,2})+Q(\|g\|_{2,2})\right), \tag{1.30}$$

where K_2 and Q are independent of $\varepsilon_0\ge\varepsilon\ge 0$.

Multiplying now equation (1.29) by $u(t)$ (t is fixed!) and integrating over Ω, we obtain, as in the proof of Lemma 1.1

$$\|u(t)\|^2_{1,2}\le C(1+\|h_u(t)\|^2_{0,2}). \tag{1.31}$$

Multiplying equation (1.29) by $\Delta_x u(t)$ (t is fixed!), integrating over Ω and using the fact that $f'(u)\ge -KId$, we have

$$\|u(t)\|^2_{2,2}\le C\|\Delta_x u(t)\|^2_{0,2}\le C_1(\|u(t)\|^2_{1,2}+\|h_u(t)\|^2_{0,2}). \tag{1.32}$$

Estimate (1.23) is then an immediate consequence of estimates (1.28), (1.30), (1.31) and (1.32) and Lemma 1.5 is proved.

Corollary 1.1. *Let the above assumptions hold and let $u(t)$ be a solution of equation* (0.2)*. Then, the following estimate is valid, for every $t \geq 0$:*

$$\|u(t)\|_{\Phi} \leq e^{Kt} \left(Q(\|u(0)\|_{\Phi}) + Q(\|g\|_{2,2})\right), \tag{1.33}$$

for some positive constant K and monotonic function Q that are independent of ε.

Indeed, for $t \leq T_0(\|u_0\|_{\Phi})$, estimate (1.33) has been proved in Lemma 1.3 and, for $t \geq T_0(\|u_0\|_{\Phi})$, it has been proved in Lemma 1.5.

We note that Corollary 1.1 gives an estimate of $u(t)$ in the $W^{2,2}$-norm that *grows* exponentially as $t \to \infty$. In order to obtain an estimate that does not *grow* as $t \to \infty$, we need the following lemma.

Lemma 1.6. *Let $u(t)$ be a solution of problem* (0.2)*. Then, the following version of the smoothing property holds uniformly with respect to ε:*

$$\|u(1)\|_{2,2} \leq Q(\varepsilon\|u(0)\|_{1,2}^2 + \|u(0)\|_{0,2}^2) + Q(\|g\|_{2,2}), \tag{1.34}$$

for some monotonic function that is independent of $\varepsilon \geq 0$.

Proof. It follows from Lemma 1.2 that

$$\int_0^1 \|u(s)\|_{2,2}^2 \, ds \leq C(\varepsilon\|u(0)\|_{1,2}^2 + \|u(0)\|_{0,2}^2 + 1 + \|g\|_{0,2}^2).$$

Consequently, there exists a time $T \in [0,1]$ such that

$$\|u(T)\|_{2,2}^2 \leq C(\varepsilon\|u(0)\|_{1,2}^2 + \|u(0)\|_{0,2}^2 + 1 + \|g\|_{0,2}^2). \tag{1.35}$$

Applying now estimate (1.33) (starting with the time $t = T$ instead of $t = 0$) and using (1.35), we obtain estimate (1.34) and Lemma 1.6 is proved.

Finally, estimate (1.1) is an immediate consequence of estimates (1.5) (1.33) and (1.34). Thus, Theorem 1.1 is proved.

Having the a priori estimate (1.1) for the solutions of (0.2), it is not difficult to verify the existence of solutions and their uniqueness.

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Then, for every $u_0 \in \Phi$, problem* (0.2) *possesses a unique solution $u(t) \in \Phi$, for every $t \geq 0$, and, consequently,* (0.2) *defines a semigroup $S_t^\varepsilon : \Phi \to \Phi$ by the expression*

$$S_t^\varepsilon u_0 = u(t), \quad \text{where } u(t) \text{ is solution of } (0.2). \tag{1.36}$$

Proof. The existence of a solution $u(t) \in \Phi$ can be proved in a standard way, using e.g. Galerkin approximations or the Leray-Schauder fixed point theorem (see [BV] and [Z]).

We now verify the uniqueness. Let $u_1(t)$ and $u_2(t)$ be two solutions of equation (0.2). We set $v(t) = u_1(t) - u_2(t)$. This function satisfies the equation

$$(\varepsilon + L)\partial_t v = a\Delta_x v - l(t)v, \quad v\big|_{\partial\Omega} = 0, \tag{1.37}$$

where $l(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds$. Since $f'(u) \geq -KId$, then, obviously, $l(t) \geq -KId$ also. Multiplying equation (1.37) by $v(t)$, integrating over Ω and arguing as in the proof of Lemma 1.5, we obtain

$$\partial_t(\varepsilon\|v(t)\|_{0,2}^2 + \|v(t)\|_{-1,2}^2) + \alpha\|v(t)\|_{1,2}^2 \leq K_1(\varepsilon\|v(t)\|_{0,2}^2 + \|v(t)\|_{-1,2}^2), \tag{1.38}$$

where K_1 and α are strictly positive constants that are independent of $\varepsilon_0 \geq \varepsilon \geq 0$. Applying Gronwall inequality to this relation, we have

$$\varepsilon\|v(t)\|_{0,2}^2 + \|v(t)\|_{-1,2}^2 + \int_t^{t+1} \|v(s)\|_{1,2}^2\,ds \leq Ce^{K_1 t}(\varepsilon\|v(0)\|_{0,2}^2 + \|v(0)\|_{-1,2}^2). \tag{1.39}$$

Moreover, K_1 and C are independent of ε.

Estimate (1.39) implies immediately the uniqueness of a solution for problem (0.2) and Theorem 1.2 is proved.

Let us now obtain some additional regularity on the solutions of equation (0.2).

Theorem 1.3. *Let the assumptions of Theorem 1.1 hold. Then, the solution $u(t)$ of problem* (0.2) *enjoys the following estimate, for every $t \geq 1$:*

$$\|\partial_t u(t)\|_{2,2}^2 + \|u(t)\|_{4,2}^2 + \int_t^{t+1} \|\partial_t^2 u(s)\|_{1,2}^2\,ds \leq Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2}), \tag{1.40}$$

where the constant $\alpha > 0$ and the function Q are independent of $\varepsilon_0 \geq \varepsilon \geq 0$.

We divide the proof of this theorem into several lemmata.

Lemma 1.7. *Let the assumptions of Theorem 1.1 hold and let $u(t)$ be a solution of problem* (0.2)*. Then, the following estimate is valid, for every $t \geq 1$:*

$$\varepsilon\|\partial_t u(t)\|_{1,2}^2 + \|\partial_t u(t)\|_{0,2}^2 + \int_t^{t+1} \|\partial_t u(s)\|_{2,2}^2\,ds \leq \\ \leq Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2}). \tag{1.41}$$

Moreover, the constants C and $\alpha > 0$ and the function Q are independent of ε.

Proof. Indeed, multiplying equation (1.24) by $(t - T_0)\Delta_x\theta(t)$ (where $\theta(t) := \partial_t u(t)$) and integrating over Ω, we obtain

$$\partial_t[(t-T_0)(\varepsilon\|\theta(t)\|_{1,2}^2 + \|\theta(t)\|_{0,2}^2)] + \\ + \alpha(t-T_0)(\varepsilon\|\theta(t)\|_{1,2}^2 + \|\theta(t)\|_{0,2}^2) + \alpha(t-T_0)\|\theta(t)\|_{2,2}^2 \leq \\ \leq \varepsilon\|\theta(t)\|_{1,2}^2 + \|\theta(t)\|_{0,2}^2 + C(t-T_0)\|f'(u(t))\theta(t)\|_{0,2}^2 := h_u(t). \tag{1.42}$$

It follows from estimates (1.1) and (1.23) and from the fact that $\Phi \subset C$ that

$$(1.43)\quad \int_t^{t+1} |h_u(s)|\,ds \le Q(\sup_{s\in[t,t+1]} \|u(s)\|_{0,\infty}) \int_t^{t+1} \|\partial_t u(s)\|_{1,2}^2\,ds \le \\ \le Q_1(\|u(0)\|_\Phi)e^{-\alpha t} + Q_1(\|g\|_{2,2}),$$

for appropriate functions Q and Q_1 which are independent of ε.

Applying now Gronwall inequality to relation (1.42) and taking into account estimate (1.43) and the fact that $T_0 \le 1/2$, we find (1.41) after simple transformations. Lemma 1.7 is proved.

Lemma 1.8. *Let the assumptions of Theorem 1.3 hold. Then, the following estimate is valid:*

$$(1.44)\quad \|\partial_t u(t)\|_{1,2}^2 + \int_t^{t+1} (\varepsilon\|\partial_t^2 u(s)\|_{0,2}^2 + \|\partial_t^2 u(s)\|_{-1,2}^2)\,ds \le \\ \le Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2}),$$

where $t \ge 2$ and the exponent α and the function Q are independent of ε.

Proof. Indeed, let us multiply equation (1.19) by $(t-T)a\partial_t\theta(t)$, $t \ge T \ge 1$, and integrate over Ω. Then, we obtain, after standard transformations

$$(1.45)\quad \varepsilon(t-T)(a\partial_t\theta(t), \partial_t\theta(t)) + (t-T)(aL^{1/2}\partial_t\theta(t), L^{1/2}\partial_t\theta(t)) + \\ + 1/2\partial_t[(t-T)(a\nabla_x\theta(t), a\nabla_x\theta(t))] = \\ = (a\nabla_x\theta(t), a\nabla_x\theta(t)) - (t-T)(f'(u(t))\theta, a\partial_t\theta) := H_u(t).$$

We then estimate the last term in the right-hand side of (1.45) as follows:

$$(1.46)\quad (t-T)(f'(u(t))\theta(t), a\partial_t\theta(t)) \le (t-T)\|f'(u(t))\theta(t)\|_{1,2}\|\partial_t\theta(t)\|_{-1,2} \le \\ \le C(t-T)(\|f'(u(t))\|_{0,\infty}\|\nabla_x\theta(t)\|_{0,2}^2 + \|f''(u(t))\|_{0,\infty}\|\theta(t)\|_{0,4}^4) + \\ + 1/2(t-T)(aL^{1/2}\partial_t\theta(t), L^{1/2}\partial_t\theta(t)).$$

Estimates (1.1), (1.41) and (1.46), together with the embeddings $\Phi \subset C$ and $W^{1,2} \subset L^6 \subset L^4$ ($n = 3!$), yield

$$(1.47)\quad \int_T^{T+s} H_u(t)\,dt \le (1+s)\left(Q(\|u_0\|_\Phi)e^{-\alpha T} + Q(\|g\|_{2,2})\right) + \\ + 1/2\int_T^{T+s} (t-T)(aL^{-1/2}\partial_t\theta(t), L^{-1/2}\partial_t\theta(t))\,dt,$$

for an appropriate function Q which is independent of ε and for $s \in [0, 2]$.

Integrating (1.45) with respect to $t \in [T, T+s]$ and using (1.47), we obtain

$$(1.48) \quad s\|\theta(T+s)\|_{1,2}^2 + \int_T^{T+s} (t-T)(\varepsilon\|\partial_t\theta(t)\|_{0,2}^2 + \|\partial_t\theta(t)\|_{-1,2}^2)\,dt \le \\ \le C\left(Q(\|u_0\|_\Phi)e^{-\alpha T} + Q(\|g\|_{2,2})\right),$$

for $T \ge 1$ and $s \in [0, 2]$.

Estimate (1.44) follows easily from (1.48). Indeed, setting $T := t-1 \ge 1$ (since $t \ge 2$) and $s = 1$ in (1.48), we obtain an estimate for $\|\partial_t u(t)\|_{1,2}$. In order to obtain an estimate for the integral in the left-hand side of (1.44), we multiply (1.19) by $a\partial_t\theta(t)$, proceed as above and integrate over $[t, t+1]$. This finishes the proof of Lemma 1.8.

Lemma 1.9. *Let the above assumptions hold. Then, the solution $u(t)$ of problem* (0.2) *enjoys the following estimate:*

$$(1.49) \qquad \|\partial_t u(t)\|_{2,2}^2 + \int_t^{t+1} \|\partial_t^2 u(s)\|_{1,2}^2\,ds \le Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2}),$$

where $t \ge 3$ and the constant $\alpha > 0$ and the function Q are independent of $\varepsilon_0 \ge \varepsilon \ge 0$.

Proof. We differentiate equation (1.24) with respect to t and set $w(t) := \partial_t\theta(t) = \partial_t^2 u(t)$. This function satisfies the equation

$$(1.50) \qquad (\varepsilon + L)\partial_t w = a\Delta_x w - [f''(u(t))\theta(t), \theta(t)] - f'(u(t))w, \quad w\big|_{\partial\Omega} = 0.$$

Multiplying this equation by $(t-2)w(t)$ and integrating with respect to $x \in \Omega$, we have the inequality

$$(1.51) \quad \partial_t[(t-2)(\varepsilon\|w(t)\|_{0,2}^2 + \|w(t)\|_{-1,2}^2)] + \alpha(t-2)\|w(t)\|_{1,2}^2 + \\ + \alpha(t-2)(\varepsilon\|w(t)\|_{0,2}^2 + \|w(t)\|_{-1,2}^2) \le \\ \le (\varepsilon\|\partial_t^2 u(t)\|_{0,2}^2 + \|\partial_t^2 u(t)\|_{-1,2}^2) + \\ + C(t-2)\|f''(u(t))\|_{0,\infty}^2\|\theta(t)\|_{1,2}^4 + 2K(t-2)\|w(t)\|_{0,2}^2.$$

It follows from Lemma 1.8 that

$$(1.52) \qquad \int_T^{T+1} (\varepsilon\|w(t)\|_{0,2}^2 + \|w(t)\|_{-1,2}^2)\,dt \le Q(\|u_0\|_\Phi)e^{-\alpha T} + Q(\|g\|_{2,2}),$$

for every $T \ge 2$ and for appropriate positive constant α and function Q which are independent of ε. Moreover, estimates (1.1) and (1.44) imply that

$$(1.53) \qquad (t-2)\|f''(u(t))\|_{0,\infty}^2\|\theta(t)\|_{1,2}^4 \le (1+t)(Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2})),$$

for $t \geq 2$ and for an appropriate function Q. Finally, the last term in the right-hand side of (1.51) can be estimated by the interpolation inequality (1.21). Inserting these estimates into (1.51) and applying Gronwall inequality, we find

$$(1.54) \qquad (t-2)(\varepsilon\|\partial_t^2 u(t)\|_{0,2}^2 + \|\partial_t^2 u(t)\|_{-1,2}^2) \leq (1+t)(Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2})),$$

for appropriate constant $\alpha > 0$ and function Q which are independent of ε. Integrating now (1.51) with respect to $t \in [T, T+1]$ and taking into account (1.52)-(1.54), one can easily obtain

$$(1.55) \qquad \int_T^{T+1} \|\partial_t^2 u(t)\|_{1,2}^2 \, dt \leq Q(\|u_0\|_\Phi)e^{-\alpha T} + Q(\|g\|_{2,2}),$$

for $T \geq 3$.

Thus, there only remains to derive an estimate for the $W^{2,2}$-norm of $\partial_t u(t)$. To this end (as in the proof of Lemma 1.5), we interpret equation (1.24) as an elliptic boundary value problem in Ω, for every fixed $t \geq 3$:

$$(1.56) \qquad a\Delta_x \theta = h_\theta(t) := (\varepsilon + L)w(t) + f'(u(t))\theta(t), \quad \theta\big|_{\partial\Omega} = 0.$$

It follows from estimates (1.1), (1.44) and (1.54) that, for $t \geq 3$

$$(1.57) \qquad \|h_\theta(t)\|_{0,2}^2 \leq Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{0,2}),$$

for appropriate constant $\alpha > 0$ and function Q which are independent of ε. Estimates (1.55) and (1.57), together with the elliptic regularity theorem, complete the proof of Lemma 1.9.

Lemma 1.10. *Let the assumptions of Theorem 1.3 hold. Then, the solution $u(t)$ of* (0.2) *satisfies the following estimate:*

$$(1.58) \qquad \|u(t)\|_{4,2}^2 \leq Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2}), \quad t \geq 3,$$

where the constant $\alpha > 0$ and the function Q are independent of $\varepsilon_0 \geq \varepsilon \geq 0$.

Proof. We rewrite equation (0.2) in the following form:

$$(1.59) \qquad a\Delta_x u(t) = \tilde{h}_u(t) := f(u(t)) + (\varepsilon + L)\theta(t) - g, \quad u\big|_{\partial\Omega} = 0.$$

We note that, due to (1.1) and (1.49), we have the estimate

$$(1.60) \qquad \|\tilde{h}_u(t)\|_{2,2} \leq Q(\|u_0\|_\Phi)e^{-\alpha t} + Q(\|g\|_{2,2}),$$

for $t \geq 3$. Applying the $(W^{4,2}, W^{2,2})$-regularity theorem for the Laplacian (see e.g. [ADN]), we obtain estimate (1.58) and Lemma 1.10 is proved.

Thus, Lemmata 1.9 and 1.10 imply estimate (1.40) for $t \geq 3$. Rescaling the time variable ($t \to \alpha t$, $\alpha > 0$) one can now prove that estimate (1.40) holds for every $t \geq \tilde{T} > 0$, where $\tilde{T} > 0$ is arbitrary (of course, with constant α and function Q depending on $\tilde{T}$). In particular, taking $\tilde{T} = 1$ we finish the proof of Theorem 1.3.

§2 Estimates for the difference of solutions.

In this section, we derive several estimates for the difference of solutions of (0.2) that will be essential in Section 4 for the study of the long time behavior of the dynamical system generated by (0.2). We start with the following estimate.

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold and let* $u_1(t)$ *and* $u_2(t)$ *be two solutions of* (0.2) *such that* $\|u_i(0)\|_\Phi \le R$, $i = 1, 2$. *Then, the following estimate is valid:*

$$\|u_1(t) - u_2(t)\|^2_{-1,2} \le C_R e^{\alpha_R t}\|u_1(0) - u_2(0)\|^2_{-1,2}, \tag{2.1}$$

where the constants C_R *and* $\alpha_R > 0$ *depend on* R *and are independent of* $\varepsilon \ge 0$.

Proof. We set $v(t) = u_1(t) - u_2(t)$. This function satisfies the equation

$$(-\Delta_x)^{-1}\partial_t v(t) = a\Delta_x(1 - \varepsilon\Delta_x)^{-1}v(t) - (1 - \varepsilon\Delta_x)^{-1}[l(t)v(t)], \quad v\big|_{\partial\Omega} = 0, \tag{2.2}$$

where $l(t) := \int_0^1 f'(su_1(t) + (1 - s)u_2(t))\, ds$. Multiplying equation (2.2) by $v(t)$ and integrating over Ω, we obtain the inequality

$$\begin{aligned} \partial_t\|v(t)\|^2_{-1,2} + \alpha\|(-\Delta_x)^{1/2}(1 - \varepsilon\Delta_x)^{-1/2}v(t)\|^2_{0,2} \le \\ \le C\|(1 - \varepsilon\Delta_x)^{-1/2}(-\Delta_x)^{-1/2}[l(t)v(t)]\|_{0,2}\|(-\Delta_x)^{1/2}(1 - \varepsilon\Delta_x)^{-1/2}v(t)\|_{0,2}. \end{aligned} \tag{2.3}$$

Estimating then the right-hand side of (2.3) by using Holder inequality and (1.10), we have

$$\partial_t\|v(t)\|^2_{-1,2} + \alpha/2\|v(t)\|^2_{0,2} \le C_1\|l(t)v(t)\|^2_{-1,2}, \tag{2.4}$$

where C_1 is independent of ε.

In order to complete the proof of the theorem, we need the following simple lemma.

Lemma 2.1. *Let the above assumptions hold. Then, the multiplication by* $l(t)$ *is bounded as an operator in* $W^{-1,2}(\Omega)$. *Moreover*

$$\|l(t)\|_{W^{-1,2}\to W^{-1,2}} \le Q_R, \tag{2.5}$$

where Q_R *only depends on the constant* R *introduced in the formulation of Theorem 2.1.*

Proof. Indeed, let $w \in W^{1,2}_0(\Omega)$. Then

$$\begin{aligned} |(l(t)v, w)| = |(v, l^*(t)w)| \le \|v\|_{-1,2}\|l^*(t)w\|_{1,2} \le \\ \le C\|v\|_{-1,2}\left(\|l(t)\|_{0,\infty}\|w\|_{1,2} + \|\nabla_x l(t)\|_{0,4}\|w\|_{1,2}\right). \end{aligned} \tag{2.6}$$

Here, we have also used the embedding $W^{1,2} \subset L^4$.

Estimate (2.6) implies that

$$\|l(t)\|_{W^{-1,2}\to W^{-1,2}} \le C\left(\|l(t)\|_{0,\infty} + \|l(t)\|_{1,4}\right). \tag{2.7}$$

We recall that, according to Theorem 1.1, $\|u_i(t)\|_\Phi \le Q(R) + Q(\|g\|_{2,2})$. Consequently, $\|l(t)\|_{0,\infty} \le C(R)$, for every $t \ge 0$. The second term in the right-hand side of (2.7) can be estimated as follows:

$$\|l(t)\|_{1,4} \le Q(\max_{i=1,2} \|u_i(t)\|_{0,\infty}) \max_{i=1,2} \|u_i(t)\|_{1,4} \le \tilde{C}(R). \tag{2.8}$$

(Here, we have used the embeddings $\Phi \subset C$ and $\Phi \subset W^{1,4}$.) Lemma 2.1 is proved.

Estimating the right-hand side of (2.4) by (2.5), we obtain the following inequality:

$$\partial_t \|v(t)\|^2_{-1,2} + \alpha/2 \|v(t)\|^2_{0,2} \le C'(R)\|v(t)\|^2_{-1,2}, \tag{2.9}$$

where $C'(R)$ is independent of ε. Applying Gronwall inequality to (2.9), we finish the proof of the theorem.

The next theorem gives the $W^{-1,2} \to W^{2,2}$-smoothing for the difference of two solutions.

Theorem 2.2. *Let the assumptions of Theorem 1.1 hold and let $u_1(t)$ and $u_2(t)$ be two solutions of* (0.2) *such that $\|u_i(0)\|_\Phi \le R$. Then, the following estimate is valid:*

$$\|u_1(1) - u_2(1)\|^2_\Phi \le C_R \|u_1(0) - u_2(0)\|^2_{-1,2}, \tag{2.10}$$

where the constant C_R depends on R and is independent of $\varepsilon_0 \ge \varepsilon \ge 0$.

We divide the proof of this theorem into several lemmata.

Lemma 2.2. *Let the above assumptions hold. Then*

$$\varepsilon\|v(t)\|^2_{0,2} + \|v(t)\|^2_{-1,2} + \int_t^{t+1} \|v(s)\|^2_{1,2}\, ds \le C_R e^{\alpha_R t}\|v(0)\|^2_{-1,2}, \tag{2.11}$$

where $t \ge 1$ and the constants C_R and $\alpha_R > 0$ are independent of ε.

Proof. Integrating relation (2.9) over $[t, t+1]$ and using estimate (2.1), we obtain the estimate

$$\int_t^{t+1} \|v(s)\|^2_{0,2}\, ds \le C_R e^{\alpha_R t}\|v(0)\|^2_{-1,2}. \tag{2.12}$$

We rewrite the equation for $v(t)$ in the equivalent form (1.37), multiply by $tv(t)$ and integrate over Ω. Then, we obtain, after simple transformations (as in the proof of Lemma 1.5)

$$\partial_t[t(\varepsilon\|v(t)\|^2_{0,2} + \|v(t)\|^2_{-1,2})] + \alpha t\|v(t)\|^2_{1,2} \le \\ \le Ct(\varepsilon\|v(t)\|^2_{0,2} + \|v(t)\|^2_{-1,2}) + C\|v(t)\|^2_{0,2}. \tag{2.13}$$

Applying Gronwall inequality to this relation and using (2.12), we obtain estimate (2.11) and Lemma 2.2 is proved.

Lemma 2.3. *Let the above assumptions hold. Then, the following estimate is valid:*

$$\|v(t)\|_{1,2}^2 + \int_t^{t+1} (\varepsilon\|\partial_t v(s)\|_{0,2}^2 + \|\partial_t v(s)\|_{-1,2}^2)\, ds \le C_R e^{\alpha_R t}\|v(0)\|_{-1,2}^2, \tag{2.14}$$

where $t \ge 2$ and the constants C_R and α_R are independent of ε.

Proof. The proof of this lemma is similar to that of Lemma 1.8: multiplying equation (1.37) by $(t-T)a\partial_t v(t)$, $t \ge T \ge 1$, and integrating over Ω, we find, as in Lemma 1.8

$$\begin{gathered}\alpha(t-T)(\varepsilon\|\partial_t v(t)\|_{0,2}^2 + \|\partial_t v(t)\|_{-1,2}^2) + \\ + \partial_t[(t-T)(a\nabla_x v(t), a\nabla_x v(t))] \le C\left(\|v(t)\|_{1,2}^2 + (t-T)\|l(t)v(t)\|_{1,2}^2\right).\end{gathered} \tag{2.15}$$

We recall that, due to (2.8) and to the fact that $\|l(t)\|_{0,\infty} \le C_1(R)$ and $\|l(t)\|_{1,4} \le C_1(R)$, we have the estimate

$$\|l(t)v(t)\|_{1,2} \le C_R\|v(t)\|_{1,2}. \tag{2.16}$$

Integrating (2.15) over t and using estimates (2.11) and (2.16), we obtain inequality (2.14) (as in the proof of Lemma 1.8) and Lemma 2.3 is proved.

Lemma 2.4. *Let the above assumptions hold. Then, the following estimate is valid:*

$$\varepsilon\|\partial_t v(t)\|_{0,2}^2 + \|\partial_t v(t)\|_{-1,2}^2 + \|v(t)\|_{2,2}^2 \le C_R e^{\alpha_R t}\|v(0)\|_{-1,2}^2, \tag{2.17}$$

where $t \ge 3$ and the constants C_R and α_R are independent of ε.

Proof. We differentiate equation (1.37) with respect to t and set $\theta(t) = \partial_t v(t)$. This function satisfies the equation

$$(\varepsilon + L)\partial_t\theta = a\Delta_x\theta - l(t)\theta - l'(t)v(t), \quad \theta\big|_{\partial\Omega} = 0. \tag{2.18}$$

Multiplying this equation by $(t-2)\theta(t)$, integrating over Ω and noting that

$$\|l'(t)v(t)\|_{0,2} \le C(R)\|v(t)\|_{1,2}, \tag{2.19}$$

we have the estimate

$$\begin{gathered}\partial_t[(t-2)(\varepsilon\|\theta(t)\|_{0,2}^2 + \|\theta(t)\|_{-1,2}^2)] + \alpha(t-2)\|\theta(t)\|_{1,2}^2 \le \\ \le (\varepsilon\|\partial_t v(t)\|_{0,2}^2 + \|\partial_t v(t)\|_{-1,2}^2) + C_R(t-2)\|v(t)\|_{1,2}^2.\end{gathered} \tag{2.20}$$

Applying Gronwall inequality (with initial value $t=2$) to (2.20) and using estimate (2.14), we prove (in a standard way) that

$$\varepsilon\|\partial_t v(t)\|_{0,2}^2 + \|\partial_t v(t)\|_{-1,2}^2 \le C_R e^{\alpha_R t}\|v(0)\|_{-1,2}^2, \tag{2.21}$$

where $t \ge 3$ and C_R and α_R are independent of ε.

Having estimate (2.21) and interpreting equation (1.37) as an elliptic equation (see Lemmata 1.5 and 1.10), one can easily obtain an appropriate estimate for $\|v(t)\|_{2,2}$ and finish the proof of Lemma 2.4.

Thus, we have proved estimate (2.10), for every $t \ge 3$. Rescaling the time ($t \to \alpha t$), we deduce that this estimate is valid for every $t \ge T_0$, $T_0 > 0$ being arbitrary. In particular, it is valid for $t \ge 1$ and Theorem 2.2 is proved.

Corollary 2.1. *Let the above assumptions hold. Then, the semigroup S_t^ε defined in Theorem 1.2 is uniformly Holder continuous on $[0,T]\times K_R$ in the topology of $W^{-1,2}$, where $K_R=\{u\in\Phi,\quad \|u\|_\Phi\le R\}$, i.e.*

$$\|S^\varepsilon_{t_1}u_0^1-S^\varepsilon_{t_2}u_0^2\|_{-1,2}\le C_{R,T}\left(\|u_0^1-u_0^2\|_{-1,2}+|t_1-t_2|^{1/2}\right), \tag{2.22}$$

for $u_0^i\in K_R$ and $t_i\le T$. Moreover, the constant $C_{R,T}$ is independent of ε.

Proof. The Lipschitz continuity with respect to the initial conditions is an immediate corollary of Theorem 2.1. In order to verify the Lipschitz continuity with respect to t, we note that, arguing as in the proof of Lemma 1.4 and using the estimate (1.1) (multiplying (0.2) by $a\partial_t u(t)$ and so on), one finds the estimate

$$\int_T^{T+1}\|\partial_t u(t)\|^2_{-1,2}\,dt\le Q(\|u(0)\|_{2,2})e^{-\alpha T}+Q(\|g\|_{2,2})\le C(R) \tag{2.23}$$

if $u(0)\in K_R$. Consequently

$$\|u(t_1)-u(t_2)\|_{-1,2}=\Big\|\int_{t_1}^{t_2}\partial_t u(s)\,ds\Big\|_{-1,2}\le\int_{t_1}^{t_2}\|\partial_t u(s)\|_{-1,2}\,ds\le$$
$$\le|t_1-t_2|^{1/2}\left(\int_{t_1}^{t_2}\|\partial_t u(s)\|^2_{-1,2}\,ds\right)^{1/2}\le C_{R,T}|t_1-t_2|^{1/2},$$

and Corollary 2.1 is proved.

The rest of this section is devoted to the study of the difference of two solutions of (0.2) for $\varepsilon=0$ and $\varepsilon>0$ small. We start with the following theorem.

Theorem 2.3. *Let the assumptions of Theorem 1.1 hold and let $u_0(t)$ and $u_\varepsilon(t)$ be two solutions of (0.2) with zero and nonzero parameters respectively. We also assume that $\|u_0(0)\|_\Phi$ and $\|u_\varepsilon(0)\|_\Phi\le R$. Then, the following estimate is valid:*

$$\|u_0(t)-u_\varepsilon(t)\|^2_{-1,2}\le C_Re^{\alpha_R t}\left(\|u_0(0)-u_\varepsilon(0)\|^2_{-1,2}+\varepsilon^2\right), \tag{2.24}$$

where $t\ge0$ and C_R and α_R are independent of ε.

Proof. Let $v_\varepsilon(t)=u_\varepsilon(t)-u_0(t)$. Then, this function obviously satisfies the equation

$$L\partial_t v_\varepsilon(t)=a\Delta_x v_\varepsilon(t)-l_\varepsilon(t)v_\varepsilon(t)+\varepsilon h(t),\quad v_\varepsilon\big|_{\partial\Omega}=0, \tag{2.25}$$

where $h(t):=\partial_t u_0(t)$ and $l_\varepsilon(t):=\int_0^1 f'(su_\varepsilon(t)+(1-s)u_0(t))\,ds$.

Arguing as in the proof of Lemma 1.4 and using estimate (1.1) (multiplying (0.2) by $a\partial_t u_0(t)$ and so on), one finds the estimate

$$\int_t^{t+1}\|\partial_t u_0(s)\|^2_{-1,2}\,ds\le Q(\|u_0(0)\|_{2,2})e^{-\alpha t}+Q(\|g\|_{2,2}), \tag{2.26}$$

and, consequently

$$\int_t^{t+1} \|\partial_t u_0(s)\|^2_{-1,2}\, ds \le C_R, \quad t \ge 0. \tag{2.27}$$

Multiplying now equation (2.25) by $v_\varepsilon(t)$ and integrating over Ω, we obtain, taking into account the estimate $l_\varepsilon(t) \ge -KId$

$$\partial_t \|v_\varepsilon(t)\|^2_{-1,2} + \alpha \|v_\varepsilon(t)\|^2_{1,2} \le 2K\|v_\varepsilon(t)\|^2_{0,2} + C\varepsilon^2 \|h(t)\|^2_{-1,2}. \tag{2.28}$$

Applying the interpolation inequality (1.21), we then have

$$\partial_t \|v_\varepsilon(t)\|^2_{-1,2} + \alpha' \|v_\varepsilon(t)\|^2_{1,2} \le C\|v_\varepsilon(t)\|^2_{-1,2} + C\varepsilon^2 \|h(t)\|^2_{-1,2}. \tag{2.29}$$

Applying Gronwall inequality together with estimate (2.27) to (2.29), we finally find

$$\|v_\varepsilon(t)\|^2_{-1,2} + \int_t^{t+1} \|v_\varepsilon(s)\|^2_{1,2}\, ds \le C_R e^{\alpha_R t} \left(\|v_\varepsilon(0)\|^2_{-1,2} + \varepsilon^2 \right), \tag{2.30}$$

and Theorem 2.3 is proved.

Theorem 2.4. *Let the assumptions of Theorem 2.3 hold and let $u_\varepsilon(t)$ and $u_0(t)$ be the same as in Theorem 2.3. Then, the following estimate is valid:*

$$\|u_\varepsilon(1) - u_0(1)\|^2_{2,2} \le C_R \left(\|u_\varepsilon(0) - u_0(0)\|^2_{-1,2} + \varepsilon^2 \right), \tag{2.31}$$

where the constant C_R is independent of ε.

Proof. The proof of this theorem is similar to that of Theorem 2.2. Indeed, according to Theorem 1.3

$$\|h(t)\|^2_{2,2} + \int_t^{t+1} \|\partial_t h(s)\|^2_{1,2}\, ds \le C(R), \tag{2.32}$$

for $t \ge 1$ and, consequently, the additional term $\varepsilon h(t)$ in the right-hand side of (2.25) does not yield additional difficulties. That is the reason why we shall only give the statements of the next two Lemmata, which are the analogues of Lemmata 2.3-2.4.

Lemma 2.5. *Let the above assumptions hold. Then, the following estimate is valid:*

$$\|v_\varepsilon(t)\|^2_{1,2} + \int_t^{t+1} (\varepsilon \|\partial_t v_\varepsilon(s)\|^2_{0,2} + \|v_\varepsilon(s)\|^2_{-1,2})\, ds \le \\ \le C_R e^{\alpha_R t} \left(\|v_\varepsilon(0)\|^2_{-1,2} + \varepsilon^2 \right), \tag{2.33}$$

where $t \ge 1$ and the constants C_R and α_R are independent of ε.

The proof of this lemma is analogous to that of Lemma 2.3 (multiplying equation (2.25) by $(t-T)a\partial_t v_\varepsilon(t)$ and so on).

Lemma 2.6. *Let the above assumptions hold. Then, the following estimate is valid:*

$$\varepsilon\|\partial_t v_\varepsilon(t)\|_{0,2}^2 + \|\partial_t v_\varepsilon(t)\|_{-1,2}^2 + \|v_\varepsilon(t)\|_{2,2}^2 \le C_R e^{\alpha_R t}\left(\|v_\varepsilon(0)\|_{-1,2}^2 + \varepsilon^2\right), \tag{2.34}$$

where $t \ge 3$ and the constants C_R and α_R are independent of ε.

The proof of this lemma is similar to that of Lemma 2.4 (differentiating equation (2.25) by t, multiplying the resulting equation by $(t-2)\theta_\varepsilon(t) := (t-2)\partial_t v_\varepsilon(t)$ and so on).

Thus, we have proved estimate (2.31), for $t \ge 3$. Rescaling the time ($t \to \alpha t$), one can easily establish that (2.31) holds for $t = 1$ as well and Theorem 2.4 is proved.

§3 Perturbations of exponential attractors: the abstract setting.

In this section, we formulate and prove an abstract theorem for the behavior of exponential attractors under perturbations. The application of this result to the Cahn-Hilliard system will be given in the next section. We start with briefly recalling the definition of exponential attractors (see [EFNT] for a detailed exposition).

Definition 3.1. Let B be a metric space and let $L : B \to B$ be a map. We define a discrete semigroup $\{S_n, n \in \mathbb{Z}_+\}$ by $S_n x := L^n x$, $x \in B$. A set $\mathcal{M} \subset B$ is an exponential attractor for the map L if the following properties hold:

1. The set $\mathcal{M}$ is compact in B and has finite fractal dimension.
2. The set $\mathcal{M}$ is semi-invariant under the map L: $L(\mathcal{M}) \subset \mathcal{M}$.
3. The set $\mathcal{M}$ is an exponentially attracting set for the semi-group S_n, i.e. there exists a constant $\alpha > 0$ such that, for every bounded subset $B_0 \subset B$, there exists a constant $C = C(B)$ such that

$$\operatorname{dist}_B(S_n B_0, \mathcal{M}) \le C(B)e^{-\alpha n},$$

where dist denotes the non-symmetric Hausdorff distance between sets.

Remark 3.1. We have given the definition of exponential attractors for discrete times ($n \in \mathbb{Z}_+$). The extension of this definition to the continuous case ($t \in \mathbb{R}_+$) is straightforward (see e.g. [EFNT]).

Remark 3.2. We note that the existence of an exponential attractor $\mathcal{M}$ for the map L automatically implies the existence of the global attractor $\mathcal{A}$ and the embedding $\mathcal{A} \subset \mathcal{M}$. We note however that, in contrast with the global attractor, an exponential attractor is not uniquely defined.

We will study in this section the behavior of exponential attractors under perturbations and, consequently, we will consider below a family of maps $L_\varepsilon : B \to B$, $\varepsilon \in [0, \varepsilon_0]$, the corresponding family of semi-groups $S_n^\varepsilon : B \to B$ and a corresponding family of exponential attractors $\mathcal{M}_\varepsilon$. The main result of this section is the following theorem.

Theorem 3.1. *Let H_1 and H be two Banach spaces such that the inclusion $H_1 \subset H$ is compact and let B be a bounded set of H. We assume that there exists a family of operators $L_\varepsilon : B \to B$, $\varepsilon \in [0, \varepsilon_0]$, which satisfies the following assumptions:*

1. For every $x_1, x_2 \in B$, the following estimate is valid:

$$\|L_\varepsilon x_1 - L_\varepsilon x_2\|_{H_1} \le L\|x_1 - x_2\|_H, \tag{3.1}$$

where the constant L is independent of ε.

2. For every $\varepsilon \in [0, \varepsilon_0]$, for every $i \in \mathbb{N}$ and for every $x \in B$, we have the estimate

$$\|L^i_\varepsilon x - L^i_0 x\|_H \le K^i \varepsilon. \tag{3.2}$$

Then, for every $\varepsilon \in [0, \varepsilon_0]$, there exists an exponential attractor $\mathcal{M}_\varepsilon$ for the map L_ε in B. Moreover, the exponential attractors $\mathcal{M}_\varepsilon$ can be chosen such that the following estimate is valid:

$$\operatorname{dist}_{sym}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C_1 \varepsilon^\kappa, \tag{3.3}$$

where the constant C_1 and the exponent $0 < \kappa < 1$ can be calculated explicitly and $\operatorname{dist}_{sym}$ *denotes the symmetric Hausdorff distance between sets in H. Finally, the fractal dimension of the exponential attractors considered above is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$:*

$$\dim_F(\mathcal{M}_\varepsilon, H) \le C = C(L), \tag{3.4}$$

where the constant C is independent of ε and can be also calculated explicitly.

Proof. We first construct an exponential attractor $\mathcal{M}_0$ for L_0. To this end (following [EfMZ1] and [EfMZ2]), we construct the $R/2^i$ coverings of the sets $(L_0)^i B$ by the following inductive procedure:

1. Since the set B is bounded in H, there exists a ball $B(x_0, R, H)$ of radius R centered at $x_0 \in H$ for the H-norm such that $B \subset B(x_0, R, H)$. We set $V^0_0 = E^0_0 := \{x_0\}$. Thus, we have constructed the initial R-covering of the set B.

2. We now assume that the $R/2^i$-covering of the set $L^i_0 B$ by the balls centered at the points of the set $V^i_0 \subset L^i_0 B$ is already constructed. Then, according to (3.1), the system of $LR/2^i$-balls $B(x_j, LR/2^i, H_1)$ for the H_1-metric centered at the points of $L_0 V^i_0$ ($x_j \in L_0 V^i_0$) covers the set $L^{i+1}_0 B$. We note that, according to our assumptions, every H_1-ball is compact for the H-metric. Consequently, we can cover each $LR/2^i$-ball for the H_1-metric from the above covering by a finite number P of $R/(4 \cdot 2^i)$-balls for the H-metric. Moreover, the number P can be computed as follows:

$$\begin{aligned} P = N_{R/2^{i+2}}(B(x_j, LR/2^i, H_1), H) = \\ = N_{R/2^{i+2}}(0, LR/2^i, H_1), H) = N_{1/(4L)}(B(0, 1, H_1), H). \end{aligned} \tag{3.5}$$

(Here, we denote by $N_\mu(V,H)$ the minimal number of μ-balls for the H-norm which is necessary to cover the compact set $V\subset H$.) We note that relation (3.5) shows that the number of $R/2^{i+2}$-balls which is necessary to cover one $LR/2^i$-ball in H_1 is independent of i. Thus, we have constructed the $R/2^{i+2}$-covering $\mathcal{U}_0^{i+1}$ of the set L_0^{i+1} by a number of balls that is not greater than

$$\#\mathcal{U}_0^{i+1}\le P\#V_0^i. \tag{3.6}$$

Moreover, increasing the radiuses of the balls twice if necessary, we may construct the $R/2^{i+1}$-covering with centers belonging to $L_0^{i+1}B$. We denote by $V_0^{i+1}\subset L_0^{i+1}B$ the set of the centers of this covering. Thus, having the set V_0^i, we have constructed the set V_0^{i+1} such that

$$\#V_0^{i+1}\le P\#V_0^i, \tag{3.7}$$

and

$$\operatorname{dist}_H(L_0^{i+1}B,V_0^{i+1})\le R/2^{i+1},\quad V_0^{i+1}\subset L_0^{i+1}B. \tag{3.8}$$

Finally, we set $E_0^{i+1}:=L_0E_0^i\cup V_0^{i+1}$. Then, due to the induction procedure, the sets E_0^i, $i\in\mathbb{N}$, enjoy the following properties:

$$\begin{cases} 1.\ \ E_0^i\subset L_0^iB,\quad 2.\ \ L_0E_0^i\subset E_0^{i+1},\quad 3.\ \ \#E_0^i\le P^{i+1},\\ 4.\ \ \operatorname{dist}_H(L_0^iB,E_0^i)\le R/2^i.\end{cases} \tag{3.9}$$

We now define the exponential attractor $\mathcal{M}_0$ as follows:

$$\mathcal{M}_0'=\cup_{i=1}^\infty E_0^i,\quad \mathcal{M}_0=\left[\mathcal{M}_0'\right]_H. \tag{3.10}$$

It is easy to verify, using (3.9), that the set (3.10) is indeed an exponential attractor for the map $L_0:B\to B$.

Let us now construct exponential attractors $\mathcal{M}_\varepsilon$ for the maps L_ε such that (3.3) is satisfied. To this end, we will essentially use the sets E_0^i constructed above.

We recall that $E_0^i\subset L_0^iB$. Consequently, there exist sets $\tilde E_0^i\subset B$ such that

$$1.\ \ \#\tilde E_0^i=\#E_0^i,\quad 2.\ \ L_0^i\tilde E_i=E_0^i. \tag{3.11}$$

Let us now fix $\varepsilon\in(0,1]$ and define the sets

$$\widehat E_\varepsilon^i:=L_\varepsilon^i\tilde E_0^i. \tag{3.12}$$

Then, according to (3.2)

$$\operatorname{dist}_{sym}(L_\varepsilon^iB,L_0^iB)\le\varepsilon K^i, \tag{3.13}$$

and, consequently

$$\operatorname{dist}(L_\varepsilon^i B, \widehat{E}_\varepsilon^i) \le 2\varepsilon K^i + R/2^i. \tag{3.14}$$

We note that, usually, $K > 1$. Therefore, the right-hand side of (3.14) tends to ∞ as $i \to \infty$ and, consequently, we cannot construct an exponential attractor $\mathcal{M}_\varepsilon$ using *only* the sets $\widehat{E}_\varepsilon^i$. Now, if i is not too large, estimate (3.14) gives us a reasonable covering of the set $L_\varepsilon^i B$. Indeed, if

$$i \le I(\varepsilon) := [\frac{\ln R/(2\varepsilon)}{\ln 2K}], \tag{3.15}$$

then $2\varepsilon K^i \le R/2^i$ and estimate (3.14) implies that

$$\operatorname{dist}(L_\varepsilon^i B, \widehat{E}_\varepsilon^i) \le R/2^{i-1}. \tag{3.16}$$

Moreover, if i satisfies (3.15), estimate (3.2) implies that

$$\operatorname{dist}_{sym}(\widehat{E}_\varepsilon^i, E_0^i) \le C(R)\varepsilon^\kappa, \quad \kappa := \frac{\ln 2}{\ln 2 + \ln K}. \tag{3.17}$$

We now define the sequence E_ε^i as follows:

1. If $i \le I(\varepsilon)$, then $E_\varepsilon^i := \widehat{E}_\varepsilon^i$.

2. If $i > I(\varepsilon)$, we forget the sets $\widehat{E}_\varepsilon^i$ and we construct the sets E_ε^i by an inductive procedure, using estimate (3.1) (proceeding as in the construction of the sets E_0^i, but starting with the initial covering generated by $\widehat{E}_\varepsilon^I$). Then, the sets E_ε^i so constructed obviously satisfy the following assumptions:

$$\begin{cases} 1.\ \ E_\varepsilon^i \subset L_\varepsilon^i B, \quad 2.\ \ L_\varepsilon E_\varepsilon^i \subset E_\varepsilon^{i+1}, \quad 3.\ \ \#E_\varepsilon^i \le P^{i+2}, \\ 4.\ \ \operatorname{dist}_H(L_\varepsilon^i B, E_\varepsilon^i) \le R/2^{i-1}. \end{cases} \tag{3.18}$$

It follows from (3.18) that the set

$$\mathcal{M}_\varepsilon := [\mathcal{M}'_\varepsilon]_H, \quad \mathcal{M}'_\varepsilon := \cup_{i=1}^\infty E_\varepsilon^i, \tag{3.19}$$

is an exponential attractor for the map $L_\varepsilon : B \to B$ (see [EfMZ1]). Moreover, the attraction property for $\mathcal{M}_\varepsilon$ is obviously uniform with respect to $\varepsilon \in [0,1]$, i.e.

$$\operatorname{dist}_H(L_\varepsilon^i B, \mathcal{M}_\varepsilon) \le R/2^{i-1}. \tag{3.20}$$

We also recall that, due to (3.17)

$$\operatorname{dist}_{sym}(E_\varepsilon^i, E_0^i) \le C\varepsilon^\kappa, \quad i \le I(\varepsilon). \tag{3.21}$$

Let us then show that inequalities (3.2), (3.20) and (3.21) imply (3.3). Indeed, let us first verify that

$$\operatorname{dist}(\mathcal{M}'_\varepsilon, \mathcal{M}_0) \le C\varepsilon^\kappa. \tag{3.22}$$

Let x_0 belong to $\mathcal{M}'_\varepsilon$. Then, by definition, $x_0 \in E^j_\varepsilon$ for some $j \in \mathbb{N}$. If $j \le I(\varepsilon)$, then everything is proved due to estimate (3.21). Let us now assume that $j > I(\varepsilon)$. Then, $x_0 \in L^j_\varepsilon B \subset L^I_\varepsilon B$, which implies that there exists a point $\tilde{x}_0 \in B$ such that $x_0 = L^I_\varepsilon \tilde{x}_0$.

We now consider the point $x_0^* := L^I_0 \tilde{x}_0$. From the one hand (due to (3.2))

$$\|x_0 - x_0^*\|_H \le \varepsilon K^I \le C_1(R)\varepsilon^\kappa, \tag{3.23}$$

where $0 < \kappa < 1$ is the same as in (3.17) and C is independent of ε and, from the other hand (due to (3.20))

$$\operatorname{dist}_H(x_0^*, \mathcal{M}_0) = \operatorname{dist}_H(L^I_0 \tilde{x}_0, \mathcal{M}_0) \le R/2^{I-1} \le C_2(R)\varepsilon^\kappa. \tag{3.24}$$

Combining (3.23) and (3.24), we obtain

$$\operatorname{dist}(x_0, \mathcal{M}_0) \le (C_1 + C_2)\varepsilon^\kappa. \tag{3.25}$$

Since $x_0 \in \mathcal{M}'_\varepsilon$ is arbitrary, then (3.25) proves (3.22). Since $\mathcal{M}_\varepsilon$ is a closure of $\mathcal{M}'_\varepsilon$, then (3.22) implies that

$$\operatorname{dist}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C\varepsilon^\kappa. \tag{3.26}$$

The opposite estimate

$$\operatorname{dist}(\mathcal{M}_0, \mathcal{M}_\varepsilon) \le C\varepsilon^\kappa, \tag{3.27}$$

can be verified similarly. Finally, the uniform estimate (3.4) for the fractal dimension is a simple corollary of the third inequality in (3.18) (see e.g. [EfMZ1]) and Theorem 3.1 is proved.

Remark 3.1. We note that the construction and results presented in this section are valid in a Banach setting (see [EfMZ1] and [EfMZ1]).

§4 Exponential attractors for the Cahn-Hilliard System.

In this section, we apply the abstract Theorem 3.1 to the study of the long time behavior of the solutions of (0.2). Using the results of Section 2, we shall construct exponential attractors for these equations and shall study their dependence on ε. The main result of this section is the following theorem.

Theorem 4.1. *Let the assumptions of Theorem 1.1 hold. Then, for every* $\varepsilon \in [0, \varepsilon_0]$, *the semi-group* S_t^ε *generated by equation* (0.2) *possesses an exponential attractor* $\mathcal{M}_\varepsilon$ *in* Φ. *Moreover, these exponential attractors can be chosen such that*

$$\dim_F(\mathcal{M}_\varepsilon, \Phi) \le C, \quad \varepsilon \in [0, \varepsilon_0], \tag{4.1}$$

$$\mathrm{dist}_{sym}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C'\varepsilon^\kappa, \quad \varepsilon \in [0, \varepsilon_0], \tag{4.2}$$

where the constants $C, C' > 0$ *and* $0 < \kappa < 1$ *are independent of* ε *and can be calculated explicitly* (dist_{sym} *denotes the symmetric distance in* Φ) *and where the rate of convergence to these attractors are also uniform with respect to* ε, *i.e. there exists a constant* $\alpha > 0$ *such that, for every bounded subset* B_0 *of* Φ, *there exists a constant* $C'' = C''(B_0)$ *such that*

$$\mathrm{dist}_\Phi(S_t^\varepsilon B_0, \mathcal{M}_\varepsilon) \le C'' e^{-\alpha t}, \tag{4.3}$$

where C'' *and* α *are also independent of* ε.

Proof. We recall that, due to Theorem 1.1, the set

$$B = \{u_0 \in \Phi, \ \|u_0\|_\Phi \le R\}, \tag{4.4}$$

where R is large enough (e.g. $R \ge 2Q(\|g\|_{2,2})$, Q being defined in (1.1)) is an absorbing set for the semi-group S_t^ε, $\varepsilon \in [0, \varepsilon_0]$. Moreover, Theorem 1.1 implies that the set (4.4) is a *uniformly* (with respect to ε) absorbing set for S_t^ε, i.e. for every bounded set $B_0 \subset \Phi$, there exists a constant T which is independent of ε such that

$$S_t^\varepsilon B_0 \subset B, \quad \text{for } t \ge T. \tag{4.5}$$

Therefore, it is sufficient to construct exponential attractors $\mathcal{M}_\varepsilon$ on B *only*. We note however that the set B may not be semi-invariant under S_t^ε. Nevertheless, it follows from Theorem 1.1 that there exists a time T_0 ($T_0 := 1/\alpha \ln \frac{Q(R)}{R}$) which is independent of ε such that

$$S_t^\varepsilon B \subset B, \quad \text{if } t \ge T_0, \quad \varepsilon \in [0, \varepsilon_0]. \tag{4.6}$$

Let us now define a family of maps $L_\varepsilon := S_{T_0}^\varepsilon : B \to B$, for $\varepsilon \in [0, \varepsilon_0]$, and let us first construct exponential attractors $\mathcal{M}_\varepsilon^d$ for the discrete semi-groups generated by these maps (then, using the results of Theorems 2.1-2.4, we construct exponential attractors $\mathcal{M}_\varepsilon^c$ for the continuous dynamics). Moreover, it will be convenient to construct these attractors on B endowed with the metric of the space $W^{-1,2}(\Omega)$ first. To this end, we apply Theorem 3.1. We set $H := W^{-1,2}(\Omega)$ and $H_1 := W^{2,2}(\Omega)$. Then, the uniform (with respect to ε) estimate (3.1) is an immediate corollary of Theorems 2.1 and 2.2 and estimate (3.2) is an immediate corollary of Theorem 2.3. Thus, all the assumptions of Theorem 3.1 are satisfied and, consequently, we have exponential attractors $\mathcal{M}_\varepsilon^d$ for the maps L_ε such that

$$\dim_F(\mathcal{M}_\varepsilon^d, H) \le C, \tag{4.7}$$

$$\operatorname{dist}_{sym,H}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C'\varepsilon^\kappa, \tag{4.8}$$

$$\operatorname{dist}_H(L_\varepsilon^n B, \mathcal{M}_\varepsilon^d) \le C'' e^{-\alpha n}, \tag{4.9}$$

for appropriate constants C, C', C'', α and κ which are independent of ε.

We now construct the exponential attractors $\mathcal{M}_\varepsilon^c$ by the standard formula (see e.g. [EFNT]):

$$\mathcal{M}_\varepsilon^c := \cup_{t\in[0,T_0]} S_t^\varepsilon \mathcal{M}_\varepsilon^d. \tag{4.10}$$

Indeed, since, thanks to Corollary 2.1, the semi-group S_t^ε is uniformly Holder continuous with respect to $(t, u_0) \in [0, T_0] \times B$ (with the Holder exponent 1/2), then $\mathcal{M}_\varepsilon^c$ is indeed an exponential attractor for S_t^ε and

$$\dim_F(\mathcal{M}_\varepsilon^c, H) \le C + 2, \tag{4.11}$$

$$\operatorname{dist}_H(S_t^\varepsilon B, \mathcal{M}_\varepsilon^d) \le \tilde{C} e^{-\alpha t/T_0}, \tag{4.12}$$

(see e.g. [EFNT]). Moreover, due to Theorem 2.3, the symmetric distance for the attractors $\mathcal{M}_\varepsilon^c$ enjoys the same type of estimate as in the discrete case $\mathcal{M}_\varepsilon^d$:

$$\operatorname{dist}_{sym,H}(\mathcal{M}_\varepsilon^c, \mathcal{M}_0^c) \le C\left(\operatorname{dist}_{sym,H}(\mathcal{M}_\varepsilon^d, \mathcal{M}_\varepsilon^d) + \varepsilon\right) \le C_1\varepsilon^\kappa. \tag{4.13}$$

Thus, we have constructed exponential attractors $\mathcal{M}_\varepsilon^c$ for the metric of $H = W^{-1,2}(\Omega)$. We now note that it follows from its definition that the set $S_1^\varepsilon \mathcal{M}_\varepsilon^c$ is also an exponential attractor. We finally set

$$\mathcal{M}_\varepsilon := S_1^\varepsilon \mathcal{M}_\varepsilon^c. \tag{4.14}$$

Then, estimates (4.1)-(4.3) are immediate consequences of (4.11)-(4.13) and of the results of Theorems 2.2 and 2.4. Theorem 4.1 is thus proved.

Remark 4.1. As a consequence of Theorem 4.1, we obtain the existence of the finite dimensional global attractor, both for the viscous Cahn-Hilliard and the limit Cahn-Hilliard systems, without growth restrictions on the potential (this result was proved directly in [EfGZ], where the authors could also consider the case of Neumann boundary conditions; the main difficulty in [EfGZ] comes from the fact that the corresponding semigroups are not differentiable without growth restrictions).

References

[ADN] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, II, Comm. Pure Appl. Math. 12, 623-727, 1959; 17, 35-92, 1964.

[BV] A. V. Babin and M. I. Vishik, Attractors of evolutionary equations, Nauka, Moscow, 1989; English translation: North-Holland, Amsterdam, 1992.

[BN] A. Babin and B. Nicolaenko, Exponential attractors of reaction-diffusion systems in an unbounded domain, J. Dyn. Diff. Equ. 7, 567-590, 1995.

[BaB] J. W. Barett and J. F. Blowey, Finite element approximation of the Cahn-Hilliard equation with concentration dependent mobility, Math. Comp. 68, 487-517, 1999.

[Bo] F. Boyer, Mathematical study of multi-phase flow under shear through order parameter formulation, Asymptotic Anal. 20 (2), 175-212, 1999.

[C] J. W. Cahn, On spinodal decomposition, Acta Metall. 9, 795-801, 1961.

[CH] J. W. Cahn and J. E. Hilliard, Free energy of a non uniform system I. Interfacial free energy, J. Chem. Phys. 2, 258-267, 1958.

[CaMP] M. Carrive, A. Miranville and A. Piétrus, The Cahn-Hilliard equation for deformable continua, Adv. Math. Sci. Appl., To appear.

[ChD] J. W. Cholewa and T. Dlotko, Global attractors of the Cahn-Hilliard system, Bull. Austral. Math. Soc. 49, 277-302, 1994.

[EFNT] A. Eden, C. Foias, B. Nicolaenko and R. Temam, Exponential attractors for dissipative evolution equations, John-Wiley, New-York, 1994.

[EFK] A. Eden, C. Foias and V. Kalantarov, A remark on the two constructions of exponential attractors for α-contractions, J. Dyn. Diff. Equ. 10 (1), 37-45, 1998.

[EfGZ] M. Efendiev, H. Gajewski and S. Zelik, The finite dimensional attractor for a 4th order system of the Cahn-Hilliard type with a supercritical nonlinearity, preprint WIAS-Berlin, 2000, No. preparation.

[EfM] M. Efendiev and A. Miranville, Finite dimensional attractors for reaction-diffusion equations in $\mathbb{R}^n$ with a strong nonlinearity, Disc. Cont. Dyn. Systems 15 (2), 399-424, 1999.

[EfMZ1] M. Efendiev, A. Miranville and S. Zelik, Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$, C.R.A.S. 330, 713-718, 2000.

[EfMZ2] M. Efendiev, A. Miranville and S. Zelik, Infinite dimensional exponential attractors for a non-autonomous reaction-diffusion system, Submitted.

[ElG] C. M. Elliott and H. Garcke, On the Cahn-Hilliard equation with degenerate mobility, SIAM J. Math. Anal. 27, 404-423, 1996.

[ElK] C. M. Elliott and I. M. Kostin, Lower semicontinuity of a non-hyperbolic attractor for the viscous Cahn-Hilliard equation, Nonlinearity 9, 687-702, 1996.

[ElS] C. M. Elliott and A. M. Stuart, The viscous Cahn-Hilliard equation. II. Analysis, J. Diff. Equ. 128 (2), 387-414, 1996.

[ElZ] C. M. Elliott and Zheng Song Mu, On the Cahn-Hilliard equation, Arch. Rat. Mech. Anal. 96, 339-357, 1986.

[Ey] J. D. Eyre, Systems of Cahn-Hilliard equations, SIAM J. Appl. Anal. 53, 1686-1712, 1993.
[FGM] P. Fabrie, C. Galusinski and A. Miranville, Uniform inertial sets for damped wawe equations, Disc. Cont. Dyn. Systems 6 (2), 393-418, 2000.
[FoST] C. Foias, G. Sell and R. Temam, Inertial manifolds for nonlinear evolution equations, J. Diff. Equ. 73, 309-353, 1988.
[G] C. Galusinski, PhD thesis, Université Bordeaux-I, 1996.
[Ga] H. Garcke, Habilitation thesis, Bonn University, 2000.
[Gu] M. Gurtin, Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance, Physica D 92, 178-192, 1996.
[H] J. Hale, Asymptotic behavior of dissipative systems, Math. Surveys and Mon., Vol. 25, Amer. Math. Soc., Providence, RI, 1987.
[L] O. Ladyzhenskaya, Attractors for semigroups of evolution equations, Cambridge University Press, 1991.
[LiZ] D. Li and C. Zhong, Global attractors for the Cahn-Hilliard equation with fast growing nonlinearity, J. Diff. Equ. 149 (2), 191-210, 1998.
[LoM] J. L. Lions and E. Magenes, Problèmes aux limites nonhomogènes et applications, Vol. I, Dunod, Paris, 1968.
[M1] A. Miranville, Exponential attractors for a class of evolution equations by a decomposition method, C.R.A.S. 328, 145-150, 1999.
[M2] A. Miranville, Some generalizations of the Cahn-Hilliard equation, Asymptotic Analysis 22, 235-259, 2000.
[M3] A. Miranville, Existence of solutions for a Cahn-Hilliard-Gurtin model, C.R.A.S., To appear.
[MPR] A. Miranville, A. Piétrus and J. M. Rakotoson, Dynamical aspects of a generalized Cahn-Hilliard equation based on a microforce balance, Asymptotic Anal. 16, 315-345, 1998.
[NST] B. Nicolaenko, B. Scheurer and R. Temam, Some global dynamical properties of a class of pattern formation equations, Comm. Partial Diff. Equ. 14 (2), 245-297, 1989.
[No] A. Novick-Cohen, On the viscous Cahn-Hilliard equation, in Material instabilities in continuum and related mathematical problems, J. M. Ball ed., Oxford University Press, Oxford, 329-342, 1988.
[R] G. Raugel, Global attractors in partial differential equations, in Handbook of dynamical systems III: towards applications, B. Fiedler, G. Iooss and N. Kopell eds., To appear.
[T] R. Temam, Infinite dimensional dynamical systems in physics and mechanics, 2d ed., Springer-Verlag, New-York, 1997.
[Tr] H. Triebel, Interpolation theory, functional spaces, differential operators, North-Holland, Amsterdam, 1978.
[Z] E. Zeidler, Nonlinear functional analysis and its applications. Part I. Fixed-point theorems, Springer-Verlag, 1985.
[Ze] S. Zelik, The attractor for a non-linear reaction-diffusion system with a supercritical nonlinearity. Rend. Accad. Naz. Lincei., 2000, (in press)