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ROBUST EXPONENTIAL ATTRACTORS FOR CAHN-HILLIARD TYPE EQUATIONS WITH SINGULAR POTENTIALS

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Dedicated to Professor Giovanni Prouse on the occasion of his seventieth birthday

Abstract. Our aim in this article is to study the long time behavior of a family of singularly perturbed Cahn-Hilliard equations with singular (and, in particular, logarithmic) potentials. In particular, we are able to construct a continuous family of exponential attractors (as the perturbation parameter goes to 0). Furthermore, using these exponential attractors, we are able to prove the existence of the finite dimensional global attractor which attracts the bounded sets of initial data for all the possible values of the spatial average of the order parameter, hence improving previous results which required strong restrictions on the size of the spatial domain and to work on spaces on which the average of the order parameter is prescribed. Finally, we are able, in one and two space dimensions, to separate the solutions from the singular values of the potential, which allows us to reduce the problem to one with a regular potential. Unfortunately, for the unperturbed problem in three space dimensions, we need additional assumptions on the potential, which prevents us from proving such a result for logarithmic potentials.

Introduction.

In this article, we are interested in the study of the long time behavior of the Cahn-Hilliard equation with singular potentials (and, in particular, with logarithmic potentials).

The Cahn-Hilliard equation (see [1] and [2]) is very important in material science. It describes important qualitative behavior of two-phase systems, namely, the transitions of phases in binary alloys. This can be observed when the alloy is cooled down sufficiently. One can then observe a partial nucleation, i.e., the apparition of nucleides, or a total

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nucleation, the so-called spinodal decomposition. In that case, the material quickly becomes inhomogeneous and forms a fine-grained structure, more or less alternating between the two components of the alloy (see [1] and [2]). From a mathematical point of view, the spinodal decomposition corresponds to an unstable homogeneous equilibrium. We refer the reader to [3] and [4] for a qualitative study of the spinodal decomposition. In a second stage, called coarsening, the microstructures observed after the spinodal decomposition coarsen. The coarsening occurs at a slower time scale and is less understood (see e.g. [5] and [6]). So, in view of a better understanding of these phenomena, it seems useful to have a good understanding of the asymptotic behavior of the equation, and also to have some understanding of the transient dynamics. In this article, we are more specifically interested in the study of finite dimensional attractors (in the sense of the Hausdorff and the fractal dimensions). In particular, the existence of such sets would indicate that the long time behavior of the system, and also, in some sense, the phenomena observed, only involve a finite number of degrees of freedom.

The starting point in the Cahn-Hilliard theory is the free energy, also called Ginzburg-Landau free energy, which goes back to van der Waals (see [7]) and is of the form

$$\psi = \psi(u, \nabla u) = \frac{\kappa}{2}|\nabla u|^2 + F(u),$$

where $u = u(t,x)$ is the order parameter (a rescaled density of atoms), F is the potential, a double-well potential whose wells correspond to the phases of the material, and $\kappa > 0$ is related to the surface tension. For simplicity, we will take $\kappa = 1$ in this article. This leads to the following fourth-order parabolic equation for the order parameter:

$$\begin{cases} \partial_t u = -\Delta_x(\Delta_x u - f(u)), \\ \partial_n u\big|_{\partial\Omega} = \partial_n \Delta_x u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \end{cases} \tag{0.1}$$

where $f(u) := F'(u)$ and Ω is a bounded smooth domain of $\mathbb{R}^3$.

A slightly more complicated model, which is based on a new balance law for microforces and which takes into account the working of internal microforces, was introduced in [8] (we can note that microforces describe forces which are associated with microscopic configurations of atoms, whereas standard forces are associated with macroscopic length scales, hence a reason to consider separate balance laws for microforces and standard forces). For an isotropic material, this leads to the following generalization of equation (0.1):

$$\begin{cases} \partial_t u = -\Delta_x(\Delta_x u - \varepsilon\partial_t u - f(u)), \\ \partial_n u\big|_{\partial\Omega} = \partial_n \Delta_x u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \end{cases} \tag{0.2}$$

where ε is a (small) positive parameter and where the term $\varepsilon\partial_t u$ describes the influence of the internal microforces. This equation can also be viewed as a viscous Cahn-Hilliard equation, see e.g. [9].

These equations have been studied intensively; see e.g. the review articles [10] and [11] and, among many references, [9], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22],

[23], [24], [25], [26] and [27]. In particular, in the case of regular potentials, the problem is well understood and one has existence and uniqueness of solutions and existence of the finite dimensional global attractor.

However, logarithmic nonlinear interaction functions $f(u) = F'(u)$ of the form

$$f(u) = f_{log}(u) := -\alpha_c u + \alpha \ln \frac{1+u}{1-u}, \quad 0 < \alpha < \alpha_c, \tag{0.3}$$

are more relevant from the physical point of view, see e.g. [1], [2] and [10]. Actually, in order to simplify the equation, one often approximates logarithmic potentials by *regular* potentials such as, in particular, polynomials of degree four. In that case, the additional condition

$$-1 < u(t,x) < 1, \quad \text{for almost every } (t,x) \in \mathbb{R}_+ \times \Omega, \tag{0.4}$$

must be imposed to the order parameter $u(t,x)$. It would be important to prove that, in the case of regular potentials, the order parameter also satisfies (0.4). At the moment, however, this seems out of reach, due to the lack of a maximum principle for fourth-order parabolic equations.

The first existence and uniqueness results for the classical Cahn-Hilliard equation with a logarithmic potential were obtained in [20]; see also [19] and [28] for mobilities which depend on the order parameter and which degenerate. In particular, in order to prove the existence of solutions, one approximates the singular potential by regular ones and then passes to the limit. The numerical analysis of such equations can be found in [12], [29] and [30].

The long time behavior of the Cahn-Hilliard equation (0.1) with a logarithmic potential was studied in [28]. In particular, the authors obtained the existence of the global attractor. In that case, however, the study of the finite dimensionality of the global attractor is much more difficult than in the case of regular potentials. Indeed, the classical scheme, based on the Lyapunov exponents and on the volume contraction method (see e.g. [27]), requires that the corresponding semigroup is (quasi)differentiable on the global attractor. This differentiability is a rather delicate problem for the Cahn-Hilliard equation with *singular* (e.g. logarithmic) potentials and, to the best of our knowledge, it can be solved only when every point of the global attractor $\mathcal{A}$ is separated from the singular points of the potential, i.e. when

$$\|\mathcal{A}\|_{L^\infty(\Omega)} \le 1 - \delta, \tag{0.5}$$

for some constant $\delta \in]0,1[$. Under this assumption, the differentiability of the semigroup can be proven exactly as in the case of regular potentials. Unfortunately, the existence of such a constant δ is also a highly nontrivial problem which has been solved in [28] only for small domains Ω. Moreover, for technical reasons related to the volume contraction method, the authors could only consider subspaces of the phase space on which the average of the order parameter is a given constant, which is not entirely satisfactory. Analogous results for equation (0.2) with a logarithmic potential of the form (0.3) have been obtained in [16].

One of the aims of this article is to improve the results of [16] and [28]. In particular, we prove that the global attractors $\mathcal{A}_\varepsilon$, $\varepsilon \geq 0$, associated with problems (0.2) with a logarithmic potential have finite dimension for *arbitrary* three-dimensional domains Ω (and not only for small domains Ω as it was the case previously). Moreover, we also construct a robust family of *exponential* attractors $\mathcal{M}_\varepsilon$ for problems (0.2) which tend to the exponential attractor $\mathcal{M}_0$ associated with the limit problem (0.1) as $\varepsilon \to 0$:

$$\text{dist}_{sym,L^\infty(\Omega)}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq C\varepsilon^\kappa, \tag{0.6}$$

where the constants C and κ can be computed explicitly. We note that this result does not require that the corresponding semigroups are quasidifferentiable, and, thus, does not require that the analogue of (0.5) is satisfied. We also consider a more general class of (singular) nonlinearities $f \in C^1(-1,1)$ which satisfy the following assumptions:

$$\begin{cases} 1. & \lim_{z\to\pm 1} f(z) = \pm\infty, \\ 2. & \lim_{z\to\pm 1} f'(z) = +\infty. \end{cases} \tag{0.7}$$

We recall that, in contrast to exponential attractors, global attractors are, in general, only upper semicontinuous with respect to perturbations. So, in general, one does not expect that estimate (0.6) is satisfied for *global* attractors. Indeed, the lower semicontinuity of global attractors is much more delicate to prove and usually requires some kind of "hyperbolicity assumption" on the unperturbed attractor $\mathcal{A}_0$. For instance, there exists a rather wide class of global attractors (the so-called *regular* attractors in the terminology of Babin and Vishik) which are upper and lower semicontinuous and satisfy estimate (0.6) under natural assumptions on the perturbations (they are also exponential), see e.g. [31] and [32]. This theory requires, however, the existence of a global Lyapunov function and the hyperbolicity of all the equilibria of the system under consideration.

In the case of the Cahn-Hilliard equation (0.1), we obviously have a global Lyapunov function, but the second assumption on the hyperbolicity of the equilibria is rather delicate to prove, even in the case of regular potentials. Indeed, even though this assumption is in some sense "generic", it is very difficult to verify for concrete values of the physical parameters (say, α, α_c and Ω) and, to the best of our knowledge, this problem has been solved only for one-dimensional domains Ω, see [33] and [34]. Moreover, even in that case, it is unclear how to estimate the constants C and κ (in (0.6)) in terms of the parameters α, α_c and Ω for regular attractors. On the contrary, estimate (0.6) for *exponential* attractors requires neither the existence of a global Lyapunov function nor any "hyperbolicity assumption" and usually holds under natural assumptions on the system under consideration, see [35], [36] and [37] for details. Moreover, as it can be seen from the proof of estimate (0.6) (see Sections 1 and 4), the constants C and κ can be estimated explicitly in terms of the physical parameters of the problem although we do not give the explicit formulae here.

Thus, exponential attractors are more robust and, perhaps, more suitable objects than global attractors for the study of dynamical systems depending on parameters. We refer the reader to [35], [36], [37] and [38] for more detailed discussions on the robustness of exponential attractors in different situations including the problem of the approximation of a PDE in an unbounded domain by the same PDE in bounded domains, which is known to be very singular at the level of *global* attractors, see [36]. Furthermore, the global attractor may be trivial, say, reduced to one point, and may thus fail to capture important transient dynamics. Again, an exponential attractor may be a better object. Indeed, the trajectories will be attracted to any exponential attractor exponentially fast and will then converge to the global attractor at a slower rate. This bears some resemblance to what can be observed in the phase separation process described by the Cahn-Hilliard equation, i.e., the spinodal decomposition followed by a slower process, the coarsening.

This article is organized as follows.

In section 1, we prove an abstract result on the construction of a continuous (with respect to perturbations) family of exponential attractors. This result generalizes previous constructions given in [35], [37] and [39]. Also, here, we do not have the continuity up to time shifts as it was the case in [38]. Such a generalization is necessary in order to be able to treat the specific difficulties induced by singular potentials.

In section 2, we derive several a priori estimates which are of fundamental significance for what follows. In particular, we prove that every solution $u(t)$ of equation (0.2) with $\varepsilon > 0$ is separated from the singular points of the nonlinearity. The latter means that, for every $\varepsilon > 0$ and $\mu > 0$, there exists a strictly positive constant $\delta_{\mu,\varepsilon}$ such that every solution $u(t)$ of (0.2) satisfies

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_{\mu,\varepsilon}, \quad \forall t \geq \mu. \tag{0.8}$$

This result shows that the additional term $\varepsilon\partial_t u$ in the right-hand side of equation (0.2) regularizes the classical Cahn-Hilliard equation (0.1) with a singular potential. We note that the constant $\delta_{\mu,\varepsilon}$ in estimate (0.8) tends to zero as $\varepsilon \to 0$ so that this estimate does not provide any information for the limit equation (0.1).

The existence and uniqueness of a solution $u(t)$ for problem (0.2), $\varepsilon \geq 0$, is obtained in Section 3. We prove the existence of solutions for (0.1) with singular potentials by using the (physically relevant) regularization (0.2). Hence, our proof is different from those in [19], [20] and [28], where the problem is regularized by approximating the singular potential by regular ones.

In Section 4, we apply the abstract scheme given in Section 1 in order to construct a continuous (as the perturbation parameter goes to 0) family of exponential attractors for our problem. As a consequence, we also obtain the existence of the finite dimensional global attractor without any assumption on the size of the spatial domain as it was the case in [16] and [28].

In Section 4, we work on spaces on which the average of the order parameter is prescribed. As a consequence, the exponential attractors (and also the global attractors) constructed depend a priori on this value of the average of the order parameter. However, in Section 5, we are able to construct a continuous family of exponential attractors that do not depend on the average of the order parameter. Furthermore, using these exponential attractors, we are able to construct the finite dimensional global attractor which attracts the bounded sets of initial data for all the possible values of the average of the order parameter, which is more satisfactory from the physical point of view. Such a result would be more difficult to prove for exponential attractors and will be studied elsewhere (see Remark 5.2).

Finally, we obtain in Section 6 uniform (with respect to the perturbation parameter) L^∞-bounds on the solutions. Such estimates allow to separate the solutions of (0.1) from the singular points of the potential and thus to reduce the problem to one with a regular potential. Unfortunately, in three space dimensions, we need additional assumptions on the potential and are thus not able to consider logarithmic potentials. However, we are able to prove such a result for logarithmic potentials in one and two space dimensions.

For the reader's convenience, several properties of the nonlinear interaction function are collected in an appendix.

§1 Perturbations of exponential attractors: the abstract setting.

In this section, we give a generalization of an abstract result on the construction of uniform exponential attractors for a singularly perturbed family of maps, which is given in [35] and [37]. This result will be applied to our problem in Section 4. In order to do so, we use the concept of the Kolmogorov ε-entropy.

Definition 1.1. Let K be a (pre)compact set in a metric space V. Then, due to the Hausdorff criterium, for every $\mu > 0$, the set K can be covered by a finite number of μ-balls in V. Let $N_\mu(K,V)$ be the minimal number of such balls. Then, by definition, the Kolmogorov μ-entropy of K is the following number:

$$\mathbb{H}_\mu(K,V) := \ln N_\mu(K,V)$$

(see e.g. [40] for details). We recall that the fractal dimension of the set K can be expressed in terms of the μ-entropy:

$$\dim_F(K,V) := \limsup_{\mu\to 0^+} \frac{\mathbb{H}_\mu(K,V)}{\ln \frac{1}{\mu}}.$$

We are now in a position to formulate our abstract scheme.

Let $\mathcal{E}(\varepsilon)$ and $\mathcal{E}^1(\varepsilon)$, $\varepsilon \in [0,1]$, be two families of Banach spaces (which are embedded into a larger topological space $\mathcal{V}$) such that $\mathcal{E}^1(\varepsilon) \Subset \mathcal{E}(\varepsilon)$, for every $\varepsilon \in [0,1]$. We also assume that these compact embeddings are uniform with respect to ε in the following sense:

$$\mathbb{H}_\mu(B(0,1,\mathcal{E}^1(\varepsilon)),\mathcal{E}(\varepsilon)) \le \mathbb{M}(\mu), \quad \forall \mu > 0, \tag{1.1}$$

where $\mathbb{M}(\mu)$ is some monotonic function which is independent of ε (here and below, we denote by $B(v,R,V)$ the R-ball in V centered at v).

We further assume that we are given a family of closed sets $\mathcal{B}_\varepsilon \subset \mathcal{E}(\varepsilon)$ and a family of maps $S_\varepsilon : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ such that

1. The set $\mathcal{B}_0$ is a subset of $\mathcal{E}(\varepsilon)$, for every $\varepsilon \in [0,1]$, and

$$\|b_0\|_{\mathcal{E}(\varepsilon)} \le C_1\|b_0\|_{\mathcal{E}(0)} + C_2\varepsilon, \quad \forall b_0 \in \mathcal{B}_0. \tag{1.2}$$

2. There exists a family of sets $\mathcal{C}_\varepsilon \subset \mathcal{B}_\varepsilon$ such that, for every $b_\varepsilon^1,\ b_\varepsilon^2 \in \mathcal{C}_\varepsilon$, the following estimate holds:

$$\|S_\varepsilon b_\varepsilon^1 - S_\varepsilon b_\varepsilon^2\|_{\mathcal{E}^1(\varepsilon)} \le K\|b_\varepsilon^1 - b_\varepsilon^2\|_{\mathcal{E}(\varepsilon)}, \tag{1.3}$$

where K is independent of ε.

3. There exist nonlinear "projectors" $\Pi_\varepsilon : \mathcal{B}_\varepsilon \to \mathcal{B}_0$ such that $\Pi_\varepsilon\mathcal{B}_\varepsilon = \mathcal{B}_0$ and

$$\|S_\varepsilon^{(k)} b_\varepsilon - S_0^{(k)}\Pi_\varepsilon b_\varepsilon\|_{\mathcal{E}(\varepsilon)} \le C\varepsilon L^k, \tag{1.4}$$

for every $b_\varepsilon \in \mathcal{B}_\varepsilon$, where the constants C and L are independent of ε (here and in what follows, $S_\varepsilon^{(k)}$ denotes the kth iteration of the map S_ε).

4. The sets $\mathcal{C}_\varepsilon$ satisfy the following uniform recurrence property: there exist $\delta > 0$ and $N \in \mathbb{N}$ (which are independent of ε) such that, for every $b_\varepsilon \in \mathcal{B}_\varepsilon$, at least one of the sets

$$S_\varepsilon(\mathcal{B}_\varepsilon \cap B(b_\varepsilon,\delta,\mathcal{E}(\varepsilon)), \cdots, S_\varepsilon^{(N-1)}(\mathcal{B}_\varepsilon \cap B(b_\varepsilon,\delta,\mathcal{E}(\varepsilon)) \tag{1.5}$$

is a subset of $\mathcal{C}_\varepsilon$.

5. The sets $\mathcal{B}_\varepsilon$ can be covered by a finite number of δ-balls and

$$\mathbb{H}_\delta(\mathcal{B}_\varepsilon, \mathcal{E}(\varepsilon)) \le M_0 < \infty, \tag{1.6}$$

where M_0 is independent of ε.

6. The maps S_ε are uniformly Lipschitz on $\mathcal{B}_\varepsilon$:

$$\|S_\varepsilon^{(k)} b_1 - S_\varepsilon^{(k)} b_2\|_{\mathcal{E}(\varepsilon)} \le Ce^{Lk}\|b_1 - b_2\|_{\mathcal{E}(\varepsilon)},\ \forall b_1,\ b_2 \in \mathcal{B}_\varepsilon,\ \forall k \in \mathbb{N}, \tag{1.7}$$

where C and L are independent of ε.

Thus, in contrast to previous results (see [35] and [37]), we now assume the smoothing property (1.3) on the "recurrent" subsets $\mathcal{C}_\varepsilon$ of the phase space $\mathcal{B}_\varepsilon$ only. This will be important when applying the abstract result given in this section to our problem, since, as we will see in the next sections below, we cannot prove that in general the solutions of the (classical) Cahn-Hilliard equation are separated from the singular values of the potential and, consequently, we cannot obtain the smoothing property (1.3) for every b_ε^1 and b_ε^2 belonging to the whole phase space $\mathcal{B}_\varepsilon$.

The main result of this section is the following theorem.

Theorem 1.1. *Let assumptions* (1.1)–(1.7) *hold. Then there exists a family of exponential attractors* $\mathcal{M}_\varepsilon \subset \mathcal{B}_\varepsilon$ *for the maps* S_ε *such that* $S_\varepsilon \mathcal{M}_\varepsilon \subset \mathcal{M}_\varepsilon$, *and the following conditions are satisfied:*

1. The rate of exponential attraction is uniform with respect to ε*:*

$$\operatorname{dist}_{\mathcal{E}(\varepsilon)}(S_\varepsilon^{(k)}\mathcal{B}_\varepsilon, \mathcal{M}_\varepsilon) \le C_3 e^{-\nu k}, \tag{1.8}$$

where the positive constants C_3 *and* ν *are independent of* ε *and where* dist *denotes the Hausdorff semidistance.*

2. The sets $\mathcal{M}_\varepsilon$ *are compact in* $\mathcal{E}(\varepsilon)$, *for every* $\varepsilon \in [0,1]$, *and their Kolmogorov* μ*-entropy is uniformly bounded with respect to* ε*:*

$$\mathbb{H}_\mu(\mathcal{M}_\varepsilon, \mathcal{E}(\varepsilon)) \le C_4 \ln \frac{1}{\mu} + C_4', \tag{1.9}$$

where the positive constants C_4 *and* C_4' *are independent of* ε *and* $\mu > 0$. *Therefore, their fractal dimension is also uniformly bounded with respect to* ε.

3. The symmetric distance between $\mathcal{M}_\varepsilon$ *and* $\mathcal{M}_0$ *satisfies*

$$\operatorname{dist}_{sym,\mathcal{E}(\varepsilon)}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C_5 \varepsilon^\tau. \tag{1.10}$$

Moreover, the constants C_i, $i = 3, 4, 5$, *and* $0 < \tau < 1$ *can be computed explicitly.*

Proof. We first construct an appropriate exponential attractor $\bar{\mathcal{M}}_0$ for $S_0^{(N)}$. To this end, we construct a family of sets $V_i \subset S_0^{(Ni)}\mathcal{B}_0$, $i = 0, 1, 2, \cdots$, by the following inductive procedure. We recall that, due to (1.6), there exists a covering of the set $\mathcal{B}_0$ by a finite number $N_0 := e^{M_0}$ of δ-balls in the metric of $\mathcal{E}(0)$. Let $V_0 := \{b_0^1, \cdots, b_0^{N_0}\} \subset \mathcal{B}_0$ be the centers of these balls. We assume that the set V_k is already constructed such that $V_k \subset S_0^{(Nk)}\mathcal{B}_0$ and V_k is an $R_k := \delta/2^k$-net of $S_0^{(Nk)}\mathcal{B}_0$. We then construct the next set V_{k+1} preserving these properties. To this end, we fix an arbitrary $b_k \in V_k$ and consider the images of the $\delta/2^k$-balls centered at b_k. Then, according to the recurrence property (1.5) and the Lipschitz continuity (1.7), there exists $N(b_k) \le N - 1$ such that

$$S_0^{(N(b_k))}(\mathcal{B}_0 \cap B(b_k, R_k, \mathcal{E}(0))) \subset \mathcal{C}_0 \cap B(S_0^{(N(b_k))} b_k, C R_k 2^{L(N-1)}, \mathcal{E}(0)). \tag{1.11}$$

Using now the smoothing property (1.3), we find

$$S_0^{(N(b_k)+1)}(\mathcal{B}_0 \cap B(b_k, R_k, \mathcal{E}(0))) \subset B(S^{(N(b_k)+1)} b_k, CKR_k 2^{L(N-1)}, \mathcal{E}^1(0)). \tag{1.12}$$

We now recall that $\mathcal{E}^1(0) \Subset \mathcal{E}(0)$. Consequently, there exists a covering of the right-hand side of (1.12) by a finite number of $R_k/(C2^{L(N-1)+2})$-balls in $\mathcal{E}(0)$. We fix a covering with a minimal number of such balls and denote by $W_k(b_k)$ the set of all the centers of this covering (without loss of generality, we can assume that $W_k(b_k) \subset \mathcal{B}_0$). Furthermore, we note that

$$
\begin{aligned}
(1.13)\quad \#W_k(b_k) = \\
&= N_{R_k/(C2^{L(N-1)+2})}\left(B(S_0^{(N(b_k)+1)}b_k, CKR_k2^{L(N-1)}, \mathcal{E}^1(0)), \mathcal{E}(0)\right) = \\
&= N_{1/(C^2K2^{2L(N-1)+2})}\left(B(0,1,\mathcal{E}^1(0)), \mathcal{E}(0)\right) \le \\
&\le \exp\left(\mathbb{M}(1/(C^2K2^{2L(N-1)+2}))\right) := P_0.
\end{aligned}
$$

It is essential for us that the number P_0 defined in (1.13) be independent of k (and of $b \in V_k$; moreover, thanks to (1.1), this number will be independent of ε as well if we replace the spaces $\mathcal{E}(0)$ and $\mathcal{E}^1(0)$ by $\mathcal{E}(\varepsilon)$ and $\mathcal{E}^1(\varepsilon)$).

Thus, due to (1.11)-(1.13), we have the following embedding:

$$
(1.14)\qquad S_0^{(N(b_k)+1)}(\mathcal{B}_0 \cap B(b_k, R_k, \mathcal{E}(0))) \subset \cup_{b \in W_k(b_k)} B(b, R_k/(C2^{L(N-1)+2}), \mathcal{E}(0)).
$$

Using (1.14), the Lipschitz continuity (1.7) and the fact that $N(b_k)+1 \le N$, we obtain

$$
(1.15)\qquad S_0^{(N)}(\mathcal{B}_0 \cap B(b_k, R_k, \mathcal{E}(0))) \subset \cup_{b \in W_k(b_k)} B(S^{(N-N(b_k)-1)}b, R_k/4), \mathcal{E}(0))
$$

and, therefore, the $R_k/4$-balls centered at the points of $\cup_{b_k \in V_k} S^{(N-N(b_k)-1)}W(b_k)$ cover the set $S_0^{((k+1)N)}\mathcal{B}_0$. Moreover, increasing the radius of the balls by a factor of two, we can assume that the centers of the covering belong to $S^{((k+1)N)}\mathcal{B}_0$. We denote by V_{k+1} the set consisting of all these centers.

Thus, we have constructed by induction a family of sets $V_k \subset S_0^{(Nk)}\mathcal{B}_0$, $k \in \mathbb{N}$, which satisfy the following properties:

$$
(1.16)\qquad \begin{cases} 1.\ \ \#V_k \le N_0 P_0^k, \\ 2.\ \ \operatorname{dist}_{\mathcal{E}(0)}\left(S_0^{(Nk)}\mathcal{B}_0, V_k\right) \le \delta 2^{-k}. \end{cases}
$$

We now define a new family of sets E_k, $k \in \mathbb{N}$, by the following inductive formula:

$$
E_1 := V_1, \quad E_{k+1} := S_0^{(N)}E_k \cup V_{k+1}.
$$

Then, obviously

$$
(1.17)\qquad \begin{cases} 1.\ \ E_k \subset S_0^{(Nk)}\mathcal{B}_0, \quad S_0^{(N)}E_k \subset E_{k+1}, \\ 2.\ \ \#E_k \le N_0 P_0^{k+1}, \\ 3.\ \ \operatorname{dist}_{\mathcal{E}(0)}\left(S_0^{(Nk)}\mathcal{B}_0, E_k\right) \le \delta 2^{-k}. \end{cases}
$$

We finally set

$$\bar{\mathcal{M}}_0' := \cup_{k\in\mathbb{N}} E_k, \quad \bar{\mathcal{M}}_0 := [\bar{\mathcal{M}}_0']_{\mathcal{E}(0)}, \tag{1.18}$$

where $[\cdot]_V$ denotes the closure in V. It is not difficult to verify that the set $\bar{\mathcal{M}}_0$ is an exponential attractor for the map $\mathcal{S}_0 := S_0^{(N)}$ on $\mathcal{B}_0$. Indeed, $(1.17)_1$ and (1.18) imply that $\mathcal{S}_0\bar{\mathcal{M}}_0 \subset \bar{\mathcal{M}}_0$ and the exponential attraction (1.8) is an immediate corollary of $(1.17)_3$. Furthermore, it follows from $(1.17)_2$, together with $(1.17)_1$, that, for every $k \in \mathbb{N}$, we have

$$N_{\delta 2^{-k}}(\bar{\mathcal{M}}_0, \mathcal{E}(0)) \le \sum_{l=1}^{k} \#E_l \le kN_0P_0^{k+1}, \tag{1.19}$$

which immediately implies (1.9) (see [39] for details).

We now construct the exponential attractors $\bar{\mathcal{M}}_\varepsilon$ fo the maps $\mathcal{S}_\varepsilon := S_\varepsilon^{(N)}$ in the case $\varepsilon > 0$, using the construction of the attractor $\bar{\mathcal{M}}_0$ as in [37]. For the reader's convenience, we briefly recall this construction.

We fix inverse images of the sets E_k under the maps $\mathcal{S}_0^{(k)}$ (this is possible due to $(1.17)_1$). To be more precise, we assume that the family of sets $\widehat{E}_k \subset \mathcal{B}_0$ is such that

$$1.\ \ \mathcal{S}_0^{(k)}\widehat{E}_k = E_k, \ \ 2.\ \ \#\widehat{E}_k = \#E_k \le N_0P_0^{k+1}. \tag{1.20}$$

We fix an arbitrary $\varepsilon \in (0, 1]$ and arbitrary liftings of the sets $\widehat{E}_k \subset \mathcal{B}_0$ to $\mathcal{B}_\varepsilon$ with respect to the projectors Π_ε, i.e., the $\widehat{E}_k(\varepsilon) \subset \mathcal{B}_\varepsilon$ are such that

$$1.\ \ \Pi_\varepsilon\widehat{E}_k(\varepsilon) = \widehat{E}_k, \quad 2.\ \ \#\widehat{E}_k(\varepsilon) = \#E_k \le N_0P_0^{k+1}. \tag{1.21}$$

Such liftings exist since $\Pi_\varepsilon\mathcal{B}_\varepsilon = \mathcal{B}_0$. We finally set $\tilde{E}_k(\varepsilon) := \mathcal{S}_\varepsilon^{(k)}\widehat{E}_k(\varepsilon)$. Then, as proven in [37], estimates (1.2), (1.4) and $(1.17)_3$ imply that

$$\mathrm{dist}_{\mathcal{E}(\varepsilon)}\left(\mathcal{S}_\varepsilon^{(k)}\mathcal{B}_\varepsilon, \tilde{E}_k(\varepsilon)\right) \le 2C\varepsilon L^{Nk} + C_1\delta 2^{-k} + C_2\varepsilon, \tag{1.22}$$

where the constants C, C_1, C_2 and L are defined in (1.2) and (1.4). Let now $k(\varepsilon)$ and $0 < \tau < 1$ be the solutions of

$$\varepsilon L^{Nk} = 2^{-k} = \varepsilon^\tau, \tag{1.23}$$

i.e., $k(\varepsilon) = \frac{\ln 1/\varepsilon}{N\ln L + \ln 2}$ and $\tau = \frac{\ln 2}{N\ln L+\ln 2}$. Then it follows from (1.4), (1.22) and (1.23) that

$$\begin{cases} 1.\ \ \mathrm{dist}_{sym,\mathcal{E}(\varepsilon)}\left(\tilde{E}_k(\varepsilon), E_k\right) \le C'\varepsilon^\tau, \\ 2.\ \ \mathrm{dist}_{\mathcal{E}(\varepsilon)}\left(\mathcal{S}_\varepsilon^{(k)}\mathcal{B}_\varepsilon, \tilde{E}_k(\varepsilon)\right) \le C''\delta 2^{-k}, \end{cases} \tag{1.24}$$

for every $1 \le k \le k(\varepsilon)$, where the constants C' and C'' are independent of k and ε.

Thus, we can take $E_k(\varepsilon) := \tilde{E}_k(\varepsilon)$, if $k \le k(\varepsilon)$. In order to construct the sets $E_k(\varepsilon)$ for $k > k(\varepsilon)$, we forget the sets $\tilde{E}_k(\varepsilon)$ and construct them by the inductive procedure described above (for the case $\varepsilon = 0$), based on (1.3) and (1.5), but starting from $E_{k(\varepsilon)}(\varepsilon)$. Finally, we have a family of sets $E_k(\varepsilon)$ which satisfy the following conditions:

$$\begin{cases} 1. \quad E_k(\varepsilon) \subset \mathcal{S}_\varepsilon^{(k)}\mathcal{B}_\varepsilon, \quad \mathcal{S}_\varepsilon E_k(\varepsilon) \subset E_{k+1}(\varepsilon), \quad \#E_k(\varepsilon) \le N_0 P_0^{k+1}, \\ 2. \quad \operatorname{dist}_{\mathcal{E}(\varepsilon)}\left(\mathcal{S}_\varepsilon^{(k)}\mathcal{B}_\varepsilon, E_k(\varepsilon)\right) \le C'\delta 2^{-k}. \end{cases} \tag{1.25}$$

Moreover, for $k \le k(\varepsilon)$, we have

$$\operatorname{dist}_{sym,\mathcal{E}(\varepsilon)}(E_k(\varepsilon), E_k) \le C''\varepsilon^\tau. \tag{1.26}$$

We then define the exponential attractor $\bar{\mathcal{M}}_\varepsilon$ as follows:

$$\mathcal{M}'_\varepsilon := \cup_{k\in\mathbb{N}} E_k(\varepsilon), \quad \bar{\mathcal{M}}_\varepsilon := [\mathcal{M}'_\varepsilon]_{\mathcal{E}(\varepsilon)}. \tag{1.27}$$

It is proven in [35] and [37] that the family of sets $\bar{\mathcal{M}}_\varepsilon$ thus defined is indeed a family of robust exponential attractors for the maps $\mathcal{S}_\varepsilon := S_\varepsilon^{(N)}$, which satisfies all the assumptions of Theorem 1.1.

The robust family of exponential attractors $\mathcal{M}_\varepsilon$ for the initial maps S_ε, $\varepsilon \in [0,1]$, can now be defined (in terms of the attractors $\bar{\mathcal{M}}_\varepsilon$ for $\mathcal{S}_\varepsilon := S_\varepsilon^{(N)}$ constructed above) by the following standard expression:

$$\mathcal{M}_\varepsilon := \cup_{i=0}^{N-1} S_\varepsilon^{(i)} \bar{\mathcal{M}}_\varepsilon, \tag{1.28}$$

and Theorem 1.1 is proven.

Remark 1.1. It is essential for the next sections below to note that $\mathcal{M}_\varepsilon$ satisfies

$$\mathcal{M}_\varepsilon \subset S_\varepsilon \mathcal{B}_\varepsilon. \tag{1.29}$$

§2 Uniform a priori estimates.

In this section, we derive several a priori estimates on the solutions of equation (0.2), which play a fundamental role for what follows. In order to obtain these results, we rewrite (following [18] and [35]) the initial problem (0.2) in the following equivalent form (see Remark 3.1 below):

$$\begin{cases} \varepsilon\partial_t u + (-\Delta_x)_N^{-1}\partial_t u = \Delta_x u - f(u) + \langle f(u)\rangle, \quad \varepsilon \ge 0, \\ \partial_n u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \end{cases} \tag{2.1}$$

where $\langle v\rangle := \frac{1}{|\Omega|}\int_\Omega v(x)\,dx$ denotes the spatial average of the function v and $(-\Delta_x)^{-1}_N$ is the inverse Laplace operator in Ω associated with Neumann boundary conditions, which is well defined on the space $L^2_0(\Omega)$ of the functions belonging to $L^2(\Omega)$ with zero average:

$$(2.2)\qquad (-\Delta_x)^{-1}_N : L^2_0(\Omega)\to L^2_0(\Omega),\quad L^2_0(\Omega) := \{v\in L^2(\Omega),\ \ \langle v\rangle = 0\}.$$

We also recall that equation (2.1) (or, equivalently, equation (0.2)) possesses a conservation law

$$(2.3)\qquad \langle u(t)\rangle \equiv \langle u_0\rangle\,,\quad \forall t\ge 0.$$

Therefore, we impose the additional condition

$$(2.4)\qquad \langle u_0\rangle = m_0,\quad |m_0|\le 1-\kappa,\quad \text{for some fixed } \kappa\in]0,1[,$$

to the initial value u_0. We also define, for every $m_0\in[-1+\kappa,1-\kappa]$, the phase space $\mathbb{D}^{m_0}_\varepsilon$ for this problem as follows:

$$(2.5)\quad \mathbb{D}^{m_0}_\varepsilon := \Big\{v\in H^2(\Omega),\ \partial_n v\big|_{\partial\Omega}=0,\ \|v\|_{L^\infty(\Omega)}\le 1,\ \langle v\rangle = m_0,\\ f(v)\in L^2(\Omega),\ \varepsilon^{1/2}\phi\in L^2(\Omega) \text{ and } \phi\in H^{-1}(\Omega),\\ \text{where}\ \ \phi := (\varepsilon + (-\Delta_x)^{-1}_N)^{-1}[\Delta_x v - f(v) + \langle f(v)\rangle]\Big\};$$

and the "norm" in the space $\mathbb{D}^{m_0}_\varepsilon$ is defined as follows:

$$(2.6)\qquad \|v\|^2_{\mathbb{D}^{m_0}_\varepsilon} := \|v\|^2_{H^2(\Omega)} + \|f(v)\|^2_{L^2(\Omega)} + \varepsilon\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{H^{-1}_N(\Omega)},$$

where $H^{-1}_N(\Omega) := [H^1(\Omega)]'$.

In this section, we only give a formal derivation of the a priori estimates for $u(t)$, assuming that $u(t)$ is a sufficiently regular function which satisfies the additional assumption

$$(2.7)\qquad \|u\|_{L^\infty(\mathbb{R}_+\times\Omega)} < 1.$$

This assumption will be relaxed in the next section below (see Definition 3.1).

The main result of this section is the following theorem which gives a dissipative estimate in the space $\mathbb{D}^{m_0}_\varepsilon$.

Theorem 2.1. *Let the nonlinearity f satisfy assumptions* (0.7) *and let $u(t)$ be a solution of equation* (2.1) *which satisfies* (2.7). *Then the following estimate holds:*

$$(2.8)\qquad \|u(t)\|^2_{\mathbb{D}^{m_0}_\varepsilon} + \int_t^{t+1}\|\partial_t u(s)\|^2_{H^1(\Omega)}\,ds \le Q_\kappa(\|u(0)\|_{\mathbb{D}^{m_0}_\varepsilon})\chi(1-t) + C_\kappa,\quad \forall t\ge 0,$$

where $\chi(z)$ is the Heaviside function and where the constant C_κ and the monotonic function Q_κ may depend on κ, but are independent of $\varepsilon\in[0,1]$ and $u(t)$.

Proof. We first prove the nondissipative analogue of estimate (2.8).

Lemma 2.1. *Let the above assumptions hold. Then, for every $T > 0$, there exists a monotonic function $Q_{\kappa,T}$ depending on κ and T such that*

$$\|u(t)\|^2_{\mathbb{D}^{m_0}_\varepsilon} + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^1(\Omega)}\,ds \le Q_{\kappa,T}(\|u(0)\|_{\mathbb{D}^{m_0}_\varepsilon}) \tag{2.9}$$

holds uniformly with respect to ε and $t \le T$.

Proof. We differentiate equation (2.1) with respect to t and set $\phi(t) := \partial_t u(t)$. Then we have

$$\varepsilon\partial_t\phi + (-\Delta_x)^{-1}_N\partial_t\phi = \Delta_x\phi - f'(u)\phi + \langle f'(u)\phi\rangle\,, \quad \partial_n\phi\big|_{\partial\Omega} = 0. \tag{2.10}$$

Multiplying this equation by $\phi(t)$, integrating over Ω and noting that $\langle\phi(t)\rangle \equiv 0$ (due to the conservation law (2.3)) and that $f'(v) \ge -K$ (due to assumption $(0.7)_2$), we obtain

$$\partial_t[\varepsilon\|\phi(t)\|^2_{L^2(\Omega)} + \|\phi(t)\|^2_{H^{-1}_N(\Omega)}] + \|\nabla_x\phi(t)\|^2_{L^2(\Omega)} \le 2K\|\phi(t)\|^2_{L^2(\Omega)}. \tag{2.11}$$

Applying the interpolation inequality $\|\phi\|^2_{L^2(\Omega)} \le C\|\nabla_x\phi\|_{L^2(\Omega)}\|\phi\|_{H^{-1}_N(\Omega)}$ and Gronwall's inequality to (2.11), we have

$$\begin{aligned}\varepsilon\|\phi(t)\|^2_{L^2(\Omega)} + \|\phi(t)\|^2_{H^{-1}_N(\Omega)} + \int_t^{t+1} \|\phi(s)\|^2_{H^1(\Omega)}\,ds \\ \le C'e^{K't}(\varepsilon\|\phi(0)\|^2_{L^2(\Omega)} + \|\phi(0)\|^2_{H^{-1}_N(\Omega)}),\end{aligned} \tag{2.12}$$

where the constants C' and K' are independent of ε and u. We note that the right-hand side of (2.12) can be uniformly (with respect to ε) estimated by $\|u(0)\|_{\mathbb{D}^{m_0}_\varepsilon}$. Thus, estimate (2.12) implies the ϕ-part of estimate (2.9). In order to obtain the u-part, we rewrite equation (2.1) as follows:

$$\Delta_x u(t) - f(u(t)) + \langle f(u(t))\rangle = h(t) := \varepsilon\phi(t) + (-\Delta_x)^{-1}_N\phi(t), \quad \partial_n u(t)\big|_{\partial\Omega} = 0, \tag{2.13}$$

and interpret this equation as an elliptic problem, for every fixed $t \ge 0$. We multiply (2.13) by $\Delta_x u(t)$ and integrate over Ω. Then, since $\langle\Delta_x u(t)\rangle = 0$ and $f'(v) \ge -K$, we find, using estimate (2.12),

$$\|\Delta_x u(t)\|^2_{L^2(\Omega)} \le K\|\nabla_x u(t)\|^2_{L^2(\Omega)} + C''e^{K't}\|u_0\|^2_{\mathbb{D}^{m_0}_\varepsilon}. \tag{2.14}$$

Using now the interpolation inequality $\|\nabla_x v\|^2_{L^2(\Omega)} \le C'''\|\Delta_x v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$ and the fact that $\|u(t)\|_{L^\infty(\Omega)} \le 1$, we deduce from (2.14) the inequality

$$\|u(t)\|^2_{H^2(\Omega)} \le C_1 e^{K't}\|u_0\|^2_{\mathbb{D}^{m_0}_\varepsilon} + C_1'. \tag{2.15}$$

It then follows from (2.12), (2.13) and (2.15) that

$$\|f(u(t)) - \langle f(u(t))\rangle\|^2_{L^2(\Omega)} \le C_2 e^{K't}\|u_0\|^2_{\mathbb{D}^{m_0}_\varepsilon} + C_2' \tag{2.16}$$

and, consequently (due to (2.3) and Proposition A.2),

$$|\langle f(u(t))\rangle| \le Q_{\kappa,T}(\|u_0\|_{\mathbb{D}^{m_0}_\varepsilon}), \quad \forall t \le T, \tag{2.17}$$

where the monotonic function $Q_{\kappa,T}$ is independent of ε. Estimates (2.12) and (2.15)-(2.17) imply (2.9), hence Lemma 2.1.

The next lemma gives the uniform (with respect to ε) smoothing property for the solutions of equation (2.1).

Lemma 2.2. *Let the above assumptions hold. Then, for every $\mu > 0$, there exists a constant $C_{\kappa,\mu}$ such that*

$$\|u(t)\|^2_{\mathbb{D}^{m_0}_\varepsilon} + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^1(\Omega)}\, ds \le C_{\kappa,\mu} \tag{2.18}$$

holds uniformly with respect to $\varepsilon \in [0,1]$, $t \ge \mu$ and $u_0 \in \mathbb{D}^{m_0}_\varepsilon$.

Proof. We first note that, rewriting equation (2.1) in the form (2.13) and arguing as in the end of the proof of Lemma 2.1, we can easily obtain estimate (2.18), provided that the following estimate holds:

$$\varepsilon\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{H^{-1}_N(\Omega)} \le C_{\kappa,\mu}, \quad \forall t \ge \mu, \tag{2.19}$$

where $C_{\kappa,\mu}$ is independent of ε and u. Moreover, it is sufficient to verify (2.19) for $t = \mu$ only (the general case can be reduced to this one by an appropriate time shift). To this end, we set $w(t) := u(t) - m_0$, where $m_0 := \langle u(t)\rangle \equiv \langle u_0\rangle$, multiply equation (2.1) by $w(t)$ and integrate over $[0,\mu] \times \Omega$. Then, employing $\langle w(t)\rangle = 0$, we have

$$\int_0^\mu \|\nabla_x u(s)\|^2_{L^2(\Omega)}\, ds + \int_0^\mu \int_\Omega f(m_0 + w(s)).w(s)\, dx\, ds \\ = -\frac{1}{2}[\varepsilon\|w(s)\|^2_{L^2(\Omega)} + \|w(s)\|^2_{H^{-1}_N(\Omega)}]_0^\mu. \tag{2.20}$$

Since $\|w(t)\|_{L^\infty(\Omega)} \le 2$ and $|m_0| \le 1-\kappa$, we find, using estimates (A.3) and (A.4)

$$\int_0^\mu \|\nabla_x w(s)\|^2_{L^2(\Omega)}\, ds + \int_0^\mu \int_\Omega F(u(s))\, dx\, ds + \int_0^\mu \int_\Omega |f(u(s))|\, dx\, ds \le C_\kappa'(\mu+1), \tag{2.21}$$

where $F(v) := \int_0^v f(z)\,dz$ is the potential. Multiplying equation (2.1) by $t\partial_t u(t)$, integrating over $[0,\mu]\times\Omega$ and using estimate (2.21), we obtain

$$
\begin{aligned}
(2.22)\quad &\int_0^\mu t(\varepsilon\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{H^{-1}_N(\Omega)})\,dt + \frac{1}{2}\mu\|\nabla_x u(\mu)\|^2_{L^2(\Omega)}\\
&+\mu\int_\Omega F(u(\mu))\,dx = \frac{1}{2}\int_0^\mu \|\nabla_x u(t)\|^2_{L^2(\Omega)}\,dt + \int_0^\mu\int_\Omega F(u(t))\,dx\,dt \le C''_{\kappa,\mu}.
\end{aligned}
$$

Differentiating now equation (2.1) with respect to t, setting $\theta(t) := \partial_t u(t)$, multiplying the equation that we obtain (i.e. (2.10)) by $t^2\theta(t)$ and integrating over $[0,\mu]\times\Omega$, we have

$$
\begin{aligned}
(2.23)\quad &\mu^2(\varepsilon\|\partial_t u(\mu)\|^2_{L^2(\Omega)} + \|\partial_t u(\mu)\|^2_{H^{-1}_N(\Omega)}) + 2\int_0^\mu t^2\|\partial_t u(t)\|^2_{H^1(\Omega)}\,dt\\
&\le 2K\int_0^\mu t^2\|\partial_t u(t)\|^2_{L^2(\Omega)}dt + \int_0^\mu t(\varepsilon\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{H^{-1}_N(\Omega)})\,dt,
\end{aligned}
$$

where we have used the inequality $f'(v)\ge -K$. Using finally (2.22) and the interpolation inequality $\|v\|^2_{L^2(\Omega)} \le C\|v\|_{H^1(\Omega)}\|v\|_{H^{-1}_N(\Omega)}$ in order to estimate the right-hand side of (2.23), we obtain estimate (2.19), which finishes the proof of Lemma 2.2.

It finally remains to note that estimate (2.8) is an immediate consequence of (2.9) and (2.18) to finish the proof of Theorem 2.1.

Remark 2.1. We note that the proof of Theorem 2.1 does not make use of the space dimension. Consequently, the a priori estimate (2.8) remains valid in any space dimension.

The next corollary gives an estimate of $f(u)$ in the space $L^2([t,t+1],L^\infty(\Omega))$, which is of fundamental significance for what follows.

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold. Then the solution $u(t)$ of problem (2.1) satisfies the following estimate:*

$$
(2.24)\qquad \int_t^{t+1}\|f(u(s))\|^2_{L^\infty(\Omega)}\,ds \le C_{\kappa,\mu},\quad \forall t\ge\mu,
$$

where the constant $C_{\kappa,\mu}$ depends on κ and μ, but is independent of ε, t and $u(t)$.

Proof. We first note that it is sufficient to prove (2.24) for $t=\mu$ only (the general case can be easily reduced to this one by an appropriate time shift). To this end, we rewrite problem (2.1) as a second-order parabolic equation:

$$
(2.25)\qquad \varepsilon\partial_t u - \Delta_x u + f(u) = \tilde h := \langle f(u)\rangle - (-\Delta_x)^{-1}_N\partial_t u.
$$

Then, according to estimate (2.18) and to the embedding $H^3(\Omega)\subset L^\infty(\Omega)$, we have

$$
(2.26)\qquad \|\tilde h\|_{L^2([t,t+1],L^\infty(\Omega))} \le C'_{\kappa,\mu},\quad \forall t\ge \mu/2,
$$

with an appropriate constant $C'_{\kappa,\mu}$. We set $h_\pm(t) := \pm\|\tilde h(t)\|_{L^\infty(\Omega)}$ and consider the following two auxiliary ODEs:

$$\varepsilon y'_\pm + f(y_\pm) = h_\pm, \quad \forall t \geq \mu/2, \quad y_\pm(\mu/2) := \pm\|u(\mu/2)\|_{L^\infty(\Omega)}. \tag{2.27}$$

The solutions $y_\pm(t)$ are well defined, due to assumption (2.7). Moreover, due to the comparison principle for second-order parabolic equations, we have

$$y_-(t) \leq u(t,x) \leq y_+(t), \quad \forall t \geq \mu/2, \ \forall x \in \Omega. \tag{2.28}$$

On the other hand, it follows from Proposition A.4 and estimate (2.26) that

$$\int_\mu^{\mu+1} |f(y_\pm(t))|^2\, dt \leq C''_{\kappa,\mu}. \tag{2.29}$$

Estimate (2.24) is then an immediate consequence of (2.28) and (2.29). This finishes the proof of Corollary 2.1.

To conclude this section, we also derive L^∞-bounds on the solutions of (2.1), with $\varepsilon > 0$. We emphasize that, in contrast to all the previous estimates, these bounds are not uniform with respect to $\varepsilon \to 0^+$.

Corollary 2.2. *Let the assumptions of Theorem 2.1 hold and let ε be strictly positive. Then $u(t)$ satisfies*

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_{\varepsilon,\kappa,\mu}, \quad \forall t \geq \mu, \tag{2.30}$$

where the constant $\delta_{\varepsilon,\kappa,\mu} > 0$ depends on ε, κ and μ, but is independent of $u(t)$.

Moreover, if, in addition, we have

$$\|u(0)\|_{L^\infty(\Omega)} \leq 1 - \delta_0, \quad \textit{for some } \delta_0 > 0, \tag{2.31}$$

then

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta'_{\varepsilon,\kappa,\delta_0}(\|u(0)\|_{\mathbb{D}^{m_0}_\varepsilon}), \quad \forall t \geq 0, \tag{2.32}$$

where the constant $\delta'_{\varepsilon,\kappa,\delta_0} > 0$ depends on ε, κ, δ_0 and $\|u(0)\|_{\mathbb{D}^{m_0}_\varepsilon}$.

Proof. It follows from (2.18) that

$$\varepsilon\|\partial_t u(t)\|^2_{L^2(\Omega)} \leq C_{\kappa,\mu}, \quad \forall t \geq \mu/2,$$

with an appropriate constant $C_{\kappa,\mu}$, and, consequently (due to the embedding $H^2(\Omega) \subset C(\Omega)$ and to the fact that $\varepsilon > 0$), we can improve (2.26) as follows:

$$\|\tilde h(t)\|_{L^\infty(\Omega)} \leq C'_{\kappa,\varepsilon,\mu}, \quad \forall t \geq \mu, \tag{2.33}$$

which, however, is not uniform with respect to ε. Then, arguing as in the proof of Corollary 2.1, but using estimate (A.16) for the solutions $y_\pm(t)$ instead of estimate (A.14), we find estimate (2.30).

We now assume that (2.31) holds. Then, using (2.8) instead of (2.18), we can prove that

$$\|\tilde{h}(t)\|_{L^\infty(\Omega)} \le C''_{\kappa,\varepsilon}(\|u(0)\|_{\mathbb{D}_\varepsilon^{m_0}}), \quad \forall t \ge 0. \tag{2.34}$$

In order to obtain estimate (2.32), it is sufficient to consider the auxiliary solutions $y_\pm(t)$ of (2.27) with the initial conditions $y_\pm(0) = \pm(1-\delta_0)$ defined on the interval $[0,\infty)$, and to use estimate (A.13) in order to obtain upper and lower bounds on these solutions. This finishes the proof of Corollary 2.2.

Remark 2.2. Using more accurate estimates on the term $W(t) := (-\Delta_x)_N^{-1}\partial_t u(t)$, we can slightly improve the result of Corollary 2.1 as follows:

$$\int_t^{t+1} \|f(u(s))\|^8_{L^\infty(\Omega)}\,ds \le C_{\kappa,\mu}, \quad \forall t \ge \mu, \tag{2.35}$$

where the constant $C_{\kappa,\mu}$ is independent of ε. Indeed, it follows from estimate (2.18) and the elliptic regularity of the Laplace operator that

$$\|W\|_{L^\infty([t,t+1],H^1(\Omega))} + \|W\|_{L^2([t,t+1],H^3(\Omega))} \le C'_\mu, \quad \forall t \ge \mu, \tag{2.36}$$

where C'_μ is independent of ε. Using an appropriate interpolation inequality, it follows from this estimate that

$$\|W\|_{L^8([t,t+1],L^\infty(\Omega))} \le C''_\mu, \quad \forall t \ge \mu.$$

Using now estimate (A.17) instead of (A.14) and arguing as in the proof of Corollary 2.1, we have estimate (2.35).

§3 Existence and uniqueness of solutions. The corresponding semigroups.

In this section, we prove the existence and uniqueness of solutions for problem (2.1) in the corresponding phase spaces and obtain several auxiliary results which will be used in the next section below for the study of exponential attractors. We first give the definition of a solution of problem (2.1).

Definition 3.1. A function $u(t)$ is a solution of equation (2.1) with $u_0 \in \mathbb{D}_\varepsilon^{m_0}$ if

$$u \in L^\infty([0,T], \mathbb{D}_\varepsilon^{m_0}) \cap C([0,T], H_N^{-1}(\Omega)), \quad \forall T > 0, \tag{3.1}$$

and equation (2.1) is satisfied in the sense of distributions.

Remark 3.1. We note that, for regular solutions $u(t)$ ($u(t) \in H^4(\Omega)$ and $\|u\|_{L^\infty(\mathbb{R}_+\times\Omega)} < 1$), equations (0.2) and (2.1) are equivalent. If $u(t)$ only belongs to $\mathbb{D}_\varepsilon^{m_0}$ (as in Definition 3.1), equation (2.1) is thus viewed as a definition of solutions for equation (0.2). It is not difficult to verify that this definition of a solution is equivalent to the standard definition of the variational solution for equation (0.2) (see e.g. [27]).

The following theorem gives the uniqueness of solutions in the above class.

Theorem 3.1. *Let the function f satisfy* (0.7) *and let $u_1(t)$ and $u_2(t)$ be two solutions of* (2.1) *(in the sense of Definition 3.1). Then the following estimate is valid:*

$$
\begin{aligned}
(3.2)\quad &\varepsilon\|u_1(t)-u_2(t)\|^2_{L^2(\Omega)}+\|u_1(t)-u_2(t)\|^2_{H_N^{-1}(\Omega)}+\\
&\qquad+\int_t^{t+1}\|u_1(s)-u_2(s)\|^2_{H^1(\Omega)}\,ds\le\\
&\le Ce^{Kt}\left(\varepsilon\|u_1(0)-u_2(0)\|^2_{L^2(\Omega)}+\|u_1(0)-u_2(0)\|^2_{H_N^{-1}(\Omega)}\right)+C_\kappa e^{Kt}|m_1-m_2|,
\end{aligned}
$$

for all $t\ge 0$, where $m_i:=\langle u_i(0)\rangle$, $i=1,2$, and the constants C, K and C_κ depend only on f and κ and are independent of $\varepsilon\in[0,1]$ and $u_i(0)$, $i=1,2$.

Proof. Since $f'(v)\ge -K$, for every $v\in(-1,1)$, then

$$
(3.3)\qquad (f(v_1)-f(v_2)).(v_1-v_2)\ge -K|v_1-v_2|^2\quad \forall v_1,\ v_2\in(-1,1).
$$

We now set $v(t):=u_1(t)-u_2(t)$. This function satisfies the equation

$$
(3.4)\qquad \varepsilon\partial_t v+(-\Delta_x)^{-1}_N\partial_t v=\Delta_x v-(f(u_1)-f(u_2))+\langle f(u_1)-f(u_2)\rangle,\quad \partial_n v\big|_{\partial\Omega}=0.
$$

Multiplying this equation by $v(t)$, integrating over Ω and using (3.3) and the conservation law (2.3), we have

$$
\begin{aligned}
(3.5)\quad &\partial_t[\varepsilon\|v(t)\|^2_{L^2(\Omega)}+\|v(t)\|^2_{H_N^{-1}(\Omega)}]+\|\nabla_x v(t)\|^2_{L^2(\Omega)}\\
&\qquad\le 2K\|v\|^2_{L^2(\Omega)}+C|m_1-m_2|\cdot|\langle f(u_1(t))-f(u_2(t))\rangle|.
\end{aligned}
$$

Applying Gronwall's inequality, together with an appropriate interpolation inequality, to (3.5), and using (2.21) in order to estimate the last term in the right-hand side of (3.5), we have estimate (3.2) (analogously to (2.11) and (2.12)) and Theorem 3.1 is proven.

We are now in a position to verify the existence of a solution for equation (2.1) in the class (3.1). We start with the case $\varepsilon>0$.

Corollary 3.1. *Let the function f satisfy* (0.7) *and let ε be strictly positive. Then, for every $u_0\in\mathbb{D}_\varepsilon^{m_0}$, there exists a unique solution $u(t)$ of equation* (2.1) *which satisfies estimates* (2.8), (2.18), (2.24) *and* (2.30).

Proof. We first assume that the initial condition $u_0\in\mathbb{D}_\varepsilon^{m_0}$ is separated from the singular points -1 and 1 of the nonlinearity f, i.e.

$$
(3.6)\qquad \|u_0\|_{L^\infty(\Omega)}\le 1-\delta,\quad \delta>0.
$$

Then, due to estimate (2.32), every solution $u(t)$ of equation (2.1) is a priori also separated from the singular points of the nonlinearity f. Consequently, the existence of such a solution can be verified exactly as in the case of regular nonlinearities f (see e.g. [13], [15], [23] and [25]; see also [12], [17], [19] and [20]).

We now assume that $u_0 \in \mathbb{D}_\varepsilon^{m_0}$, but that assumption (3.6) is not satisfied. Then we approximate the initial condition $u_0 = u_0(x)$ by a sequence $u_0^n(x)$ as follows:

$$u_0^n(x) := \theta_n u_0(x), \tag{3.7}$$

where $0 < \theta_n < 1$ is an arbitrary sequence such that $\lim_{n\to\infty} \theta_n = 1$. Then, obviously, $u_0^n \in \mathbb{D}_\varepsilon^{\theta_n m_0}$ and u_0^n satisfies (3.6), with $\delta = 1 - \theta_n > 0$. Moreover, it is not difficult to verify, noting that $\varepsilon > 0$ and using Lebesgue's theorem, that

$$\|u_0^n\|_{\mathbb{D}_\varepsilon^{\theta_n m_0}} \to \|u_0\|_{\mathbb{D}_\varepsilon^{m_0}} \quad \text{as} \quad n \to \infty. \tag{3.8}$$

Indeed, since $\varepsilon > 0$, the operator $(\varepsilon + (-\Delta_x)_N^{-1})^{-1}$ is bounded in $L^2(\Omega)$ and, consequently, in order to prove (3.8), it is sufficient to verify only that $\|u_0^n\|_{H^2(\Omega)} \to \|u_0\|_{H^2(\Omega)}$ and $\|f(u_0^n)\|_{L^2(\Omega)} \to \|f(u_0)\|_{L^2(\Omega)}$ as $n \to \infty$. The first convergence is obvious, thanks to (2.7) and the convergence $\theta_n \to 1$. The second one is also obvious, due to the inequalities $|u_0^n(x)| \le |u_0(x)|$ and $\|f(u_0)\|_{L^2(\Omega)} < \infty$ and Lebesgue's theorem.

Let $u^n(t)$ be the corresponding solution of equation (2.1) (whose existence is proven above). Then it follows from (3.8) that these solutions satisfy estimates (2.8), (2.18), (2.24) and (2.30) *uniformly* with respect to n. Passing to the limit $n \to \infty$ in these estimates and in equations (2.1) for u_n and using (2.30) and (3.8), we obtain the desired solution $u(t)$. The passage to the limit $n \to \infty$ in equation (2.1) arises no difficulty since, due to (2.30), the solutions $u_n(t)$ are uniformly (with respect to n) separated from the singular points $u = \pm 1$ of the nonlinearity f if $t > 0$. This finishes the proof of Corollary 3.1.

We now prove the existence of a solution of equation (2.1) in the limit case $\varepsilon = 0$.

Corollary 3.2. *Let the function f satisfy* (0.7)*. Then the limit equation* (2.1) *with* $\varepsilon = 0$ *possesses a unique solution $u(t)$ in the class* (3.1) *which satisfies estimates* (2.8), (2.18) *and* (2.24) *(with $\varepsilon = 0$).*

Proof. Let u_0 belong to $\mathbb{D}_0^{m_0}$. Then, obviously, $u_0 \in \mathbb{D}_\varepsilon^{m_0}$, for every $\varepsilon > 0$, and

$$\|u_0\|_{\mathbb{D}_\varepsilon^{m_0}} \le C\|u_0\|_{\mathbb{D}_0^{m_0}}, \tag{3.9}$$

where C is independent of $\varepsilon \to 0$. We consider a sequence $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and the corresponding sequence $u_{\varepsilon_n}(t)$ of solutions of equation (2.1) with $\varepsilon = \varepsilon_n$, whose existence is proven in Corollary 3.1. Then, due to (3.9), estimates (2.8), (2.18) and (2.24) are satisfied *uniformly* with respect to ε_n. The desired solution $u(t)$ can be obtained by passing to the limit $\varepsilon_n \to 0$. Indeed, since the u_{ε_n} are uniformly bounded in the space $L^\infty([0,T], H^2(\Omega)) \cap H^1([0,T] \times \Omega)$, for every fixed T (due to estimate (2.8)), we can assume, without loss of generality, that this sequence converges $*$-weakly to some function u in this space. Moreover, arguing in a standard manner (see e.g. [19], [20] and [28]), we can prove that $f(u_{\varepsilon_n}(t))$ converges $*$-weakly to the limit function $f(u(t))$ in $L^\infty([0,T],L^2(\Omega))$. Passing now to the limit $\varepsilon_n \to 0$ in equations (2.1) for $u_{\varepsilon_n}(t)$, we verify that the limit function $u(t)$ solves equation (2.1) with $\varepsilon = 0$. Passing finally to the limit $\varepsilon_n \to 0$ in estimates (2.8), (2.18) and (2.24), it follows that the limit function $u(t)$ also satisfies these estimates and belongs, in particular, to the space (3.1). This finishes the proof of Corollary 3.2.

Remark 3.2. We note that, in contrast to the case $\varepsilon > 0$, we cannot prove that the solution $u(t)$ of equation (2.1) is separated from the singular points -1 and 1 of the nonlinearity f in the limit case $\varepsilon = 0$ in general (even when (3.6) is satisfied; see Section 6 for a discussion on some particular cases), which is the main difficulty in the study of global and exponential attractors. Nevertheless, it follows from estimate (2.24) that

$$\|u(t)\|_{L^\infty(\Omega)} < 1, \quad \text{for almost every } t \in \mathbb{R}_+. \tag{3.10}$$

As shown in the next section, this will be sufficient to develop the theory of exponential attractors for this class of equations. We note, however, that the term $\varepsilon\partial_t u$, $\varepsilon > 0$, in (2.1) regularizes the classical Cahn-Hilliard equation ($\varepsilon = 0$) with respect to the separation from the singular points (see also Section 6 below).

It follows from Corollaries 3.1 and 3.2 that equation (2.1) defines solving semigroups $S_t(\varepsilon)$, $\varepsilon \in [0,1]$, in the phase spaces $\mathbb{D}^{m_0}_\varepsilon$:

$$S_t(\varepsilon) : \mathbb{D}^{m_0}_\varepsilon \to \mathbb{D}^{m_0}_\varepsilon, \quad S_t(\varepsilon)u_0 := u_\varepsilon(t), \tag{3.11}$$

where $u_\varepsilon(t)$ is the solution of equation (2.1) with $u_\varepsilon(0) = u_0$. Moreover, according to Theorem 3.1, these semigroups are uniformly Lipschitz (with respect to the initial data). We consider the spaces $\mathcal{E}(\varepsilon)$ defined by the following norms:

$$\|v\|^2_{\mathcal{E}(\varepsilon)} := \varepsilon\|v\|^2_{L^2(\Omega)} + \|v\|^2_{H^{-1}_N(\Omega)} \tag{3.12}$$

(i.e., $\mathcal{E}(\varepsilon) = L^2(\Omega)$ if $\varepsilon > 0$ and $\mathcal{E}(0) = H^{-1}_N(\Omega)$). By continuity, the semigroups $S_t(\varepsilon)$ can be extended in a unique way to semigroups acting on the larger phase space $\mathbb{L}^{m_0}$:

$$\begin{aligned} &S_t(\varepsilon) : \mathbb{L}^{m_0} \to \mathbb{L}^{m_0}, \\ &\qquad \mathbb{L}^{m_0} := \left[\mathbb{D}^{m_0}_\varepsilon\right]_{\mathcal{E}(\varepsilon)} = \{u_0 \in L^\infty(\Omega),\ \|u_0\|_{L^\infty(\Omega)} \le 1,\ \langle u_0\rangle = m_0\}, \end{aligned} \tag{3.13}$$

where, for $u_0 \notin \mathbb{D}^{m_0}_\varepsilon$, we set

$$S_t(\varepsilon)u_0 := \mathcal{E}(\varepsilon)\text{--}\lim_{n\to\infty} S_t(\varepsilon)u_0^n, \tag{3.14}$$

and where $u_0^n \in \mathbb{D}^{m_0}_\varepsilon$ is an arbitrary sequence satisfying the condition $\|u_0^n - u_0\|_{\mathcal{E}(\varepsilon)} \to 0$ as $n \to \infty$. Such a sequence exists, since $\mathbb{D}^{m_0}_\varepsilon$ contains regular functions $w_0(x)$ (e.g. such that $w_0 \in C^3(\Omega)$, $\|w_0\|_{L^\infty(\Omega)} < 1$ and $\langle w_0\rangle = m_0$) and these regular functions are dense in $\mathbb{L}^{m_0}$ for the topology of $\mathcal{E}(\varepsilon)$. Moreover, $u_\varepsilon(t) := S_t(\varepsilon)u_0$ belongs to $C([0,T],\mathcal{E}(\varepsilon))$, for every $u_0 \in \mathbb{L}^{m_0}$, and, due to (2.18), we have

$$S_t(\varepsilon) : \mathbb{L}^{m_0} \to \mathbb{D}^{m_0}_\varepsilon, \quad \forall t > 0. \tag{3.15}$$

Thus, by passing to the limit $n \to \infty$, it is not difficult to verify the formula $u(t) := S_t(\varepsilon)u_0$, which allows us to define a solution of equation (2.1) (in the sense of distributions) for every initial datum $u_0 \in \mathbb{L}^{m_0}$.

To conclude this section, we derive several useful properties of the semigroups constructed above. We start with the uniform Hölder continuity in the C-norm.

Corollary 3.3. *Let the function f satisfy* (0.7) *and let $u_1(t) := S_t(\varepsilon)u_0^1$ and $u_2(t) := S_t(\varepsilon)u_0^2$ be two solutions of* (2.1), *with initial data in $\mathbb{L}^{m_0}$. Then, for every $\mu > 0$, the following estimate is valid:*

$$\|u_1(t) - u_2(t)\|_{L^\infty(\Omega)} \le C_\mu e^{Kt}\|u_1(0) - u_2(0)\|^\beta_{\mathcal{E}(\varepsilon)}, \quad \forall t \ge \mu, \tag{3.16}$$

where the constants C_μ, K and β are independent of ε, u_0^1 and u_0^2.

Proof. According to (2.18), we have

$$\|u_1(t) - u_2(t)\|_{H^2(\Omega)} \le C_{\kappa,\mu}, \quad \forall t \ge \mu. \tag{3.17}$$

Estimate (3.16) is then an immediate consequence of Theorem 3.1, estimate (3.17) and the following interpolation inequality:

$$\|v\|_{L^\infty(\Omega)} \le C\|v\|^{1-\beta}_{H^2(\Omega)}\|v\|^\beta_{H^{-1}_N(\Omega)}, \tag{3.18}$$

for an appropriate exponent $0 < \beta < 1$, and Corollary 3.3 is proven.

The next corollary gives the uniform Lipschitz continuity of the solutions with respect to t in the $H^{-1}_N(\Omega)$-norm.

Corollary 3.4. *Let the nonlinearity f satisfy* (0.7) *and let $u_\varepsilon(t) := S_t(\varepsilon)u_0$, $u_0 \in \mathbb{L}^{m_0}$, be a solution of* (2.1). *Then, for every $\mu > 0$, the following estimate holds:*

$$\|u_\varepsilon(t+s) - u_\varepsilon(t)\|_{\mathcal{E}(\varepsilon)} \le C_\mu s, \quad \forall t \ge \mu, \quad \forall s \in [0,1], \tag{3.19}$$

where the constant C_μ is independent of ε, t, u_0 and s. Moreover

$$\|u_\varepsilon(t+s) - u_\varepsilon(t)\|_{L^\infty(\Omega)} \le C'_\mu s^\beta, \quad \forall t \ge \mu, \quad \forall s \in [0,1], \tag{3.20}$$

where β is the same as in Corollary 3.3 and C'_μ is independent of ε, t, u_0 and s.

Proof. According to (2.18), we have

$$\|\partial_t u_\varepsilon(t)\|_{\mathcal{E}(\varepsilon)} \le C^1_\mu, \quad \forall t \ge \mu,$$

where C^1_μ is independent of t, ε and u_0, which immediately implies estimate (3.19). Estimate (3.20) now follows from (3.19), the interpolation inequality (3.18) and the fact that the $H^2(\Omega)$-norm of the solutions is uniformly bounded on $[\mu, +\infty)$. This finishes the proof of Corollary 3.4.

We finally give an estimate on the difference of solutions of (2.1) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, in terms of the parameter ε.

Theorem 3.2. *Let f satisfy (0.7) and the initial datum $u^0 \in \mathbb{L}^{m_0}$ be such that*

$$\|u^0\|^2_{H^1(\Omega)} + 2\int_\Omega F(u^0)\,dx < \infty. \tag{3.21}$$

Then the following estimate is valid for the solutions $u_\varepsilon(t) := S_t(\varepsilon)u^0$ and $u_0(t) := S_t(0)u^0$:

$$\|u_\varepsilon(t) - u_0(t)\|^2_{H_N^{-1}(\Omega)} \le Ce^{Kt}\left(\|u^0\|^2_{H^1(\Omega)} + 2\int_\Omega F(u^0)\,dx\right)\varepsilon^2, \tag{3.22}$$

where the constants C and K are independent of ε and u^0 and F is the potential.

Proof. Thanks to definition (3.14), we can assume, without loss of generality, that $u^0 \in \mathbb{D}_0^{m_0}$. Multiplying then equation (2.1) for $u_\varepsilon(t)$ by $\partial_t u_\varepsilon(t)$ and integrating over $[0,T]\times\Omega$, we obtain, after standard transformations

$$\|\nabla_x u_\varepsilon(T)\|^2_{L^2(\Omega)} + 2\int_\Omega F(u_\varepsilon(T))\,dx \\ + 2\int_0^T (\varepsilon\|\partial_t u_\varepsilon(t)\|^2_{L^2(\Omega)} + \|\partial_t u_\varepsilon(t)\|^2_{H_N^{-1}(\Omega)})\,dt = \|\nabla_x u_\varepsilon(0)\|^2_{L^2(\Omega)} + 2\int_\Omega F(u_\varepsilon(0))\,dx. \tag{3.23}$$

We now set $v_\varepsilon(t) := u_\varepsilon(t) - u_0(t)$. This function satisfies the equation

$$(-\Delta_x)_N^{-1}\partial_t v_\varepsilon = \Delta_x v_\varepsilon - (f(u_\varepsilon) - f(u_0)) + \langle f(u_\varepsilon) - f(u_0)\rangle + \varepsilon\partial_t u_\varepsilon(t), \\ v_\varepsilon\big|_{t=0} = 0. \tag{3.24}$$

Multiplying (3.24) by $v_\varepsilon(t)$, integrating over Ω and using inequality (3.3) and the fact that $\langle v_\varepsilon\rangle = 0$, we obtain

$$\frac{1}{2}\partial_t\|v_\varepsilon(t)\|^2_{H_N^{-1}(\Omega)} + \frac{1}{2}\|\nabla_x v_\varepsilon(t)\|^2_{L^2(\Omega)} \le K\|v_\varepsilon(t)\|^2_{L^2(\Omega)} + \varepsilon^2\|\partial_t u_\varepsilon(t)\|^2_{H_N^{-1}(\Omega)}. \tag{3.25}$$

Applying Gronwall's inequality to (3.25) and using (3.23) and a proper interpolation inequality, we finally derive (analogously to (2.11) and (2.12)) estimate (3.22). Theorem 3.2 is proven.

§4 Exponential attractors for the Cahn-Hilliard equation.

In this section, we apply the abstract scheme of Section 1 to construct a robust family of exponential attractors for the Cahn-Hilliard equation (2.1) with singular potentials as $\varepsilon \to 0$. The main result of the section is the following theorem.

Theorem 4.1. *Let the nonlinearity f satisfy assumptions* (0.7)*. Then, for every m_0 satisfying $|m_0| \leq 1 - \kappa$, $0 < \kappa < 1$, there exists a robust family of exponential attractors $\mathcal{M}_\varepsilon = \mathcal{M}_\varepsilon(m_0) \subset \mathbb{D}_\varepsilon^{m_0}$, $\varepsilon \in [0,1]$, for the semigroups* (3.13) *associated with equations* (2.1)*, which satisfies the following properties:*

1. Semi-invariance: $S_t(\varepsilon)\mathcal{M}_\varepsilon \subset \mathcal{M}_\varepsilon$, for every $t \geq 0$.

2. Uniform exponential attraction:

$$\operatorname{dist}_{\mathcal{E}(\varepsilon)} \left(S_t(\varepsilon)\mathbb{L}^{m_0}, \mathcal{M}_\varepsilon\right) \leq Ce^{-\gamma t}, \tag{4.1}$$

where the positive constants C and γ are independent of ε.

3. Finite dimensionality:

$$\mathbb{H}_\mu \left(\mathcal{M}_\varepsilon, \mathcal{E}(\varepsilon)\right) \leq M \ln \frac{1}{\mu} + M', \tag{4.2}$$

where the constants M and M' are independent of ε and $\mu > 0$.

4. Continuity as $\varepsilon \to 0$:

$$\operatorname{dist}_{sym,\mathcal{E}(\varepsilon)} \left(\mathcal{M}_\varepsilon, \mathcal{M}_0\right) \leq C\varepsilon^\tau, \tag{4.3}$$

where $C > 0$ and $0 < \tau < 1$ are independent of ε.

Proof. We apply Theorem 1.1 in order to construct the desired family $\mathcal{M}_\varepsilon$. To this end, we set

$$\mathcal{B}_\varepsilon = \mathcal{B}_0 \equiv \mathcal{B}(R) := \{u_0 \in \mathbb{L}^{m_0}, \ \ \|\nabla_x u_0\|^2_{L^2(\Omega)} + 2\int_\Omega F(u_0)\, dx \leq R\}, \tag{4.4}$$

where $R > 0$ is large enough so that, due to (2.18)

$$S_1(\varepsilon)\mathbb{L}^{m_0} \subset \mathcal{B}(R), \quad \forall \varepsilon \in [0,1]. \tag{4.5}$$

Thus, it is sufficient to construct the exponential attractors $\mathcal{M}_\varepsilon$ in the phase space $\mathcal{B}(R)$ instead of $\mathbb{L}^{m_0}$. Moreover, it follows from estimate (3.23) that

$$S_t(\varepsilon)\mathcal{B}(R) \subset \mathcal{B}(R), \quad \text{for every } \varepsilon \in [0,1] \text{ and } t \geq 0. \tag{4.6}$$

We now set $\mathcal{S}_\varepsilon := S_{1/N}(\varepsilon)$, where $N \in \mathbb{N}$ is sufficiently large and will be specified below, and verify the conditions of Theorem 1.1 for this family of maps. According to (4.6), the maps $\mathcal{S}_\varepsilon$ are well defined on $\mathcal{B}_\varepsilon := \mathcal{B}(R)$. Moreover, since the H^1-norm of u_0 is uniformly bounded on $\mathcal{B}(R)$, estimate (1.2) (with ε replaced by $\varepsilon^{1/2}$) is an immediate consequence of definition (3.12) of the norm in $\mathcal{E}(\varepsilon)$. The Lipschitz continuity (1.7) is verified in Theorem 3.1. Furthermore, if we set $\Pi_\varepsilon := \mathrm{Id}$, then estimate (1.4) is satisfied (with ε replaced by $\varepsilon^{1/2}$), due to Theorem 3.2 and estimate (1.2). Property (1.6) is also obviously satisfied for every $\delta > 0$, since the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact.

Thus, it remains to construct the sets $\mathcal{C}_\varepsilon \subset \mathcal{B}_\varepsilon$ which possess the recurrence property (1.5) and find the family of spaces $\mathcal{E}^1(\varepsilon)$ which satisfy (1.1) and for which we have estimate (1.3).

Let $u_0 \in \mathcal{B}(R)$ be an arbitrary point and let $u_\varepsilon(t) := S_t(\varepsilon)u_0$ be the corresponding trajectory. Then, according to estimate (2.24), we have

$$\int_{1/4}^{3/4} \|f(u_\varepsilon(t))\|^2_{L^\infty(\Omega)}\,dt \le C, \tag{4.7}$$

where C is independent of u_0 and ε. Consequently, there exists a time $T = T(u_0,\varepsilon) \in [1/4, 3/4]$ such that

$$\|f(u_\varepsilon(T))\|_{L^\infty(\Omega)} \le (2C)^{1/2}$$

and, consequently,

$$\|u_\varepsilon(T)\|_{L^\infty(\Omega)} \le 1 - 4\delta_f, \tag{4.8}$$

where the strictly positive constant δ_f depends only on C and f. Moreover, it follows from the Hölder continuities (3.16) and (3.20) that there exist sufficiently small constants $s > 0$ and $\delta > 0$ which are independent of ε, T and u_0 such that

$$\|S_t(\varepsilon)u_0'\|_{L^\infty(\Omega)} \le 1 - 2\delta_f,\ \ \forall t \in [T, T+s],\ \ \forall u_0' \in B(u_0,\delta,\mathcal{E}(\varepsilon)) \cap \mathcal{B}_\varepsilon. \tag{4.9}$$

Setting now

$$\mathcal{C}_\varepsilon := S_{1/4}(\varepsilon)\mathcal{B}(R) \cap \{v \in \mathbb{L}^{m_0},\ \|v\|_{L^\infty(\Omega)} \le 1 - 2\delta_f\} \tag{4.10}$$

and fixing $N \in \mathbb{N}$ large enough, we can prove that there exists $M = M(u_0,\varepsilon) \in \mathbb{N}$, $M \le N-1$, such that

$$S_{M/N}(\varepsilon)(\mathcal{B}_\varepsilon \cap B(u_0,\delta,\mathcal{E}(\varepsilon))) \subset \mathcal{C}_\varepsilon. \tag{4.11}$$

Since $u_0 \in \mathcal{B}(R)$ is arbitrary, (4.11) implies the recurrence property (1.5) for the maps $\mathcal{S}_\varepsilon := S_{1/N}(\varepsilon)$. We finally define the family of spaces $\mathcal{E}^1(\varepsilon)$ by the following norms:

$$\|u_0\|^2_{\mathcal{E}^1(\varepsilon)} := \varepsilon\|v\|^2_{H^1(\Omega)} + \|v\|^2_{L^2(\Omega)} \tag{4.12}$$

(i.e., $\mathcal{E}^1(\varepsilon) = H^1(\Omega)$ if $\varepsilon > 0$ and $\mathcal{E}^1(0) = L^2(\Omega)$). Then the entropy condition (1.1) is also, obviously, satisfied and it only remains to verify the uniform smoothing property (1.3). Let $u_0 \in \mathcal{C}_\varepsilon$ be an arbitrary point and let $u_\varepsilon(t) := S_t(\varepsilon)u_0$ be the corresponding solution. Then, since $\mathcal{C}_\varepsilon \subset S_{1/4}(\varepsilon)\mathcal{B}(R)$, we have, thanks to the Hölder continuity (3.20) and assuming that N is large enough

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le 1 - \delta_f,\quad \forall t \in [0, 1/N]. \tag{4.13}$$

Thus, in order to verify assumption (1.3), it suffices to prove the following lemma.

Lemma 4.1. *Let the above assumptions hold and let $u_\varepsilon^1(t)$ and $u_\varepsilon^2(t)$ be two solutions of* (2.1) *which satisfy the inequality*

$$\|u_\varepsilon^i(t)\|_{L^\infty(\Omega)} \le 1-\delta, \quad \forall t\in[0,1/N], \tag{4.14}$$

for $i=1,2$ and for some constant $\delta\in]0,1[$. Then the following estimate holds:

$$t\|u_\varepsilon^1(t)-u_\varepsilon^2(t)\|^2_{\mathcal{E}^1(\varepsilon)} \le C_\delta e^{Kt}\|u_\varepsilon^1(0)-u_\varepsilon^2(0)\|^2_{\mathcal{E}(\varepsilon)}, \quad \forall t\in[0,1/N], \tag{4.15}$$

where the constants C_δ and K are independent of ε.

Proof. The function $v_\varepsilon(t) := u_\varepsilon^1(t)-u_\varepsilon^2(t)$ is solution of the equation

$$\varepsilon\partial_t v_\varepsilon + (-\Delta_x)_N^{-1}\partial_t v_\varepsilon = \Delta_x v_\varepsilon - l_\varepsilon v_\varepsilon + \langle l_\varepsilon v_\varepsilon\rangle\,, \tag{4.16}$$

where $l_\varepsilon(t) := \int_0^1 f'(su_1(t)+(1-s)u_2(t))\,ds$ satisfies, in view of (4.14)

$$\|l_\varepsilon(t)\|_{L^\infty(\Omega)} \le C_\delta, \quad \forall t\in[0,1/N], \tag{4.17}$$

for some constant C_δ which is independent of ε. Multiplying equation (4.16) by $t\Delta_x v_\varepsilon(t)$, integrating over Ω and using (4.17), we find, after simple calculations

$$\frac{1}{2}\partial_t[t\|v_\varepsilon(t)\|^2_{\mathcal{E}^1(\varepsilon)}] \le C'_\delta t\|v_\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_\varepsilon(t)\|^2_{\mathcal{E}^1(\varepsilon)}, \tag{4.18}$$

where $t\le 1/N$ and C'_δ is independent of ε. Applying Gronwall's inequality to (4.18) and using (3.2) in order to estimate the right-hand side of (4.18), we then obtain estimate (4.15) and Lemma 4.1 is proven.

Thus, all the assumptions of Theorem 4.1 are satisfied for the discrete maps $\mathcal{S}_\varepsilon := S_{1/N}(\varepsilon)$ and, consequently, there exists a uniform family of discrete exponential attractors $\mathcal{M}_\varepsilon^d \subset S_{1/N}\mathcal{B}_\varepsilon \subset \mathbb{D}_\varepsilon^{m_0}$ for the maps $\mathcal{S}_\varepsilon$, which satisifies all the assertions of Theorem 4.1. The desired family of exponential attractors for the continuous semigroups $S_t(\varepsilon)$ can be now constructed by the following standard expression:

$$\mathcal{M}_\varepsilon := \cup_{t\in[0,1/N]} S_t(\varepsilon)\mathcal{M}_\varepsilon^d. \tag{4.19}$$

Since the semigroups $S_t(\varepsilon)$ are uniformly (with respect to t and u_0) Lipschitz on $\mathbb{D}_\varepsilon^{m_0}$, (4.19) defines indeed the desired family of exponential attractors (see [38] for details) and Theorem 4.1 is proven.

Remark 4.1. Since all the exponential attractors $\mathcal{M}_\varepsilon$ are uniformly bounded in $H^2(\Omega)$, it follows from the interpolation inequality (3.18) that estimates (4.1)–(4.3) remain valid (with different constants) if we replace the spaces $\mathcal{E}(\varepsilon)$ by the space $L^\infty(\Omega)$. Thus, the family $\mathcal{M}_\varepsilon$, $\varepsilon\in[0,1]$, is a robust family of exponential attractors for problems (2.1) in the space $L^\infty(\Omega)$ as well.

Remark 4.2. We recall that an exponential attractor always contains the global attractor. Consequently, Theorem 4.1 implies the existence of the finite dimensional global attractors $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon(m_0)$ for the semigroups (3.13) associated with equations (2.1) and the following uniform estimate on their fractal dimension holds:

$$\dim_F(\mathcal{A}_\varepsilon, L^\infty(\Omega)) \le C < \infty, \tag{4.20}$$

where the constant C is independent of $\varepsilon\in[0,1]$.

§5 Dependence of the attractors on the average m_0.

In this section, we study the dependence of the global attractors $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon(m_0)$ and of the exponential attractors $\mathcal{M}_\varepsilon = \mathcal{M}_\varepsilon(m_0)$ of the semigroups (3.13) associated with problems (2.1) on the average m_0 of the initial datum (defined in (2.4)). We first note that all the estimates obtained in Sections 2 and 3 depend on the parameter $\kappa > 0$ introduced in (2.4). Furthermore, all these estimates diverge as $\kappa \to 0$. Therefore, estimates (4.1)–(4.3) for the exponential attractors $\mathcal{M}_\varepsilon(m_0)$ also a priori diverge as $|m_0| \to 1$. Nevertheless, it is possible to obtain a uniform (with respect to m_0) family of exponential attractors $\mathcal{M}_\varepsilon(m_0)$ based on the following simple proposition which shows that equation (2.1) is uniformly exponentially stable if $|m_0|$ is large enough.

Proposition 5.1. *Let the nonlinearity f satisfy assumptions* (0.7) *and let, in addition, the average $m_0 := \langle u_0 \rangle$ of the initial datum u_0 satisfy the condition $|m_0| \geq M_0$, where $M_0 < 1$ is defined by formula* (A.2)*. Then there holds the following estimate on the solution $u(t) := S_t(\varepsilon)u_0$ of problem* (2.1)*:*

$$\|u(t) - m_0\|^2_{\mathcal{E}(\varepsilon)} \leq Ce^{-\alpha t}\|u_0 - m_0\|^2_{\mathcal{E}(\varepsilon)}, \tag{5.1}$$

where the positive constants C and α are independent of u_0, ε and m_0.

Proof. Multiplying equation (2.1) by $w(t) := u(t) - m_0$, integrating over Ω and employing $\langle w(t) \rangle = 0$, we have

$$\frac{1}{2}\partial_t\|w(t)\|^2_{\mathcal{E}(\varepsilon)} + \|\nabla_x w(t)\|^2_{L^2(\Omega)} + \int_\Omega (f(u(t)) - f(m_0)).(u(t) - m_0)\,dx = 0. \tag{5.2}$$

Since $|m_0| \geq M_0$, estimate (A.1) implies that the last term in (5.2) is nonnegative and, consequently

$$\frac{1}{2}\partial_t\|w(t)\|^2_{\mathcal{E}(\varepsilon)} + \|\nabla_x w(t)\|^2_{L^2(\Omega)} \leq 0. \tag{5.3}$$

Moreover, since $\langle w \rangle = 0$, then $\|w\|_{\mathcal{E}(\varepsilon)} \leq \lambda\|\nabla_x w\|_{L^2(\Omega)}$, for some positive constant λ depending only on Ω. Applying Gronwall's inequality to (5.3), we finally obtain estimate (5.1) and Proposition 5.1 is proven.

We note that, obviously, $u(t) \equiv m_0$ solves (2.1). Thus, Proposition 5.1 implies that every solution $u(t)$ of this equation converges to this spatially homogeneous equilibrium exponentially if $|m_0| \geq M_0$. Therefore

$$\mathcal{A}_\varepsilon(m_0) = \{m_0\} \ \text{ if } \ |m_0| \geq M_0. \tag{5.4}$$

Moreover, this attractor is exponential and is independent of ε. Consequently, we can redefine the family of exponential attractors $\mathcal{M}_\varepsilon(m_0)$ constructed in Theorem 4.1 as follows:

$$\bar{\mathcal{M}}_\varepsilon(m_0) := \begin{cases} \mathcal{M}_\varepsilon(m_0) & \text{if } |m_0| \leq M_0, \\ \{m_0\} & \text{if } |m_0| > M_0, \end{cases} \tag{5.5}$$

and the family of exponential attractors that we obtain is uniform with respect to m_0 (i.e. all the constants in estimates (4.1)–(4.3) are independent of $m_0 \in (-1,1)$). In order to simplify the notations, we write below $\mathcal{M}_\varepsilon(m_0)$ again instead of $\bar{\mathcal{M}}_\varepsilon(m_0)$.

Using the exponential attractors constructed above, we are now in a position to prove the existence of the universal (global) attractor $\widetilde{\mathcal{A}_\varepsilon}$ for equation (2.1), which attracts all the solutions of this equation with all possible averages $m_0 \in [-1,1]$. To this end, we first extend the action of the semigroups (3.13) to the points $u_0 \equiv \pm 1$ by the following natural formula:

$$S_t(\varepsilon)(\pm 1) := \pm 1. \tag{5.6}$$

Employing this extension, the semigroups (3.13) will act on the whole unit ball of $L^\infty(\Omega)$:

$$S_t(\varepsilon) : \mathbb{L} \to \mathbb{L}, \quad \mathbb{L} := B(0,1,L^\infty(\Omega)) = \cup_{m_0\in[-1,1]}\mathbb{L}^{m_0}. \tag{5.7}$$

The main result of this section is the following theorem which establishes the existence of the finite dimensional global attractor $\widetilde{\mathcal{A}_\varepsilon}$ for $S_t(\varepsilon)$.

Theorem 5.1. *Let the nonlinearity f satisfy assumptions* (0.7) *and let the set $\mathbb{L}$ be endowed with the topology of $\mathcal{E}(\varepsilon)$. Then the semigroup* (5.7) *possesses the global attractor $\widetilde{\mathcal{A}_\varepsilon}$ which has the following structure:*

$$\widetilde{\mathcal{A}_\varepsilon} = \cup_{m_0\in[-1,1]}\mathcal{A}_\varepsilon(m_0). \tag{5.8}$$

Moreover, its fractal dimension is finite and uniformly bounded with respect to ε:

$$\dim_F(\widetilde{\mathcal{A}_\varepsilon}, L^\infty(\Omega)) \le C, \tag{5.9}$$

where C is independent of $\varepsilon \in [0,1]$.

Proof. We first prove the existence of the global attractor $\widetilde{\mathcal{A}_\varepsilon}$ for the semigroup $S_t(\varepsilon)$ defined by (5.7). To this end, we note that, due to Proposition 5.1, it is sufficient to verify the existence of the global attractor only for the following restriction of the semigroup (5.7):

$$S_t(\varepsilon) : \mathbb{L}_{M_0} \to \mathbb{L}_{M_0}, \quad \mathbb{L}_{M_0} := \cup_{|m_0|\le M_0}\mathbb{L}^{m_0}, \tag{5.10}$$

where $M_0 < 1$ is defined in (A.2). It remains to note that the semigroup (5.10) is Hölder continuous for the $\mathcal{E}(\varepsilon)$-metric (due to estimate (3.2)) and possesses a compact absorbing set (due to estimate (2.18)). Consequently, according to the standard existence theorem (see e.g. [17], [27], [31], [41], [42] and [43]), the semigroup possesses a compact global attractor. Thus, the existence of $\widetilde{\mathcal{A}_\varepsilon}$ is verified. Description (5.8) now follows from the general fact that the global attractor is generated by all the complete bounded trajectories of the corresponding semigroup. So, it only remains to verify the finite dimensionality (5.9). To this end, we need the following lemma.

Lemma 5.1. *Let the above assumptions hold. Then, for every $m_1, m_2 \in [-M_0, M_0]$ with $m_1 \neq m_2$, the following estimate holds:*

$$\operatorname{dist}_{L^\infty(\Omega)}\left(S_t(\varepsilon)\mathbb{L}^{m_1}, \mathcal{M}_\varepsilon(m_2)\right) \le C|m_1 - m_2|^\beta, \quad \forall t \ge 1 + L \ln \frac{2}{|m_1 - m_2|}, \tag{5.11}$$

where the positive constants β, C and L are independent of t, ε and m_1, m_2.

Proof. Let $u_0 \in \mathbb{L}^{m_1}$ be an arbitrary point. Then, obviously, $v_0 := u_0 + m_2 - m_1 \in \mathbb{L}^{m_2}$ and, according to estimate (3.2), we have

$$\|u(t) - v(t)\|_{\mathcal{E}(\varepsilon)} \le Ce^{Kt}|m_1 - m_2|^{1/2}, \tag{5.12}$$

where $u(t) := S_t(\varepsilon)u_0$, $v(t) := S_t(\varepsilon)v_0$ and the constants K and C are independent of ε. On the other hand, since $\mathcal{M}_\varepsilon(m_2)$ is an exponential attractor in $\mathbb{L}^{m_2}$, then, due to (4.1), we have

$$\operatorname{dist}_{\mathcal{E}(\varepsilon)}(v(t), \mathcal{M}_\varepsilon(m_2)) \le Ce^{-\gamma t}, \quad \gamma > 0, \tag{5.13}$$

and, consequently,

$$\operatorname{dist}_{\mathcal{E}(\varepsilon)}(S_t(\varepsilon)\mathbb{L}^{m_1}, \mathcal{M}_\varepsilon(m_2)) \le C(e^{Kt}|m_1 - m_2|^{1/2} + e^{-\gamma t}). \tag{5.14}$$

Setting now $t = T := \frac{1}{2(K+\gamma)} \ln \frac{2}{|m_1 - m_2|} + 1$, we obtain

$$\operatorname{dist}_{\mathcal{E}(\varepsilon)}(S_T(\varepsilon)\mathbb{L}^{m_1}, \mathcal{M}_\varepsilon(m_2)) \le C'|m_1 - m_2|^{\beta_1}, \tag{5.15}$$

where $\beta_1 := \frac{\gamma}{2(K+\gamma)}$ and the constant C' is independent of ε and m_1, m_2. Moreover, due to estimate (2.18) and to the interpolation inequality (3.18), it follows from (5.15) that

$$\operatorname{dist}_{L^\infty(\Omega)}(S_T(\varepsilon)\mathbb{L}^{m_1}, \mathcal{M}_\varepsilon(m_2)) \le C''|m_1 - m_2|^{\beta\beta_1}, \tag{5.16}$$

where β is defined in (3.18). Finally, since $S_t(\varepsilon)\mathbb{L}^{m_1} \subset \mathbb{L}^{m_1}$ for $t \ge 0$, (5.16) implies estimate (5.11), for every $t \ge T$, which finishes the proof of Lemma 5.1.

We are now in a position to verify the finite dimensionality of the attractor $\widetilde{\mathcal{A}}_\varepsilon$ and finish the proof of Theorem 5.1. To this end, we note that, due to Proposition 5.1 and formula (5.4), it is sufficient to verify the finite dimensionality of the following set:

$$\widetilde{\mathcal{A}}_\varepsilon(M_0) := \cup_{|m_0| \le M_0} \mathcal{A}_\varepsilon(m_0). \tag{5.17}$$

Moreover, it follows from estimate (5.11) and the embedding $\mathcal{A}_\varepsilon(m_1) \subset S_t(\varepsilon)\mathbb{L}^{m_1}$ for every $t \ge 0$, that

$$\operatorname{dist}_{L^\infty(\Omega)}(\mathcal{A}_\varepsilon(m_1), \mathcal{M}_\varepsilon(m_2)) \le C|m_1 - m_2|^\beta, \tag{5.18}$$

for every $m_1, m_2 \in [-M_0, M_0]$. Now, let us fix an arbitrary $\mu > 0$ and let $V_\mu \subset [-M_0, M_0]$ be a $(\mu/2C)^{1/\beta}$-net of the interval $[-M_0, M_0]$. For every $m \in V_\mu$, we also fix a $\mu/2$-net $W_{\mu,m} \subset \mathcal{M}_\varepsilon(m)$ of the exponential attractor $\mathcal{M}_\varepsilon(m)$ for the metric of $L^\infty(\Omega)$ (which exists, thanks to Theorem 4.1 and Remark 4.1). Then, according to (5.18), the system of μ-balls centered at the points of $\cup_{m\in V_\mu} W_{\mu,m}$ covers the set (5.17). Therefore, due to estimate (4.2) and Remark 4.1, we have

$$\mathbb{H}_\mu\left(\widetilde{\mathcal{A}_\varepsilon}(M_0), L^\infty(\Omega)\right) \le \mathbb{H}_{(\mu/2C)^{1/\beta}}([-M_0, M_0], \mathbb{R}) + M \ln\frac{1}{2\mu} + M', \tag{5.19}$$

which implies the uniform (with respect to $\varepsilon \in [0,1]$) finite dimensionality of the set (5.17) and finishes the proof of Theorem 5.1.

Remark 5.1. We recall that Theorem 3.1 gives the Lipschitz continuity of the semigroup $S_t(\varepsilon)$ with respect to initial data belonging to $\mathbb{L}^{m_0}$ with the same average only, whereas for initial data with *different* averages, we only have Hölder continuity with Hölder exponent 1/2. That is the reason why we cannot apply the results of Section 1 in order to construct exponential attractors for the semigroup (5.10) and verify in this way that the global attractor is finite dimensional. Therefore, we have to use different arguments in the proof of Theorem 5.1 in order to verify this property.

Remark 5.2. Analogously to formula (5.8), we can try to construct a robust (with respect to ε) family of exponential attractors for the semigroups (5.7) in the following "naive" way:

$$\widetilde{\mathcal{M}_\varepsilon} := \cup_{m_0\in[-1,1]}\mathcal{M}_\varepsilon(m_0). \tag{5.20}$$

Then the uniform exponential attraction property provided by estimates (4.1) and (4.3), i.e., continuity as $\varepsilon \to 0$, will obviously be satisfied for this family of "exponential attractors". Unfortunately, definition (5.20) does not allow to control the fractal dimension of these attractors and, therefore, does not allow to construct exponential attractors for the semigroups (5.7). Nevertheless, there exists a slightly more sophisticated way to construct a robust (with respect to ε) family of exponential attractors $\widetilde{\mathcal{M}_\varepsilon}$ for the semigroups (5.7), based on the exponential attractors $\mathcal{M}_\varepsilon(m_0)$ constructed above, namely, by employing

$$\widetilde{\mathcal{M}_\varepsilon}' := \cup_{n=1}^\infty S_n(\varepsilon)\left\{\cup_{l=-2^{Kn}}^{2^{Kn}}\mathcal{M}_\varepsilon(\frac{l}{2^{Kn}})\right\}, \quad \widetilde{\mathcal{M}_\varepsilon} := \left[\widetilde{\mathcal{M}_\varepsilon}'\right]_{L^\infty(\Omega)}. \tag{5.21}$$

Indeed, by standard arguments, it can be verified that definition (5.21) gives a family of exponential attractors for the semigroups (5.7), which satisfies (4.1)–(4.3), uniformly with respect to ε if the constant K is large enough. We will give the details of the proof of this result in a forthcoming article.

§6 Uniform L^∞-bounds on the exponential attractors.

In this concluding section, we discuss the problem of obtaining *uniform* (with respect to ε) L^∞-bounds on the solutions of problem (2.1) as in (2.32), which allow to separate the solutions of equation (2.1) from the singular points ± 1 of the nonlinearity f. Estimates of this type are crucial for the study of equations of the type (0.1) with singular potentials, since they allow to reduce the problem to one with regular potentials, whose theory is now developed (see e.g. [12], [17], [19], [22], [26], [27] and the references therein). Unfortunately, in the three-dimensional case (even for the logarithmic nonlinearities (0.3)), we are not able to obtain such estimates for general nonlinearities f satisfying (0.7). Nevertheless, in this section, we obtain such estimates when f has a sufficiently strong singularity at $u = \pm 1$ and we also discuss the case of lower dimensions $\dim \Omega < 3$. We begin with the following result.

Theorem 6.1. *Let the nonlinearity f satisfy (0.7) together with the following additional assumption:*

$$|f'(u)| \le C(|f(u)|^2 + 1). \tag{6.1}$$

Then, for every $\mu > 0$, there exists a positive constant $\delta = \delta_{\mu,\kappa}$ which is independent of $\varepsilon \in [0, 1]$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \le 1 - \delta, \quad \forall t \ge \mu, \tag{6.2}$$

for every solution $u(t)$ of (2.1) with $| \langle u_0 \rangle | \le 1 - \kappa$.

Proof. Due to estimate (2.18), we can assume, without loss of generality, that $u_0 \in \mathbb{D}_\varepsilon^{m_0}$ and that its norm in this space is uniformly bounded. Moreover, due to estimate (2.35), we can also assume that

$$\int_t^{t+1} \|f(u(s))\|^8_{L^\infty(\Omega)}\, ds \le C_{\mu,\kappa}, \quad \forall t \ge 0. \tag{6.3}$$

Then assumption (6.1) implies that

$$\|f'(u)\|_{L^4([t,t+1],L^\infty(\Omega))} \le C'_{\mu,\kappa}, \quad \forall t \ge 0. \tag{6.4}$$

We now rewrite equation (2.10) for $\phi(t) := \partial_t u(t)$ as follows:

$$\varepsilon\partial_t\phi + (-\Delta_x)^{-1}_N \partial_t\phi - \Delta_x\phi = H_u := \langle f'(u)\phi\rangle - f'(u)\phi, \tag{6.5}$$
$$\phi\big|_{t=0} = \phi_0, \quad \partial_n\phi\big|_{\partial\Omega} = 0.$$

We then estimate the L^2-norm of the function $H_u(t)$. To this end, we note that, due to estimate (2.8), we have

$$\|\phi\|_{L^\infty([t,t+1],H^{-1}_N(\Omega))} + \|\phi\|_{L^2([t,t+1],H^1(\Omega))} \le C''_{\mu,\kappa}, \quad \forall t \ge 0, \tag{6.6}$$

and, consequently,

$$(6.7)\qquad \|\phi\|^2_{L^4([t,t+1],L^2(\Omega))} \le C\|\phi\|_{L^\infty([t,t+1],H^{-1}_N(\Omega))}\|\phi\|_{L^2([t,t+1],H^1(\Omega))} \le C'''_{\mu,\kappa}, \quad \forall t \ge 0,$$

due to a proper interpolation inequality. Estimates (6.4) and (6.7), together with Hölder's inequality, imply that

$$(6.8)\qquad \|H_u\|_{L^2([t,t+1]\times\Omega)} \le K_{\mu,\kappa}, \quad \forall t \ge 0,$$

where the constant $K_{\mu,\kappa}$ is independent of ε and t. Multiplying equation (6.5) by $t\Delta_x\phi(t)$, integrating over Ω and using estimates (6.6) and (6.8), we obtain, after some calculations

$$(6.9)\qquad t\|\partial_t u(t)\|^2_{L^2(\Omega)} \le K'_{\mu,\kappa}, \quad \forall t \ge 0,$$

where $K'_{\mu,\kappa}$ is also independent of ε and t. Arguing finally as in the proof of Corollary 2.2, but using (6.9) in order to estimate the function $\tilde{h}(t)$ (defined in (2.25)), we obtain the analogue of estimate (2.33), but with a constant $C'_{\kappa,\mu}$ which is *independent* of ε. Consequently, estimates (2.28) and (A.16) imply (6.2), which finishes the proof of Theorem 6.1.

Corollary 6.1. *Let the assumptions of Theorem 6.1 be satisfied. Then the exponential attractors $\mathcal{M}_\varepsilon = \mathcal{M}_\varepsilon(m_0)$, $\varepsilon \in [0,1]$, $|m_0| \le 1-\kappa$, constructed in Theorem 4.1 satisfy*

$$(6.10)\qquad \|\mathcal{M}_\varepsilon\|_{L^\infty(\Omega)} \le 1 - \delta_\kappa,$$

where the positive constant δ_κ is independent of $\varepsilon \in [0,1]$.

Inequality (6.10) is an immediate consequence of (6.2) and of the fact that $\mathcal{M}_\varepsilon(m_0) \subset S_{1/N}(\varepsilon)\mathbb{L}^{m_0}$.

Remark 6.1. Assumption (6.1) is satisfied, for instance, for the following class of nonlinearities:

$$(6.11)\qquad f(u) = \frac{\phi(u)}{(1-u^2)^\alpha}, \quad \text{where } \phi \in C^1([-1,1]), \quad \phi(\pm 1) \ne 0 \quad \text{and} \quad \alpha \ge 1.$$

Consequently, we have estimate (6.2) in that case. Moreover, we can slightly relax assumption (6.11) (say, up to $\alpha > \frac{3}{7}$ instead of $\alpha \ge 1$). Nevertheless, we do not know how to adapt this scheme for obtaining estimate (6.2) for the logarithmic nonlinearities (0.3) which are the most important ones from the physical point of view (see [2]).

To conclude, we briefly consider the case of lower dimensions $\dim\Omega < 3$. We start with the simplest case, $\dim\Omega = 1$.

Proposition 6.1. *Let* $\dim\Omega = 1$ *and let the nonlinearity* f *satisfy assumptions* (0.7)*. Then every solution* $u(t)$ *of equation* (2.1) *satisfies estimate* (6.2)*.*

Proof. Since in one space dimension we have the Sobolev embedding $H^1(\Omega) \subset C(\Omega)$, the uniform (with respect to ε) analogue of estimate (2.33) follows from estimate (2.18) which gives uniform bounds on $\|\partial_t u(t)\|_{H_N^{-1}(\Omega)}$. Arguing now as in the proof of Corollary 2.2, we derive estimate (6.2) and finish the proof of Proposition 6.1.

Thus, in one space dimension we have estimates (6.2) and (6.10) for all the admissible nonlinearities f, in particular, for the logarithmic nonlinearities (0.3).

We now consider the case $\dim\Omega = 2$. In that case, we do not have the embedding $H^1(\Omega) \subset C(\Omega)$ and, consequently, we are not able to obtain estimate (6.2) *for all* the nonlinearities satisfying (0.7). Nevertheless, using the embedding of $H^1(\Omega)$ into an appropriate Orlicz space, we obtain this result for a wide class of nonlinearities, which includes the logarithmic nonlinearities (0.3).

Theorem 6.2. *We assume that* $\dim\Omega = 2$ *and that the nonlinearity* f *satisfies assumptions* (0.7) *and the following additional condition:*

$$|f'(u)| \le e^{C_1|f(u)|+C_2}, \tag{6.12}$$

with some positive constants C_1 *and* C_2*. Then every solution* $u(t)$ *of equation* (2.1) *satisfies estimate* (6.2)*.*

The proof of this result is based on the following lemma.

Lemma 6.1. *We assume that* $\dim\Omega = 2$ *and that the nonlinearity* f *satisfies assumptions* (0.7)*. Then, for every* $L > 0$ *and every* $\mu > 0$*, the following estimate holds:*

$$\int_{[t,t+1]\times\Omega} e^{L|f(u(s))|}\,dx\,ds \le C_{L,\kappa,\mu}, \tag{6.13}$$

where $t \ge \mu$ *and the constant* C *is independent of* ε *and of the initial datum* u_0 *satisfying* $|\langle u_0\rangle| \le 1-\kappa$*.*

Proof. Let $u(t)$ be an arbitrary solution of (2.1). Then, due to estimates (2.18), (2.24) and (3.10), there exists a strictly positive constant $\beta = \beta_u \le \mu$ such that

$$\|u(\beta)\|_{\mathbb{D}_\varepsilon^{m_0}} \le C_{\kappa,\mu} \quad\text{and}\quad \|u(\beta)\|_{L^\infty(\Omega)} \le 1-\delta_{\kappa,\mu}, \tag{6.14}$$

where the positive constants $\delta_{\kappa,\mu}$ and $C_{\kappa,\mu}$ are independent of ε and $u(t)$. We now rewrite equation (2.1) in the form (2.25). Then, due to estimate (2.8), the function $\tilde h(t)$ in (2.25) satisfies

$$\|\tilde h(t)\|_{H^1(\Omega)} \le C'_{\kappa,\mu}, \quad \forall t \ge \beta, \tag{6.15}$$

where $C'_{\kappa,\mu}$ is independent of ε. We can also assume, without loss of generality, that

$$f'(v) \geq 0, \ \forall v \in (-1,1) \tag{6.16}$$

since otherwise it is sufficient to add the term Ku to both parts of equation (2.25), where K is sufficiently large, and consider the new function $\tilde{f}(u) := f(u) + Ku$ which now satisfies assumption (6.16).

Let $L > 0$ be an arbitrary positive number. We multiply (2.25) by $f(u(t))e^{L|f(u(t))|}$, integrate over $[\beta, \mu+1] \times \Omega$ and, after elementary transformations, have

$$\begin{aligned} \varepsilon \int_\Omega F_L(u(\mu+1))\,dx + \int_{[\beta,\mu+1]\times\Omega} |\nabla_x u(t)|^2 f'(u(t))[1 + L|f(u(t))|]e^{L|f(u(t))|}\,dx\,dt \\ + \int_{[\beta,\mu+1]\times\Omega} |f(u(t))|^2 e^{L|f(u(t))|}\,dx\,dt \\ = \varepsilon \int_\Omega F_L(u(\beta))\,dx + \int_{[\beta,\mu+1]\times\Omega} \tilde{h}(t) f(u(t)) e^{L|f(u(t))|}\,dx\,dt, \end{aligned} \tag{6.17}$$

where $F_L(v) := \int_0^v z e^{L|z|}\,dz$. Since $f'(v) \geq 0$, (6.14) and (6.17) imply the estimate

$$\begin{aligned} \int_{[\beta,\mu+1]\times\Omega} |f(u(t))|^2 e^{L|f(u(t))|}\,dx\,dt \\ \leq C_{L,\kappa,\mu} + \int_{[\beta,\mu+1]\times\Omega} |\tilde{h}(t)| \cdot |f(u(t))| e^{L|f(u(t))|}\,dx\,dt, \end{aligned} \tag{6.18}$$

where the constant $C_{L,\kappa,\mu}$ is independent of ε. In order to estimate the last term in the right-hand side of (6.18), we use the following version of Young's inequality:

$$a \cdot b \leq \Phi(a) + \Psi(b), \quad \forall a, b \geq 0, \tag{6.19}$$

where

$$\Phi(s) := e^s - s - 1, \quad \Psi(s) := (s+1)\ln(1+s) - s, \tag{6.20}$$

see e.g. [44], [45] and [46]. Applying this inequality to $a := N|\tilde{h}(t)|$ and

$$b := N^{-1}|f(u(t))|e^{L|f(u(t))|},$$

where $N = N(L)$ is sufficiently large, we obtain

$$|\tilde{h}(t)| \cdot |f(u(t))| e^{L|f(u(t))|} \leq \frac{1}{2}|f(u(t))|^2 e^{L|f(u(t))|} + e^{N|\tilde{h}(t)|} + C, \tag{6.21}$$

where C is independent of ε and $u(t)$. Thus, inserting (6.21) into the right-hand side of (6.18), we have

$$\int_{[\beta,\mu+1]\times\Omega} |f(u(t))|^2 e^{L|f(u(t))|}\,dx\,dt \le 2\int_{[\beta,\mu+1]\times\Omega} e^{N|\tilde h(t)|}\,dx\,dt + C'_{\kappa,\mu,L}. \tag{6.22}$$

We now recall that, due to the Orlicz embedding theorem (see e.g. [44] and [46]), we have the estimate

$$\int_\Omega e^{N|v(x)|}\,dx \le e^{\alpha_N(\|v\|^2_{H^1(\Omega)}+1)}, \tag{6.23}$$

for every $v \in H^1(\Omega)$, where α_N only depends on N and Ω. Estimates (6.15), (6.22) and (6.23) finally imply that

$$\int_{[\beta,\mu+1]\times\Omega} |f(u(t))|^2 e^{L|f(u(t))|}\,dx\,dt \le C''_{\kappa,\mu,L}, \tag{6.24}$$

where the constant $C''_{\kappa,\mu,L}$ is independent of $\varepsilon \in [0,1]$. Estimate (6.24) implies (6.13) for $t = \mu$, which, in turn, implies estimate (6.13) for every $t \ge \mu$ by employing an appropriate time shift. This finishes the proof of Lemma 6.1.

We are now in a position to prove Theorem 6.2. Indeed, estimates (6.12) and (6.13) imply that, for every $p \ge 1$, there exists a constant $C_{p,\kappa,\mu}$ which is independent of ε such that

$$\|f'(u)\|_{L^p([t,t+1]\times\Omega)} \le C_{p,\kappa,\mu}, \quad \forall t \ge \mu. \tag{6.25}$$

Arguing as in the proof of Theorem 6.1, but using now estimate (6.25) instead of (6.4), we can easily verify the analogues of estimates (6.8) and (6.9) and finish the proof of Theorem 6.2.

Corollary 6.2. *In the case* $\dim\Omega = 2$, *every solution* $u(t)$ *of equation* (2.1) *with a logarithmic nonlinearity* (0.3) *satisfies estimate* (6.2) *and, consequently, the exponential attractors* $\mathcal{M}_\varepsilon(m_0)$ *for this equation constructed in Theorem 4.1 satisfy estimate* (6.10).

Indeed, for $\dim\Omega = 2$, any function of the form (0.3) satisfies condition (6.12) of Theorem 6.2.

Remark 6.1. Since the nonlinearities (0.3) belong to $C^\infty(-1,1)$ and the domain Ω with $\dim\Omega = 2$ is smooth, estimate (6.2) implies the smoothing property

$$S_t(\varepsilon) : \mathbb{L}^{m_0} \to C^\infty(\Omega) \tag{6.26}$$

and, consequently, $\mathcal{M}_\varepsilon(m_0)$ is uniformly bounded (with respect to ε) in $C^\infty(\Omega)$.

Appendix. Some auxiliary estimates.

In this appendix, we formulate and prove several properties on the singular interaction function $f(u)$, which are essential in our study of the Cahn-Hilliard equation (0.2). We begin with the following proposition.

Proposition A.1. *Let the function $f \in C^1(-1,1)$ satisfy assumptions (0.7). Then:*

1. For every $m \in (-1,1)$, there exists a constant C_m, $C_m \le C$ (where C is independent of m) such that

$$[f(m+v) - f(m)].v \ge -C_m|v|^2, \quad \text{for every } v \in (-1-m, 1-m). \tag{A.1}$$

In particular, if $|m| \ge M_0$, where

$$M_0 := \max\{|w| : \exists z \in (-1,1), \quad f(z) = f(w) \quad \text{and} \quad f'(z) = 0\} < 1, \tag{A.2}$$

then $C_m = 0$.

2. Let $F(u) := \int_0^u f(v)\, dv$ be the potential. Then, for every $m \in (-1,1)$ and $w \in (-1-m, 1-m)$, the following estimate holds:

$$F(m+w) \le f(m+w).w + F(m) + C, \tag{A.3}$$

where the constant C is independent of m and w.

3. For every $m \in (-1,1)$, there exist positive constants C_m and C'_m such that

$$|f(v+m)| \le C_m f(v+m) \cdot v + C'_m, \tag{A.4}$$

for every $v \in (-1-m, 1-m)$.

Proof. Due to assumption $(0.7)_2$, we have $f'(v) \ge -C$, for every $v \in (-1,1)$, which immediately implies estimate (A.1). Let us now verify that $C_m = 0$, if $|m| \ge M_0$. We first note that $M_0 < 1$, due to assumption $(0.7)_2$. Let us assume that $m \ge M_0$ (the case $m \le -M_0$ can be treated analogously). Then the function f is monotonic ($f' \ge 0$) on the interval $[m, 1)$ and, consequently,

$$f(m+v) - f(m) \ge 0, \quad \forall v \ge 0, \quad m+v < 1. \tag{A.5}$$

On the other hand, it follows from the definition of M_0 that $f(m) \ge f(z)$ for every $z \le m$, if $m \ge M_0$, and, consequently,

$$f(m+v) - f(m) \le 0, \ \forall v \le 0, \quad v+m > -1. \tag{A.6}$$

Estimates (A.5) and (A.6) imply that (A.1) holds with $C_m = 0$.

Let us now verify estimate (A.3). To this end, we recall that, due to conditions (0.7), the function f can be split as follows:

$$f = f_0 + \phi, \tag{A.7}$$

where f_0 is monotonic ($f_0' \geq 0$) and ϕ is bounded on $(-1,1)$. We can thus assume, without loss of generality, that $f'(u) \geq 0$, $\forall u \in (-1,1)$. In that case, (A.3) is an immediate consequence of the following formula:

$$F(m+w) - F(m) - f(m+w).w = \int_0^1 sf'(m+sw)|w|^2\,ds \geq 0.$$

Let us now verify estimate (A.4). We only consider the case $m \geq 0$ (the case $m < 0$ can be treated analogously). Moreover, due to the splitting (A.7), we can also assume that f is monotonic and that $f(0) = 0$. Then we have two possibilities:

$$1.\ \ m+v \in [-1+m/2, (1+m)/2], \quad 2.\ \ m+v \notin [-1+m/2, (1+m)/2].$$

In the first case, the value $m+v$ is separated from the singular points of the nonlinearity f and, consequently,

$$|f(v+m)| \leq C_m',$$

where C_m' is independent of v. In the second case, we have $|v| > (1-m)/2$ and $v \cdot (m+v) > 0$. Noting that $v \cdot f(v+m) \geq 0$, we find the estimate

$$|f(v+m)| \leq \frac{2}{1-m} f(v+m) \cdot v,$$

which finishes the proof of estimate (A.4) and Proposition A.1 is proven.

The next proposition allows us to estimate the average of $f(v(x))$ in terms of L^1-bounds of $f(v(x)) - \langle f(v)\rangle$ and of the average of v.

Proposition A.2. *Let the function f satisfy* (0.7)*, Ω be a bounded domain in $\mathbb{R}^n$ and the functions $v \in L^\infty(\Omega)$, $\|v\|_{L^\infty(\Omega)} \leq 1$, $f(v) \in L^1(\Omega)$, and $\theta \in L^1(\Omega)$ satisfy*

$$f(v(x)) - \langle f(v)\rangle = \theta(x), \quad |\langle v\rangle| \leq 1-\delta, \quad x \in \Omega, \tag{A.8}$$

with some $\delta > 0$. Then there exist positive constants C_δ and C_δ' (depending only on δ, f and Ω) such that

$$|\langle f(v)\rangle| \leq C_\delta \|\theta\|_{L^1(\Omega)} + C_\delta'. \tag{A.9}$$

Proof. We set $m := \langle v\rangle$ and $w := v - m$. Multiplying equation (A.8) by w, integrating over Ω and employing $\|w\|_{L^\infty(\Omega)} \leq 2$, we have

$$\int_\Omega f(m+w(x)) \cdot w(x)\,dx \leq 2\|\theta\|_{L^1(\Omega)}. \tag{A.10}$$

Estimate (A.9) is then an immediate consequence of (A.4) and (A.10) and Proposition A.2 is proven.

To conclude, we obtain several estimates on the solutions of the following ODE, which are necessary to apply the comparison principle to the Cahn-Hilliard equation (see Section 2):

$$\varepsilon y' + f(y) = h, \quad y(0) = y_0, \quad |y_0| < 1. \tag{A.11}$$

We begin with the following result.

Proposition A.3. *Let the function f satisfy* (0.7) *and let us assume that $\varepsilon \geq 0$ and that*

$$|y_0| \leq 1 - \delta_0 \quad \text{and} \quad h \in L^\infty([0,T]), \tag{A.12}$$

with some positive constants δ_0 and T. Then there exists a constant $\delta = \delta(\delta_0, \|h\|_{L^\infty([0,T])}) >$ 0 which is independent of ε and T, such that

$$|y(t)| \leq 1 - \delta, \quad \forall t \in [0,T]. \tag{A.13}$$

This assertion is a consequence of $(0.7)_1$ and of the comparison principle for the solutions of first-order ODEs.

We also formulate the following version of the "smoothing" property.

Proposition A.4. *Let the nonlinearity f satisfy* (0.7) *and let, in addition, h belong to $L^2([0,T])$. Then every solution $y(t)$ of equation* (A.11) *satisfies*

$$\int_0^T t|f(y(t))|^2\,dt \leq C_T(1 + \|h\|^2_{L^2([0,T])}), \tag{A.14}$$

where the constant C_T is independent of $\varepsilon \geq 0$ and $y(0)$.

Proof. Multiplying equation (A.11) by $tf(y(t))$ and integrating over $[0,T]$, we have

$$\varepsilon T F(y(T)) + \frac{1}{2}\int_0^T t|f(y(t))|^2\,dt \leq \varepsilon \int_0^T F(y(t))\,dt + \frac{1}{2}\int_0^T t|h(t)|^2\,dt. \tag{A.15}$$

We note that, due to assumptions (0.7), we have $F(u) \geq -C$. So, it only remains to estimate the first integral in the right-hand side of (A.15). To this end, we multiply equation (A.11) by $y(t)$, integrate over $[0,T]$ and note that $|y(t)| \leq 1$ to obtain the estimate

$$\int_0^T f(y(t)).y(t)\,dt = \frac{\varepsilon}{2}(y(0)^2 - y(T)^2) + \int_0^T h(t).y(t)\,dt \leq C'_T(1 + \|h\|^2_{L^2([0,T])}),$$

where C'_T is independent of ε. Combining this estimate with (A.3) and (A.15), we derive (A.14) and Proposition A.4 is proven.

Combining Propositions A.3 and A.4, we have the following result.

Corollary A.1. *Let the nonlinearity f satisfy* (0.7) *and let h belong to $L^\infty([0,T])$. Then, for every $\mu > 0$, there exists a constant $\delta = \delta(\mu, \|h\|_{L^\infty([0,T])}) > 0$ which is independent of ε, T and $y(0)$, such that*

$$|y(t)| < 1 - \delta, \quad \forall t \in [\mu,T]. \tag{A.16}$$

Indeed, it follows from estimate (A.14) that there exist a time $T_\mu \in [\mu/2, \mu]$ and a constant $K = K(\mu, \|h\|_{L^2([0,T])})$ such that

$$|f(y(T_\mu))| \leq K$$

and, consequently, there exists $\delta_0 = \delta_0(f, K) > 0$ such that $|y(T_\mu)| \leq 1 - \delta_0$. Estimate (A.16) is now a consequence of Proposition A.3.

Remark A.1. It is not difficult to verify that, for every $y(0)$ satisfying $|y(0)| < 1$, there exists a unique solution $y(t)$ of problem (A.11), which satisfies $|y(t)| < 1$ for every $t \in \mathbb{R}$.

Remark A.2. If, under the assumptions of Proposition A.4, we have, in addition, $h \in L^p([0,T])$ for some $p > 2$, then, multiplying equation (A.11) by $f(y(t))|f(y(t))|^{p-2}$ and arguing analogously, we can prove that, for every $\mu > 0$, the following estimate holds:

$$\int_\mu^T |f(y(t))|^p \, dt \leq C_\mu (1 + \|h\|^p_{L^p([0,T])}), \tag{A.17}$$

where the constant C_μ is independent of ε and $y(t)$.

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