Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$.

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Version française abrégée :

Nous nous intéressons dans cette note à l'existence d'attracteurs exponentiels (voir le définition ci-après) pour des équations de réaction-diffusion dans $\mathbb{R}^3$ de la forme (1). La difficulté essentielle est qu'ici nous ne pouvons pas utiliser les constructions classiques (voir [1] et [4]) ; la raison étant que les espaces dans lesquels nous travaillons (définis dans la Section 1) n'ont pas de structure hilbertienne. En effet, toutes les constructions connues font appel de manière essentielle à des projecteurs orthogonaux de rang fini. Afin de contourner cette difficulté, nous donnons dans cette note (voir la Proposition 1 ci-dessous) une construction d'attracteurs exponentiels, valable dans des espaces de Banach, qui généralise celle de [4] pour des opérateurs s'écrivant comme somme d'une contraction et d'un opérateur compact. On en déduit alors l'existence d'un attracteur exponentiel pour (1).

Introduction :

Our aim in this note is to prove the existence of exponential attractors for reaction-diffusion equations in $\mathbb{R}^3$ of the form (1) below. In [5], the authors obtained the existence of the global attractor for such systems. They also proved that under proper assumptions (see Section 1 below), the global attractor has finite fractal and Hausdorff dimensions. Compared to an exponential attractor, the global attractor presents two defaults for practical purposes. Indeed, it is very sensitive to perturbations and the rate of attraction of the trajectories may be very small. An exponential attractor however, as its name indicates, attracts exponentially the trajectories and will thus be more stable. Furthermore, in some situations, the global attractor can be very simple (say, reduced to one point) and thus fails to capture interesting transcient behaviors. Again, in such situations, an exponential attractor seems to be a more suitable object. In [1], the authors proposed a construction for exponential attractors for equations in unbounded domains. However, as it is the case for the usual construction of [4], this construction is only valid in Hilbert spaces ; indeed, it makes an essential use of orthogonal projectors with finite rank. This construction will thus not apply to our problem (we shall see below that the phase space for our problem is not a Hilbert space). We propose, for maps that can be decomposed into the sum of a contraction and of a compact map, a construction that is not based on projectors and that is therefore valid in Banach spaces. As an application, we obtain the existence of an exponential attractor for (1).

1. Setting of the problem.

This note is devoted to the study of the longtime behavior of solutions of the following problem :

$$(1)\qquad \begin{cases} \partial_t u = \Delta_x u - f(u, \nabla_x u) - \lambda_0 u + g(t),\ x \in \mathbb{R}^3, \\ u\big|_{t=\tau} = u_\tau. \end{cases}$$

Here $u = (u^1, \cdots, u^k)$ is an unknown vector valued function, $f = (f_1, \cdots, f_k)$ and $g(t) = g(t,x) = (g^1(t,x), \cdots, g^k(t,x))$ are given functions, Δ_x is a Laplacian with respect to the variables $x = (x_1, x_2, x_3)$ and λ_0 is a fixed positive number.

It is assumed that the nonlinear term f satisfies the conditions

$$\begin{cases} 1.\ f \in C^1(\mathbb{R}^k \times \mathbb{R}^{3k}, \mathbb{R}^k)\ ; \\ 2.\ f(v,p).v \geq 0\ ; \\ 3.\ |f(v,p)| \leq |v| Q(|v|)(1+|p|^q),\ q<2\ ; \end{cases} \tag{2}$$

for every $v \in \mathbb{R}^k$, $p \in \mathbb{R}^{3k}$ and for some monotonous function Q. (Here and below, we denote by $u.v$ the inner product in $\mathbb{R}^k$.)

In order to introduce the phase space for our problem and to impose the assumptions on the right-hand side g, we need the following definition :

Definition 1. Let $B^R_{x_0}$ be an open R-ball in $\mathbb{R}^3$ centered at x_0 and let as usual $W^{l,p}(B^R_{x_0})$ denote the Sobolev space of functions on $B^R_{x_0}$ whose derivatives up to the order l belong to $L^p(B^R_{x_0})$ ($\|u, B^R_{x_0}\|_{l,p} \equiv \|u\|_{W^{l,p}(B^R_{x_0})}$). For every $l \geq 0$ and $1 \leq p \leq \infty$, we define the space

$$W^{l,p}_b(\mathbb{R}^3) \equiv \{u \in D'(\mathbb{R}^3) : \|u\|_{b,l,p} = \sup_{x_0 \in \mathbb{R}^3} \|u, B^1_{x_0}\|_{l,p} < \infty\}, \tag{3}$$

(roughly speaking, the space $W^{l,p}_b$ consists of functions whose derivatives up to the order l are bounded as $|x| \to \infty$) and the space

$$W^{l,p}_{b,0}(\mathbb{R}^3) \equiv \{u \in W^{l,p}_b : \lim_{|x_0| \to \infty} \|u, B^1_{x_0}\|_{l,p} = 0\}. \tag{4}$$

In other words, the functions in $W^{l,p}_{b,0}$ decay as $|x| \to \infty$.

We assume that the right-hand side $g \in C^1_b(\mathbb{R}, L^2_{b,0}(\mathbb{R}^3))$ and is quasiperiodic with respect to t with l independent frequencies, i.e. there exist a function

$$G \in C^1(\mathbb{T}^l, L^2_{b,0}(\mathbb{R}^3)), \tag{5}$$

$\mathbb{T}^l$ being the l-dimensional torus, rationally independent frequencies $\alpha = (\alpha^1, \cdots, \alpha^l)$ and the initial phase $\phi_0 = (\phi^1_0, \cdots, \phi^l_0) \in \mathbb{T}^l$ such that

$$g(t,x) = G(\phi_0 + \alpha t, x). \tag{6}$$

The phase space for problem (1) will be the space

$$\Phi = W^{2-\delta,2}_b(\mathbb{R}^3), \tag{7}$$

where $\delta > 0$ is chosen such that $\delta < \min\{\frac{1}{2}, \frac{1}{q} - \frac{1}{2}\}$ and the exponent q is the same as in (2). Consequently, the solution $u(t)$ of (1) belongs to the space Φ, for every fixed $t \geq \tau$ and initial value $u_\tau \in \Phi$. Thus, we shall consider below only *bounded* as $|x| \to \infty$ solutions of problem (1).

Remark 1. Since the exponent δ in the definition of the phase space is small enough, then one can easily verify, using the third assumption of (2) and Sobolev embedding theorems, that $f(v, \nabla_x v) \in L^2_b(\mathbb{R}^3)$ if $v \in \Phi$ and consequently equation (1) can be understood in the sense of distributions.

Remark 2. Recall that we require that $g(t) \in L^2_{b,0}(\mathbb{R}^3)$ for every $t \geq 0$. It is worth emphasizing here that $L^2(\mathbb{R}^3) \subset L^2_{b_0}(\mathbb{R}^3)$. Consequently, right-hand sides g belonging to $L^2(\mathbb{R}^3)$ are also admissible. Note also that the space $L^2_{b,0}(\mathbb{R}^3)$ is *essentially* larger than $L^2(\mathbb{R}^n)$, since *arbitrary* decay rates as $|x| \to 0$ are allowed. For example, the function

$$g(x) = \frac{1}{\ln(|x|^2 + 2)},$$

belongs to the space $L^2_{b,0}(\mathbb{R}^3)$, but evidently $g \notin L^2(\mathbb{R}^3)$.

Remark 3. It is worth emphasizing also that such decay rates of the right-hand side g as $|x| \to \infty$ ($g(t) \in L^2_{b,0}(\mathbb{R}^3)$) is *essential* to prove the finite dimensionality of the global attractor. Indeed, it is known (see for instance [5] or [7]) that even in the autonomous case $g(t) = g$, the global attractor may have infinite fractal dimension if $g \in L^2_b(\mathbb{R}^3)$ but $g \notin L^2_{b,0}(\mathbb{R}^3)$.

Theorem 1. *Let the above assumptions hold. Then, problem* (1) *has a unique solution* $u(t) \in \Phi$ *for every* $u_\tau \in \Phi$. *Moreover, the following estimate holds :*

$$\|u(t)\|_\Phi \leq Q_1(\|u_\tau\|_\Phi)e^{-\varepsilon(t-\tau)} + Q_1(\|G\|_{C^1(\mathbb{T}^l, L^2_b)}), \tag{8}$$

where $\varepsilon > 0$ *and* Q_1 *is some monotonous function depending only on the equation.*

Corollary 1. *Theorem 1 implies that the family of operators (called the process associated with the equation)*

$$U_g(t,\tau) : \Phi \to \Phi, \quad u(t) = U_g(t,\tau)u_\tau, \tag{9}$$

are well defined and are bounded as $t - \tau \to \infty$.

2. Existence of an exponential attractor.

In order to study the longtime behavior of the nonautonomous equation (1), we consider, following [3], the family of equations of type (1) obtained from the initial one by shifting along the t axis and by taking the closure in the corresponding topology. To be more precise, instead of studying the sole equation (1), we shall actually study the family of equations

$$\begin{cases} \partial_t u = \Delta_x u - f(u, \nabla_x u) - \lambda_0 u + \xi(t), \ \forall \xi \in \mathcal{H}(g), \\ u\big|_{t=\tau} = u_\tau, \end{cases} \tag{10}$$

where the hull $\mathcal{H}(g)$ can be defined in the following way :

$$\mathcal{H}(g) = \{G(\phi + \alpha t, x) : \phi \in \mathbb{T}^l\}. \tag{11}$$

Since the functions $\xi \in \mathcal{H}(g)$ can be parametrized by points ϕ of the l-dimensional torus, we shall denote by $U_\phi(t,\tau)$ the family of processes associated with (10) (instead of $U_\xi(t,\tau)$ with $\xi(t) = G(\phi + \alpha t)$).

It is known (see for instance [3]) that the family of processes $\{U_\phi(t,\tau), \phi \in \mathbb{T}^l\}$ can be extended to a semigroup $\mathbb{S}_t$ acting on the space $\Phi \times \mathbb{T}^l$ by formula

$$\mathbb{S}_t(v,\phi) \equiv (U_\phi(t,0), T_t\phi), \quad T_t\phi \equiv \phi + \alpha t. \tag{12}$$

Thus, instead of studying the longtime behavior of the single equation (1), we shall actually study the long time behavior of the trajectories of the semigroup $\mathbb{S}_t : \Phi \times \mathbb{T}^l \to \Phi \times \mathbb{T}^l$.

Recall that the set $\mathcal{A}$ is called the global attractor of the semigroup $\mathbb{S}_t$ if

1. The set $\mathcal{A}$ is a compact set of $\Phi \times \mathbb{T}^l$.

2. The set $\mathcal{A}$ is invariant by $\mathbb{S}_t$, i.e

$$\mathbb{S}_t\mathcal{A} = \mathcal{A}, \quad \text{for } t \geq 0. \tag{13}$$

3. The set $\mathcal{A}$ attracts the bounded subsets of $\Phi \times \mathbb{T}^l$ as $t \to \infty$, i.e. for every bounded $B \subset \Phi \times \mathbb{T}^l$

$$\lim_{t\to\infty} dist_{\Phi\times\mathbb{T}^l}\{\mathbb{S}_tB, \mathcal{A}\} = 0, \tag{14}$$

where $dist_V\{X, Y\} \equiv \inf_{x\in X} \sup_{y\in Y} \|x - y\|_V$, is the nonsymmetric Hausdorff distance between the sets X and Y in the space V. (See [2] and [6] for details).

Theorem 2. *Let the above assumptions hold. Then, the semigroup $\mathbb{S}_t$ defined by* (12) *possesses the global attractor $\mathcal{A}$ in the space $\Phi \times \mathbb{T}^l$. Moreover, this attractor has finite fractal dimension*

$$\dim_F(\mathcal{A}, \Phi \times \mathbb{T}^l) < \infty. \tag{15}$$

Note that although we have constructed the global attractor $\mathcal{A}$ which attracts the bounded subsets of $\Phi \times \mathbb{T}^l$, we have no information on the rate of attraction in (14). Furthermore, examples show that this rate of attraction may be arbitrarily slow. So, in order to control this rate of attraction, we shall use the concept of an exponential attractor, introduced in [4].

Recall that a set $\mathcal{M}$ is called an exponential attractor for the semigroup $\mathbb{S}_t$ on $\Phi \times \mathbb{T}^l$ if the following conditions hold :

1. The set $\mathcal{M}$ is a compact set of $\Phi \times \mathbb{T}^l$.

2. The set $\mathcal{M}$ is semi-invariant by $\mathbb{S}_t$, i.e.

$$\mathbb{S}_t\mathcal{M} \subset \mathcal{M} \quad \text{for } t \geq 0. \tag{16}$$

3. The set $\mathcal{M}$ attracts *exponentially* all bounded subsets of $\Phi \times \mathbb{T}^l$, i.e. there exists a positive constant $\mu > 0$ such that for every bounded $B \subset \Phi \times \mathbb{T}^l$

$$\text{dist}\{S_tB, \mathcal{M}\} \leq C\,(\|B\|_{\Phi\times\mathbb{T}^l})\, e^{-\mu t}. \tag{17}$$

4. The set $\mathcal{M}$ has finite fractal dimension in $\Phi \times \mathbb{T}^l$:

$$\dim_F(\mathcal{M}, \Phi \times \mathbb{T}^l) < \infty. \tag{18}$$

Remark 4. Note that since we lose the invariance (assumption (16) instead of (13)), then, contrarily to the global attractor, an exponential attractor is not necessarily unique. However, we always have

$$\mathcal{A} \subset \mathcal{M}. \tag{19}$$

Theorem 3. *Let the above assumptions hold. Then, the semigroup* $\mathbb{S}_t$ *defined by* (12) *possesses an exponential attractor* $\mathcal{M}$ *in the space* $\Phi \times \mathbb{T}^l$.

The proof of this Theorem is based on the following sufficient conditions for the existence of an exponential attractor for maps in *Banach* spaces which generalize those given in [1] and [2] that are valid in Hilbert spaces only :

Proposition 1. *Let* H *and* H_1 *be two Banach spaces such that* H_1 *is compactly embedded in* H. *Let also* X *be a bounded subset of* H. *We consider a nonlinear map*

$$L : X \to X,$$

such that L *can be decomposed into a sum of two maps*

$$L = L_0 + K, \quad L_0 : X \to H, \quad K : X \to H, \tag{20}$$

in such a way that L_0 *is a contraction, i.e.*

$$\|L_0 x_1 - L_0 x_2\|_H \le \alpha \|x_1 - x_2\|_H \quad \text{with } \alpha < 1/2, \tag{21}$$

and K *satisfies the condition*

$$\|K x_1 - K x_2\|_{H_1} \le C \|x_1 - x_2\|_H. \tag{22}$$

Then, the map $L : X \to X$ *possesses a finite dimensional exponential attractor.*

Sketch of the proof. Let us fix positive $\theta > 0$ in such a way then $2(\alpha+\theta) < 1$. Since X is bounded then there exists a ball $B(R, x_0, H)$ of radius R centered in $x_0 \in X$ in the space H which contains X. Let $E_0 = V_0 = \{x_0\}$. It follows from (22) that the H_1-ball $B(CR, Kx_0, H_1)$ covers the image $K(X)$. Let us cover now this ball by the finite number of αR balls in H (it is possible to do since the embedding $H_1 \subset H$ is compact). Moreover the minimal number of balls in this covering can be estimated in the following way

$$\begin{aligned} N_{\theta R}(B(CR, K(x_0), H_1), H) = N_{\theta R}(B(CR, 0, H_1), H) = \\ = N_{\theta/C}(B(1, K(x_0), H_1), H) \equiv N(\theta) \end{aligned} \tag{23}$$

(It is very essential for us that this number is independent of R.) Thus we have constructed the αR-covering for the set $K(X)$ It follows now from the assumption (21) that the system of balls with the same centers but with radiuses $(\alpha + \theta)R$ covers $L(X)$. But the centers of balls of this covering may be out of $L(X)$ and even out of X. To avoid this difficulty we increase twicely the radiuses and construct the new $2(\alpha + \theta)$-covering $\{B(2(\alpha + \theta)R, x_1^i, H)\}$, $i = 1, \cdots N(\theta)$, of $L(X)$ in such a way that $x_1^i \in L(X)$. Define now $V_1 = \{x_1^i, i = 1, \cdots N(\theta)\}$.

Applying now the above procedure to every ball of this new covering we obtain the $(2(\alpha + \theta))^2 R$-covering of $L^2(X)$ with the number of balls $N(\theta)^2$. Denote the set of their centers by V_2. Repeating this procedure we construct finally a sequence of sets $V_k \subset L^k(X)$ such that

$$\operatorname{dist}(L^k(X), V_k) \le R(2(\alpha + \theta))^k \quad \text{and } \#V_k \le N(\theta)^k \tag{24}$$

To obtain the invariantness we define now another sequence of sets $E_k = L(E_{k-1}) \cup V_k$ and

$$E_\infty = \cup_{k=1}^{\infty} E_k; \quad \mathcal{M} = [E_\infty]_H \tag{25}$$

where $[\cdot]_H$ means the closure in H. Let us verify that $\mathcal{M}$ is an exponential attractor for L on X. Indeed, the invarianness follows immediately from our construction. Since $V_k \subset \mathcal{M}$ and $2(\alpha+\theta) < 1$ then the exponential attracting property is a corollary of (24). Thus, it remains to estimate the dimension of $\mathcal{M}$ or which is the same the dimension of E_∞.

Note that $LX \subset X$ then

$$\cup_{k\geq n} E_k \subset L^n X \subset \cup_{v\in V_n} B(v, R(2(\theta+\alpha))^n, H)$$

Let us fix now $\varepsilon > 0$ and choose the minimal integer n such $R(2(\alpha+\theta))^n \leq \varepsilon$. Then

$$N_\varepsilon(E_\infty, H) \leq N_\varepsilon(\cup_{k\leq n} E_k) + N_\varepsilon(\cup_{k>n} E_k) \leq \sum_{k\leq n} \#E_k + \#V_{n+1} \leq C_2 N(\theta)^n$$

here we have used the fact that $\#E_k \leq C_1 N(\theta)^n$ which can be easily deduced from the recurrent formula

$$\#E_n \leq \#E_{n-1} + N(\theta)^n$$

Thus,

$$\dim_F(X, H) \leq \frac{\log_2 N(\theta)}{\log_2 \frac{1}{2(\theta+\alpha)}}$$

Proposition 1 is proved.

Remark 5. Analogous sufficient conditions are given in [7] for the existence of the global attractor.

References : [1] A. Babin and B. Nicolaenko, Exponential attractors of reaction-diffusion systems in an unbounded domain, J. Dyn. Diff. Equ. 7 (4) (1995), 567-590. [2] A. V. Babin and M. I. Vishik, Attractors of evolution equations, North-Holland, Amsterdam, 1991. [3] V. V. Chepyzhov and M. I. Vishik, Attractors of nonautonomous dynamical systems and their dimension, J. Math. Pures Appl. 73 (1994), 279-333. [4] A. Eden, C. Foias, B. Nicolaenko and R. Temam, Exponential attractors for dissipative evolution equations, Research in Applied Mathematics, Vol. 37, John-Wiley, New-York, 1994. [5] M. Efendiev and S. Zelik, The attractor for a nonlinear reaction-diffusion system in an unbounded domain, Preprint. [6] R. Temam, Infinite dimensional dynamical systems in mechanics and physics, 2d ed., Springer-Verlag, 1997. [7] S. Zelik. [8] In preparation.