FINITE-DIMENSIONALITY OF ATTRACTORS FOR DEGENERATE EQUATIONS OF ELLIPTIC-PARABOLIC TYPE

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Abstract. Our aim in this article is to study the long time behavior, in terms of finite-dimensional attractors, of degenerate triply nonlinear equations. In particular, we are interested in the case where the equation becomes elliptic in some region.

Introduction.

We are interested in this article in the study of the long time behavior (in terms of finite-dimensional attractors) of triply nonlinear parabolic equations of the form

$$\partial_t B(u) = \operatorname{div}(a(\nabla_x u)) - f(u) + g, \tag{0.1}$$

in a bounded regular domain of $\mathbb{R}^3$. Such equations occur, e.g., in the study of phase separation, and, in particular, in models of Allen-Cahn equations based on a microforce balance and an anisotropic free energy (see [Gu], [Mi] and [TC]).

The study of equations of the form (0.1) can be found in [AL], [Ba], [D1], [DG], [DS], [DV], [EMR], [ER], [EfZ2], [GM], [Mi], [R] and [S] (actually, some of these works also consider the more general case of differential inclusions).

Here, we are more particularly interested in the case where the equation is elliptic when $u \le 0$. Such a situation has been considered, e.g., in [DG], [DV] and [O]. However, there is, to the best of our knowledge, no result concerning the long time behavior of the solutions of (0.1) in that case; more precisely, our aim here is to prove the existence of finite-dimensional attractors.

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It is worth noting that, in spite of a very large number of results concerning the finite-dimensionality of attractors (see, e.g., [BV], [T] and the references therein), the validity of any finite-dimensional reduction for equations with *singularities* or *degenerations* in the leading terms (such as porous media type equations, elliptic-parabolic problems, ...) has been completely unclear for a long time. The main obstacle here is the lack of regularity (and of smoothing) near the degeneration points, which prevents from using classical methods. Furthermore, as it has recently been established, the problem is far from being just technical and the degenerations can lead to essentially new types of attractors which are not observable in "regular" equations in bounded domains. Indeed, as shown in [EfZ1], the global attractor of the simplest degenerate analogue of the real Ginzburg-Landau equation in a bounded domain Ω, namely,

$$\partial_t u = \Delta_x(u^3) + u - u^3, \quad u\big|_{\partial\Omega} = 0,$$

is infinite-dimensional (to the best of our knowledge, this is the first example of a physically relevant dissipative system in a bounded domain with an infinite-dimensional global attractor). Furthermore, the "thickness" of this attractor (in the sense of Kolmogorov's ε-entropy) is typical of Sobolev spaces embeddings and is of the order of that of compact absorbing sets. Thus, this attractor is "huge", even in comparison with the infinite-dimensional global attractors of "regular" systems in *unbounded* domains for which the typical thickness is usually of the order of spaces of analytic functions embeddings, see, e.g., [Z2] and the references therein.

Nevertheless, a satisfactory finite-dimensional reduction still seems possible under proper restrictions on the structure of the equations. Roughly speaking, these conditions should prevent the energy income near the degenerations (the equations must be exponentially stable near all the degeneration points). In particular, this condition is violated for the above degenerate Ginzburg-Landau equation, since the "linearization" near $u = 0$ reads

$$\partial_t u = u$$

and is, obviously, not stable.

The validity of the finite-dimensional reduction (in terms of global and exponential attractors) under such additional restrictions has been verified in [EfZ1] for porous media equations. Furthermore, analogous results for degenerate doubly nonlinear equations of the form

$$B(\partial_t u) = \Delta_x u - f(u) + g \tag{0.2}$$

(here, B degenerates) have recently been obtained in [EfZ2].

It is however worth emphasizing that, surprisingly, the semilinear equation (0.1) considered in this article appears to be much more complicated than the, at least formally, more difficult fully nonlinear problem (0.2). Indeed, even *classical* solutions are available for equation (0.2) with a finite number of "reasonable" degeneration points for B, see [EfZ2]. In contrast to this, only Hölder continuous solutions are to be expected for equation (0.1) with a finite number of degeneration points and, e.g., in the elliptic-parabolic case, discontinuities are even to be expected. This lack of regularity prevents from directly applying the techniques devised in [EfZ1] and [EfZ2]. In particular, the lack of information on the time derivative $\partial_t u$ in the regions where $u \leq 0$ (in the elliptic-parabolic case) is crucial here.

Nevertheless, by using some proper combination of the results of [MZ] and the so-called l-trajectories method (see [MP]), we are able to overcome the above mentioned difficulties and justify the finite-dimensional reduction under some natural assumptions on B, a and f (point or elliptic-parabolic degenerations for B, standard ellipticity and non-degeneracy assumptions on a, plus some restrictions on f yielding that there is no energy income near the degenerations, see Section 1 for details). Thus, the main result of this article is the existence (in that case) of finite-dimensional global and exponential attractors for the semigroup associated with problem (0.1), see Theorem 2.2.

This article is organized as follows. In Section 1, we give the main assumptions and prove the existence and uniqueness of solutions. Then, in Section 2, we prove the existence of the global attractor and, under some additional assumptions on B, we prove, in Section 3, the existence of an exponential attractor, which yields that the global attractor has finite fractal dimension. Finally, we give, in Section 4, some remarks and possible extensions.

§1 A priori estimates, existence and uniqueness of solutions.

We consider the following problem in a bounded smooth domain $\Omega \subset \mathbb{R}^3$:

$$(1.1) \qquad \begin{cases} \partial_t B(u) = \operatorname{div}(a(\nabla_x u)) - f(u) + g, \\ u\big|_{\partial\Omega} = 0, \quad B(u)\big|_{t=0} = b_0, \end{cases}$$

where $u = u(t,x)$ is an unknown function, B, a and f are given functions and $g \in L^\infty(\Omega)$ corresponds to given external forces.

We assume that the nonlinearity a derives from a strictly convex potential $A \in C^2(\mathbb{R}^3)$, i.e.,

$$(1.2) \qquad \begin{cases} 1. \ \ a(z) := \nabla_z A(z), \quad a(0) = 0, \\ 2. \ \ \kappa_1 \le A''(z) \le \kappa_2, \quad \kappa_1, \ \kappa_2 > 0. \end{cases}$$

We also assume that the second nonlinearity $f \in C^1(\mathbb{R})$ is dissipative,

$$(1.3) \qquad \liminf_{|z|\to\infty} \frac{f(z)}{z} \ge \alpha_0 > 0,$$

and has the following structure:

$$(1.4) \qquad f(z) = f_0(z) + \phi(B(z)),$$

where the functions f_0 and ϕ also belong to $C^1(\mathbb{R})$ and f_0 is monotone,

$$(1.5) \qquad f_0'(z) \ge 0.$$

Finally, the third nonlinearity B is assumed to be smooth enough, namely, $B \in C^1(\mathbb{R})$, and monotone, $B'(z) \ge 0$, and to satisfy one of the following classes of assumptions:

$$(\textbf{Assumptions (A)}) \qquad \begin{cases} 1. \ \ B(0) = 0, \quad B(z) \neq 0, \ z \neq 0, \\ 2. \ \ \kappa_1 |z|^p \le B'(z) \le \kappa_2 |z|^p, \quad \kappa_1, \ \kappa_2 > 0; \end{cases}$$

or

$$(\textbf{Assumptions (B)})\qquad \begin{cases} 1.\ \ B(z)=0,\ z\le 0,\quad B'(z)>0,\ z>0,\\ 2.\ \ \kappa_1|z|^p\le B'(z)\le \kappa_2|z|^p,\quad z\ge 0,\quad \kappa_1,\ \kappa_2>0; \end{cases}$$

where $p\ge 0$ is some fixed number. Thus, when assumptions **(A)** are satisfied, we have a parabolic system (1.1) with at most one degeneration point for B at $z=0$ and, when assumptions **(B)** are satisfied, (1.1) is parabolic for $u>0$ and elliptic for $u\le 0$.

As usual, in order to study the degenerate case, we approximate problem (1.1) by non-degenerate ones,

$$\begin{cases} \partial_t B(u)+\varepsilon\partial_t u=\operatorname{div}(a(\nabla_x u))-f(u)+g,\\ u\big|_{\partial\Omega}=0,\quad (B(u)+\varepsilon u)\big|_{t=0}=b_0, \end{cases}\tag{1.6}$$

where $0<\varepsilon\ll 1$ is a small parameter. Equation (1.6) is a non-degenerate second-order parabolic problem which, obviously, has a unique solution $u=u_\varepsilon\in W^{(1,2),q}([0,T]\times\Omega)$, for every $q<\infty$, see, e.g., [LSU] (if B and b_0 are smooth enough, say, B, b_0 of class C^2; the existence of solutions for less regular initial data then follows from the a priori estimates obtained below and standard approximation arguments). Our aim is now to obtain uniform with respect to ε a priori estimates on u and then obtain a solution of (1.1) by passing to the limit $\varepsilon\to 0$.

We start with a uniform dissipative L^∞-estimate for the solutions of (1.6).

Theorem 1.1. *Let the above assumptions hold and the initial datum b_0 be such that there exists $u_0\in L^\infty(\Omega)$ such that*

$$B(u_0)=b_0,\ \textit{i.e.},\ b_0\in L^\infty(\Omega),\ \textit{and, when assumptions } \textbf{(B)} \textit{ hold,}\\ \textit{we assume that, in addition, } b_0(x)\ge 0.\tag{1.7}$$

Then, the solution $u=u_\varepsilon$ of equation (1.6) *satisfies the following estimate:*

$$\|u(t)\|_{L^\infty(\Omega)}\le C(1+\|g\|_{L^\infty(\Omega)})+Q(\|b_0\|_{L^\infty(\Omega)})e^{-\alpha t},\tag{1.8}$$

where the positive constants C and α and the monotonic function Q are independent of $\varepsilon\to 0$.

Proof. As usual, the proof of estimate (1.8) is based on the comparison principle. We first derive the upper L^∞-bound on the solution u. To this end, we note that, due to the dissipativity assumption (1.3), there exists a sufficiently large constant $K>0$ such that

$$f(u)\ge 1/2\alpha_0(u-K),\ u\ge 0;\quad f(u)\le 1/2\alpha_0(u+K),\ u\le 0.\tag{1.9}$$

Let now $y=y_+(t)$ be solution of the following first-order ODE:

$$\frac{d}{dt}(B(y)+\varepsilon y)+1/2\alpha_0(y-K)=\|g\|_{L^\infty(\Omega)},\quad y(0)=\max\{K,\sup_{x\in\Omega}u(0,x)\}.\tag{1.10}$$

Then, $y(t)\ge K$ and, consequently, $y(t)$ is a supersolution for equation (1.6). The comparison principle (for the non-degenerate second-order parabolic problem (1.6)) reads

$$u(t,x)\le y(t),\quad (t,x)\in\mathbb{R}_+\times\Omega.\tag{1.11}$$

Using now the fact that, when both **(A)** and **(B)** hold, the function $B(z)$ grows monotonically as $z \to +\infty$, one can easily deduce from (1.10) that

$$y(t) \le C(1 + \|g\|_{L^\infty(\Omega)}) + Q(\|b_0\|_{L^\infty(\Omega)})e^{-\alpha t},\ t \ge 0, \tag{1.12}$$

for proper positive constants C and α and monotonic function Q which are independent of ε. This gives the upper L^∞-bound on the solution u of the form (1.8).

We now check the lower bound. Arguing analogously, we establish that the solution $y = y_-(t)$ of the following ODE:

$$\frac{d}{dt}(B(y)+\varepsilon y)+1/2\alpha_0(y+K) = -\|g\|_{L^\infty(\Omega)},\ \ y(0) = \min\{-K, \inf_{x\in\Omega} u(0,x)\}, \tag{1.13}$$

gives a subsolution of problem (1.8) if K is large enough and we have

$$u(t,x) \ge y(t),\ (t,x) \in \mathbb{R}_+ \times \Omega. \tag{1.14}$$

Then, when assumptions **(A)** hold, the situation is completely analogous to the previous case and we have the analogue of (1.12) for the solution $-y(t)$, which gives (1.8) and finishes the proof in that case.

Let us now assume that conditions **(B)** hold. In that case, we have $B(y(t)) \equiv 0$, since $y(t) \le -K < 0$. Moreover, due to (1.7), we have $b_0 \ge 0$, which, in turn, implies that $u(0) \ge 0$ and $y(0) = -K$. So, (1.13) reads

$$\varepsilon\frac{d}{dt}y(t) + 1/2\alpha_0(y(t) + K) = -\|g\|_{L^\infty(\Omega)},\ \ y(0) = -K, \tag{1.15}$$

which can be solved explicitly,

$$y(t) = -K - \|g\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{\alpha_0}{2\varepsilon}t}\right),\ t \ge 0.$$

Thus, $y(t) \ge -K - \|g\|_{L^\infty(\Omega)}$, $\forall t \ge 0$, and estimate (1.8) is also verified under assumptions **(B)**. This finishes the proof of Theorem 1.1.

Our next aim is to obtain uniform estimates on the derivatives of u. We state them in three simple Lemmata below.

Lemma 1.1. *Let the assumptions of Theorem 1.1 hold. Then, the solution u of (1.6) satisfies*

$$\int_t^{t+1} \|\nabla_x u(s)\|^2_{L^2(\Omega)}\,ds \le C(1 + \|g\|_{L^\infty(\Omega)}) + Q(\|b_0\|_{L^\infty(\Omega)})e^{-\alpha t}, \tag{1.16}$$

where the positive constants C and α and the monotonic function Q are independent of ε and t.

Proof. Multiplying equation (1.6) by u and integrating over $[t, t+1] \times \Omega$, we have

$$\begin{aligned}(\mathcal{B}(u(t+1)) - \mathcal{B}(u(t)), 1)_{L^2(\Omega)} + \varepsilon/2[\|u(t+1)\|^2_{L^2(\Omega)} - \|u(t)\|^2_{L^2(\Omega)}]+ \\ + \int_t^{t+1} (a(\nabla_x u(s)), \nabla_x u(s))_{L^2(\Omega)}\,ds = \\ = \int_t^{t+1} [(g, u(s))_{L^2(\Omega)} - (f(u(s)), u(s))_{L^2(\Omega)}]\,ds,\end{aligned} \tag{1.17}$$

where $\mathcal{B}(v) := \int_0^v B'(u)u\,du$. Using now the L^∞-estimates for u obtained in the previous theorem and the fact that $a = \nabla_u A$, for a strictly convex potential A (see assumptions (1.2)), we obtain (1.16) and finish the proof of the lemma.

The next lemma gives a gradient-like energy inequality.

Lemma 1.2. *Let the above assumptions hold. Then, the solution u of problem* (1.6) *satisfies the following estimates:*

$$\int_t^{t+1}[(B'(u(s))\partial_t u(s),\partial_t u(s))_{L^2(\Omega)}+\varepsilon/2\|\partial_t u(s)\|^2_{L^2(\Omega)}]\,ds+ \tag{1.18}$$

$$\|\nabla_x u(t)\|^2_{L^2(\Omega)}\le\frac{t+1}{t}\left(C(1+\|g\|^2_{L^\infty(\Omega)})+Q(\|b_0\|_{L^\infty(\Omega)})e^{-\alpha t}\right),\ t>0,\ \alpha>0,$$

where all the constants and the monotonic function Q are independent of ε. If, in addition, $u(0)\in W_0^{1,2}(\Omega)$, then

$$\int_0^1[(B'(u(s))\partial_t u(s),\partial_t u(s))_{L^2(\Omega)}+\varepsilon\|\partial_t u(s)\|^2_{L^2(\Omega)}]\,ds+\|\nabla_x u(t)\|^2_{L^2(\Omega)}\le \tag{1.19}$$

$$\le C\left(1+\|g\|^2_{L^\infty(\Omega)}+Q(\|b_0\|_{L^\infty(\Omega)})+\|\nabla_x u(0)\|^2_{L^2(\Omega)}\right).$$

Proof. Multiplying equation (1.6) by $(t-T)\partial_t u$ and integrating over $[T,T+2]\times\Omega$, we have

$$\int_T^{T+2}[(t-T)(B'(u(t))\partial_t u(t),\partial_t u(t))_{L^2(\Omega)}+\varepsilon/2\|\partial_t u(t)\|^2_{L^2(\Omega)}]\,dt+ \tag{1.20}$$

$$+(t-T)\|\nabla_x u(t)\|^2_{L^2(\Omega)}+(t-T)(F(u(t)),1)_{L^2(\Omega)}\le$$

$$\le C\int_T^{T+2}[\|\nabla_x u(t)\|^2_{L^2(\Omega)}+(F(u(t)),1)_{L^2(\Omega)}]\,dt+C\|g\|^2_{L^\infty(\Omega)},$$

for $t\in[T,T+2]$. This internal estimate, together with the dissipative estimates for $\|u(t)\|_{L^\infty(\Omega)}$ and $\|u\|_{L^2([T,T+1],W^{1,2}(\Omega))}$ obtained in Theorem 1.1 and Lemma 1.1, respectively, give the desired estimate (1.18). Estimate (1.19) can be proven analogously, but is much simpler, since we only need to multiply the equation by $\partial_t u$. This finishes the proof of Lemma 1.2.

In the third lemma, we state the $W^{2,2}$-regularity result.

Lemma 1.3. *Let the above assumptions hold. Then, the solution $u(t)$ of problem* (1.6) *satisfies the following estimate:*

$$\int_t^{t+1}\|u(s)\|^2_{W^{2,2}(\Omega)}\,ds\le\frac{t+1}{t}\left(C(1+\|g\|^2_{L^\infty(\Omega)})+Q(\|b_0\|_{L^\infty(\Omega)})e^{-\alpha t}\right), \tag{1.21}$$

$t>0$, where C, $\alpha>0$ and Q are independent of ε.

Proof. We rewrite (1.6) as an elliptic boundary value problem for every fixed t,

$$\operatorname{div}(a(\nabla_x u(t)))=\partial_t B(u(t))+\varepsilon\partial_t u(t)+f(u(t))-g:=H_u(t),\quad u(t)\big|_{\partial\Omega}=0. \tag{1.22}$$

Then, according to the H^2-L^2-regularity result for second-order quasilinear elliptic equations (see, e.g., [Mi]), we have

$$\|u(t)\|_{W^{2,2}(\Omega)}\le C\|H_u(t)\|_{L^2(\Omega)}, \tag{1.23}$$

where the constant C is independent of u. Using now estimates (1.8) and (1.18) to estimate the L^2-norm of H_u, we obtain (1.21) and finish the proof of the lemma.

Finally, we formulate an L^1-Lipschitz continuity result based on the Kato inequality.

Lemma 1.4. *Let the above assumptions hold and let $u_1(t)$ and $u_2(t)$ be two solutions of problem* (1.6)*. Then, the following estimates hold:*

$$
\begin{aligned}
(1.24)\quad &\|B(u_1(t))-B(u_2(t))\|_{L^1(\Omega)}+\varepsilon\|u_1(t)-u_2(t)\|_{L^1(\Omega)}\le\\
&\le Ce^{Kt}\left(\|B(u_1(0))-B(u_2(0))\|_{L^1(\Omega)}+\varepsilon\|u_1(0)-u_2(0)\|_{L^1(\Omega)}\right),
\end{aligned}
$$

where the constants C and K depend only on the L^∞-norms of u_1 and u_2.

Proof. We set $v(t)=u_1(t)-u_2(t)$. This function solves

$$
\begin{aligned}
(1.25)\quad \partial_t[B(u_1(t))-B(u_2(t))+\varepsilon v(t)]&=\operatorname{div}[a(\nabla_x u_1(t))-a(\nabla_x u_2(t))]-\\
&-[f_0(u_1(t))-f_0(u_2(t))]-[\phi(B(u_1(t))-\phi(B(u_2(t))].
\end{aligned}
$$

Multiplying this equation by $\operatorname{sgn} v(t)=\operatorname{sgn}(B(u_1(t))-B(u_2(t)))$ and using the Kato inequality (this multiplication can be easily justified in a standard way, since (1.6) is non-degenerate and the solutions u_1 and u_2 are sufficiently regular), we have

$$
\begin{aligned}
(1.26)\quad &\partial_t[\|B(u_1(t))-B(u_2(t))\|_{L^1(\Omega)}+\varepsilon\|v(t)\|_{L^1(\Omega)}]+\\
&+(f_0(u_1(t))-f_0(u_2(t)),\operatorname{sgn}(u_1(t))-\operatorname{sgn}(u_2(t)))_{L^2(\Omega)}\le\\
&\le(\phi(B(u_1(t)))-\phi(B(u_2(t))),\operatorname{sgn}(B(u_1(t))-B(u_2(t))))_{L^2(\Omega)}.
\end{aligned}
$$

Using now assumptions (1.4) and (1.5) for the functions f_0 and ϕ, together with the L^∞-bounds for u_1 and u_2, we deduce that

$$
\partial_t[\|B(u_1(t))-B(u_2(t))\|_{L^1(\Omega)}+\varepsilon\|v(t)\|_{L^1(\Omega)}]\le K\|B(u_1(t))-B(u_2(t))\|_{L^1(\Omega)},
$$

which, together with the Gronwall inequality, give (1.24) and finish the proof of the lemma.

We are now able to formulate the solvability result for the limit degenerate problem (1.1), which can be considered as the main result of this section.

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Then, for every $b_0\in L^\infty(\Omega)$, $b_0\ge 0$, problem* (1.1) *has at least one solution $u(t)$ belonging to the following class:*

$$
\begin{aligned}
(1.27)\quad &u\in L^\infty([0,T]\times\Omega),\quad B(u)\in C([0,T],L^1(\Omega)),\\
&u\in L^\infty([0,T],W_0^{1,2}(\Omega))\cap L^2([t,T],W^{2,2}(\Omega)),\\
&\partial_t R(u)\in L^2([t,T]\times\Omega),\ t>0,\quad R(v):=\int_0^v\sqrt{B'(u)}\,du.
\end{aligned}
$$

Furthermore, this solution satisfies all the estimates obtained in Lemmata 1.1–1.3 and Theorem 1.1 and can be obtained in a unique way *as the limit of the corresponding solutions u_ε of the regularized problems* (1.6) *as $\varepsilon\to 0$. Finally, for every two such solutions $u_1(t)$ and $u_2(t)$ (corresponding to different initial data b_0^1 and b_0^2), the following global L^1-Lipschitz continuity holds:*

$$
\begin{aligned}
(1.28)\quad &\|B(u_1(t))-B(u_2(t))\|_{L^1(\Omega)}\le\\
&\le Ce^{Kt}\|B(u_1(0))-B(u_2(0))\|_{L^1(\Omega)}=Ce^{Kt}\|b_0^1-b_0^2\|_{L^1(\Omega)},
\end{aligned}
$$

where the constants C and K only depend on the L^∞-norms of b_0^1 and b_0^2.

Proof. Let $u_n(t) := u_{\varepsilon_n}(t)$ be a sequence of solutions of the approximate problems (1.6) with $\varepsilon_n \to 0$ and with the same initial datum b_0. Then, due to Theorem 1.1 and Lemmata 1.1–1.3, we can assume, without loss of generality, that

$$(1.29)\quad u_n \to u \text{ weakly-}* \text{ in}$$
$$L^\infty([0,T]\times\Omega)\cap L^\infty([t,T],W_0^{1,2}(\Omega))\cap L^2([t,T],W^{2,2}(\Omega)).$$

The main problem is, however, that, when assumptions **(B)** hold, we do not control the time derivative $\partial_t u$ in the region $u \le 0$ and, consequently, we cannot directly extract the *strong* convergence $u_n \to u$ in a proper space from (1.29) (which is essential for the passage to the limit $n\to\infty$ in the nonlinear terms of equation (1.6)). In order to overcome this difficulty, we use monotonicity arguments. We first note that Lemma 1.2 allows to control the L^2-norm of the time derivative of the functions $\psi_n(t) := B(u_n(t))$ on every interval $[t,T]$. Furthermore, its x-gradient can also be easily controlled, since $\|\nabla_x u_n(t)\|_{L^2(\Omega)}$ is uniformly bounded on $[t,T]$. Thus, the sequence ψ_n is precompact in the *strong* topology of $L^2([t,T]\times\Omega)$ and, without loss of generality, we can assume, in addition, that

$$(1.30)\qquad \psi_n \to \psi \text{ strongly in } C([t,T],L^2(\Omega)).$$

Let us prove that

$$(1.31)\qquad \psi = B(u).$$

To this end, we use the standard fact that the operator $z \mapsto B(z)$ is maximal monotone in $L^2([t,T]\times\Omega)$, since $B'(z)\ge 0$ (being pedants, we should first cut off the function B for large z in order to make it well-defined as an operator in $L^2([t,T]\times\Omega)$, but, since we control the L^∞-norm of the solutions, this procedure is not essential and is omitted). Thus, in order to verify (1.31), we only need to check that

$$(1.32)\qquad (\psi - B(w), u-w)_{L^2([t,T]\times\Omega)} \ge 0, \quad \forall w\in L^2([t,T]\times\Omega),$$

see, e.g., [Li]. There remains to note that the strong convergence (1.30) allows to obtain (1.32) by a direct passage to the limit $n\to\infty$ in the following obvious inequality:

$$(1.33)\quad (B(u_n)-B(w),u_n-w)_{L^2([t,T]\times\Omega)} = (\psi_n - B(w), u_n-w)_{L^2([t,T]\times\Omega)} \ge 0.$$

Thus, (1.31) is verified and, consequently,

$$B(u_n)\to B(u) \textit{ strongly} \text{ in } C([t,T],L^2(\Omega)),$$

which, in turn, implies that

$$(1.34)\quad \partial_t B(u_n) \to \partial_t B(u) \text{ weakly in } L^2([t,T]\times\Omega),$$
$$\phi(B(u_n))\to\phi(B(u)) \text{ strongly in } C([t,T],L^2(\Omega)).$$

Moreover, arguing analogously, we have

$$(1.35)\qquad R(u_n)\to R(u) \text{ strongly in } C([t,T],L^2(\Omega)).$$

Consequently, $\partial_t R(u_n) = \sqrt{B'(u_n)}\partial_t u_n \to \partial_t R(u)$ weakly in $L^2([t,T]\times\Omega)$ and

$$(1.36)\qquad \|\partial_t R(u)\|_{L^2([t,T]\times\Omega)} \le \liminf_{n\to\infty}\|\partial_t R(u_n)\|_{L^2([t,T]\times\Omega)}.$$

In order to pass to the limit in the right-hand side of (1.6), we need the following lemma.

Lemma 1.5. *Let the above assumptions hold and let u_n and u be as above. Then,*

$$\partial_t B(u) = \partial_t R(u) \cdot \sqrt{B'(u)} \tag{1.37}$$

and, for every $t > 0$,

$$\lim_{n\to\infty} (\partial_t B(u_n), u_n)_{L^2([t,T]\times\Omega)} = (\partial_t B(u), u)_{L^2([t,T]\times\Omega)}. \tag{1.38}$$

Proof of the lemma. Since $\partial_t u_n$ is regular enough, we have

$$\partial_t B(u_n) = \partial_t R(u_n) \cdot \sqrt{B'(u_n)}. \tag{1.39}$$

We now recall that the weak convergences $\partial_t B(u_n) \to \partial_t B(u)$ and $\partial_t R(u_n) \to \partial_t R(u)$ have already been established. Thus, (1.37) will be proven provided that we check that $\sqrt{B'(u_n)} \to \sqrt{B'(u)}$ *strongly* in $L^2([t,T]\times\Omega)$. Let us first assume that assumptions **(A)** hold. Then, the inverse function $v \mapsto B^{-1}(v)$ exists and is even Hölder continuous. Consequenlty, the strong convergence of $B(u_n)$ to $B(u)$ implies the strong convergence of u_n to u and, therefore, $\sqrt{B'(u_n)}$ also converges strongly to $\sqrt{B'(u)}$, which, in turn, implies (1.37). Let now assumptions **(B)** be satisfied. Then, since $B(u) \equiv 0$ for $u \le 0$ and is strictly monotone for $u > 0$, we have a Hölder continuous partial inverse function $v \mapsto \mathcal{T}(v)$ such that $\mathcal{T}(B(u)) = u^+ := \max\{u, 0\}$. Thus, in that case, the strong convergence $B(u_n)$ to $B(u)$ only implies that u_n^+ converges strongly to u^+. Nevertheless, since now $B'(u) = B'(u^+)$, this convergence is sufficient to conclude that $\sqrt{B'(u_n)}$ converges strongly to $\sqrt{B'(u)}$ and finish the proof of equality (1.37) for both assumptions **(A)** and **(B)**.

In order to check (1.38), it is now sufficient to rewrite it in the form

$$\lim_{n\to\infty} (\partial_t R(u_n), \sqrt{B'(u_n)} \cdot u_n)_{L^2([t,T]\times\Omega)} = (\partial_t R(u), \sqrt{B'(u)} \cdot u)_{L^2([t,T]\times\Omega)}$$

and note that, analogously to the arguments given above, $\sqrt{B'(u_n)} \cdot u_n$ converges *strongly* to $\sqrt{B'(u)} \cdot u$. This finishes the proof of Lemma 1.5.

It is now not difficult to finish the passage to the limit $n \to \infty$ in equations (1.6) for u_n and verify that u solves indeed the limit degenerate problem (1.1). To this end, we use the standard fact that the quasilinear differential operator

$$\mathcal{A}(u) := -\operatorname{div}(a(\nabla_x u)) + f_0(u) \tag{1.40}$$

is maximal monotone in $L^2([t,T], W_0^{1,2}(\Omega))$ (we recall that f_0 is monotone). Then, we rewrite equation (1.6) in the form

$$\mathcal{A}(u_n) = \theta_n := g - \partial_t B(u_n) - \phi(B(u_n)). \tag{1.41}$$

According to the above convergences, we have

$$\nabla_x u_n \to \nabla_x u, \quad \theta_n \to \theta := g - \partial_t B(u) - \phi(B(u)) \text{ weakly in } L^2([t,T]\times\Omega).$$

Moreover, using (1.34) and (1.38), we see that

$$\lim_{n\to\infty} (\theta_n, u_n)_{L^2([t,T]\times\Omega)} = (\theta, u)_{L^2([t,T]\times\Omega)},$$

which, by monotonicity arguments, implies that $\mathcal{A}(u) = \theta$. Thus, the function u solves indeed the limit degenerate problem (1.1).

Passing to the limit $n \to \infty$ in the estimates of Theorem 1.1 and Lemmata 1.1–1.4, it follows that these estimates hold for the solution of the limit problem as well.

Thus, there only remains to check the uniqueness and the fact that the limit solution u is such that $B(u) \in C([0,T], L^1(\Omega))$. To this end, we take the difference between equations (1.6) for u_n and u_m, respectively, multiply the resulting equation by $\operatorname{sgn}(u_n - u_m)$, use the fact that $u_n(0) = u_m(0)$ and argue as in Lemma 1.4 to infer

$$\begin{aligned}(1.42)\quad \partial_t \|B(u_n(t)) - B(u_m(t))\|_{L^1(\Omega)} \le K \|B(u_n(t)) - B(u_m(t))\|_{L^1(\Omega)} + \\ + C(\varepsilon_n + \varepsilon_m)(\|\partial_t u_n(t)\|_{L^1(\Omega)} + \|\partial_t u_m(t)\|_{L^1(\Omega)}).\end{aligned}$$

Assume first that b_0 is chosen in such a way that, in addition, $u(0) \in W_0^{1,2}(\Omega)$. Then, according to estimates (1.18) and (1.19), we can control the derivatives in the right-hand side of (1.42) and, using the Gronwall inequality, deduce that

$$(1.43)\qquad \|B(u_n(t)) - B(u_m(t))\|_{L^1(\Omega)} \le Ce^{Kt}(\varepsilon_n + \varepsilon_m)^{1/2}.$$

Thus, $B(u_n)$ is a Cauchy sequence in $C([t,T], L^1(\Omega))$ and, consequently, $B(u) = \lim_{\varepsilon\to\infty} B(u_\varepsilon)$ belongs to $C([t,T], L^1(\Omega))$ and is determined in a unique way by the solutions of the approximate equations (1.6). In the general case, i.e., $b_0 \in L^\infty(\Omega)$, $b_0 \ge 0$, it is sufficient to approximate $u(0)$ by smooth initial data $u_n(0)$ in $L^1(\Omega)$ and pass to the limit $n \to \infty$. This yields that $B(u) \in C([0,T], L^1(\Omega))$ for general initial data as well and finishes the proof of the theorem.

Remark 1.1.
(i) It is worth emphasizing that we have proven the uniqueness of a solution u *only* in the *subclass* of (1.27) of the solutions which can be obtained by passing to the limit $\varepsilon \to 0$ in the non-degenerate approximate equations. The uniqueness of a solution in the whole class (1.27) is much more delicate, since, for degenerate equations, the validity of the Kato inequality is nontrivial and must be verified. Since, everywhere in the sequel, we will only consider the solutions of equation (1.1) which can be obtained by the above limit procedure, this uniqueness is not important for what follows and we refer the reader to [O] for a more detailed exposition.
(ii) We also mention that we only consider initial data $b_0 \in L^\infty(\Omega)$ in order to exclude from the very beginning the "pathological" singular solutions which may appear in doubly nonlinear equations with less regular initial data, see, e.g., [D2] and [EfZ2]. Moreover, it is worth noting that our assumption $B'(z) \sim z^p$ is necessary *only* near the degeneration point $z = 0$ and should not be considered as some growth assumption as $z \to \infty$, for which we only need $B'(z) \ge C > 0$.

We conclude this section by some kind of additional regularity for the time derivative $\partial_t u$ which will be crucial for our theory.

Proposition 1.1. *Let the above assumptions hold and let $u(t)$ be a solution of (1.1) as constructed in the previous theorem. Assume also that the nonlinearity B belongs to $C^2(\mathbb{R})$ and satisfies the additional condition*

$$(1.44)\qquad |B''(z)|^{4/3} \le CB'(z), \quad z \in I, \quad \forall I \subset \mathbb{R} \text{ bounded.}$$

Then, there exists a positive constant τ, $0<\tau<1$, depending only on the L^∞-norm of the initial datum b_0, such that, for every time interval $[T,T+1]$, there exists $T_0\in[T,T+1]$ (depending on the solution u) such that

$$\|\partial_t R(u(t))\|^2_{L^2(\Omega)}+\int_{T_0}^{T_0+\tau}\|\partial_t\nabla_x u(s)\|^2_{L^2(\Omega)}\,ds\le Q(\|b_0\|_{L^\infty(\Omega)}), \tag{1.45}$$

for all $t\in[T_0,T_0+\tau]$, where the monotonic function Q is also independent of the concrete choice of u.

Proof. We only give below the formal derivation of estimate (1.44), which can be justified in a standard way by considering the approximate solutions of (1.6) and passing to the limit $\varepsilon\to0$.

We first note that, according to Lemma 1.2, we have

$$\int_T^{T+1}(B'(u(s))\partial_t u(s),\partial_t u(s))_{L^2(\Omega)}\,ds\le Q(\|b_0\|_{L^\infty(\Omega)}),\quad T\ge 1/2.$$

Consequently, $(B'(u(t))\partial_t u(t),\partial_t u(t))_{L^2(\Omega)}$ is finite for almost all t and, for every time interval $[T,T+1]$, there exists at least one point $T_0=T_0(u,T)\in[T,T+1]$ such that

$$(B'(u(T_0))\partial_t u(T_0),\partial_t u(T_0))_{L^2(\Omega)}\le 2Q(\|b_0\|_{L^\infty(\Omega)}). \tag{1.46}$$

Since equation (1.1) is autonomous, then, without loss of generality, we may assume that $T_0=0$.

We now differentiate equation (1.1) with respect to t and set $v=\partial_t u$. Then, we have

$$B'(u(t))\partial_t v(t)+B''(u(t))|v(t)|^2=\operatorname{div}(a'(\nabla_x u(t))\nabla_x v(t))-\\ -f_0'(u(t))v(t)-\phi'(B(u(t))B'(u(t))v(t).$$

Multiplying this equation by v, integrating with respect to $x\in\Omega$ and using the fact that a and f are monotonic, we have

$$\partial_t(B'(u(t))v(t),v(t))_{L^2(\Omega)}+2\theta\|\nabla_x v(t)\|^2_{L^2(\Omega)}\le\\ \le C(B'(u(t))v(t),v(t))_{L^2(\Omega)}+(|B''(u(t))|,|v(t)|^3)_{L^2(\Omega)}, \tag{1.47}$$

for some positive constant θ. Let $I_u(t):=(B'(u(t))v(t),v(t))_{L^2(\Omega)}$. Then, using Theorem 1.1, assumption (1.44) and the Sobolev embedding $W^{1,2}(\Omega)\subset L^6(\Omega)$, we can estimate the last term in the right-hand side of (1.47) as follows:

$$(|B''(u)|,|v|^3)_{L^2(\Omega)}=(|B''(u)|\cdot|v|^{3/2},|v|^{3/2})_{L^2(\Omega)}\le\\ \le(|B''(u)|^{4/3},|v|^2)^{3/4}_{L^2(\Omega)}\|v\|^{3/2}_{L^6(\Omega)}\le\\ \le C(B'(u),|v|^2)^{3/4}_{L^2(\Omega)}\|\nabla_x v\|^{3/2}_{L^2(\Omega)}\le\theta\|\nabla_x v\|^2_{L^2(\Omega)}+CI_u^3,$$

where $C=C(\|b_0\|_{L^\infty(\Omega)})$. Thus, (1.47) reads

$$\partial_t I_u(t)+\theta\|\nabla_x v(t)\|^2_{L^2(\Omega)}\le C(I_u(t)+I_u(t)^3). \tag{1.48}$$

Moreover, due to (1.46) and owing to the fact that $T_0 = 0$, we have

$$0 \le I_u(0) \le 2Q(\|b_0\|_{L^\infty(\Omega)}). \tag{1.49}$$

We can note that the differential inequality (1.48) is not strong enough in order to obtain global in time estimates for $I_u(t)$. Nevertheless, it is sufficient for the required local in time ones. Indeed, due to the comparison principle, we have

$$I_u(t) \le y(t), \tag{1.50}$$

where y solves

$$y' = C(y + y^3), \quad y(0) = 2Q(\|b_0\|_{L^\infty(\Omega)}). \tag{1.51}$$

Therefore, the *local* solvability result for the ODE (1.51) gives the existence of a time interval $[0, \tau]$, with $\tau > 0$ only depending on $\|b_0\|_{L^\infty(\Omega)}$, such that

$$I_u(t) \le y(t) \le Q_1(\|b_0\|_{L^\infty(\Omega)}), \tag{1.52}$$

where Q_1 is also independent of the concrete choice of u. Integrating now (1.48) with respect to $t \in [0, \tau]$ and using (1.52), we deduce the required estimate for the integral norm of $\nabla_x v$ and finish the proof of Proposition 1.1.

Remark 1.2. Obviously, assumption (1.44) is automatically satisfied in the non-degenerate case (which corresponds to assumptions **(A)** and $p = 0$). However, in the degenerate case $p > 0$, this gives rather essential restrictions on the regularity of the function B near the degeneration point. In particular, it is not difficult to verify that (1.44) implies that $p \ge 4$ if $p > 0$. This assumption will be satisfied, e.g., if the function B is of class C^5 near the degeneration point $z = 0$.

§2 Semigroups and attractors.

In this section, we show that the semigroup associated with the degenerate equation (1.1) possesses the global attractor in an appropriate phase space and formulate the main result of the article, namely, the existence of an exponential attractor for this semigroup, which will be proven in the next section.

We first define the phase space Φ for problem (1.1) as follows:

$$\Phi := \{b_0 \in L^\infty(\Omega), \text{ and, when assumptions } \mathbf{(B)} \text{ hold, } b_0(x) \ge 0,\ x \in \Omega, \text{ also}\} \tag{2.1}$$

and we define the semigroup $S(t)$ associated with equation (1.1) by the following natural expression:

$$S(t)b_0 := B(u(t)), \text{ where } u(t) \text{ solves (1.1) with } B(u(0)) = b_0. \tag{2.2}$$

Remark 2.1. We see that, in contrast to the usual situation, the semigroup $S(t)$ does not map $u(0)$ onto $u(t)$, but $B(u(0))$ onto $B(u(t))$. This naturally reflects the fact that the solution $u(t)$ is uniquely defined by $B(u(0))$ and that the equation may become elliptic in some regions; when assumptions **(A)** hold, we can actually consider the usual framework. We also emphasize once more that, by a "solution" of equation (1.1), we always mean a solution constructed in Theorem 1.2 by the limit procedure, no matter whether or not problem (1.1) has other "pathological" solutions which are automatically dropped out of our analysis.

We now recall the definition of the global attractor for the semigroup $S(t)$ adapted to our framework.

Definition 2.1. A set $\mathcal{A} \subset \Phi$ is the global attractor for the semigroup $S(t)$ associated with the degenerate problem (1.1) if

1) it is compact in $L^1(\Omega)$ and bounded in $L^\infty(\Omega)$;

2) it is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$;

3) it attracts the images of all bounded (in the L^∞-topology) subsets of Φ in the topology of $L^1(\Omega)$, i.e., for every bounded subset $\mathcal{B}$ of Φ and every neighborhood $\mathcal{O}(\mathcal{A})$ of the set $\mathcal{A}$ in $L^1(\Omega)$, there exists $T = T(\mathcal{B}, \mathcal{A})$ such that

$$S(t)\mathcal{B} \subset \mathcal{O}(\mathcal{A}), \quad \text{for all } t \geq T. \tag{2.3}$$

Remark 2.2.
(i) According to Definition 2.1, the attractor $\mathcal{A}$ attracts the bounded subsets of $\Phi = L^\infty(\Omega)$ in the *weaker* topology of $L^1(\Omega)$ and, thus, coincides with the so-called $(L^1(\Omega), L^\infty(\Omega))$-attractor in the terminology of Babin and Vishik, see [BV]. We also note that, since the trajectories of $S(t)$ are bounded in $L^\infty(\Omega)$, the space $L^1(\Omega)$ in the formulation of the attraction property can be replaced by $L^p(\Omega)$, for every *finite* p. However, the case $p = \infty$, which coincides with the standard definition of the global attractor, is more delicate and requires estimates on the solutions of the degenerate system (1.1) in Hölder spaces which, to the best of our knowledge, are not known for the elliptic-parabolic problem when assumptions **(B)** hold.
(ii) The attraction property can also be formulated via the Hausdorff semi-distance between subsets of Φ. More precisely, let

$$\operatorname{dist}_V(X, Y) := \sup_{x \in X} \inf_{y \in Y} \|x - y\|_V \tag{2.4}$$

be the non-symmetric Hausdorff distance between X and Y in a Banach space V. Then, the attraction property reads: for every bounded subset $\mathcal{B} \subset \Phi$,

$$\lim_{t \to \infty} \operatorname{dist}_{L^1(\Omega)}(S(t)\mathcal{B}, \mathcal{A}) = 0. \tag{2.5}$$

The next theorem gives the existence of the above global attractor for the semigroup $S(t)$ associated with the degenerate problem (1.1).

Theorem 2.1. *Let the assumptions of Theorem 1.2 hold. Then, the semigroup $S(t)$ defined by* (2.2) *possesses the global attractor $\mathcal{A}$ in the sense of Definition 2.1 which is bounded in $L^\infty(\Omega) \cap W^{1,2}(\Omega)$ and possesses the following standard description:*

$$\mathcal{A} = B(\mathcal{K}\big|_{t=0}), \tag{2.6}$$

where $\mathcal{K} \subset L^\infty(\mathbb{R} \times \Omega)$ is the set of all solutions of (1.1) *which are defined for all $t \in \mathbb{R}$ and are globally bounded.*

Proof. According to standard results on the existence of the global attractor (see, e.g, [BV] and [T]), we need to check that

1) the semigroup $S(t)$ is continuous in the L^1-topology on every bounded subset of Φ;

2) the semigroup $S(t)$ possesses a bounded in $L^\infty(\Omega)$ and compact in $L^1(\Omega)$ absorbing set.

The first assumption is an immediate corollary of the global Lipschitz continuity of the semigroup $S(t)$, see estimate (1.28). Moreover, it follows from estimates (1.8) and (1.18) that the R-ball in the space $L^\infty(\Omega)\cap W^{1,2}(\Omega)$ is an absorbing set for the semigroup $S(t)$ if R is large enough. Since this ball is, obviously, compact in the topology of $L^1(\Omega)$, the existence of the global attractor $\mathcal{A}$ follows, see [BV] and [T]. Its boundedness is now a consequence of the fact that the global attractor is contained in any absorbing set and description (2.6) follows from the standard description of the global attractor via bounded complete trajectories of the associated semigroup, see [BV]. This finishes the proof of Theorem 2.1.

Our next task is to establish the existence of an *exponential* attractor for the semigroup $S(t)$ associated with equation (1.1), which implies, in particular, the finite-dimensionality of the global attractor constructed in the previous theorem. We first give the definition of an exponential attractor adapted to our framework.

Definition 2.2. A set $\mathcal{M}\subset\Phi$ is an exponential attractor for the semigroup $S(t)$ associated with problem (1.1) if the following conditions are satisfied:

1) it is bounded in Φ and compact in $L^1(\Omega)$;

2) it is *semi*-invariant, $S(t)\mathcal{M}\subset\mathcal{M},\ \forall t\ge 0$;

3) it has finite fractal dimension in $L^1(\Omega)$,

$$\dim_f(\mathcal{M},L^1(\Omega))\le C<\infty;$$

4) it attracts *exponentially* the images of all bounded subsets of Φ, i.e., there exists a positive constant α and a monotonic function Q such that, for every bounded subset $\mathcal{B}$ of the phase space Φ, there holds

$$\operatorname{dist}_{L^1(\Omega)}(S(t)\mathcal{B},\mathcal{M})\le Q(\|\mathcal{B}\|_{L^\infty(\Omega)})e^{-\alpha t}, \tag{2.7}$$

for all $t\ge 0$.

The following theorem can be considered as the main result of this article.

Theorem 2.2. *Let the assumptions of Theorem 1.1 hold and let, in addition, the nonlinearity B belong to $C^2(\mathbb{R})$ and satisfy assumption* (1.44)*. Then, the semigroup $S(t)$ associated with the degenerate problem* (1.1) *possesses a finite-dimensional exponential attractor $\mathcal{M}$ in the sense of Definition 2.2 which is bounded in $L^\infty(\Omega)\cap W^{1,2}(\Omega)$.*

The proof of this theorem will be completed in the next section. In the remaining of this section, we formulate an abstract result on the existence of an exponential attractor which is close to that given in [MZ] (see also [EfMZ]) and is the main technical tool to prove Theorem 2.2.

Proposition 2.1. *Let $\mathcal{H}_1$ and $\mathcal{H}$, $\mathcal{H}_1\subset\mathcal{H}$, be two Banach spaces such that the embedding $\mathcal{H}_1\subset\mathcal{H}$ is compact and let $\mathcal{C}$ be a closed bounded subset of $\mathcal{H}$. Assume also that there exists a map $\mathcal{S}:\mathcal{C}\to\mathcal{C}$ which satisfies the following properties:*

1) it is globally Lipschitz continuous on $\mathcal{C}$, i.e., for every $c_1,\ c_2\in\mathcal{C}$, there holds

$$\|\mathcal{S}c_1-\mathcal{S}c_2\|_{\mathcal{H}}\le L\|c_1-c_2\|_{\mathcal{H}}, \tag{2.8}$$

where the Lipschitz constant L is independent of the choice of c_1 and c_2 belonging to $\mathcal{C}$;

2) there exists an integer N_0 such that, for every $c \in \mathcal{C}$, there exists $n = n(c) \in \{0, \cdots, N_0 - 1\}$ such that, for every $c_1 \in \mathcal{C}$, there holds

$$\|\mathcal{S}c_1 - \mathcal{S}c_2\|_{\mathcal{H}_1} \le K\|c_1 - c_2\|_{\mathcal{H}}, \quad c_2 := \mathcal{S}(n)c, \tag{2.9}$$

where the discrete semigroup generated on $\mathcal{C}$ by the iterations of $\mathcal{S}$ is denoted by $\{\mathcal{S}(l),\ l \in \mathbb{N}\}$ and the constant K is independent of c and c_1.

Then, the discrete semigroup $S(l)$ possesses an exponential attractor $\mathcal{M}$ on $\mathcal{C}$, i.e., there exists a set $\mathcal{M} \subset \mathcal{C}$ which satisfies the following properties:

1) it is a compact subset of $\mathcal{C}$;

2) it is semi-invariant, $S(l)\mathcal{M} \subset \mathcal{M},\ \forall l \in \mathbb{N}$;

3) it has finite fractal dimension in $\mathcal{H}$,

$$\dim_f(\mathcal{M}, \mathcal{H}) \le C_1; \tag{2.10}$$

4) it attracts exponentially the images of $\mathcal{C}$ in the metric of $\mathcal{H}$,

$$\operatorname{dist}_{\mathcal{H}}(S(l)\mathcal{C}, \mathcal{M}) \le C_2 e^{-\alpha l},\ \forall l \in \mathbb{N}. \tag{2.11}$$

Moreover, the positive constants C_1, C_2 and α can be expressed explicitly in terms of K, L, N_0 and some qualitative characteristics of the embedding $\mathcal{H}_1 \subset \mathcal{H}$.

The proof of this proposition repeats word by word that given in [MZ] and is therefore omitted.

§3 Proof of the main result.

In this section, we complete the proof of Theorem 2.2 and establish the existence of a finite-dimensional exponential attractor for the semigroup $S(t)$ associated with the degenerate equation (1.1). To this end, we need the following result.

Proposition 3.1. *Let the assumptions of Theorem 2.2 hold, let u be a solution of problem* (1.1) *and let $[T_0, T_0 + \mu]$ belong to one of the regularity intervals found in Proposition 1.1. The latter means that, on this time interval, we can control the L^2-norm of $\partial_t \nabla_x u$ by* (1.44)*. Then, for every other solution $\bar{u}(t)$, $t \ge T_0$, of equation* (1.1)*, the following estimate holds:*

$$\int_{T_0+\mu/2}^{T_0+\mu} \|u(t) - \bar{u}(t)\|^2_{W^{1,2}(\Omega)}\, dt \le C\|B(u(T_0)) - B(\bar{u}(T_0))\|^2_{L^1(\Omega)}, \tag{3.1}$$

where the constant C only depends on μ and the L^∞-norms of u and $\bar{u}$ and is independent of the concrete choice of u and $\bar{u}$.

Proof. As in Proposition 1.1, we only give below the formal derivation of estimate (3.1), which can be easily justified by using the approximate equations (1.6).

We set $v(t) := \bar{u}(t) - u(t)$. Then, this function obviously solves

$$\partial_t[B(\bar{u}(t)) - B(u(t))] = \operatorname{div}[a(\nabla_x \bar{u}(t)) - a(\nabla_x u(t))] - [f(\bar{u}(t)) - f(u(t))]. \tag{3.2}$$

Multiplying this equation by $v(t)$, integrating with respect to $x \in \Omega$ and using the monotonicity of a and assumption (1.4), we have

$$(3.3)\quad (\partial_t(B(\bar{u}(t)) - B(u(t))), v(t))_{L^2(\Omega)} + 1/2(a(\nabla_x \bar{u}) - a(\nabla_x u), \nabla_x v)_{L^2(\Omega)} + \\ + 2\theta \|\nabla_x v(t)\|^2_{L^2(\Omega)} \le C(|B(\bar{u}(t)) - B(u(t))|, |v(t)|)_{L^2(\Omega)},\ \theta > 0.$$

The right-hand side of (3.3) can be estimated as follows:

$$(3.4)\quad (|B(\bar{u}(t)) - B(u(t))|, |v(t)|)_{L^2(\Omega)} = \\ = (|B(\bar{u}(t)) - B(u(t))|^{1/2}, |B(\bar{u}(t)) - B(u(t))|^{1/2} \cdot |v(t)|)_{L^2(\Omega)} \le \\ \le C\|B(\bar{u}(t)) - B(u(t))\|^{1/2}_{L^1(\Omega)} \|v(t)\|^{3/2}_{L^3(\Omega)} \le \\ \le \theta \|\nabla_x v(t)\|^2_{L^2(\Omega)} + C_1 \|B(\bar{u}(t)) - B(u(t))\|^2_{L^1(\Omega)}.$$

In order to transform the left-hand side of (3.3), we use the following identity:

$$(3.5)\qquad \partial_t[B(\bar{u}(t)) - B(u(t))] \cdot v(t) = \partial_t \mathcal{I}_{u,\bar{u}}(t) + \partial_t u(t) \cdot \mathcal{J}_{u,\bar{u}}(t),$$

where

$$(3.6)\qquad \begin{aligned} \mathcal{I}_{u,\bar{u}}(t) &:= G(u(t)) - G(\bar{u}(t)) + B(\bar{u}(t))v(t), \\ \mathcal{J}_{u,\bar{u}}(t) &:= B(\bar{u}(t)) - B(u(t)) - B'(u(t))v(t), \end{aligned}$$

with $G(v) := \int_0^v B(u)\, du$. Inequality (3.3) reads, in view of (3.4) and (3.5),

$$(3.7)\quad \partial_t(\mathcal{I}_{u,\bar{u}}(t), 1)_{L^2(\Omega)} + \theta\|\nabla_x v(t)\|^2_{L^2(\Omega)} \le \\ \le C\|B(\bar{u}(t)) - B(u(t))\|^2_{L^1(\Omega)} + C(|\partial_t u(t)|, |\mathcal{J}_{u,\bar{u}}(t)|)_{L^2(\Omega)}.$$

In order to estimate the terms $\mathcal{I}$ and $\mathcal{J}$, we need the following lemma.

Lemma 3.1. *Let the above assumptions hold. Then, the functions $\mathcal{I}_{u,\bar{u}}$ and $\mathcal{J}_{u,\bar{u}}$ satisfy the following estimates:*

$$(3.8)\qquad \begin{cases} 1)\ \ \mathcal{I}_{u,\bar{u}} \ge 0, \\ 2)\ \ |\mathcal{J}_{u,\bar{u}}| \le C\mathcal{I}^{1/2}_{u,\bar{u}} \cdot |u - \bar{u}|, \\ 3)\ \ \mathcal{I}_{u,\bar{u}} \le C|B(u) - B(\bar{u})|^{1/2} \cdot |u - \bar{u}|^{3/2}, \end{cases}$$

where $|u| + |\bar{u}| \le R$ and the constant $C = C_R$ depends on R, but is independent of $u,\ \bar{u} \in \mathbb{R}$.

Proof. Since G is of class C^2 and $G''(z) = B'(z) \ge 0$, we have

$$(3.9)\quad \mathcal{I}_{u,\bar{u}} = \int_0^1 [B(su + (1-s)\bar{u}) - B(\bar{u})]\, ds \cdot (u - \bar{u}) = \\ = \int_0^1 \int_0^1 B'(s_1(su + (1-s)\bar{u}) + s_1\bar{u})\, ds_1\, ds \cdot |u - \bar{u}|^2 \ge 0.$$

Thus, $(3.8)_1$ is verified. Let us now check $(3.8)_2$ and $(3.8)_3$. Let first assumptions **(A)** for the nonlinearity B be satisfied. Then, since $B'(u) \sim |u|^p$, estimate (3.9) can be rewritten as follows:

$$(3.10)\quad C_2(B'(u)+B'(\bar u))\cdot|u-\bar u|^2 \ge \mathcal{I}_{u,\bar u} \ge \\ \ge C(|u|^p+|\bar u|^p)|\cdot|u-\bar u|^2 \ge C_1(B'(u)+B'(\bar u))\cdot|u-\bar u|^2,$$

see [Z1] for details. Analogously, using, in addition, (1.44), we can estimate $\mathcal{J}_{u,\bar u}$ as follows:

$$(3.11)\quad |\mathcal{J}_{u,\bar u}| = \int_0^1\int_0^1 |B''(s_1(su+(1-s)\bar u)+s_1\bar u)|\,ds_1\,ds\cdot|u-\bar u|^2 \le \\ \le C\int_0^1\int_0^1 |B'(s_1(su+(1-s)\bar u)+s_1\bar u)|^{3/4}\,ds_1\,ds \le C_1(B'(u)+B'(\bar u))^{3/4}\cdot|u-\bar u|^2$$

and, concerning the difference $B(u)-B(\bar u)$, we have

$$(3.12)\qquad |B(u)-B(\bar u)| \ge C(B'(u)+B'(\bar u))\cdot|u-\bar u|.$$

Since $1/2 < 3/4$, estimates (3.10)–(3.12), together with the fact that $|u|+|\bar u| \le R$, imply estimates $(3.8)_2$ and $(3.8)_3$. Thus, when assumptions **(A)** hold, Lemma 3.1 is proven.

Let us now consider assumptions **(B)**. To this end, we note that, if $u > 0$ and $\bar u > 0$, we have exactly the same situation as with assumptions **(A)**, so that all the estimates of the lemma are already verified. The case $u < 0$ and $\bar u < 0$ is also obvious since, in that case, both sides of inequalities (3.8) are identically equal to zero. So, we only need to consider the following two cases:

1) $u > 0$ and $\bar u < 0$;
2) $u < 0$ and $\bar u > 0$.

Let us consider case 1). Then, (3.8) reads

$$(3.13)\qquad \begin{cases} 2)\ \ B'(u)(u-\bar u)-B(u) \le G(u)^{1/2}\cdot|u-\bar u|,\\ 3)\ \ G(u) \le B(u)^{1/2}\cdot|u-\bar u|^{3/2}.\end{cases}$$

We note that, in that case, $|u-\bar u| \ge |u|$. Moreover, since $B'(u) \sim |u|^p$, $p \ge 0$, then, $G(u) \le CB(u)^{1/2}u^{3/2}$ near $u=0$, which implies $(3.13)_3$. In order to verify $(3.13)_2$, it suffices to note that (1.44) implies

$$(3.14)\qquad B'(u) \le C[B'(u)]^{3/4}\cdot|u|,\quad G(u) \ge CB'(u)\cdot|u|^2.$$

This inequality, together with the fact that $|u-\bar u| \ge |u|$, imply $(3.13)_2$. Thus, Lemma 3.1 is also verified in case 1).

Let us now consider case 2). In that case, (3.8) reads

$$(3.15)\qquad \begin{cases} 2)\ \ B(\bar u) \le C[B(\bar u)(\bar u-u)-G(\bar u)]^{1/2}\cdot|\bar u-u|,\\ 3)\ \ B(\bar u)(\bar u-u)-G(\bar u) \le B(\bar u)^{1/2}\cdot|\bar u-u|^{3/2}.\end{cases}$$

Since $B'(\bar u) \sim \bar u^p$ and $\bar u - u \ge \bar u$, we have

$$(3.16)\quad B(\bar u)(\bar u-u)-G(\bar u) \ge B(\bar u)\bar u - G(u) = \\ = \int_0^1 sB'(s\bar u)\,ds\cdot u^2 \ge C\bar u^{p+2} \ge C_1G(\bar u).$$

Moreover, analogously to (3.14),

$$C_1[B'(\bar u)]^{3/4}\bar u^2\le B(\bar u)\le C_2[B'(\bar u)]^{3/4}\bar u^2,\\ C_3B'(\bar u)\bar u\le B(\bar u)\le C_4B'(\bar u)\bar u,\ \ C_5B'(\bar u)\bar u^2\le G(\bar u)\le C_6B'(\bar u)\bar u^2. \tag{3.17}$$

Estimates (3.16) and (3.17) imply (3.15). Thus, estimates (3.8) are verified in all cases and Lemma 3.1 is proven.

It is now not difficult to finish the proof of the proposition. To this end, we multiply equation (3.7) by $(t-T_0)^4$ and set $\mathcal Z_{u,\bar u}(t):=(t-T_0)^4(\mathcal I_{u,\bar u}(t),1)_{L^2(\Omega)}$. Then, we have

$$\partial_t\mathcal Z_{u,\bar u}(t)+\theta(t-T_0)^4\|\nabla_x v(t)\|^2_{L^2(\Omega)}\le 4((t-T_0)^3,\mathcal I_{u,\bar u}(t))_{L^2(\Omega)}+\\ +(|\partial_t u(t)|,(t-T_0)^4|J_{u,\bar u}(t)|)_{L^2(\Omega)}+(t-T_0)^4\|B(u(t))-B(\bar u(t))\|^2_{L^1(\Omega)}. \tag{3.18}$$

Using $(3.8)_3$, we can estimate the first term in the right-hand side of (3.18) as follows:

$$((t-T_0)^3,\mathcal I_{u,\bar u})_{L^2(\Omega)}\le C(|B(u)-B(\bar u)|^{1/2},(t-T_0)^3|v|^{3/2})_{L^2(\Omega)}\le\\ \le C_1\|B(u)-B(\bar u)\|^{1/2}_{L^1(\Omega)}(t-T_0)^3\|v\|^{3/2}_{L^3(\Omega)}\le\\ \le C_2\|B(u)-B(\bar u)\|^2_{L^1(\Omega)}+\theta/4(t-T_0)^4\|\nabla_x v\|^2_{L^2(\Omega)}. \tag{3.19}$$

Analogously, using $(3.8)_2$ and the embedding $W^{1,2}(\Omega)\subset L^6(\Omega)$, we can estimate the second term in the right-hand side of (3.18),

$$(|\partial_t u|,(t-T_0)^4\mathcal J_{u,\bar u})_{L^2(\Omega)}\le C(|\partial_t u|,(t-T_0)^4\mathcal I_{u,\bar u}\cdot|v|)_{L^2(\Omega)}\le\\ \le C\|\partial_t u\|_{L^6(\Omega)}[(t-T_0)^4(\mathcal I_{u,\bar u},1)_{L^2(\Omega)}]^{1/2}(t-T_0)^2\|v\|_{L^3(\Omega)}\le\\ \le C\|\partial_t\nabla_x u\|^2_{L^2(\Omega)}\mathcal Z_{u,\bar u}+\theta/4(t-T_0)^4\|\nabla_x v\|^2_{L^2(\Omega)}. \tag{3.20}$$

Inserting these estimates into (3.18), we finally have

$$\partial_t\mathcal Z_{u,\bar u}(t)-C\|\partial_t\nabla_x u(t)\|^2_{L^2(\Omega)}\mathcal Z_{u,\bar u}(t)+\theta/2(t-T_0)^4\|\nabla_x v(t)\|^2_{L^2(\Omega)}\le\\ \le C'(1+(t-T_0)^4)\|B(u(t))-B(\bar u(t))\|^2_{L^1(\Omega)}. \tag{3.21}$$

We recall that, due to our assumptions, the time interval $[T_0,T_0+\tau]$ is a regular interval with respect to the solution u, i.e., on this interval, Proposition 1.1 allows to control the L^2-norm of $\partial_t\nabla_x u$,

$$\int_{T_0}^{T_0+\mu}\|\partial_t\nabla_x u(t)\|^2_{L^2(\Omega)}\,dt\le Q(\|u\|_{L^\infty([0,T]\times\Omega)}). \tag{3.22}$$

Moreover, according to estimate (1.28), we have

$$\|B(u(t))-B(\bar u(t))\|_{L^1(\Omega)}\le Ce^{K(t-T_0)}\|B(u(T_0))-B(\bar u(T_0))\|_{L^1(\Omega)}. \tag{3.23}$$

Applying the Gronwall inequality to (3.21) and using (3.22), (3.23) and the fact that $\mathcal{Z}_{u,\bar{u}}(T_0) = 0$, we deduce that

$$
\begin{aligned}
(3.24)\quad & \mathcal{Z}_{u,\bar{u}}(t) \leq \\
& \leq Q(\|u\|_{L^\infty([0,T]\times\Omega)} + \|\bar{u}\|_{L^\infty([0,T]\times\Omega)})\|B(u(T_0)) - B(\bar{u}(T_0))\|^2_{L^1(\Omega)}, \\
& \qquad\qquad t \in [T_0, T_0 + \mu],
\end{aligned}
$$

for some monotonic function Q which is independent of the concrete choice of u and $\bar{u}$. Integrating now inequality (3.21) with respect to $t \in [T_0 + \mu/2, T_0 + \mu]$ and using (3.22–3.24), we obtain estimate (3.1) for the $L^2(W^{1,2})$-norm of v and finish the proof of Proposition 3.1.

The next corollary is crucial in order to verify the second assumption of Proposition 2.1 in our situation.

Corollary 3.1. *Let the assumptions of Proposition 3.1 hold and let u, $\bar{u}$ and $[T_0, T_0 + \mu]$ be the same as in this proposition. Then, the following estimates hold:*

$$
\begin{aligned}
(3.25)\quad & \|\partial_t B(u) - \partial_t B(\bar{u})\|_{L^2([T_0+\mu/2,T_0+\mu],W^{-1,2}(\Omega))} + \\
& + \|B(u) - B(\bar{u})\|_{L^2([T_0+\mu/2,T_0+\mu],W^{1,2}(\Omega))} \leq K\|B(u) - B(\bar{u})\|_{L^1([T_0,T_0+\mu/2]\times\Omega)},
\end{aligned}
$$

where the constant K only depends on μ and the L^∞-norms of u and $\bar{u}$, but is independent of the concrete choice of u and $\bar{u}$.

Proof. We first note that, for every two solutions u and $\bar{u}$ of equation (1.1) and every $\delta > 0$, the following estimate holds:

$$
(3.26)\qquad \|B(u(\delta)) - B(\bar{u}(\delta))\|_{L^1(\Omega)} \leq C_\delta\|B(u) - B(\bar{u})\|_{L^1([0,\delta]\times\Omega)},
$$

where the constant C_δ only depends on δ and the L^∞-norms of u and $\bar{u}$. Indeed, in order to prove this estimate, it suffices to multiply equation (3.2) by $t\,\mathrm{sgn}(u(t) - \bar{u}(t))$, integrate over $[T_0, T_0 + \mu/2] \times \Omega$ and use the Kato inequality.

Combining the smoothing property (3.26) with Proposition 3.1, we check that the following estimate holds:

$$
(3.27)\qquad \int_{T_0+\mu/2}^{T_0+\mu} \|u(t) - \bar{u}(t)\|^2_{W^{1,2}(\Omega)}\,dt \leq C\|B(u) - B(\bar{u})\|^2_{L^1([T_0,T_0+\mu/2]\times\Omega)}.
$$

In order to deduce (3.25) from (3.27), it is sufficient to note that, expressing $\partial_t(B(u) - B(\bar{u}))$ from equation (3.2) and using the fact that the L^∞-norms of u and $\bar{u}$ can be controlled, we have

$$
\begin{aligned}
(3.28)\quad & \|\partial_t(B(u(t)) - B(\bar{u}(t)))\|^2_{W^{-1,2}(\Omega)} + \\
& \qquad + \|\nabla_x(B(u(t)) - B(\bar{u}(t)))\|^2_{L^2(\Omega)} \leq C\|u(t) - \bar{u}(t)\|^2_{W^{1,2}(\Omega)},
\end{aligned}
$$

where the constant C only depends on the L^∞-norms of u and $\bar{u}$. Thus, Corollary 3.1 is proven.

We are now ready to finish the proof of Theorem 2.2 by verifying the assumptions of Proposition 2.1 for some proper discrete semigroup associated with equation

(1.1). In order to construct it, we first construct a semi-invariant absorbing set $\mathcal{B}$ for the semigroup $S(t)$ associated with equation (1.1). As shown in the proof of Theorem 2.1, the ball

$$\mathcal{B}_0 := \{b_0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega),\ \|b_0\|_{L^\infty(\Omega)} + \|b_0\|_{W^{1,2}(\Omega)} \le R\} \tag{3.29}$$

is an absorbing set for this semigroup if R is large enough, but it is not necessarily semi-invariant. In order to overcome this difficulty, we transform this set in the following standard way:

$$\mathcal{B} = \left[\cup_{t\ge 0} S(t)\mathcal{B}_0\right]_{L^1(\Omega)}, \tag{3.30}$$

where $[\cdot]_V$ denotes the closure in the space V. Then, on the one hand, this new absorbing set remains bounded in $L^\infty(\Omega) \cap W^{1,2}(\Omega)$ (due to Theorem 1.1 and Lemmata 1.2–1.3), i.e., for every trajectory $u(t)$ starting from $B(u(0)) = b_0 \in \mathcal{B}$,

$$\|u(t)\|_{L^\infty(\Omega)} + \|u(t)\|_{W^{1,2}(\Omega)} + \|\partial_t B(u)\|_{L^2([t,t+1]\times\Omega)} \le C, \tag{3.31}$$

where the constant C is independent of u and $t \ge 0$. On the other hand, this set is, obviously, semi-invariant with respect to $S(t)$,

$$S(t)\mathcal{B} \subset \mathcal{B}. \tag{3.32}$$

Then, according to Proposition 1.1, there exists $\tau > 0$ such that, for every trajectory $u(t)$ starting from $\mathcal{B}$ and every time interval $[T, T+1]$ of length one, there exists a subinterval $[T_0, T_0+\tau] \subset [T, T+1]$ of length τ on which the L^2-norm of $\partial_t \nabla_x u$ is controlled as follows:

$$\int_{T_0}^{T_0+\tau} \|\partial_t \nabla_x u(t)\|^2_{L^2(\Omega)}\, dt \le C, \tag{3.33}$$

where C is independent of the trajectory u starting from $\mathcal{B}$. Thus, it is sufficient to construct the required exponential attractor on the absorbing set $\mathcal{B}$ only.

Let us now fix $\mu = 1/N$, where $N \in \mathbb{N}$ is large enough so that

$$1/N \le \tau/3, \tag{3.34}$$

and introduce the following spaces of functions depending on x and t:

$$\begin{aligned} &\mathcal{H} := L^1([0,\mu/2]\times\Omega),\\ &\mathcal{H}_1 := W^{1,2}([0,\mu/2], W^{-1,2}(\Omega)) \cap L^2([0,\mu/2], W^{1,2}(\Omega)). \end{aligned} \tag{3.35}$$

Then, $\mathcal{H}_1$ is compactly embedded into $\mathcal{H}$ (see, e.g., [LSU]). We also introduce the trajectory analogue of the absorbing set $\mathcal{B}$ as follows:

$$\mathcal{B}_{tr} := \{B(u(t)),\ t \in [0,\mu/2],\ u(t) \text{ solves (1.1) with } B(u(0)) \in \mathcal{B}\} \subset \mathcal{H} \tag{3.36}$$

and define the $\mu/2$-shift map $\mathcal{S}$ on $\mathcal{B}_{tr}$ by

$$(\mathcal{S}v)(t) := S(\mu/2)v(t), \quad v \in \mathcal{B}_{tr}. \tag{3.37}$$

Then, the semi-invariance (3.32) implies that the set $\mathcal{B}_{tr}$ is also semi-invariant with respect to the shift map $\mathcal{S}$,

$$\mathcal{S}:\mathcal{B}_{tr}\to\mathcal{B}_{tr}. \tag{3.38}$$

Our next task is to verify the conditions of Proposition 2.1 for the map (3.38). Indeed, estimate (1.28) immediately implies that the map $\mathcal{S}$ is globally Lipschitz continuous on $\mathcal{B}_{tr}$ and we only need to verify the second assumption of Proposition 2.1 and inequality (2.9). Indeed, due to our choice of the number μ, for every trajectory $u(t)$ starting from $\mathcal{B}$ (or, equivalently, for every trajectory of the discrete semigroup $\mathcal{S}(n)$ starting from $\bar{b}_0 := \{B(u(t)),\, t\in[0,\mu/2]\}$), at least one of the intervals

$$[0,\mu],\ [\mu,2\mu],\ \ \cdots,\ [(N-1)\mu,N\mu] \tag{3.39}$$

(let it be the interval $[n_0\mu,(n_0+1)\mu]$) belongs to the regularity interval of u, i.e., estimate (3.33) is satisfied on $[n_0\mu,(n_0+1)\mu]$. Thus, due to Corollary 3.1, we have

$$\|\mathcal{S}w-\mathcal{S}\bar{v}\|_{\mathcal{H}_1}\le K\|w-\bar{v}\|_{\mathcal{H}},\ \ \forall\bar{v}\in\mathcal{B}_{tr}, \tag{3.40}$$

where $w=\mathcal{S}(2n_0)\bar{b}_0$. So, the second assumption of Proposition 2.1 holds for $\mathcal{S}$ with $N_0=2N$.

Thus, we have proven that the discrete semigroup $\mathcal{S}(n)$ generated by the iterations of the shift operator $\mathcal{S}$ possesses an exponential attractor $\mathcal{M}_{tr}\subset\mathcal{B}_{tr}$ which is finite-dimensional and satisfies properties 1)–4) of Proposition 2.1.

We now recall that, due to estimate (3.26), the projection map Π,

$$\Pi:\mathcal{B}_{tr}\to\mathcal{B},\quad \Pi v=v(\mu/2),\ \ v\in\mathcal{B}_{tr}, \tag{3.41}$$

is globally Lipschitz continuous. Consequently, projecting the trajectory attractor $\mathcal{M}_{tr}$ onto $\mathcal{B}$, we obtain an exponential attractor $\mathcal{M}_d:=\Pi\mathcal{M}_{tr}$ for the discrete semigroup $\{S(n\mu/2),\, n\in\mathbb{N}\}$ on $\mathcal{B}$ which satisfies all properties 1)–4) of Proposition 2.1 with $\mathcal{H}=L^1(\Omega)$.

Thus, there only remains to pass from the exponential attractor $\mathcal{M}_d$ of the semigroup $S(n\mu/2)$ with discrete times $n\in\mathbb{N}$ to the semigroup $S(t)$ with continuous times $t\in\mathbb{R}_+$. To this end, we note that the map $(t,b_0)\mapsto S(t)b_0$ is uniformly Hölder continuous with respect to $(t,b_0)\in[0,\mu/2]\times\mathcal{B}$ with Hölder exponent $1/2$. Indeed, the Hölder (and even the Lipschitz) continuity of $S(t)b_0$ with respect to b_0 is an immediate consequence of (1.28) and the Hölder continuity with respect to t follows from the following simple estimates:

$$\begin{aligned}\|B(u(t+s))-B(u(t))\|_{L^1(\Omega)}&=\|\int_t^{t+s}B'(u(\kappa))\partial_t u(\kappa)\,d\kappa\|_{L^1(\Omega)}\le\\ &\le\int_t^{t+s}(B'(u(\kappa)),|\partial_t u(\kappa)|)_{L^2(\Omega)}\,d\kappa\le\left(\int_t^{t+s}B'(u(\kappa))\,d\kappa\right)^{1/2}\times\\ &\times\left(\int_0^{\mu/2}(B'(u(\kappa))\partial_t u(\kappa),\partial_t u(\kappa))_{L^2(\Omega)}\,d\kappa\right)^{1/2}\le Cs^{1/2}.\end{aligned} \tag{3.42}$$

Thus, the required exponential attractor $\mathcal{M}$ for continuous times can be constructed by the following standard formula:

$$\mathcal{M}:=\left[\cup_{t\in[0,\mu/2]}S(t)\mathcal{M}_d\right]_{L^1(\Omega)}\subset\mathcal{B}, \tag{3.43}$$

see [EFNT] for more details. So, our main theorem on the existence of an exponential attractor for the degenerate equation (1.1) is proven.

§4 Generalizations and concluding remarks.

In this concluding section, we discuss possible generalizations of the results obtained above and indicate several alternative methods to prove the finite-dimensionality of attractors.

Remark 4.1. To start with, we note that assumption $(1.2)_2$ requires the nonlinearity $a(\nabla_x u)$ to have a linear growth. However, this assumption is not essential and can be replaced by a standard polynomial growth of order p:

$$\kappa_1(1+|z|^{p-2}) \le A''(z) \le \kappa_2(1+|z|^{p-2}), \tag{4.1}$$

for some fixed $p \ge 2$ and positive constants κ_1 and κ_2. In that case, of course, we will, thanks to energy inequalities, control the $W^{1,p}$-norm of the solution u (instead of the usual $W^{1,2}$-norm). Indeed, an accurate analysis shows that the global boundedness of $A''(u)$ has been used only in the proof of Corollary 3.1 and only in order to obtain the control of the $W^{-1,2}$-norm of $\partial_t(B(u)-B(\bar u))$, see estimate (3.28). In the general case $p>2$, this estimate fails and should be replaced by an appropriate estimate of the L^q-norm, with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}(4.2)\quad &\|\partial_t(B(u)-B(\bar u))\|^2_{L^q([S,T],W^{-1,q}(\Omega))} \le \|a(\nabla_x u)-a(\nabla_x \bar u)\|^2_{L^q([S,T]\times\Omega)}+\\ &+\|f(u)-f(\bar u)\|^2_{L^q([S,T]\times\Omega)} \le C(\|u\|_{L^p([S,T],W^{1,p}(\Omega))}+\|\bar u\|_{L^p([S,T],W^{1,p}(\Omega))})^{p-2}\times\\ &\times(a(\nabla_x u)-a(\nabla_x\bar u),\nabla_x u-\nabla_x\bar u)_{L^2([S,T]\times\Omega)}+C\|u-\bar u\|^2_{L^q([S,T]\times\Omega)}\le\\ &\le C_1(1+|T-S|^{p-2})(a(\nabla_x u)-a(\nabla_x u),\nabla_x u-\nabla_x\bar u)_{L^2([S,T]\times\Omega)},\end{aligned}$$

where we have used estimates (4.1), together with the fact that the $L^\infty(W^{1,p})$-norms of u and $\bar u$ can be controlled. There remains to note that the scalar product in the right-hand side of (4.2) can be controlled by an analogue of Proposition 3.1, see estimate (3.3). Thus, the estimate of Corollary 3.1 remains true if we replace the $L^2(W^{-1,2})$-norm by the $L^q(W^{-1,q})$-norm and, consequently, the space $\mathcal{H}_1$ in (3.35) should be replaced by

$$\mathcal{H}_1 := W^{1,q}([0,\mu/2],W^{-1,q}(\Omega))\cap L^2([0,\mu/2],W^{1,2}(\Omega)).$$

Since this change does not destroy the compactness of the embedding $\mathcal{H}_1\subset\mathcal{H}$, the remaining of the proof of Theorem 2.2 does not change as well. Therefore, the main result of this article (Theorem 2.2 on the existence of a finite-dimensional exponential attractor) remains true under the more general assumption (4.1).

Remark 4.2. We now discuss the regularity assumptions of the domain Ω. Indeed, although we have assumed the boundary $\partial\Omega$ to be smooth, this assumption has been used only in Lemma 1.3 (in order to verify the $L^2(W^{2,2})$-regularity of the solutions) and in Theorem 1.2 (in order to make sure that the solutions u_n of the approximate problem (1.6) are regular enough). However, the $W^{2,2}$-regulartity of the solutions is, in fact, nowhere used in the sequel and all the other estimates do not require the domain Ω to be regular. Indeed, we have factually only used the Sobolev embedding $W^{1,2}_0(\Omega)\subset L^6(\Omega)$ and some interpolation inequalities which do not require *any* regularity of the boundary (due to our choice of Dirichlet boundary conditions; for Neumann boundary conditions, the Lipschitz continuity of the boundary is required). Thus, analyzing the solutions of the approximate problems (1.6) in a more accurate way, we see that the main results of the article hold, e.g., for Lipschitz domains (and even for some non-Lipschitz ones).

Remark 4.3. We now note that the above results are also valid (with a lot of simplifications) in the non-degenerate case

$$B'(u) \geq \kappa > 0 \tag{4.3}$$

as well. Indeed, in that case, assumption (1.44) is automatically satisfied (for B of class C^2) and assumption (1.4) also holds automatically, since, now, $B(z) \sim z$ near zero. However, our method may seem artificial in this situation. Indeed, under assumption (4.3), equation (1.1) can be rewritten in the form of a quasilinear second-order parabolic equation,

$$\partial_t u = [B'(u)]^{-1} A''_{ij}(\nabla_x u)\partial_{x_i}\partial_{x_j} u - [B'(u)]^{-1}(f(u) - g). \tag{4.4}$$

The analytic properties of such equations in the non-degenerate case are very well understood, see, e.g., [LSU], and we can use the classical and powerful regularity theory of such equations. Indeed, in particular, if $B,\ a \in C^2(\mathbb{R})$ and $g \in C^\alpha(\Omega)$, for some $\alpha > 0$, then, due to the interior regularity estimates, equation (4.4) (or, equivalently, equation (1.1)) possesses an absorbing ball in the space $C^{1+\alpha/2,2+\alpha}([T, T+1] \times \Omega)$,

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([T+1,T+2]\times\Omega)} \leq Q(\|u(T)\|_{L^\infty(\Omega)}), \tag{4.5}$$

see, e.g., [LSU], Chapter 6, Sections 1-6. Having this estimate, one can verify the finite-dimensionality of the global attractor, e.g., by the classical volume contraction method and verify the existence of an exponential attractor by proving the following simpler smoothing property for the difference of two solutions:

$$\|S(t)b_0^1 - S(t)b_0^2\|_{H^1(\Omega)} \leq C\frac{e^{Kt}}{\sqrt{t}}\|b_0^1 - b_0^2\|_{L^2(\Omega)},\ t > 0, \tag{4.6}$$

instead of the complicated version of such an inequality formulated in Proposition 2.1. Thus, our article is mainly oriented towards the degenerate case when (4.3) fails and when the reduction to (4.4) and the regularity (4.5) also fail (see also the next remark).

Remark 4.4. We now discuss an alternative method to prove the finite-dimensionality of the global attractor in the degenerate case (when assumption (4.3) is not satisfied). In contrast to the regular case, one cannot expect the existence of *classical* solutions or/and the smothing property (4.5) to hold in the degenerate case (even if all the terms are of class C^∞) and the best regularity which can be expected for our equation is the following Hölder continuity:

$$\|u\|_{C^{\alpha/2,\alpha}([T+1,T+2]\times\Omega)} \leq Q(\|u(T)\|_{L^\infty(\Omega)}),\ T > 0,\ \alpha > 0, \tag{4.7}$$

see [D2], [DUV] and the references therein for precise conditions which guarantee Hölder continuity results for degenerate second-order parabolic equations.

However, there exists a general method (suggested in [EfZ1]) which allows to extract the finite-dimensionality (and the existence of an exponential attractor) from this Hölder continuity and the L^1-Lipschitz continuity with respect to the initial data. The application of this method to our problem gives the following result.

Proposition 4.1. *Let the assumptions of Theorem 1.1 hold and let, in addition, a, B be of class C^2, the Hölder continuity estimate* (4.7) *be satisfied and the following monotonicity assumption:*

$$|f(z_1) - f(z_2)| \geq \kappa |B(z_1) - B(z_2)|, \quad \kappa > 0, \tag{4.8}$$

hold, for every z_1 and z_2 in a small neighborhood of all the degeneration points of B. Assume also that $g \in C^\alpha(\Omega)$, for some $\alpha > 0$. Then, the global attractor $\mathcal{A}$ of problem (1.1) *is finite-dimensional and there exists an exponential attractor for this problem in the sense of Definition 2.2.*

Sketch of the proof. We briefly recall here the main idea of [EfZ1] by considering, for simplicity, the case of one degeneration point for B at $z = 0$, i.e., assumptions **(A)** hold. In that case, we can consider the usual framework, i.e., $S(t)$ maps $u(0)$ onto $u(t)$. Let $\mathcal{B}$ be an absorbing set of the semigroup $S(t)$ (for which the uniform Hölder continuity holds due to (4.7)) and let $\mathcal{B}_\varepsilon(u_0)$ be an ε-ball in the metric of $L^1(\Omega)$ centered at $u_0 \in \mathcal{B}$. Then, since u_0 is continuous, we can split the domain Ω into the union of two subdomains,

$$\Omega = \Omega_+(u_0) \cup \Omega_-(u_0), \quad \Omega_+ := \{x \in \Omega, \ |u_0(x)| > \beta\}, \\ \Omega_- := \{x \in \Omega, \ |u_0(x)| < 2\beta\}, \tag{4.9}$$

where β is a sufficiently small positive number. Moreover, since the semigroup $S(t)$ is globally Lipschitz continuous in the L^1-metric and the norm $\|u\|_{C^{\alpha/2}([0,T]\times\Omega)}$ is uniformly bounded with respect to all trajectories starting from $\mathcal{B}$, then, for ε sufficiently small, there exists $T > 0$ (which is independent of u_0 and ε) such that, for every trajectory $u(t)$ such that $u(0) \in \mathcal{B}_\varepsilon(u_0) \cap \mathcal{B}$, the following estimates hold:

$$\begin{cases} |u(t,x)| > \beta/2, \ (t,x) \in [0,T] \times \Omega_+, \\ |u(t,x)| < 3\beta, \ (t,x) \in [0,T] \times \Omega_-. \end{cases} \tag{4.10}$$

In other words, all the trajectories starting from $\mathcal{B}_\varepsilon(u_0) \cap \mathcal{B}$ remain uniformly close to the degeneration point $z = 0$ for $x \in \Omega_-$ and are uniformly non-degenerate for $x \in \Omega_+$ if $t \leq T$. This simple observation is the key point of the method and is the precise reason why we need an assumption on the Hölder continuity.

Furthermore, since the above trajectories are (uniformly) non-degenerate on $[0,T] \times \Omega_+$, then, (4.3) holds on this domain and we can rewrite equation (1.1) in the form (4.4) on the domain Ω_+. Then, the parabolic regularity theorem mentioned above yields that $u \in C^{1+\alpha/2,2+\alpha}([0,T] \times \Omega_+)$ and the norm in this space is uniform with respect to all trajectories starting from $\mathcal{B}_\varepsilon(u_0) \cap \mathcal{B}$. Therefore, the difference $v(t) := u_1(t) - u_2(t)$ between two such solutions restricted to Ω_+ satisfies a linear second-order parabolic equation with regular coefficients and, thus (roughy speaking, see [EfZ1] for a precise formulation), we have, for the Ω_+-component of the difference $v(t)$, a smoothing property which is analogous to (4.6) (of course, $v\big|_{\partial\Omega_+} \neq 0$, so that the precise formulation should contain some interior estimates and cut-off functions).

On the other hand, fixing $\beta > 0$ small enough so that inequality (4.8) holds, for all z_1 and z_2 with $|z_i| \leq 3\beta, \ i = 1, \ 2$, and applying the L^1-Lipschitz continuity estimate in the domain Ω_- to the equation for the difference $v(t)$, we have

$$\|B(u_1(t)) - B(u_2(t))\|_{L^1(\Omega_-)} \leq e^{-\kappa t} \|B(u_1(0)) - B(u_2(0))\|_{L^1(\Omega_-)}, \quad t \leq T \tag{4.11}$$

(again, roughly speaking, since $v\big|_{\partial\Omega_-} \neq 0$, and some cut-off functions are necessary).

Thus, we have a uniform (with respect to all initial data belonging to $\mathcal{B}_\varepsilon(u_0)\cap\mathcal{B}$) contraction of the Ω_--component of the difference of two solutions and a uniform smoothing property for their Ω_+-component. As shown in [EfZ1], this decomposition is sufficient to verify that the image $S(T)(\mathcal{B}_\varepsilon(u_0)\cap B)$ of an ε-ball centered at u_0 can be covered by a finite number N of $\gamma\varepsilon$-balls in $L^1(\Omega)$, with $\gamma < 1$ and N independent of ε and u_0, and this last assumption implies the finite-dimensionality and the existence of an exponential attractor, see [EfZ1] for details. Thus, Proposition 4.1 is proven.

The essential advantage of the method introduced above is that, in contrast to our scheme developed in Sections 2 and 3, where we factually need the nonlinearity B to be of class C^5 near the degeneration points, this method does not require *any* regularity on B near the degeneration points and B of class C^2 is necessary only outside the degeneration points (i.e., in the domain Ω_+). Near the degeneration points (i.e., in Ω_-), we do not need *any* regularity assumption on B and only need the monotonicity estimate (4.8) to be satisfied.

However, in contrast to the classical De Giorgi theory for non-degenerate second-order parabolic equations, the Hölder continuity for degenerate equations is a non-trivial and delicate fact which can be even violated, e.g., for some elliptic-parabolic equations. In fact, we do not know whether or not the Hölder continuity holds under assumptions **(B)** in the elliptic-parabolic case. That is the reason why we chose to give an alternative proof which is based on relatively simple energy type estimates in this article.

Remark 4.5. To conclude, we note that it would be very interesting to prove the existence of exponential attractors when the function $B(u)$ has *singularities* or discontinuities (e.g., for Stefan-like problems). However, the above methods do not work in this situation, since inequalities of the form (4.8) cannot be satisfied if B has singularities and f is regular. The only type of singularities which we are able to treat are the discontinuities of the derivatives of B.

Indeed, let us consider the following particular case of equation (1.1):

$$\begin{cases} \partial_t B(u) = a\Delta_x u - f(u) + g, \\ u\big|_{\partial\Omega} = 0, \quad B(u)\big|_{t=0} = b_0, \end{cases} \tag{4.12}$$

where $a > 0$ is some fixed number and the function B is only Lipschitz continuous,

$$\kappa_1|z_1 - z_2|^2 \le [B(z_1) - B(z_2)].(z_1 - z_2) \le \kappa_2|z_1 - z_2|^2, \tag{4.13}$$

with positive constants κ_1 and κ_2.

Proposition 4.2. *Let the nonlinearity B satisfy* (4.13) *and the nonlinearity f be Lipschitz and satisfy the dissipativity assumption* (1.3). *Then, the semigroup $S(t)$ associated with equation* (4.12) *via* (2.2) *possesses an exponential attractor $\mathcal{M}$ in the sense of Definition 2.2.*

Sketch of the proof. Let $u_1(t)$ and $u_2(t)$ be two solutions of (4.12) starting from an absorbing set in $L^\infty(\Omega)$. Then, multiplying the equation for the difference of two solutions u_1 and u_2 by $(-\Delta_x)^{-1}(B(u_1)-B(u_2))$ and using (4.13), together with the Lipschitz continuity of f, we have

$$\begin{aligned}(4.14)\quad &\frac{d}{dt}\|B(u_1)-B(u_2)\|^2_{H^{-1}(\Omega)}+\\ &\qquad +\theta\|B(u_1)-B(u_2)\|^2_{L^2(\Omega)}\le C\|B(u_1)-B(u_2)\|^2_{H^{-1}(\Omega)},\end{aligned}$$

with some positive constants C and θ which are independent of u_1 and u_2. Fixing now some $T>0$, multiplying (4.14) by t and integrating over $[0,T]$, we deduce that

$$(4.15)\qquad \|B(u_1(T))-B(u_2(T))\|^2_{H^{-1}(\Omega)}\le C_T\|B(u_1)-B(u_2)\|^2_{L^2([0,T],H^{-1}(\Omega))}.$$

Integrating then relation (4.14) with respect to $t\in[T,2T]$ and using (4.15), we infer

$$(4.16)\qquad \|B(u_1)-B(u_2)\|_{L^2([T,2T],L^2(\Omega))}\le C_T\|B(u_1)-B(u_2)\|_{L^2([0,T],H^{-1}(\Omega))},$$

with some constant C_T depending on T. Finally, we deduce, from the equation for the difference between u_1 and u_2, together with (4.13) and the Lipschitz continuity of f, that

$$(4.17)\quad \|\partial_t B(u_1)-\partial_t B(u_2)\|_{L^2([T,2T],H^{-2}(\Omega))}\le C\|B(u_1)-B(u_2)\|_{L^2([T,2T],L^2(\Omega))}.$$

Thus, combining (4.16) and (4.17), we have

$$\begin{aligned}(4.18)\quad &\|B(u_1)-B(u_2)\|_{W^{1,2}([T,2T],H^{-2}(\Omega))\cap L^2([T,2T],L^2(\Omega))}\le\\ &\qquad\qquad\le C_T\|B(u_1)-B(u_2)\|_{L^2([0,T],H^{-1}(\Omega))},\end{aligned}$$

with some positive constant C_T which is independent of the choice of the trajectories u_1 and u_2 starting from the absorbing set.

Since the embedding

$$W^{1,2}([0,T],H^{-2}(\Omega))\cap L^2([0,T],L^2(\Omega))\subset L^2([0,T],H^{-1}(\Omega))$$

is compact, inequality (4.18) allows indeed to construct an exponential attractor for the semigroup $S(t)$ in the topology of $H^{-1}(\Omega)$ by the l-trajectories method, see [MP]. Since the H^1-norm can be controlled on the absorbing set, we obtain, by interpolating between $H^{-1}(\Omega)$ and $H^1(\Omega)$, the existence of an exponential attractor in the topology of $L^2(\Omega)$ (and even $L^p(\Omega)$, p finite) as well and Proposition 4.2 is proven.

Example 4.1. Although the result of Proposition 4.2 does not apply to the Stefan problem, it gives however the finite-dimensionality of attractors for some free boundary problems. In particular, the following free boundary problem:

$$(4.19)\qquad \begin{cases}\partial_t u=a_1\Delta_x u-f_1(u), & u>0,\\ \partial_t u=a_2\Delta_x u-f_2(u), & u<0,\\ u\big|_+=u\big|_-,\quad \partial_n u\big|_+ +\partial_n u\big|_-=0, & u=0,\end{cases}$$

where $a_1,\ a_2>0$, the functions f_i, $i=1,\ 2$, satisfy the dissipativity assumption (1.3) and $f_1(0)=f_2(0)=0$, can be rewritten in the form (4.12), with

$$B(z)=\begin{cases}a_1^{-1}z, & z>0,\\ a_2^{-1}z, & z<0,\end{cases}\qquad f(z)=\begin{cases}a_1^{-1}f_1(z), & z>0,\\ a_2^{-1}f_2(z), & z<0.\end{cases}$$

Thus, all the assumptions of Proposition 4.2 are satisfied and, consequently, problem (4.19) possesses a finite-dimensional exponential attractor in the phase space $L^2(\Omega)$.

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References

[AL] H.W. Alt and S. Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.* **183**, 311-341, 1983.

[BV] A.V. Babin and M.I. Vishik, *Attractors of evolutionary equations*, Nauka, Moscow, 1989; English translation: North-Holland, Amsterdam, 1992.

[Ba] A. Bamberger, Etude d'une équation doublement non linéaire, *J. Funct. Anal.* **24**, 148-155, 1977.

[D1] E. DiBenedetto, A boundary modulus of continuity for a class of singular parabolic equations, *J. Diff. Eqns.* **63** (3), 418-447, 1986.

[D2] E. DiBenedetto, *Degenerate parabolic equations*, Springer-Verlag, New York, 1993.

[DG] E. DiBenedetto and R. Gariepy, Local behavior of solutions of an elliptic-parabolic equation, *Arch. Rational Mech. Anal.* **97** (1), 1-17, 1987.

[DS] E. DiBenedetto and R.E. Showalter, Implicit degenerate evolution equations and applications, *S.I.A.M. J. Math. Anal.* **12**, 731-751, 1981.

[DUV] E. DiBenedetto, J.M. Urbano and V. Vespri, Current issues on singular and degenerate evolution equations, *Evolutionary equations*, Vol. 1, 169-286, Handb. Diff. Eqns., North-Holland, Amsterdam, 2004.

[DV] E. DiBenedetto and V. Vespri, On the singular equation $\beta(u)_t = \Delta u$, *Arch. Rational Mech. Anal.* **132** (3), 247-309, 1995.

[EFNT] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential attractors for dissipative evolution equations*, John-Wiley, New York, 1994.

[EMR] A. Eden, B. Michaux and J.-M. Rakotoson, Doubly nonlinear parabolic-type equations as dynamical systems, *J. Dyn. Diff. Eqns.* **3**, 87-131, 1991.

[ER] A. Eden and J.-M. Rakotoson, Exponential attractors for a doubly nonlinear equation, *J. Math. Anal. Appl.* **185** (2), 321-339, 1994.

[EfMZ] M. Efendiev, A. Miranville and S. Zelik, Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$, C. R. Acad. Sci. Paris **330** (8), 713-718, 2000.

[EfZ1] M. Efendiev and S. Zelik, Finite and infinite dimensional attractors for porous media equations, Submitted.

[EfZ2] M. Efendiev and S. Zelik, Finite dimensional attractors and exponential attractors for degenerate doubly nonlinear equations, Preprint.

[GM] O. Grange and F. Mignot, Sur la résolution d'une équation et d'une inéquation paraboliques non linéaires, *J. Funct. Anal.* **11**, 77-92, 1972.

[Gu] M. Gurtin, Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance, *Physica D* **92**, 178-192, 1996.

[LSU] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI, 1968.

[Li] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Gauthier-Villars, Paris, 1969.

[MP] J. Málek and D. Pražák, Large time behavior via the method of l-trajectories, *J. Diff. Eqns.* **181** (2), 243-279, 2002.

[MZ] A. Miranville and S. Zelik, Robust exponential attractors for Cahn-Hilliard type equations with singular potentials, *Math. Meth. Appl. Sci.* **27** (5), 545-582, 2004.

[Mi] A. Miranville, Finite dimensional global attractor for a class of doubly nonlinear parabolic equations, *Cent. Eur. J. Math.* **4** (1), 163-182, 2006.
[O] F. Otto, L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations, *J. Diff. Eqns.* **131** (1), 20-38, 1996.
[R] P.-A. Raviart, Sur la résolution de certaines équations paraboliques non linéaires, *J. Funct. Anal.* **5**, 299-328, 1970.
[S] K. Shirakawa, Large time behavior for doubly nonlinear systems generated by subdifferentials, *Adv. Math. Sci. Appl.* **10**, 77-92, 2000.
[TC] J.E. Taylor and J.W. Cahn, Linking anisotropic sharp and diffuse surface motion laws via gradient flows, *J. Statist. Phys.* **77** (1-2), 183-197, 1993.
[T] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, 2d ed., Springer-Verlag, New York, 1997.
[Z1] S. Zelik, Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities, *Disc. Cont. Dyn. Systems* **11** (2-3), 351-392, 2004.
[Z2] S. Zelik, Attractors of reaction-diffusion systems in unbounded domains and their spatial complexity, *Comm. Pure. Appl. Math.* **56** (5), 584-637, 2003.