REGULAR ATTRACTORS AND THEIR NONAUTONOMOUS PERTURBATIONS.

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ABSTRACT. The paper is devoted to the study of regular attractors of dissipative semigroups with discrete and continuous time and their nonautonomous perturbations. The results obtained are applied to the model example of a reaction diffusion system in a bounded domain of $\mathbb{R}^3$ with the nonautonomous external forces.

INTRODUCTION.

It is well known that longtime behavior of dissipative systems generated by the PDEs of mathematical physics can be described in terms of global attractors which are, by definition, compact invariant sets of the phase space attracting the images of all bounded subsets of the initial data, see [2-4], [14] and references therein. Moreover, very often these attractors occur (in a sense) finite-dimensional although the initial phase space of such dynamical systems is infinite dimensional (e.g., $\Phi = L^2(\Omega)$ or $\Phi = L^\infty(\Omega)$). Thus, the concept of a global attractor allows to reduce the study of the initial infinite dimensional dynamics to the associated finite-dimensional dynamics on the attractor. We however note that usually the attractor $\mathcal{A}$ is not a submanifold of the phase space and can have rather complicated fractal structure, so, obtaining a reasonable description of this structure becomes crusial for further investigation of the associated dynamics. Unfortunately, up to the moment, very few is known about the structure of the global attractors in general. Moreover, as elementary examples show, this structure can be very sensitive to the perturbations of the dynamical system considered and the rate of attraction to the global attractor can be arbitrarily slow.

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Nevertheless, the exists rather wide model class of dynamical systems (which includes reaction-diffusion equations, damped wave equations, Cahn-Hilliard equation, etc.) whose attractors possess more or less complete description. These are the so-called *regular* (global) attractors introduced in [2-3] (see also [11] for the particular cases). According to the theory developed there, the regularity of the global attractor requires the existence of a Lyapunov function for the dynamical system $\{S_t, t \geq 0\}$ considered and the finitness and hyperbolicity of all its equilibria $z_0 \in \mathcal{R}_0$. Then, under the additional technical assumptions, its global attractor $\mathcal{A}$ will be a finite union of of the unstable manifolds $\mathcal{M}^+_{0,z_0}$ of that equilibria:

$$\mathcal{A} = \cup_{z_0 \in \mathcal{R}_0} \mathcal{M}^+_{0,z_0}. \tag{0.1}$$

The regular attractors are exponential in the following sense: for every bounded subset B of the phase space, we have

$$\operatorname{dist}(S_t B, \mathcal{A}) \leq Q(\|B\|) e^{-\alpha t}, \tag{0.2}$$

where the monotonic function Q and the positive constant α are independent of t and B and $\operatorname{dist}(\cdot,\cdot)$ denotes the nonsymmetric Hausdorff distance between sets.

Moreover, the regular attractors are robust with respect to the perturbations, i.e., if $\mathcal{A}_\varepsilon$ is the global attractor of the perturbed semigroup $\{S^\varepsilon_t, t \geq 0\}$, $\varepsilon \in [0,1]$ and the global attractor $\mathcal{A}_0$ is regular then, under natural assumptions on the form of the pertubation, the attractors $\mathcal{A}_\varepsilon$ are also regular for sufficiently small ε and

$$\operatorname{dist}^{sym}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq C\varepsilon^\kappa, \tag{0.3}$$

where the positive constants C and κ are independent of ε and $\operatorname{dist}^{sym}(\cdot,\cdot)$ denotes the symmetric Hausdorff distance between sets.

In the present paper, we consider the nonautonomous perturbations of regular attractors. To be more precise, we study the longtime behavior of the trajectories of the *nonautonomous* dynamical processes $\{U_\varepsilon(t,\tau), t \geq \tau\}$ which tend as $\varepsilon \to 0$ to the autonomous dynamical semigroup S_t under the assumption that this limit semigroup possesses a regular attractor $\mathcal{A}_0$. The main aim of the paper is to verify that the regular structure preserves under the nonautonomous perturbations as well. Some particular results in this direction have been obtained in [6] and [8-9].

We however note that the concept of a global attractor is not directly applicable to the nonautonomous problems and sholuld be modified. Up to the moment, there exist two main possibilities to generalize this concept to the nonautonomous problems. The first one is based on the reducing the initial nonautonomous problem to the autonomous one acting on the (properly) extended phase space. This approach naturally leads to the so-called *uniform* attractor $\mathcal{A}^{un}_\varepsilon$ which remains time-independent (as in the autonomous case, see [4]). The alternative approach treats the attractor of the nonautonomous problem as a time-dependent set as well ($t \to \mathcal{A}_\varepsilon(t)$, $t \in \mathbb{R}$). This assumption leads to the so-called *pullback* attractor (or which is the same the theory of kernel sections in the terminology of [4], see also [5] and [13]). Analogously to the autonomous case, the attractors $t \to \mathcal{A}_\varepsilon(t)$ are generated by all bounded trajectories of the process $U_\varepsilon(t,\tau)$ concidered. Namely, let $\mathcal{K}_\varepsilon$ be the set of all bounded trajectories of $U_\varepsilon(t,\tau)$. Then

$$\mathcal{A}_\varepsilon(\tau) = \mathcal{K}_\varepsilon\big|_{t=\tau}, \tag{0.4}$$

i.e., $\mathcal{A}_\varepsilon(\tau)$ consists of values at $t=\tau$ of all bounded trajectories of $U_\varepsilon(t,\tau)$, see [4].

As we show below (see Section 4 and 5), the generalization of the regular attractors theory to the nonautonomous case can be naturally formulated in terms of the time-dependent attractors $t\to\mathcal{A}_\varepsilon(t)$. To be more precise, we assume that the limit semigroup S_t possesses the regular attractor $\mathcal{A}_0$. Then, under natural assumptions on the perturbation (see Sections 4 and 5), for all sufficiently small ε and every equilibrium $z_0\in\mathcal{R}_0$ of the limit semigroup S_t, there exists a unique trajectory $\bar u_{\varepsilon,z_0}(t)$ of the process $U_\varepsilon(t,\tau)$ belonging to small neighborhood of z_0. The solutions $\bar u_{\varepsilon,z_0}(t)$, $z_0\in\mathcal{R}_0$, play the role of the equilibria for the nonautonomous dynamical process $U_\varepsilon(t,\tau)$. Moreover, the (nonautonomous) unstable sets $t\to\mathcal{M}^+_{\varepsilon,z_0}(t)$ of that solutions are the C^1-submanifolds for every $t\in\mathbb{R}$ diffeomorphed to $\mathcal{M}^+_{0,z_0}$ and the following generalization of (0.1) holds:

$$\mathcal{A}_\varepsilon(\tau)=\cup_{z_0\in\mathcal{R}_0}\mathcal{M}^+_{\varepsilon,z_0}(\tau),\quad \tau\in\mathbb{R}, \tag{0.5}$$

see Sections 4 and 5. The attractors $\mathcal{A}_\varepsilon(\tau)$ are uniformly exponential, i.e., for every bounded subset B of the phase space and every sufficiently small ε, we have

$$\operatorname{dist}(U_\varepsilon(t+\tau,\tau)B,\mathcal{A}_\varepsilon(t+\tau))\le Q(\|B\|)e^{-\alpha t},\quad \tau\in\mathbb{R},\ t\ge 0, \tag{0.6}$$

where the positive constant C and the monotonic function Q are independent of ε, t and τ (compare with (0.2)) and close to the limit regular attractor $\mathcal{A}_0$:

$$\operatorname{dist}^{sym}(\mathcal{A}_\varepsilon(t),\mathcal{A}_0)\le C\varepsilon^\kappa,\quad t\in\mathbb{R}, \tag{0.7}$$

where positive constants C and κ are independent of ε and t (compare with (0.3)).

In order to obtain these results, we first consider (analogously to (3)) the case of discrete time ($\{U_\varepsilon(t,\tau),\ t,\tau\in\mathbb{Z}\}$) and obtain, using the abstract generalization of the method suggested in [6], the discrete version $n\to\mathcal{A}_\varepsilon(n)$, $n\in\mathbb{Z}$, of the nonautonomous regular attractor. After that (in Section 5), we extend this result to the case of continuous time.

The paper is organized as follows. Some auxiliary results on the exponential dichotomy of linear difference equations generated by linear hyperbolic maps are recalled in Section 1. The local autonomous and nonautonomous unstable manifolds for the dicrete cascades and the associated difference equations are inverstigated in Section 2. The regular attractor for the limit autonomous semigroup $\{S_n,n\in\mathbb{N}\}$ is constructed in Section 3 and its nonautonomous perturbations are studied in Section 4. These results are extended to the case of continuous time in Section 5.

The above results are illustrated (in Section 6) on a model example of a reaction-diffusion system

$$\partial_t u=a\Delta_x u-f(u)+g_0+\varepsilon g_\varepsilon(t),\quad u\big|_{t=0}=u_0,\quad u\big|_{\partial\Omega}=0 \tag{0.8}$$

in a bounded domain Ω of $\mathbb{R}^3$ which has been also investigated in [8-9]. More complicated applications of the above theory will be considered in the forthcoming papers [15] and [17].

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§1 Exponential dichotomy of linear difference equations.

In this section, we formulate several results on difference equations associated with a linear hyperbolic map L of a Banach space H which will be used in the next section for constructing the invariant manifolds in the nonlinear map. To be more precise, the hyperbolicity of the map $L \in \mathcal{L}(H,H)$ means that its spectrum does not intersect the unit circle:

$$\sigma(L) \cap \{\lambda \in \mathbb{C},\ |\lambda| = 1\} = \varnothing. \tag{1.1}$$

Then, as known, the space H can be splited in a direct sum $H = H_+ + H_-$ of two invariant spectral subspaces $H_\pm$ such that

$$\sigma(L\big|_{H_+}) = \sigma(L) \cap \{|\lambda| > 1\},\ \ \sigma(L\big|_{H_-}) = \sigma(L) \cap \{|\lambda| < 1\} \tag{1.2}$$

and, consequently, the map L is invertible on H_+ and the following estimates are valid:

$$\begin{cases} \|L^n h_-\|_H \le C\|h_-\|_H, & \forall h_- \in H_-, \quad n \in \mathbb{N}, \\ \|L^{-n} h_+\|_H \le C\|h_+\|_H, & \forall h_+ \in H_+, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where the positive constants C and α are independent of n and $h_\pm$. We denote by $\Pi_\pm : H \to H_\pm$ the spectral projectors associated with the spectral subspaces $H_\pm$ and by $L_\pm := \Pi_\pm L$ the restrictions of L to its spectral subspaces.

We now study the following linear difference equation, associated with the operator L:

$$u(n) - Lu(n-1) = h(n-1), \quad n \in \mathbb{Z} \tag{1.4}$$

in the space $l^\infty(\mathbb{Z}, H)$ of bounded H-valued sequences. The following proposition gives the solvability of problem (1.4) in this space for every $h \in l^\infty(\mathbb{Z}, H)$.

Proposition 1.1. *Let the operator $L \in \mathcal{L}(H,H)$ be hyperbolic. Then, for every $h \in l^\infty(\mathbb{Z}, H)$, equation* (1.4) *possesses a unique bounded solution $u \in l^\infty(\mathbb{Z}, H)$ and, thus the linear solving operator $\mathbb{T} \in \mathcal{L}(l^\infty, l^\infty)$ is well defined via $(\mathbb{T}h)(n) := u(n)$. Moreover, the solution $u \in l^\infty(\mathbb{Z}, H)$ of equation* (1.4) *satisfies the following estimate:*

$$\|u(n)\|_H \le C' \sup\nolimits_{l \in \mathbb{Z}} \left\{ e^{-\beta|n-l|} \|h(l)\|_H \right\} \tag{1.5}$$

where the positive constants C and $\beta < \alpha$ depend only only on constants C and α introduced in (1.4) *(and independent of n, l and h).*

Proof. Indeed the solution $u(n) := (u_+(n), u_-(n)) \in H_+ \times H_-$ can be defined by the following standard expression:

$$u_+(n) := \sum_{l=n}^{+\infty} (L_+)^{n-1-l} h_+(l), \quad u_-(n) := \sum_{l=-\infty}^{n} (L_-)^{n-1-l} h_-(l). \tag{1.6}$$

Obviously, the sequence $u(n)$ thus defined solves (1.4) and we only need to verify estimate (1.5). To this end, we note that, due to (1.3)(1),

$$\|u_-(n)\|_H \le C\sum_{l=-\infty}^{n} e^{-\alpha(n-1-l)}\|h(l)\|_H \le$$
$$\le Ce^{-\beta}\left(\sum_{l=-\infty}^{n} e^{-(\alpha-\beta)(n-1-l)}\right)\sup_{l\le n}\left\{e^{-\beta(n-l)}\|h(l)\|_H\right\} \le$$
$$\le C'\sup_{l\in\mathbb{Z}}\left\{e^{-\beta|n-l|}\|h(l)\|_H\right\}.$$

The estimate for the H_+ component of u can be obtained analogously and Proposition 1.1 is proven.

We now consider the analogue of equation (1.4) on a finite segment $\{M,N\} := \mathbb{Z}\cap[M,N]$, $M,N\in\bar{\mathbb{Z}}$. To this end, we need to add the 'boundary conditions' at $n=M$ and $n=N$ as follows:

$$\begin{cases} u(n) - Lu(n-1) = h(n-1), & n=\{M+1,\cdots,N\},\\ \Pi_-u(M)=v_-, \quad \Pi_+u(N)=v_+, & v_\pm\in H_\pm. \end{cases} \tag{1.7}$$

The following proposition is the analogue of Proposition 1.1 for problem (1.7).

Proposition 1.2. *Let $L\in\mathcal{L}(H,H)$ be a hyperbolic map. Then, for every initial data $(v_+,v_-)\in H$ and every $h\in l^\infty(\{M,N\},H)$, problem (1.7) has a unique solution $u\in l^\infty(\{M,N\},H)$ which can be presented in the following form:*

$$u = \mathbb{T}_{M,N}h + \mathbb{T}^l_M v_- + \mathbb{T}^r_N v_+, \tag{1.8}$$

where the linear operator $\mathbb{T}_{M,N}\in\mathcal{L}(l^\infty(\{M,N\},H),l^\infty(\{M,N\},H))$ gives the solution of (1.7) with zero boundary conditions ($v_+=v_-=0$), the operator $\mathbb{T}^l_M:\mathcal{L}(H_-,l^\infty(\{M,N\},H))$ solves problem (1.7) with $h=v_+=0$ and the operator $\mathbb{T}^r_N:\mathcal{L}(H_+,l^\infty(\{M,N\},H))$ is the solving operator for (1.7) with $h=v_-=0$. Moreover, the following estimates hold:

$$\begin{aligned} &1.\ \|(\mathbb{T}_{M,N}h)(n)\|_H \le C'\sup_{l\in\{M,N\}}\left\{e^{-\beta|n-l|}\|h(l)\|_H\right\},\\ &2.\ \|(\mathbb{T}^l_M v_-)(n)\|_H \le Ce^{-\beta(n-M)}\|v_-\|_H,\\ &3.\ \|(\mathbb{T}^r_N v_+)(n)\|_H \le Ce^{-\beta(N-n)}\|v_+\|_H, \end{aligned} \tag{1.9}$$

where $n\in\{M,N\}$ and the positive constants C' and β depend only on C and α introduced in (1.3) (and independent of $v_\pm$, h, M, N and n).

Proof. Indeed, it is not difficult to verify that the operators in (1.8) can be expressed as follows:

$$(\mathbb{T}^l_M v_-)(n) := (L_-)^{n-M}v_-, \quad (\mathbb{T}^r_N v_+)(n) := (L_+)^{N-n}v_+ \tag{1.10}$$

and, denoting by $\bar h$ the extension by zero of the sequence $h\in l^\infty(\{M,N\},H)$ to the space $l^\infty(\mathbb{Z},H)$,

$$(\mathbb{T}_{M,N}h)(n) := (\mathbb{T}\bar h)(n) - (L_-)^{n-M}(\mathbb{T}\bar h)(M) - (L_+)^{N-n}(\mathbb{T}\bar h)(N), \tag{1.11}$$

where the operator $\mathbb{T}$ is defined in Proposition 1.1. Applying estimates (1.3) and (1.5) to (1.10) and (1.11), we deduce estimates (1.9) and finish the proof of Proposition 1.2.

We will also need (in the next section) to consider operators in (1.8) in weighted spaces of H-valued sequences. To this end, we define, for every $\gamma \in \mathbb{R}$, the space $L^\infty_\gamma(\{M,N\},H)$ by the following norm

$$\|u\|_{l^\infty_\gamma(\{M,N\},H)} := \sup_{l\in\{M,N\}} \left\{e^{-\gamma l}\|u(l)\|_H\right\}. \tag{1.12}$$

The following corollary gives the norms of operators introduced in (1.8) in the weighted spaces (1.12).

Corollary 1.1. *Let L be a hyperbolic map and operators $\mathbb{T}_{M,N}$, $\mathbb{T}^l_M$ and $\mathbb{T}^r_N$ and the constant β be as in Proposition 1.2. Then,*

1. For every γ such that $|\gamma| \le \beta$, the operator $\mathbb{T}_{M,N}$ maps $l^\infty_\gamma(\{M,N\},H)$ to itself and

$$\|\mathbb{T}_{M,N}\|_{\mathcal{L}(l^\infty_\gamma(\{M,N\},H),l^\infty_\gamma(\{M,N\},H))} \le C'' \tag{1.13}$$

where the constant C'' depend only on C and α from (1.3) *and is independent of $M,N\in\bar{\mathbb{Z}}$ and $|\gamma|\le\beta$.*

2. For every $0\le\gamma\le\beta$, we have

$$\begin{aligned} &\|\mathbb{T}^r_N\|_{\mathcal{L}(H_-,l^\infty_\gamma(\{M,N\},H))} \le C'e^{-\gamma N},\\ &\|\mathbb{T}^l_M\|_{\mathcal{L}(H_+,l^\infty_{-\gamma}(\{M,N\},H))} \le C'e^{\gamma M}, \end{aligned} \tag{1.14}$$

where the constant C'' depend only on C and α from (1.3) *and is independent of $M,N\in\bar{\mathbb{Z}}$ and $0\le\gamma\le\beta$.*

Indeed, estimates (1.13) and (1.14) are immediate corollaries of (1.9).

§2 Local invariant manifolds near the hyperbolic equilibrium.

In this section, we consider the nonlinear and nonautonomous analogue of equation (1.4) in a small neighborhood of the hyperbolic equilibrium $z_0\in H$ (without loss of generality, we may assume that $z_0 = 0$). To be more precise, we study the following nonlinear difference equation in a small neighborhood of zero:

$$u(n) - Lu(n-1) = S_0(u(n-1)) + P_\varepsilon(n,u(n-1)) \tag{2.1}$$

where $L\in\mathcal{L}(H,H)$ is a hyperbolic linear map, the map $S_0\in C^1(H,H)$ satisfies

$$S_0(0) = S_0'(0) = 0 \tag{2.2}$$

and its Frechet derivative $v\to S_0'(v)$ is uniformly continuous on bounded subsets of H and the map $P_\varepsilon(n,v)$ belongs to $C^1(H,H)$, for every $n\in\mathbb{Z}$ and every $\varepsilon\in[0,\varepsilon_0]$, and for every $v\in H$, $\|v\|_H\le 1$, these operators satisfy the following estimate

$$\|P_\varepsilon(n,v)\|_H + \|P_\varepsilon'(n,v)\|_{\mathcal{L}(H,H)} \le C\varepsilon, \tag{2.3}$$

where the constant C is independent of ε, n and v and the Frechet derivatives $v \to P'_\varepsilon(n,v)$ are uniformly (with respect to n and v) continuous on bounded subsets of H and $n \in \mathbb{Z}$. In particular, (2.3) implies that $P_0(n,v) \equiv P'_0(n,v) \equiv 0$, consequently, (2.1) can be interpreted as a small (if $\varepsilon \ll 1$) nonautonomous perturbation of the limit autonomous equation

$$u(n) - Lu(n-1) = S_0(u(n-1)). \tag{2.4}$$

We recall that, due to our assumptions the limit autonomous equation (2.4) has the (hyperbolic) equilibrium $\bar{u}_0(n) \equiv 0$. Our first task is to construct the analogue of this equilibrium for the nonautonomous equation (2.1) for small *positive* ε.

Theorem 2.1. *Let L be a hyperbolic linear map and the nonlinear operators S_0 and P_ε satisfy* (2.2) *and* (2.3)*. Then, there exist a positive ε_0 and a small positive R_0 such that, for every $\varepsilon \le \varepsilon_0$, equation* (2.1) *possesses a unique solution $\bar{u}_\varepsilon \in l^\infty(\mathbb{Z}, H)$ satisfying*

$$\|\bar{u}_\varepsilon(n)\|_H \le R_0, \quad \forall n \in \mathbb{Z}. \tag{2.5}$$

Moreover, the following estimate holds:

$$\|\bar{u}_\varepsilon(n)\|_H \le C'\varepsilon \tag{2.6}$$

where C' is independent of n and ε and, thus, the solution $\bar{u}_\varepsilon(n)$ tends to the limit equilibrium $\bar{u}_0(n) \equiv 0$ as $\varepsilon \to 0$.

Proof. In order to solve equation (2.1) in $l^\infty(\mathbb{Z}, H)$, we first invert the linear part of it using Proposition 1.1. Then, we have

$$u = \mathbb{T} \circ \mathcal{S}_0(u) + \mathbb{T} \circ \mathcal{P}_\varepsilon(u), \tag{2.7}$$

where $\mathcal{S}_0(u)(n-1) := S_0(u(n-1))$, $\mathcal{P}_\varepsilon(u)(n-1) := P_\varepsilon(n, u(n-1))$ and the operator $\mathbb{T}$ is defined in Proposition 1.1. Then, conditions (2.2) and (2.3) imply that $\mathcal{S}_0, \mathcal{P}_\varepsilon \in C^1(l^\infty(\mathbb{Z}, H), l^\infty(\mathbb{Z}, H))$ (to this end, we need the assumption on the uniform continuity of Frechet derivatives), $\mathcal{S}_0(0) = \mathcal{S}'_0(0) = 0$ and

$$\|\mathcal{P}_\varepsilon(v)\|_{l^\infty(\mathbb{Z})} + \|\mathcal{P}'_\varepsilon(v)\|_{\mathcal{L}(l^\infty(\mathbb{Z}), l^\infty(\mathbb{Z}))} \le C\varepsilon, \tag{2.8}$$

for all v such that $\|v\|_{l^\infty(\mathbb{Z})} \le 1$. We are going to solve (2.7) using the implicit function theorem. To this end, for every $\varepsilon \in [0,1]$, we set

$$\Phi_\varepsilon(v) := v - \mathbb{T} \circ S_0(v) - \mathbb{T} \circ \mathcal{P}_\varepsilon(v), \quad v \in l^\infty(\mathbb{Z}, H) \tag{2.9}$$

Then, the desired trajectory $\bar{u}_\varepsilon$ solves the equation $\Phi_\varepsilon(\bar{u}_\varepsilon) \equiv 0$. Moreover, obviously, $\Phi_0(0) = 0$. In order to construct this solution for positive ε, we need to verify that 1) $\Phi_\varepsilon \in C^1(l^\infty, l^\infty)$; 2) The linear operator $\Phi'_0(0)$ is invertible in H; 3) Functions $\Phi_\varepsilon(v)$ and $\Phi'_\varepsilon(v)$ are Lipschitz continuous with respect to ε at $\varepsilon = 0$. Then, thanks to the implicit function theorem (see e.g., [16]), for every sufficiently small $\varepsilon > 0$ we will have a unique solution $v = \bar{u}_\varepsilon$ of equation $\Phi_\varepsilon(v) = 0$ which will be also Lipschitz continuous with respect to ε.

Let us verify that assumptions. Indeed $\Phi_\varepsilon \in C^1(l^\infty(\mathbb{Z}), l^\infty(\mathbb{Z}))$, since $\mathcal{S}_0$ and $\mathcal{P}_\varepsilon$ belong to this space; definition (2.9) immediately implies that $\Phi_0'(0) = \mathrm{Id}$ and, thus, the second condition is also verified. Moreover, due to (2.8), we have

$$\|\Phi_\varepsilon(v) - \Phi_0(v)\|_{l^\infty(\mathbb{Z})} + \|\Phi_\varepsilon'(v) - \Phi_0'(v)\|_{\mathcal{L}(l^\infty(\mathbb{Z}), l^\infty(\mathbb{Z}))} \le C'\varepsilon \tag{2.10}$$

and the third condition is also verified. Consequently, the implicit function theorem is indeed applicable to equation (2.7) and, consequently, for every $\varepsilon \in [0, \varepsilon_0]$, there exists a solution $\bar u_\varepsilon$ of (2.7) which is unique in a sufficiently small neighborhood of zero in $l^\infty(\mathbb{Z}, H)$. Moreover, obviously, $\bar u_0 = 0$ and, therefore, (2.10) implies estimate (2.6) and Theorem 2.1 is proven.

Remark 2.1. It is not difficult to verify that, if the operator $P_\varepsilon(n, v)$ is autonomous, periodic, quasiperiodic or almost-periodic with respect to n, the same will be true for the solution $\bar u_\varepsilon(n)$ constructed in Theorem 2.1.

Our next task is to construct the local unstable manifolds for problem (2.1) near zero diffeomorphed to linear space H_+. We start our exposition with the autonomous case $\varepsilon = 0$ (which corresponds to equation (2.4)).

Definition 2.1. For a given $\delta > 0$, the local unstable set $\mathcal{M}^{+,loc} = \mathcal{M}^{+,loc}_\delta$ is defined as follows:

$$\begin{aligned}\mathcal{M}^{+,loc} := \{u_0 \in H,\ \ \exists u \in l^\infty(\mathbb{Z}_-, H),\ \ \|u\|_{l^\infty(\mathbb{Z}_-)} \le \delta,\\ u(n) \text{ solves (2.4) for } n \in \mathbb{Z}_-,\ u(0) = u_0 \text{ and } \lim_{n\to-\infty} u(n) = 0\},\end{aligned} \tag{2.11}$$

i.e., $\mathcal{M}^{+,loc}$ consists of all $u_0 \in H$ for which there exists a backward solution $u(n)$, $n \in \mathbb{Z}_-$ of problem (2.4) belonging to the δ-neighborhood of zero for every $n \in \mathbb{Z}_-$ and tending to zero as $n \to -\infty$ such that $u(0) = u_0$.

It is well known that, if the map L is hyperbolic and the $\delta > 0$ is small enough, then sets $\mathcal{M}^{+,loc}$ are, in fact, C^1-submanifolds of H diffeomorphed to H_+. To be more precise, the following result holds.

Theorem 2.2. *Let L be a linear hyperbolic map, $\varepsilon = 0$ and the map S_0 satisfy the assumptions of Theorem 2.1. Then, there exist $\delta > 0$, a neighborhood $\mathcal{V}_+$ of zero in the space H_+ and a C^1-map $\mathbb{M} : \mathcal{V}_+ \to H_-$ such that $\mathbb{M}(0) = \mathbb{M}'(0) = 0$ and*

$$\mathcal{M}^{+,loc}_\delta \cap (\mathcal{V}_+ \times H_-) = \{v_+ + \mathbb{M}(v_+),\ \ v_+ \in \mathcal{V}_+\}, \tag{2.12}$$

i.e., $\mathcal{M}^{+,loc} \cap (\mathcal{V}_+ \times H_-)$ is a graph of the function $\mathbb{M} : \mathcal{V}_+ \to H_-$.

The proof of this theorem can be found, e.g. in [3]. Nevertheless, we below give an independent proof of this fact in more general nonautonomous case.

We are now ready to introduce the unstable manifolds for problem (2.1) with sufficiently small *positive* ε. To this end, we recall that, in contrast to (2.4) this problem is *nonautonomous*, consequently, the unstable manifolds should also depend explicitly on n.

Definition 2.2. Let $\varepsilon \le \varepsilon_0$ where ε_0 is defined in Theorem 2.1 and let $\bar u_\varepsilon(n)$ be a bounded solution of (2.1) defined in Theorem 2.1. For a given neighborhood $\delta > 0$ and every $l \in \mathbb{Z}$, the local unstable set $\mathcal{M}^{+,loc}_\varepsilon(l) = \mathcal{M}^{+,loc}_{\varepsilon,\delta}(l)$ is defined as follows:

$$\begin{aligned}\mathcal{M}^{+,loc}_\varepsilon(l) := \{u_l \in H,\ \ \exists u \in l^\infty(\{-\infty, l\}, H),\ \|u\|_{l^\infty(\{-\infty,l\})} \le \delta,\\ u(n) \text{ solves (2.1) for } n \le l,\ u(l) = u_l,\\ \text{and } \lim_{n\to-\infty} \|u(n) - \bar u_\varepsilon(n)\|_H = 0\},\end{aligned} \tag{2.13}$$

i.e., $\mathcal{M}_\varepsilon^{+,loc}(l)$ consists of all $u_l \in H$ for which there exists a backward solution $u(n)$, $n \le l$ of problem (2.1) belonging to δ-neighborhood of zero for every $n \le l$ and tending to $\bar u_\varepsilon(n)$ as $n \to -\infty$ such that $u(l) = u_l$.

The following theorem is the nonautonomous analogue of Theorem 2.2.

Theorem 2.3. *Let the assumptions of Theorem 2.1 hold. Then, there exist positive $\varepsilon_0' \le \varepsilon_0$ and $\delta > 0$, a neighborhood $\mathcal{V}_+ \sim H_+$ of zero in the space H_+ such that, for every $\varepsilon \le \varepsilon_0'$, there exists a family of C^1-maps $\mathbb{M}_{\varepsilon,n} : \mathcal{V}_+ \to H_-$ satisfying $\mathbb{M}_{0,n}(0) = \mathbb{M}'_{0,n}(0) = 0$ and*

$$\|\mathbb{M}_{\varepsilon,n}(v_+) - \mathbb{M}(v_+)\|_{H_-} + \|\mathbb{M}'_{\varepsilon,n}(v_+) - \mathbb{M}'(v_+)\|_{\mathcal{L}(H_+,H_-)} \le C\varepsilon, \tag{2.14}$$

where $\mathbb{M}(v_+) := \mathbb{M}_{0,n}(v_+)$, $v_+ \in \mathcal{V}_+$, the constant C is independent of ε, n and v. Moreover, the set $\mathcal{M}_{\varepsilon,\delta}^{+,loc}(l) \cap (\bar u_\varepsilon(l) + \mathcal{V}_+ \times H_-)$ is a graph of the function $\mathbb{M}_{\varepsilon,l} : \mathcal{V}_+ \to H_-$:

$$\mathcal{M}_{\varepsilon,\delta}^{+,loc}(l) \cap (\bar u_\varepsilon(l) + \mathcal{V}_+ \times H_-) = \left\{\bar u_\varepsilon(l) + v_+ + \mathbb{M}_{\varepsilon,l}(v_+),\ \ v_+ \in \mathcal{V}_+\right\}. \tag{2.15}$$

Proof. We first give the proof of Theorem 2.3 for the case $l = 0$. The general case will be considered below. We now recall that, according to Definition 2.2, in order to construct the unstable set $\mathcal{M}_{\varepsilon,\delta}^{+,loc}(0)$, one needs to find all backward solutions $u \in l^\infty(\mathbb{Z}_-)$, $\|u\|_{l^\infty(\mathbb{Z}_-)} \le \delta$ of problem (2.1) which tend to $\bar u_\varepsilon(n)$ as $n \to -\infty$. To this end, it is more convenient to change the independent variable u in (2.1) as follows: $w = u - \bar u_\varepsilon$. Then, the sequence w satisfies the following difference equation:

$$\begin{aligned} w(n) - Lw(n-1) = [S_0(\bar u_\varepsilon(n-1) + w(n-1)) - S_0(\bar u_\varepsilon(n-1))]+ \\ + [P_\varepsilon(n, \bar u_\varepsilon(n-1) + w(n-1)) - P_\varepsilon(\bar u_\varepsilon(n-1))]. \end{aligned} \tag{2.16}$$

In order to solve equation (2.16), we invert the linear part of it using Proposition 1.2 with $M = -\infty$ and $N = 0$. To this end, we need to impose the additional 'boundary condition' $\Pi_+ w(0) = v_0$, $v_0 \in H_+$, at $n = 0$. Then, (2.16) reads

$$w = \mathbb{T}_0^r v_0 + \mathbb{T}_{-\infty,0} \circ [\mathcal{S}_0(w + \bar u_\varepsilon) - \mathcal{S}_0(\bar u_\varepsilon)] + \mathbb{T}_{-\infty,0} \circ [\mathcal{P}_\varepsilon(w + \bar u_\varepsilon) - \mathcal{P}_\varepsilon(\bar u_\varepsilon)] \tag{2.17}$$

where the operators $\mathbb{T}_{-\infty,0}$ and $\mathbb{T}_0^r$ are defined in Proposition 1.2 and the operators $\mathcal{S}_0$ and $\mathcal{P}_\varepsilon$ are the same as in (2.7). We are going to apply the implicit function theorem to equation (2.17). To this end, we set

$$\begin{aligned} \Phi_\varepsilon(v_0, w) := w - \mathbb{T}_0^r v_0 - \mathbb{T}_{-\infty,0} \circ [\mathcal{S}_0(w + \bar u_\varepsilon) - \mathcal{S}_0(\bar u_\varepsilon)]- \\ - \mathbb{T}_{-\infty,0} \circ [\mathcal{P}_\varepsilon(w + \bar u_\varepsilon) - \mathcal{P}_\varepsilon(\bar u_\varepsilon)] \end{aligned} \tag{2.18}$$

Then, since the operators $\mathcal{S}_0$ and $\mathcal{P}_\varepsilon$ are C^1-differentiable, we have $\Phi_\varepsilon \in C^1(H_+ \times l^\infty(\mathbb{Z}_-), l^\infty(\mathbb{Z}_-))$. Moreover, due to (2.10), (2.8) and the fact that $\mathcal{S}_0(0) = \mathcal{S}_0'(0) = 0$, we have $\Phi_\varepsilon(0,0) = 0$, $\partial_w \Phi_0(0,0) = \mathrm{Id}$ and

$$\|\Phi_\varepsilon(v_0, w) - \Phi_0(v_0, w)\|_{l^\infty(\mathbb{Z}_-)} \le C''\varepsilon. \tag{2.19}$$

Thus, the implicit function theorem is indeed applicable to equation (2.17) and, due to this theorem, for every $\varepsilon \le \varepsilon_0'$ and every v_0 belonging to a sufficiently small (but independent of ε) neighborhood $\mathcal{V}_+ \sim H_+$ equation (2.17) possesses a unique solution $u = U_\varepsilon(v_0)$ belonging to the δ_0-neighborhood of zero in $l^\infty(\mathbb{Z})$. Moreover, the function $v_0 \to U_\varepsilon(v_0)$ belongs to $C^1(\mathcal{V}_+, l^\infty(\mathbb{Z}_-))$ and, thanks to (2.19)

$$(2.20) \qquad \|U_\varepsilon(v_0) - U_0(v_0)\|_{l^\infty(\mathbb{Z})} + \|U'_\varepsilon(v_0) - U'_0(v_0)\|_{\mathcal{L}(H_+, l^\infty(\mathbb{Z}))} \le C_1\varepsilon$$

Moreover since $\Phi_\varepsilon(0,0) = 0$, then $U_\varepsilon(0) = 0$ and, consequently,

$$(2.21) \qquad \|U_\varepsilon(v_0)\|_{l^\infty(\mathbb{Z}_-)} \le C\|v_0\|_{H_+},$$

where the constant C is independent of ε. Let us now consider the function $v_0 \to U^0_\varepsilon(v_0) := U_\varepsilon(v_0)(0)$ which gives the initial value at $n = 0$ for the sequence $U_\varepsilon(v_0)$. Then, due to equation (2.17) and the definition of the operator $\mathbb{T}_{-\infty,0}$, we have $\Pi_+ U^0_\varepsilon(v_0) = v_0$ and $\Pi_+ U^0_\varepsilon(0) = 0$, consequently, we may define the desired function $\mathbb{M}_{\varepsilon,0} : H_+ \to H_-$ as follows:

$$(2.22) \qquad \mathbb{M}_{\varepsilon,0}(v_0) := \Pi_- U^0_\varepsilon(v_0).$$

Furthermore, since $\partial_{v_0}\Phi_0(0,0)(0) = \Pi_+$, then $\mathbb{M}'_{0,0}(0) = 0$ and estimates (2.14) follow from (2.20). Let us now define

$$(2.23) \qquad \bar{\mathcal{M}}_\varepsilon(0) := \bar{u}_\varepsilon(0) + U^0_\varepsilon(\mathcal{V}_+) = \bar{u}_\varepsilon(0) + \{v_0 + \mathbb{M}_{\varepsilon,0}(v_0), \ \ v_0 \in \mathcal{V}_+\}.$$

Then, on the one hand (2.23) is a C^1-submanifold of H diffeomorphed to $\mathcal{V}_+ \sim H_-$ (since it is a graph of the C^1-function $\mathbb{M}_{\varepsilon,0}$) and, on the other hand set (2.22) consists of all initial data at $n = 0$ for which there exists a backward solution $u(n)$ of (2.1), belonging to the δ_0-neighborhood of zero for every $n \in \mathbb{Z}_-$ and satisfying $u(0) \in \bar{u}_\varepsilon(0) + \mathcal{V}_+ \times H_-$. Thus, in order to verify the equality

$$(2.24) \qquad \mathcal{M}^{+,loc}_\varepsilon(0) \cap (\bar{u}_\varepsilon(0) + \mathcal{V}_+ \times H_-) = \bar{\mathcal{M}}_\varepsilon(0)$$

and finish the proof of Theorem 2.3, it is sufficient to verify that the solution $U_\varepsilon(v_0)$ of equation (2.16) tends to zero as $n \to -\infty$. In fact, we verify more strong assertion that $U_\varepsilon(v_0) \in l^\infty_\beta(\mathbb{Z}_-)$ where $\beta > 0$ is the same as in Proposition 2.2 and the weighted space l^∞_β is defined by (1.12). To this end, we note that , due to our assumptions on S_0 and P_ε, the operators $w \to \mathcal{S}_0(w + \bar{u}_\varepsilon) - \mathcal{S}_0(u_\varepsilon)$ and $w \to \mathcal{P}_\varepsilon(w + \bar{u}_\varepsilon) - \mathcal{P}_\varepsilon(\bar{u}_\varepsilon)$ are well defined not only in the space $l^\infty(\mathbb{Z}_-)$ of bounded sequences, but also in the space $l^\infty_\beta(\mathbb{Z}_-)$ of exponentially decaying sequences and belong to the class $C^1(l^\infty_\beta(\mathbb{Z}_-), l^\infty_\beta(\mathbb{Z}_-))$. Thus, thanks to Corollary 1.1, the function $\Phi_\varepsilon(v_0, w)$ is also belongs to $C^1(H_+ \times l^\infty_\beta(\mathbb{Z}_-), l^\infty_\beta(\mathbb{Z}_-))$. Therefore, we may apply the implicit function theorem to this function not only in the space $l^\infty(\mathbb{Z}_-)$, but also in the space $l^\infty_\beta(\mathbb{Z}_-)$ which, for every $v_0 \in \tilde{\mathcal{V}}_+$, gives a solution $\tilde{w} = \tilde{U}_\varepsilon(v_0)$ of equation (2.16) which belong to $l^\infty_\beta(\mathbb{Z}_-)$. Since bounded backward solution of (2.16) is unique, then necessarily $U_\varepsilon(v_0) \equiv \tilde{U}_\varepsilon(v_0)$ and estimate (2.21) can be improved as follows:

$$(2.25) \qquad \|U_\varepsilon(v_0)\|_{l^\infty_\beta(\mathbb{Z}_-)} \le C\|v_0\|_{H_+},$$

therefore the sequence $U_\varepsilon(v_0)$ exponentially tends to zero as $n \to -\infty$ and, in the case $l = 0$, Theorem 2.3 is proven. In order to reduce the case of general $l \in \mathbb{Z}$

to that one, we change the independent variable $n' := n - l$. Then, equation (2.1) reads

$$u(n') - Lu(n'-1) = S_0(u(n'-1)) + \bar{P}_\varepsilon(n', u(n'-1)), \tag{2.26}$$

where $\bar{P}_\varepsilon(n', v) := P_\varepsilon(n - l, v)$. Then, constructing as above the unstable manifold at $n' = 0$ for this equation, we obtain the desired unstable manifold at $n = l$ for the initial equation (2.1). Moreover, since our assumptions on $P_\varepsilon(n, v)$ are uniform with respect to $n \in \mathbb{Z}$, the operator $\bar{P}_\varepsilon(n', v)$ satisfies the same conditions as $P_\varepsilon(n, v)$. Thus, all the constants and estimates of Theorem 2.3 also will be uniform with respect to $l \in \mathbb{Z}$ and Theorem 2.3 is proven.

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold. Then, there exists $\delta_0 > 0$ such that, for every bounded backward solution $u(n)$, $n \leq l$, of (2.1) with $\varepsilon \leq \varepsilon_0'$ which satisfies $\|u\|_{l^\infty(\{-\infty, l\})} \leq \delta_0$, $u(l)$ necessarily belongs to the unstable manifold $\mathcal{M}_\varepsilon^{+,loc}(l)$. Thus, $u(n)$ converges exponentially to $\bar{u}_\varepsilon(n)$ as $n \to -\infty$:*

$$\|u(n) - \bar{u}(n)\|_H \leq Ce^{\beta(n-l)}\|u(l) - \bar{u}_\varepsilon(l)\|_H, \ n \leq l \tag{2.27}$$

where the positive constants C and β are independent of u, ε, n and l.

Indeed, without loss of generality we may assume that $l = 0$ (see the end of the proof of Theorem 2.3) and, in this case the equality $u = \bar{u}_\varepsilon + U_\varepsilon(\Pi_+ u(0))$ follows from the fact that the bounded backward solution of (2.16) is unique in a small neighborhood of zero. Consequently, $u(0) \in \mathcal{M}_\varepsilon^{+,loc}(0)$. Estimate (2.27) is now an immediate corollary of (2.25).

Remark 2.2. Analogously to Remark 2.1, if the operator $P_\varepsilon(n, v)$ is autonomous, periodic, quasiperiodic or almost-periodic with respect to n the same will be true for the operators $\mathbb{M}_{\varepsilon,n}$. In particular, if $\varepsilon = 0$ then the corresponding equation (2.4) is autonomous and, consequently, $\mathbb{M}_{0,n}$ is independent of n and, thus, coincide with the map $\mathbb{M}$ defined in Theorem 2.2.

Remark 2.3. Analogously to unstable manifolds $\mathcal{M}_\varepsilon^{+,loc}(l)$, one can consider the (local) stable manifolds $\mathcal{M}_\varepsilon^{-,loc}(l)$ which consist of the initial values at $n = l$ of all forward solutions of (2.1) (defined for $n \geq l$) which belong to the small neighborhood of zero for all $n \geq l$ and tend to $\bar{u}_\varepsilon(n)$ as $n \to +\infty$. Obviously, this manifold will be diffeomorphed to H_-. We however note that only the *unstable* manifolds are important for regular attractors, that is why we do not consider the stable manifolds here.

We now note that the manifolds $\mathcal{M}_\varepsilon^{+,loc}(l)$ obtained above consist of the backward solutions of equation (2.1). To be more precise, the following result holds.

Remark 2.4. Let the above assumptions hold. Then, there exists $\delta_0 > 0$ such that, for every $v_l \in \mathcal{M}_\varepsilon^{+,loc}(l)$ satisfying $\|v_l\|_H \leq \delta_0$ there exists a unique backward solution $v(n)$, $n \leq l$ of problem (2.1) which tends to $\bar{u}_\varepsilon(n)$ as $n \to -\infty$ and $v(l) = v_l$. Moreover,

$$v(n) \in \mathcal{M}_\varepsilon^{+,loc}(n), \quad \forall n \leq l.$$

Indeed, this assertion is an immediate corollary of Definition 2.2 and Theorem 2.3.

We conclude this section by proving that every (forward) solution of (2.1) tends exponentially to the appropriate unstable manifold while it remains in a δ-neighbourhood of z_0. This result is crucial for establishing the exponential rate of attraction to the regular attractors (see Sections 3 and 4).

Theorem 2.4. *Let the assumptions of Theorem 2.1 hold. Then, there exist $\delta > 0$ and positive $\varepsilon_0'' \le \varepsilon_0$ such that for every $\varepsilon \le \varepsilon_0''$, every $l \in \mathbb{Z}$ and every (forward) solution $u(n)$, $n \ge l$ of equation (2.1), belonging to the δ-neighborhood of zero in H for $l \le n \le l+N$, $n \in \bar{\mathbb{N}}$, there exists a backward solution $v(n)$, $n \le l+N$, of equation (2.1) belonging to the unstable manifold of $\bar{u}_\varepsilon$ (i.e., $v(n) \in \mathcal{M}_\varepsilon^{+,loc}(n)$, $n \le l+N$) such that*

$$\|u(n) - v(n)\|_H \le Ce^{-\beta(n-l)}\|u(l) - v(l)\|_H, \quad l \le n \le l+N, \tag{2.28}$$

where the positive constants C and β are independent of l, n, N, u and ε

Proof. We first note that, without loss of generality we may assume that $l = 0$ (see the trick at the end of the proof of Theorem 2.3). We now recall that, according to Theorem 2.3 (and Remark 2.4), there exists a unique backward solution $v(n)$, $n \le l+N$ such that

$$v(n) \in \mathcal{M}_\varepsilon^{+,loc}(n), \quad \forall n \le l+N \quad \text{and} \quad \Pi_+ v(l+N) = \Pi_+ u(l+N) \tag{2.29}$$

if $\varepsilon_0'' > 0$ and $\delta > 0$ is small enough (in the case $N = \infty$ we set $v(n) := \bar{u}_\varepsilon(n)$). We claim that the solution $v(n)$ thus defined satisfies (2.28). Indeed, since the sequences $u(n)$ and $v(n)$ solve (2.1), then the difference $\theta := u - v$ satisfies

$$\begin{aligned}\theta = \mathbb{T}_{0,N} \circ [\mathcal{S}_0(u) - \mathcal{S}_0(v)] +& \\ + \mathbb{T}_{0,N} \circ [\mathcal{P}_\varepsilon(u) - \mathcal{P}_\varepsilon(v)] &+ \mathbb{T}_0^l \circ \Pi_-(u(0) - v(0)).\end{aligned} \tag{2.30}$$

(we recall that, due to boundary condition (2.29), $\Pi_+ u(N) = \Pi_+ v(N)$). Computing the $l^\infty_{-\beta}(\{0,N\})$-norm of the both parts of (2.30) (where β is the same as in Proposition 1.1) and using Corollary 1.1, we have

$$\begin{aligned}\|\theta\|_{l^\infty_{-\beta}(\{0,N\})} \le C(\|\mathcal{S}_0(u) - \mathcal{S}_0(v)\|_{l^\infty_{-\beta}(\{0,N\})} +& \\ + \|\mathcal{P}_\varepsilon(u) - \mathcal{P}_\varepsilon(v)\|_{l^\infty_{-\beta}(\{0,N\})} &+ \|\Pi_-(u(0) - v(0))\|_H)\end{aligned} \tag{2.31}$$

where the constant C is independent of ε, N u and v. We also recall that, thanks to (2.6), (2.27) and our assumption that $\|u\|_{l^\infty(\{0,N\})} \le \delta$, we have

$$\|u\|_{l^\infty(\{0,N\})} + \|v\|_{l^\infty(\{0,N\})} \le C_1(\delta + \varepsilon_0'') \tag{2.32}$$

and, consequently, using that $\mathcal{S}_0'(0) = 0$, we can fix δ and ε_0'' so small that

$$\begin{aligned}&\|\mathcal{S}_0(u) - \mathcal{S}_0(v)\|_{l^\infty_{-\beta}(\{0,N\})} \le \\ &\le \int_0^1 \|\mathcal{S}_0'(su + (1-s)v)\|_{\mathcal{L}(l^\infty, l^\infty)}\, ds \cdot \|u - v\|_{l^\infty_{-\beta}(\{0,N\})} \le \frac{1}{4C}\|\theta\|_{l^\infty_{-\beta}(\{0,N\})}.\end{aligned} \tag{2.33}$$

Estimating the term with $\mathcal{P}_\varepsilon$ analogously and using (2.32) and (2.3), we can finally fix δ and ε_0'' so small that

$$\|\mathcal{P}_\varepsilon(u) - \mathcal{P}_\varepsilon(v)\|_{l^\infty_{-\beta}(\{0,N\})} \le \frac{1}{4C}\|\theta\|_{l^\infty_{-\beta}(\{0,N\})} \tag{2.34}$$

inserting (2.33) and (2.34) into the right-hand side of (2.31), we deduce that

$$\|\theta\|_{l^\infty_{-\beta}(\{0,N\})} \le 2C\|\Pi_-(u(0) - v(0))\|_H. \tag{2.35}$$

Estimate (2.28) is an immediate corollary of (2.35) and Theorem 2.4 is proven.

In the following corollary, we study the particular case $N = \infty$ of Theorem 2.4.

Corollary 2.2. *Let the assumptions of Theorem 2.4 hold, $\varepsilon \le \varepsilon_0''$ and let $u(n)$, $n \ge l$ be a solution of* (2.1) *which remains in the δ-neighborhood of zero for all $n \ge l$. Then, this solution stabilizes exponentially to $\bar u_\varepsilon(n)$ as $n \to +\infty$:*

$$\|u(n) - \bar u_\varepsilon(n)\|_H \le Ce^{-\beta(n-l)}\|u(l) - \bar u_\varepsilon(l)\|_H, \quad n \ge l \tag{2.36}$$

where the positive constants C and β are independent of n, l, ε and u.

Indeed, by definition $v(n) = \bar u_\varepsilon(n)$ in the case $N = \infty$.

§3 Regular attractors: the autonomous case.

In this section, we start to study the *global* behavior of solutions of difference equations of the form (2.1) as $n \to \infty$. For simplicity, we first consider the autonomous case:

$$u(n) = S(u(n-1)), \quad u(0) = u_0, \quad n \in \mathbb{N} \tag{3.1}$$

and the nonautonomous case will be considered in the next section.

We impose three groups of assumptions on the nonlinear map S:

I) Analytic properties: a) $S \in C^1(H, H)$ and its Frechet derivative $v \to S'(v)$ is uniformly continuous on bounded subsets of H; b) the map $S : H \to H$ is injective and $\ker S'(v) = \{0\}$ for all $v \in H$.

II. Hyperbolicity assumptions: we assume that equation

$$S(v) = v \tag{3.3}$$

possesses only finite number of solutions $\mathcal{R}_0 = \{v_1, \cdots, v_N\} \in H$ (which are the equilibria for problem (3.1)) and all of them are hyperbolic, i.e. the maps $S'(v_i) \in \mathcal{L}(H, H)$ are hyperbolic for all $i = 1, \cdots, N$ (see (1.1)).

III. Some kind of regularity assumptions on the associated dynamics. Let us consider the solving semigroup $S_n : H \to H$, $n \in \mathbb{Z}_+$, generated by equation (3.1) via

$$S_n u_0 := u(n) \ \text{ where } u(n) \text{ solves (3.1) with } u(0) = u_0. \tag{3.4}$$

a) We suppose that semigroup (3.4) possesses a global attractor $\mathcal{A}$ in the phase space H. The latter means that

1) The set $\mathcal{A}$ is compact in H;

2) It is strictly invariant: $S_n\mathcal{A} = \mathcal{A}$ for all $n \in \mathbb{Z}_+$;

3) $\mathcal{A}$ is an attracting set for the semigroup $\{S_n,\, n \in \mathbb{Z}_+\}$, i.e., for every bounded subset B of H and every neighborhood $\mathcal{O}(\mathcal{A})$ there exists $N = N(\mathcal{A}, \mathcal{O})$ such that

$$S_n B \subset \mathcal{O}(\mathcal{A}), \quad \text{for all } \ n \ge N. \tag{3.5}$$

b) We also assume that every solution $u(n)$, $u(0) = u_0 \in \mathcal{A}$, of problem (3.1) *belonging to the attractor* stabilizes as $n \to \infty$ to some equilibrium $v = v_u \in \mathcal{R}_0$:

$$\lim_{n\to\infty} \|u(n) - v\|_H = 0. \tag{3.6}$$

c) Equation (3.1) does not possess any homoclinic structures, i.e., if $u_1,\cdots,u_k\in l^\infty(\mathbb{Z})$ be arbitrary sequence of bounded solutions of (3.1) such that

(3.7) $\lim_{n\to-\infty}\|u_i(n)-v_i\|_H=0,\quad \lim_{n\to+\infty}\|u_i(n)-v_{i+1}\|_H=0,\ i=1,\cdots,k,$

for some equilibria $\{v_i\}_{i=1}^{k+1}\subset\mathcal{R}_0$, then necessarily all of v_i *are different.*

Remark 3.1. It is well known that conditions III. b) and III. c) are satisfied if equation (3.1) possesses a global Lyapunov function.

We also note that, due to assumptions I and II, all of the assumptions of Section 2 (with $\varepsilon=0$) are satisfied in sufficiently small neighborhood of *every* equilibrium $v_0\in\mathcal{R}_0$. Consequently, we have the *locall* description of the dynamics associated with (3.1) in a small neighborhood of every equilibrium $v_0\in\mathcal{R}_0$, see Theorems 2.1–2.4. Our task in this section is to give the detailed description of the *global* dynamics associated with of (3.1) (under the assumptions I, II and III). The following standard lemmata are crucial for that purposes.

Lemma 3.1. *Let the assumptions II, III a) and III b) be satisfied. Then, for every bounded subset $B\subset H$ and every positive δ, there exists $T=T(B,\delta)\in\mathbb{N}$ such that every solution $u\in l^\infty(\{0,T\})$ defined on the interval $n\in\{0,T\}$ with the initial data belonging to B ($u(0)\in B$) necessarily visits the δ-neighborhood $\mathcal{O}_\delta(\mathcal{R}_0)$ of the equilibria $\mathcal{R}_0$:*

$$\left(\cup_{n=0}^T\{u(n)\}\right)\cap\mathcal{O}_\delta(\mathcal{R}_0)\neq\varnothing,\quad \mathcal{O}_\delta(\mathcal{R}_0)=\cup_{i=1}^N\mathcal{O}_\delta(v_i) \tag{3.8}$$

Proof. Assume that the assertion of the lemma is wrong. Then, there exists positive δ_0, bounded $B_0\subset H$, a sequence $T_k\to+\infty$ and a sequence of solutions $u_k\in l^\infty(\{0,T_k\})$ of equation (3.1) such that $u_k(0)\in B_0$ and

$$\operatorname{dist}_H(u_k(n),\mathcal{R}_0)\ge\delta_0,\quad n\in\{0,T_k\}. \tag{3.9}$$

We now define a new sequence of solutions $\tilde u_k(n):=u_k(n+[T_k/2])$ on the interval $k\in\{-[T_n/2],[T_n/2]\}$. Then, thanks to (3.9),

$$\operatorname{dist}_H(\tilde u_k(n),\mathcal{R}_0)\ge\delta_0,\quad n\in\{-[T_k/2],[T_k/2]\}. \tag{3.10}$$

On the other hand, since semigroup (3.4) associated with equation (3.1) possesses the global attractor $\mathcal{A}$ and $T_k\to+\infty$, then, for every fixed $n\in\mathbb{Z}$, we have

$$\lim_{k\to\infty}\operatorname{dist}_H(\tilde u_k(n),\mathcal{A})=0. \tag{3.11}$$

Consequently, since $\mathcal{A}$ is compact in H, the sequence $\{\tilde u_k(n)\}_{k=1}^\infty$ is precompact for every fixed $n\in\mathbb{Z}$. Then, using the Cantor's diagonal procedure, we may assume without loss of generality that $\tilde u_k(n)\to\tilde u(n)\in\mathcal{A}$, for every fixed $n\in\mathbb{Z}$. Moreover, passing to the limit $k\to\infty$ in equations (3.1) for u_k, we obtain that the sequence $\tilde u\in l^\infty(\mathbb{Z})$ solves (3.1) (and belongs to the attractor $\mathcal{A}$). Finally, passing to the limit $k\to\infty$ in (3.10), we have

$$\operatorname{dist}_H(\tilde u(n),\mathcal{R}_0)\ge\delta_0,\quad n\in\mathbb{Z}.$$

which contradicts assumption III b). Thus, Lemma 3.1 is proven.

Lemma 3.2. *Let the assumptions II and III be satisfied. Then, for every (sufficiently small) $\delta > 0$ there exists positive $\delta' < \delta$ such that every solution $u(n)$ of equation* (3.1) *which starts from the $\mathcal{O}_{\delta'}(v_i)$ of some equilibrium $v_i \in \mathcal{R}_0$ (i.e., $u(0) \in \mathcal{O}_{\delta'}(v_i)$) and such that $u(T) \notin \mathcal{O}_\delta(v_i)$ never returns again in the neighborhood $\mathcal{O}_{\delta'}(v_i)$ of the same equilibrium for $n \geq T$, i.e.*

$$u(n) \notin \mathcal{O}_{\delta'}(v_i), \quad \forall n \geq T. \tag{3.12}$$

Proof. Let us assume that the assertion of the lemma is wrong. Then, there exist an equilibrium v_0 its neighborhood $\mathcal{O}_{\delta_0}(v_0)$, sequences $T_k \in \mathbb{N}$ and $T'_k > T_k$ and a sequence u^k of solutions of equation (3.1) such that

$$u^k(0), u^k(T'_k) \in \mathcal{O}_{1/k}(v_0), \quad u^k(T_k) \notin \mathcal{O}_{\delta_0}(v_0). \tag{3.13}$$

Moreover, without loss of generality we may assume that T_k is the first time when $u^k(n)$ is out of $\mathcal{O}_{\delta_0}(v_0)$, i.e.,

$$u^k(n) \in \mathcal{O}_{\delta_0}(v_0), n \leq T_k. \tag{3.14}$$

We recall that v_0 is equilibrium of equation (3.1), consequently, since $S \in C^1$, we have $T_k \to +\infty$. We now define another sequence $\tilde{u}^k(n) := \tilde{u}^k(n + T_k)$. Then, due to (3.13) and (3.14), we have

$$\begin{aligned} &1.\ \tilde{u}^k(0) \notin \mathcal{O}_{\delta_0}(v_0),\\ &2.\ \tilde{u}^k(n) \in \mathcal{O}_{\delta_0}(v_0), \quad -T_k \leq n \leq 0,\\ &3.\ \tilde{u}^k(\tilde{T}'_k) \in \mathcal{O}_{1/k}(v_0), \quad \tilde{T}'_k := T'_k - T_k > 0. \end{aligned} \tag{3.15}$$

Arguing as in the proof of Lemma 3.1, we can assume without loss of generality that $\tilde{u}^k(n)$ converges as $k \to \infty$ to some solution $u_1(n)$, $n \in \mathbb{Z}$, belonging to the attractor $\mathcal{A}$:

$$\lim_{k\to\infty} \|\tilde{u}^k(n) - u_1(n)\|_H = 0, \quad \forall n \in \mathbb{Z} \tag{3.16}$$

and conditions 1) and 2) of (3.15) imply that $u_1(n) \in \mathcal{O}_{\delta_0}(v_0)$ for $n \leq 0$ and, consequently, due to Corollary 2.2 with $\varepsilon = 0$ (without loss of generality, we may assume that $\delta_0 > 0$ is small enough that the assumptions of Corollary 2.2 be satisfied), we have

$$\lim_{n\to-\infty} \|u_1(n) - v_0\|_H = 0. \tag{3.17}$$

Moreover, due to assumption III b), we also have

$$\lim_{n\to+\infty} \|u_1(n) - v_0^1\|_H = 0, \tag{3.18}$$

for some $v_0^1 \in \mathcal{R}_0$. Furthermore, if $v_0 = v_0^1$, then u_1 is a homoclinic orbit to the equilibrium v_0 which is not allowed due to assumption III c). Thus, $v_0 \neq v_0^1$. Then, due to (3.15)(3) and (3.16), we also have a sequence $T_k^1 \leq \tilde{T}'_k$ such that

$$\mathrm{dist}_H(\tilde{u}^k(T_k^1), v_0^1) \leq \delta_k \tag{3.19}$$

such that $\delta_k \to 0$ as $k \to \infty$ and taking a subsequence if necessary, we may assume without loss of generality that $\delta_k = 1/k$. Moreover, due to (3.15) and the fact that $T_k^1 \le \tilde{T}_k'$, the solution $u^k(n)$ cannot stay in $\mathcal{O}_{\delta_0}(v_0^1)$ for all $n \ge T_k^{(1)}$. Consequently, we also have a sequence $T_k^{(2)}$, $T_k^{(1)} < T_k^{(2)} < \tilde{T}_k$ such that

$$\tilde{u}^k(T_k^{(2)}) \notin \mathcal{O}_{\delta_0}(v_0^1) \tag{3.20}$$

Thus, scaling the time $n \to n + T_k^{(1)}$ in the sequence $\tilde{u}^k(n)$, we obtain a new sequence of solutions which satisfies (3.13) (with v_0 replaced by v_0^1) and, consequently, repeating the above arguments, we construct the second heteroclinic orbit $u_2(n)$:

$$\lim_{n\to-\infty} \|u^2(n) - v_0^1\|_H = 0, \quad \lim_{n\to+\infty} \|u^2(n) - v_0^2\|_H = 0 \tag{3.21}$$

for some equilibrium $v_0^2 \in \mathcal{R}_0$ and again $v_0^2 = v_0$ is not allowed thanks to III c). Therefore, we can repeat the above procedure once more and so on. Finally, we can construct arbitrarily long sequence v_0^i of equilibria and a sequence u_i of the heteroclinic orbits

$$v_0 \to v_0^1 \to v_0^2 \to \cdots \to v_0^M. \tag{3.22}$$

Moreover, thanks to III c), all of v_0^i should be different which contradicts the fact that the set $\mathcal{R}_0$ of equilibria is finite. This contradiction proves Lemma 3.2.

We now verify that every solution of (3.1) (with the initial data u_0 belonging to all H and not only for $u_0 \in \mathcal{A}$) stabilizes as $n \to +\infty$ to one of the equilibria.

Corollary 3.1. *Let the assumptions I, II and III hold. Then, for every $u_0 \in H$, the corresponding solution $u(n)$, $n \ge 0$ of problem* (3.1) *stabilizes to one of the equilibria $\mathcal{R}_0$ as $n \to +\infty$:*

$$\lim_{n\to+\infty} \|u(n) - v_u\|_H = 0, \quad v_u \in \mathcal{R}_0. \tag{3.23}$$

Moreover, every complete solution $u \in l^\infty(\mathbb{Z})$ belonging to the attractor $\mathcal{A}$ is a heteroclinic orbit between two different equilibria of $\mathcal{R}_0$:

$$\lim_{n\to+\infty} \|u(n) - v_+\|_H = 0, \quad \lim_{n\to-\infty} \|u(n) - v_-\|_H = 0, \tag{3.24}$$

for some $v_\pm \in \mathcal{R}_0$, $v_+ \ne v_-$.

Proof. Indeed, let $\delta > 0$ be so small that in the δ-neighborhood of every $v_0 \in \mathcal{R}_0$ the assumptions of Corollaries 2.1 and 2.2 (with $\varepsilon = 0$) be satisfied (this δ exists since the number of equilibria is finite). We also fix δ' in such way that the assertion of Lemma 3.2 be satisfied.

Let now $u(n)$ be an arbitrary solution of (3.1). Then, thanks to Lemma 3.1, there exists positive T such that solution $u(n)$ visits the neighborhood $\mathcal{O}_{\delta'}(v_0)$ of some equilibrium $v_0 \in \mathcal{R}_0$ at every time interval $n \in \{K, K+T\}$ of length T. On the other hand, due to Lemma 3.2, if that solution goes out from the δ-neighborhood of v_0 it never visits $\mathcal{O}_{\delta'}(v_0)$ again. Since the number of equilibria is finite, these two assertions imply that there exist $v_u \in \mathcal{R}_0$ such that

$$u(n) \in \mathcal{O}_\delta(v_u), \quad \forall n \ge T_+ \tag{3.25}$$

and, thanks to Corollary 2.2, we have convergence (3.23).

Let us now $u(n)$, $n \in \mathbb{Z}$ be a complete solution of (3.1) belonging to the attractor. Then, arguing analogously, we deduce that there exist two equilibria $v_+, v_- \in \mathcal{R}_0$ such that

$$u(n) \in \mathcal{O}_\delta(v_-), \quad \forall n \le T_-; \quad u(n) \in \mathcal{O}_\delta(v_+), \quad \forall n \ge T_+; \tag{3.26}$$

and, thanks to Corollaries 2.1 and 2.2, we have convergences (3.23). Corollary 3.1 is proven.

In order to describe the global structure of the attractor $\mathcal{A}$, we need the following global version of the unstable sets $\mathcal{M}^{+,loc} = \mathcal{M}^{+,loc}_{v_i}$ introduced in Definition 2.1.

Definition 3.1. For a given equilibrium $v \in \mathcal{R}_0$, its (global) unstable set $\mathcal{M}^+_{0,v}$ is defined as follows:

$$\begin{aligned} \mathcal{M}^+_{0,v} := \{u_0 \in H,\ \ \exists u \in l^\infty(\{-\infty, 0\}, H),\ \text{such that} \\ u(n)\ \text{solves (3.1) for}\ n \le 0,\ u(l) = u_0\ \text{and}\ \lim_{n\to-\infty} \|u(n) - v\|_H = 0\}, \end{aligned} \tag{3.27}$$

i.e., $\mathcal{M}^+_{0,v}$ consists of all $u_0 \in H$ for which there exists a backward solution $u(n)$, $n \le 0$ of problem (2.1) tending to v as $n \to -\infty$ such that $u(l) = u_l$ (the difference with Definition 2.1 is we do not require now that this solution belongs to the small neighborhood of v.

Corollary 3.2. *Let the assumptions I, II and III hold. The the attractor $\mathcal{A}$ of problem* (3.1) *possesses the following description:*

$$\mathcal{A} = \cup_{v \in \mathcal{R}_0} \mathcal{M}^+_{0,v}. \tag{3.28}$$

Indeed, due to the classical description of global attractors, we have

$$\mathcal{A} = \mathcal{K}\big|_{n=0}, \tag{3.29}$$

where $\mathcal{K}$ consists of all solutions $u \in l^\infty(\mathbb{Z})$ of equation (3.1). Then, (3.24) is an immediate corollary of (3.28) and Definition 3.1 of the unstable sets.

The following theorem is a global version of Theorem 2.2.

Theorem 3.1. *Let the assumptions I, II and III hold. Then, for every $v \in \mathcal{R}_0$, the corresponding unstable set $\mathcal{M}^+_{0,v}$ is a finite-dimensional C^1-submanifold of H diffeomorphed to $H_+(v)$ where $H_+(v)$ is the unstable subspace of the linear hyperbolic map $S'(v)$. In particular,*

$$\operatorname{ind}_+(v) := \dim H_+(v) < \infty, \quad \forall v \in \mathcal{R}_0. \tag{3.30}$$

Proof. We first prove assertion (3.30). Indeed, let $v \in \mathcal{R}_0$, then thanks to Theorem 2.2, the corresponding *local* unstable set $\bar{\mathcal{M}}(v) := \mathcal{M}^{+,loc}_\delta \cap (\mathcal{V}_+ \times H_-)$ is a C^1-submanifold of H diffeomorphed to $\mathcal{V}_+ \sim H_+$. On the other hand, thanks to (2.29), this set belongs to the attractor $\mathcal{A}$. Moreover, since the attractor $\mathcal{A}$ is compact, then the manifold $\bar{\mathcal{M}}(v)$ is also (locally) compact and, consequently, it should be finite-dimensional. Thus, (3.30) is verified.

Let us now verify that $\mathcal{M}^+_{0,v}$ is a manifold. To this end, we recall that, due to Definitions 2.1 and 3.1, we have

$$\mathcal{M}^+_{0,v} = \cup_{k=0}^{\infty} S_k \bar{\mathcal{M}}(v). \tag{3.31}$$

Moreover, since every trajectory belonging to $\bar{\mathcal{M}}(v)$ diverges exponentially, see Corollary 2.1, then (decreasing the neighborhood $\mathcal{V}_+$ if necessary), we deduce the existence of $k_0 \in \mathbb{N}$ such that

$$\bar{\mathcal{M}}(v) \subset S_{k_0} \bar{\mathcal{M}}(v) \tag{3.32}$$

For simplicity, we below assume that $k_0 = 1$ (in fact, it is possible to fix the neighborhood $\mathcal{V}_+ \sim H_+$ in the definition of $\bar{\mathcal{M}}(v)$ such that (3.32) be satisfied with $k_0 = 1$, but we do not give the proof of this fact here).

Moreover, since $\bar{\mathcal{M}}(v)$ is a finite-dimensional C^1-submanifold of H (thanks to Theorem 2.2), consequently, since the map S is injective and the kernel $\ker S'(u)$ is trivial for any $u \in H$ (thanks to assumption I), then the restriction of the map S to the manifold $\bar{\mathcal{M}}(v)$ is a diffeomorphism:

$$S : \bar{\mathcal{M}}(v) \to S\bar{\mathcal{M}}(v) \tag{3.33}$$

Thus, all of the sets $\bar{\mathcal{M}}_k(v) := S_k \bar{\mathcal{M}}(v)$ are C^1-submanifolds of H diffeomorphed to $H_+ \sim \mathbb{R}^{\operatorname{ind}_+(v)}$. Moreover, according to (3.31) and (3.32) with $k_0 = 1$, we have

$$\mathcal{M}^+_{0,v} = \cup_{k=1}^{\infty} \bar{\mathcal{M}}_k(v), \quad \bar{\mathcal{M}}_k(v) \subset \bar{\mathcal{M}}_{k+1}(v) \tag{3.34}$$

and, consequently, the set $\mathcal{M}^+_{0,v}$ possesses the structure of a C^1-manifold diffeomorphed to $H_+ \sim \mathbb{R}^{\operatorname{ind}_+(v)}$. Thus, we only need to verify that $\mathcal{M}^+_{0,v}$ is a *submanifold* of H, i.e., that the topology induced on $\mathcal{M}^+_{0,v}$ by that C^1-structure coincides with the topology induced by the embedding $\mathcal{M}^+_{0,v}(v) \subset H$. To this end, it is sufficient to verify that, for every $n \geq 0$

$$\liminf_{k\to\infty} d_H(\bar{\mathcal{M}}_n(v), \bar{\mathcal{M}}_{k+1}(v) \setminus \bar{\mathcal{M}}_k(v)) > 0. \tag{3.35}$$

where $d_H(X,Y) := \inf_{(x,y)\in H^2} \|x-y\|_H$ is a (usual) distance between sets in the space H. In order to verify (3.35), we recall that the map S is injective and the global attractor $\mathcal{A}$ is compact in H, consequently, the restriction of S to attractor is a homeomorphism and, therefore, there exists a (uniformly) continuous inverse S_{-1}:

$$S^{-1} : \mathcal{A} \to \mathcal{A} \tag{3.36}$$

Thus, the restriction of semigroup $\{S_n, n \in \mathbb{Z}_+\}$ associated with equation (3.1) to the attractor can be extended to a group $\{S_n, n \in \mathbb{Z}\}$ of homeomorphisms acting on the attractor. In particular, since $\mathcal{M}^+_{0,v}$ is an invariant subset of the attractor, this group of homeomorphisms also acts on $\mathcal{M}^+_{0,v}$.

We also recall that, according to the definition of the sets $\bar{\mathcal{M}}_k(v)$, we have

$$S_l \bar{\mathcal{M}}_n(v) = \bar{\mathcal{M}}_{n+l}(v), \quad \forall l \in \mathbb{Z},\ n \in \mathbb{N},\ n + l \geq 0 \tag{3.37}$$

and, consequently, applying the homeomorphism S_{n-M} to the sets $\bar{\mathcal{M}}_n(v)$ and $\bar{\mathcal{M}}_{k+1}(v)\backslash\bar{\mathcal{M}}_k(v)$, we deduce that assertion (3.35) is equivalent to the following one: for every $M \in \mathbb{N}$,

$$\liminf_{k\to\infty} d_H(S_{-M}\bar{\mathcal{M}}(v), \bar{\mathcal{M}}_{k+1}(v)\backslash\bar{\mathcal{M}}_k(v)) > 0. \tag{3.38}$$

We now note that, due to Lemma 3.2, there exists δ'-neighborhood of v in H such that any solution u belonging to the unstable manifold $\mathcal{M}^+_{0,v}$ never returns to $\mathcal{O}_{\delta'}$ after its exiting of the set $\bar{\mathcal{M}}(v)$. On the other hand, according to Corollary 2.1 (with $\varepsilon = 0$), there exists $M > 0$ such that

$$S_{-M}\bar{\mathcal{M}}_n(v) \subset \mathcal{O}_{\delta'/2}(v) \tag{3.39}$$

and, consequently, since all of the sets $\bar{\mathcal{M}}_k(v)$ consists of the solutions of (3.1) which stabilize to v as $n \to -\infty$, we have the following estimate:

$$d_H(S_{-M}\bar{\mathcal{M}}(v), \bar{\mathcal{M}}_{k+1}(v)\backslash\bar{\mathcal{M}}_k(v)) \ge \delta'/2, \quad \forall k \ge 0.$$

Thus, estimate (3.38) (or which is the same (3.35)) is proven and, consequently, $\mathcal{M}^+_{0,v}$ is indeed a C^1-submanifold of H diffeomorphed to H_+. Theorem 3.1 is proven.

Remark 3.2. As it follows from the proof of Theorem 3.1, we need not the assumptions III b) and III c) in order to introduce the structure of a C^1-manifold on the global unstable set $\mathcal{M}^+_{0,v}$ (and only the hyperbolicity of v is essential for that fact). In contrast to this, these assumptions are essential for proving that this structure gives a C^1-submanifold of H. Indeed, if $\mathcal{M}^+_{0,v}$ contains a *homoclinic* orbit to v then, obviously, it cannot be a C^1-submanifold in H.

Corollary 3.3. *Let the assumptions of Theorem 3.1 hold. Then, the attractor $\mathcal{A}$ consists of the finite collection* (3.28) *of finite-dimensional C^1-submanifolds in H. The latter means that the attractor $\mathcal{A}$ is regular in the terminology of* [3].

Indeed, this fact is an immediate corollary of (3.28) and Theorem 3.1.

We conclude this section by proving that the rate of attraction of bounded sets to the attractor $\mathcal{A}$ is, in fact, exponential.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. Then there exists a positive number α and a monotonic function Q such that, for every bounded subset B of H, the following estimate is valid:*

$$\operatorname{dist}_H(S_nB, \mathcal{A}) \le Q(\|B\|_H)e^{-\alpha n}, \quad n \in \mathbb{N} \tag{3.40}$$

where $\operatorname{dist}_H(X,Y)$ is a nonsymmetric Hausdorff (semi)distance between the sets X and Y in H.

Proof. We first note that, since system (3.40) possesses a global attractor, then every neighborhood $\mathcal{O}_\delta(\mathcal{A})$ will be an absorbing set for the associated semigroup. Consequently, it is sufficient to prove estimate (3.40) for $B = B_0 := \mathcal{O}_{\delta_0}(\mathcal{A})$ only (where $\delta_0 > 0$ is some fixed number). In order to verify that estimate, we need the following lemma which improves the assertions of Lemmata 3.1 and 3.2.

Lemma 3.3. *Let the above assumptions hold. Then, for every sufficiently small* $\delta > 0$*, there exists* $T = T(\delta) > 0$ *such that, for every solution* $u(n)$*,* $n \in \mathbb{Z}_+$*, one can find two sequences of numbers* $\{T_-^i\}_{i=0}^M$ *and* $\{T_+^i\}_{i=1}^M$*, where* $M = M_u \le N := \#\mathcal{R}_0$ *and a sequence of different equilibria* $\{v_i\}_{i=1}^M \subset \mathcal{R}_0$ *such that*

$$T_-^0 = 0, \quad T_+^i - T_-^{i-1} \le T, \quad T_-^M = +\infty, \quad i = 1, \cdots, M \tag{3.41}$$

and

$$u(n) \in \mathcal{O}_\delta(v_i), \quad \forall n \in \{T_+^i, T_-^i\}, \quad i = 1, \cdots, M. \tag{3.42}$$

Proof. Let positive constant $\delta' < \delta$ be the same as in Lemma 3.2, i.e. every trajectory which starts from the δ'-neighborhood of some equilibrium $v \in \mathcal{R}_0$ and goes out from its δ-neighborhood never returns again to the initial neighborhood. We also fix, according to Lemma 3.1, $T = T(\delta', B_0)$ such that every trajectory starting from B_0 visits the δ'-neighborhood of $\mathcal{R}_0$ at every time period $\{K, K+T\}$ of length T. We claim that T satisfies all the assumptions of the lemma.

Indeed, let $u(n)$, $n \ge 0$, be an arbitrary solution of equation (3.1) such that $u(0) \in B_0$. Then, according to Lemma 3.1, this trajectory visits $\mathcal{O}_{\delta'}(\mathcal{R}_0)$ on the time interval $\{0, T\}$. We denote the equilibrium which satisfies this property by v_1. Then, by definition T_+^1 will be the first time, for which $u(n) \in \mathcal{O}_{\delta'}(v_1)$. Obviously, $T_+^1 - T_0^- \le T$. If the trajectory $u(n)$ stays in $\mathcal{O}_\delta(v_1)$ for all $n \ge T_+^1$, then we set $T_-^1 = +\infty$ and the assertion of the lemma holds with $M = 1$. Otherwise, we denote by T_-^1 the first time for which $u(T_-^1) \in \mathcal{O}_{\delta'}(v_1)$ and $u(T_-^1 + 1) \notin \mathcal{O}_\delta(v_1)$. Applying Lemma 3.1 again, we find $T_2^+ \in \{T_1^-, T_1^- + T\}$ and the equilibrium $v_2 \in \mathcal{R}_0$ such that $u(T_2^+) \in \mathcal{O}_{\delta'}(v_2)$ and so on. Moreover, all of the equilibria thus obtained should be *different*, due to Lemma 3.2 and our choice of δ'. Consequently, iterating the above arguments, we can construct only finite number of equilibria $\{v_i\}_{i=1}^M$ with $M \le \#\mathcal{R}_0$ and the trajectory stays inside of $\mathcal{O}_\delta(v_M)$ for all $n \ge T_+^M$ (i.e., $T_-^M = +\infty$) and Lemma 3.3 is proven.

Thus, in order to verify (3.40), we need to control the distance to the attractor inside of $\mathcal{O}_\delta(\mathcal{R}_0)$ and outside of it. Moreover, since the time which the trajectory spend outside of $\mathcal{O}_\delta(\mathcal{R}_0)$ is finite and uniformly bounded by $T\#\mathcal{R}_0$, we can use exponentially divergent estimate for this part of the trajectory which is formulated in the next lemma.

Lemma 3.4. *Let the above assumptions hold. Then, there exists positive constants* C *and* K *such that for every solution* $u(n)$ *starting from* B_0 *and every* $l \ge 0$*,*

$$\operatorname{dist}_H(u(n+l), \mathcal{A}) \le Ce^{Kn} \operatorname{dist}_H(u(l), \mathcal{A}), \quad n \ge 0 \tag{3.43}$$

where the constants C *and* K *are independent of* n*,* l *and* u*.*

Proof. Indeed, since (thanks to assumption I) the map S belongs to the class C^1 and (thanks to the existence of a global attractor) all the trajectories starting from the bounded set B_0 are uniformly bounded in $l^\infty(\mathbb{Z}_+)$, we have the Lipschitz continuity of the semigroup S_n on B_0, i.e., there exist positive constants C and K such that, for every two solutions $u(n)$ and $u_1(n)$ starting from B_0 and every $l \ge 0$, we have

$$\|u(n+l) - u_1(n+l)\|_H \le Ce^{Kn}\|u(l) - u_1(l)\|_H, \quad n \ge 0 \tag{3.44}$$

where the constants C and K are independent of l, u, u_1 and n. Since $\mathcal{A} \subset B_0$ and $\mathcal{A}$ is invariant with respect to S_n, then (3.44) implies (3.43) and Lemma 3.4 is proven.

In order to control the distance to the attractor for the part of the trajectory belonging to $\mathcal{O}_\delta(\mathcal{R}_0)$, we will essentially use the local description of the dynamics near the hyperbolic equilibrium obtained in the previous section.

Lemma 3.5. *Let the above assumptions hold and let the constant δ is chosen in such way that the assumptions of Theorem 2.4 (with $\varepsilon = 0$) be satisfied (in the δ-neighborhood) for every equilibrium $v \in \mathcal{R}_0$. Then, for every solution $u(n)$ starting from B_0 and satisfying*

$$u(n) \in \mathcal{O}_\delta(v), \quad n \in \{l, l+T_0\}, \quad T_0 \in \{0, \infty\}, \tag{3.45}$$

the following estimate is valid:

$$\operatorname{dist}_H(v(l+n), \mathcal{A}) \le Ce^{-\gamma n} \left(\operatorname{dist}_H(u(l), \mathcal{A})\right)^\theta \tag{3.46}$$

where the positive constants C, γ and $0 < \theta < 1$ are independent of l, n, T_0, v and u.

Proof. Indeed, due to Theorem 2.4 (with $\varepsilon = 0$) estimate (2.28) and our choice of the constant δ, we have

$$\operatorname{dist}_H(u(n+l), \mathcal{A}) \le \operatorname{dist}_H(u(n+l), \mathcal{M}^+_{0,v}) \le Ce^{-\beta n} \tag{3.47}$$

where the positive constants C and β are independent of n, l T_0, v and u. On the other hand, thanks to Lemma 3.4, we have

$$\operatorname{dist}_H(u(n+l), \mathcal{A}) \le Ce^{Kn} \operatorname{dist}_H(u(l), \mathcal{A}). \tag{3.48}$$

Combining estimates (3.47) and (3.48), we deduce that

$$\operatorname{dist}_H(u(n+l), \mathcal{A}) \le Ce^{-\beta n/2} \min\{e^{(K+\beta/2)n} \operatorname{dist}_H(u(l), \mathcal{A}), e^{-\beta n/2}\} \tag{3.49}$$

Computing the minimum into the right-hand side of (3.49), we obtain

$$\min\{e^{(K+\beta/2)n} \operatorname{dist}_H(u(l), \mathcal{A}), e^{-\beta n/2}\} \le e^{K+\beta} \left(\operatorname{dist}_H(u(l), \mathcal{A})\right)^\theta, \tag{3.50}$$

where $\theta := \frac{\beta}{2(K+\beta)}$. Inserting (3.50) into the right-hand side of (3.49), we derive (3.46) and finish the proof of Lemma 3.5.

We are now ready to finish the proof of Theorem 3.2. Indeed, let $u(n)$, $n \ge 0$, be an arbitrary solution of (3.1) starting from B_0 and let δ be the same as in Lemma 3.5 and the sequences $T^i_- = T^i_-(u)$ and $T^i_+ = T^i_+(u)$ be the same as in Lemma 3.3. Thus, according to Lemma 3.5, on the intervals $\{T^i_+, T^i_-\}$ $i = 1, \cdots, M$, we have

$$\operatorname{dist}_H(u(n), \mathcal{A}) \le Ce^{-\gamma(n-T^i_+)} \left(\operatorname{dist}_H(u(T^i_+), \mathcal{A})\right)^\theta. \tag{3.51}$$

On the other hand, since $T^i_+ - T^{i-1}_- \le T$, then, for $n \in \{T^{i-1}_-, T^i_+\}$, we have (due to Lemma 3.4) the following estimate:

$$\operatorname{dist}_H(u(n), \mathcal{A}) \le Ce^{KT} \operatorname{dist}_H(u(T^{i-1}_-), \mathcal{A}). \tag{3.52}$$

Iterating formulae (3.51) and (3.52) and using the evident estimates

$$\text{dist}_H(u(0),\mathcal{A}) \le \delta_0, \quad e^{-\theta^i\gamma T_+^i-\beta(n-T_+^i)} \le e^{-\gamma\theta^i n}, \quad n \ge T_+^i, \tag{3.53}$$

we finally have

$$\text{dist}_H(u(n),\mathcal{A}) \le \delta_0(C^2e^{KT})^M e^{-\gamma\theta^M n}. \tag{3.54}$$

Since all of the constants in (3.54) is independent of u (we recall that $M \le \#\mathcal{R}_0$), then (3.54) implies (3.40) and finish the proof of Theorem 3.2.

To conclude this section, we formulate the improved version of the exponential attraction property (3.40) which gives some kind of the asymptotical completeness of regular attractors. In order to do so we need to recall the following definition.

Definition 3.2. A sequence $u \in l^\infty(\mathbb{Z})$ is an N-composed trajectory of the semigroup S_n if there exists a sequence $\{T_i\}_{i=1} \subset \mathbb{Z}$, $T_1 < T_2 < \cdots < T_N$, such that $u(n)$ solves (3.1) at every interval $\{-\infty, T_1-1\}$, $\{T_i+1, T_{i+1}-1\}$ for $i = 1,\cdots,N$ and $\{T_N+1,\infty\}$. Thus, every N-composed trajectory $u(n)$ of the semigroup S_n consists of $N+1$ pieces of 'ordinary' trajectories of that semigroup with jump points at $T_1,\cdots,T_N$.

Corollary 3.4. *Let the above assumptions hold. Then for every bounded subset B of H and every trajectory $u(n) := S_nu_0$ with $u_0 \in B$ there exists an N-composed trajectory $v = v_u(n)$, $n \in \mathbb{Z}$, of that semigroup belonging to the attractor $\mathcal{A}$ (i.e., $v(n) \in \mathcal{A}$ for all $n \in \mathbb{N}$) such that $N \le \#\mathcal{R}_0$ and*

$$\|u(n)-v(n)\|_H \le C\|u(0)-v(0)\|_H e^{-\alpha n}, \quad n \in \mathbb{N} \tag{3.55}$$

where the positive constants C and α depend on B, but are independent of the concrete choice of $u_0 \in B$.

Indeed, the assertion of Corollary 3.4 can be obtained in a standard way from Lemma 3.3 and Theorem 2.4 (with $\varepsilon = 0$) (slightly modifying the proof of Theorem 3.4, see [3]).

§4 The nonautonomous case: perturbations of regular attractors.

In this section, we consider the following nonautonomous perturbation of equation (3.1)

$$u(n) = S(u(n-1)) + P_\varepsilon(n,u(n-1)), \quad u(l) = u_l, \quad n \ge l \tag{4.1}$$

and obtain the nonautonomous analogue of Theorems 3.1 and 3.2.

We assume that the operator S satisfies assumptions I–III of the previous section and the family of nonautonomous perturbations $U_\varepsilon(n,v)$ satisfy the following additional assumptions:

IV. Analytic properties: a) Regularity. We assume that $P_\varepsilon(n,\cdot)$ belongs to $C^1(H,H)$ for every fixed $\varepsilon \in [0,1]$ and every $n \in \mathbb{N}$. Moreover, the Frechet derivative $v \to P'_\varepsilon(n,v)$ is uniformly continuous on every bounded subset of H and its modulus of continuity is also uniform with respect to $n \in \mathbb{Z}$.

b) Convergence as $\varepsilon \to 0$. We suppose that the following estimate holds:

$$\|P_\varepsilon(n,v)\|_H + \|P'_\varepsilon(n,v)\|_{\mathcal{L}(H,H)} \le C\varepsilon \tag{4.2}$$

where the constant C depends (monotonically) on $\|v\|_H$, but is independent of ε and n.

c) Injectivity. We assume that, for every $\varepsilon \in [0,1]$ and every $n \in \mathbb{Z}$, the map $S(v) + P_\varepsilon(n,v)$ is injective in H and the kernel of it Frechet derivative is trivial for every $v \in H$:

$$\ker\{S'(v) + P'_\varepsilon(n,v)\} = \{0\}, \quad \forall n \in \mathbb{Z}, \quad \varepsilon \in [0,1]. \tag{4.3}$$

V. Uniform dissipativity. Let us consider the dynamical (semi)process $\{U_\varepsilon(n,l),\ l \in \mathbb{Z},\ n \ge l\}$ associated with problem (4.1) via

$$U_\varepsilon(n,l)u_l := u(n), \quad \text{where } u(n) \text{ solves (4.1) with } u(l) = u_l. \tag{4.4}$$

We finally assume that process (4.4) possesses a compact uniformly attracting set $\mathcal{B} \subset\subset H$. The latter means that, for every bounded subset $B \subset H$ and every neighborhood $\mathcal{O}(\mathcal{B})$ of the set $\mathcal{B}$, there exists $T = T(B,\mathcal{O})$ such that, for every $l \in \mathbb{Z}$ and $\varepsilon \in [0,1]$,

$$U_\varepsilon(l+n,l)B \subset \mathcal{O}(\mathcal{B}), \quad \forall n \ge T. \tag{4.5}$$

We now formulate the nonautonomous analogues of Lemmata 3.1 and 3.2.

Lemma 4.1. *Let the assumptions I–V be satisfied. Then, for every bounded subset $B \subset H$ and every $\delta > 0$, there exist $\varepsilon_0 = \varepsilon_0(B,\delta)$ and $T = T(B,\delta)$ such that, for every $\varepsilon \le \varepsilon_0$ and every $l \in \mathbb{Z}$, every solution $u(n)$, $n \ge l$, visit the δ-neighborhood $\mathcal{O}_\delta(\mathcal{R}_0)$ of the equilibria $\mathcal{R}_0$ at every time interval $n \in \{K, K+T\}$ of length T:*

$$(\cup_{n=K}^{K+T} u(n)) \cap \mathcal{O}_\delta(\mathcal{R}_0) \ne \varnothing. \tag{4.6}$$

Proof. Assume, analogously to the proof of Lemma 3.1, that the assertion of the lemma is wrong. Then, there exist $\delta_0 > 0$, a sequence of $\varepsilon_k \to 0$, sequences $l_k, K_k \in \mathbb{Z}, l_k \le K_k$, a sequence $T_k \to \infty$ and a sequence u_k of solutions of equation (4.1) with $\varepsilon = \varepsilon_k$ such that

$$u_k(l_k) \in B, \quad \operatorname{dist}_H(u_k(n), \mathcal{O}_\delta(\mathcal{R}_0)) \ge \delta_0, \quad k \in \{K_n, K_n + T_n\}. \tag{4.7}$$

As in the proof of Lemma 3.1, we define a new sequence of solutions $\tilde u_k(n) := u_k(n + K_k + [T_k/2])$. To be more precise, $\tilde u_k(n)$ satisfies the following 'shifted' version of equation (4.1):

$$\tilde u_k(n) = S(\tilde u_k(n-1)) + \tilde P_{\varepsilon_k}(n, \tilde u_k(n-1)), \quad \tilde P_{\varepsilon_k}(n,v) := P_{\varepsilon_k}(n + K_k + [T_k/2], v). \tag{4.8}$$

Moreover, assumption (4.7) implies that

$$\operatorname{dist}_H(\tilde u_k(n), \mathcal{O}_\delta(\mathcal{R}_0) \ge \delta_0, \quad n \in \{-[T_k/2], [T_k/2]\}. \tag{4.9}$$

We also note that, since $T_k \to \infty$ and the dynamical process (4.5) associated with (4.1) possesses the uniform attracting set $\mathcal{B}$, then

$$\lim_{k\to\infty} \operatorname{dist}_H(\tilde{u}_k(n), \mathcal{B}) = 0, \quad \text{for every fixed } n \in \mathbb{Z}. \tag{4.10}$$

Thus, the sequence $\{\tilde{u}_k(n)\}_{k=1}^{\infty}$ is precompact in H, for every fixed n and, consequently, due to Cantor's diagonal procedure, we may assume without loss of generality that $u_k(n) \to u(n)$ for some $u(n) \in \mathcal{B}$, for every $n \in \mathbb{Z}$. Then, on the one hand, passing to the limit $k \to \infty$ (using (4.2)), we find that $u \in l^\infty(\mathbb{Z})$ solves the limit autonomous equation (3.1) and, consequently, $u(n) \in \mathcal{A}$, for every $n \in \mathbb{Z}$. On the other hand passing to the limit in (4.9), we obtain that

$$\operatorname{dist}_H(u(n), \mathcal{O}_\delta(\mathcal{R}_0)) \geq \delta_0, \quad \forall n \in \mathbb{Z}$$

which contradicts assumption III b) of Section 3. Thus, Lemma 4.1 is proven.

Lemma 4.2. *Let the assumptions of Lemma 4.1 hold. Then, for every sufficiently small $\delta > 0$, there exist $\varepsilon_0 = \varepsilon_0(\delta)$ and $\delta' = \delta'(\delta)$ such that, for every $\varepsilon \leq \varepsilon_0$, every solution $u(n)$ of equation* (4.1) *which belongs to the δ'-neighborhood of some equilibrium $v \in \mathcal{R}_0$ at time $n = l$ and which goes out from the δ-neighborhood of v at time $n = l' > l$, never again returns to the initial neighborhood:*

$$u(n) \notin \mathcal{O}_{\delta'}(v), \quad \forall n \geq l_0'. \tag{4.11}$$

Proof. Let us assume that the assertion of the lemma is wrong. Then, there exist $\delta_0 > 0$, a sequence $\varepsilon_k \to 0$, three sequences $l_k, l_k', l_k'' \in \mathbb{Z}$, $l_k \leq l_k' \leq l_k''$, and a sequence of solutions $u_k(n)$, $n \geq l_k$, of equation (4.1) with $\varepsilon = \varepsilon_k$ such that

$$u_k(l_k), u_k(l_k'') \in \mathcal{O}_{1/k}(v), \quad u_k(l_k') \notin \mathcal{O}_{\delta_0}(v). \tag{4.12}$$

Passing to the limit $k \to \infty$ in (4.12), exactly as in the proof of Lemma 3.2 and 4.1, we obtain, for every $M \in \mathbb{N}$, a sequence of heteroclinic connections

$$v \to v_1 \to \cdots \to v_M, \quad v, v_i \in \mathcal{R}_0,$$

for the limit autonomous equation (3.1). Since $\#\mathcal{R}_0 < \infty$, then (for $M > \#\mathcal{R}_0$), we should have $v_i = v_j$, for some $i \neq j$ which gives a homoclinic structure for equation (3.1). This contradicts assumption III c) and Lemma 4.2 is proven.

In order to formulate the analogue of Theorem 3.1 for the nonautonomous case, we first need the analogue of equilibria set $\mathcal{R}_0$ for the case of positive ε. To this end, we assume from now on that $\varepsilon \leq \varepsilon_0$ is small enough that the assumptions of Theorem 2.1 hold for every equilibrium $v \in \mathcal{R}_0$. Then, for every $v \in \mathcal{R}_0$ and every $\varepsilon \leq \varepsilon_0$, there exists a unique solution $\bar{u}_{\varepsilon,v} \in l^\infty(\mathbb{Z})$ of equation (4.1) which belongs to a small neighborhood of v for every $n \in \mathbb{Z}$. We interpret the solutions $\{\bar{u}_{\varepsilon,v},\ v \in \mathcal{R}_0\}$ as the 'equilibria' of the nonautonomous equation (4.1) and, consequently, we set

$$\mathcal{R}_\varepsilon := \{\bar{u}_{\varepsilon,v},\ v \in \mathcal{R}_0\} \subset l^\infty(\mathbb{Z}). \tag{4.13}$$

Then the nonautonomous analogue of Corollary 3.1 can be formulated as follows.

Theorem 4.1. *Let the assumptions I–V hold. Then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \le \varepsilon_0$, the following assertions hold:*

1) Every solution $u(n)$, $n \ge l$ of equation (4.1) *stabilizes as $n \to \infty$ to one of the solutions $\mathcal{R}_\varepsilon$:*

$$\lim_{n\to+\infty} \|u(n) - w_\varepsilon(n)\|_H = 0, \tag{4.14}$$

for some 'equilibrium' $w_\varepsilon \in \mathcal{R}_\varepsilon$.

2) Every complete solution $u \in l^\infty(\mathbb{Z})$ is a heteroclinic orbit between two different 'equilibria' $w_-, w_+ \in \mathcal{R}_\varepsilon$:

$$\lim_{n\to-\infty} \|u(n)-w_-(n)\|_H = 0,\ \lim_{n\to+\infty} \|u(n)-w_+(n)\|_H = 0,\ w_- \ne w_+. \tag{4.15}$$

Proof. Let us verify the first assertion of the theorem. To this end, we first note that, every bounded neighborhood $\mathcal{O}(\mathcal{B})$ is a uniform absorbing set for dynamical process associated with equation (4.2), consequently, it is sufficient to verify (4.14) only for the solutions u of this equation whose initial data $u(l)$ belong to some fixed neighborhood $B_0 := \mathcal{O}(\mathcal{B})$ of the attracting set $\mathcal{B}$. The rest of the proof of Theorem 4.1 is analogous to the proof of Corollary 3.1. Indeed, let $\delta > 0$ and $\varepsilon_0 > 0$ be so small that in the δ-neighborhood of every $v_0 \in \mathcal{R}_0$ the assumptions of Corollaries 2.1 and 2.2 (with $\varepsilon \le \varepsilon_0$) be satisfied (this δ exists since the number of equilibria is finite). We also fix δ' in such way that the assertion of Lemma 4.2 be satisfied. Then, thanks to Lemma 4.1, there exist positive $T = T(B_0)$ and $\varepsilon_0 = \varepsilon_0(B_0)$ such that, for $\varepsilon \le \varepsilon_0$, every solution $u(n)$, $n \ge l$ of (4.1) with $u(l) \in B_0$ visits the neighborhood $\mathcal{O}_{\delta'}(v_0)$ of some equilibrium $v_0 \in \mathcal{R}_0$ at every time interval $n \in \{K, K+T\}$ of length T. On the other hand, due to Lemma 4.2, if that solution goes out from the δ-neighborhood of v_0 it never visits $\mathcal{O}_{\delta'}(v_0)$ again. Since the number of equilibria is finite, these two assertions imply that, for every such u, there exist $v_u \in \mathcal{R}_0$ such that

$$u(n) \in \mathcal{O}_\delta(v_u), \quad \forall n \ge T_+ \tag{4.16}$$

and, thanks to Corollary 2.2, we have convergence (4.14) where $w_\varepsilon := \bar u_{\varepsilon,v_u} \in \mathcal{R}_\varepsilon$.

Let us now $u(n)$, $n \in \mathbb{Z}$ be a complete bounded solution of (4.1) Then, arguing analogously, we deduce that there exist two equilibria $v_+, v_- \in \mathcal{R}_0$ such that

$$u(n) \in \mathcal{O}_\delta(v_-), \quad \forall n \le T_-; \quad u(n) \in \mathcal{O}_\delta(v_+), \quad \forall n \ge T_+; \tag{4.17}$$

and, thanks to Corollaries 2.1 and 2.2, we have convergences (4.15) with $w_\pm := \bar u_{\varepsilon,v_\pm}$. Theorem 4.1 is proven.

In order to formulate the nonautonomous analogue of formula (3.28), we first need to find the nonautonomous analogue of the global attractor $\mathcal{A}$ and the global unstable manifolds $\mathcal{M}^+_{0,v}$. Since, in contrast to (3.1), equation (4.1) depends explicitly on time n, it is natural to seek for the analogues of the sets $\mathcal{A}$ and $\mathcal{M}^+_{0,v}$ for equation (4.1) in the class of time dependent sets (i.e., in the form $\mathcal{A}_\varepsilon(l)$ and $\mathcal{M}^+_{\varepsilon,v}(l)$, $l \in \mathbb{Z}$). Then, analogously to (3.29), we give the following definition.

Definition 4.1. A kernel $\mathcal{K}_\varepsilon \subset l^\infty(\mathbb{Z})$ of equation (4.1) (or which is the same, of the dynamical process (4.4) associated with this equation) is a set of all bounded solutions $u \in l^\infty(\mathbb{Z})$ of (4.1). Let us define the nonautonomous analogue $\mathcal{A}_\varepsilon(l)$ of the attractor $\mathcal{A}$ for equation (4.1) as follows:

$$\mathcal{A}_\varepsilon(l) := \{u(l), \quad u \in \mathcal{K}_\varepsilon\}, \tag{4.18}$$

i.e., the attractor $\mathcal{A}_\varepsilon(l)$ is a section of the kernel $\mathcal{K}_\varepsilon$ at time $n = l$ in the terminology of [4].

As we will show below, (4.18) gives indeed a natural generalization of the global attractor's concept to the case of nonautonomous perturbations of regular attractors.

The following definition generalizes Definition 3.1 to the nonautonomous case.

Definition 4.2. Let $\varepsilon \le \varepsilon_0$ and ε_0 be small enough that assumptions of Theorem 2.1 are satisfied for every $v \in \mathcal{R}_0$. Then, for every $v \in \mathcal{R}_0$ (and the corresponding solution $\bar{u}_{\varepsilon,v} \in \mathcal{R}_\varepsilon$) and and every $l \in \mathbb{Z}$, the (global) unstable set $\mathcal{M}^+_{\varepsilon,v}(l)$ is defined as follows:

$$\begin{aligned} \mathcal{M}^+_{\varepsilon,v}(l) := \{u_l \in H, \quad \exists u \in l^\infty(\{-\infty, l\}, H), u(n) \text{ solves } (4.1) \text{ for } n \le l, \\ u(l) = u_l \text{ and } \lim_{n\to-\infty} \|u(n) - \bar{u}_{\varepsilon,v}(n)\|_H = 0\}, \end{aligned} \tag{4.19}$$

i.e., $\mathcal{M}^+_{\varepsilon,v}(l)$ consists of all $u_l \in H$ for which there exists a backward solution $u(n)$, $n \le l$ of problem (4.1) tending to $\bar{u}_{\varepsilon,v}(n)$ as $n \to -\infty$ such that $u(l) = u_l$. The difference with Definition 2.2 is that we do not require in (4.19) the solution u to belong to the small neighborhood of v.

Corollary 4.1. *Let the assumptions I–V be satisfied. Then, there exists positive ε_0 such that, for every $\varepsilon \le \varepsilon_0$ and every $l \in \mathbb{Z}$ the attractor $\mathcal{A}_\varepsilon(l)$ of equation (4.1) (in the sense of Definition 4.1) has the following structure:*

$$\mathcal{A}_\varepsilon(l) = \cup_{v\in\mathcal{R}_0} \mathcal{M}^+_{\varepsilon,v}(l) \tag{4.20}$$

Indeed, (4.20) is an immediate corollary of the second assertion of Theorem 4.1 and Definitions 4.1 and 4.2.

The next theorem shows that the sets $\mathcal{M}^+_{\varepsilon,v}(l)$ remain to be C^1-submanifolds if $\varepsilon > 0$ is small enough.

Theorem 4.2. *Let the assumptions I–V hold. Then, there exists small $\varepsilon_0 > 0$ such that, for every $\varepsilon \le \varepsilon_0$ and every $v \in \mathcal{R}_0$, the sets $\mathcal{M}^+_{\varepsilon,v}(l)$ are the finite dimensional C^1-submanifolds of H diffeomorphed to $H_+(v)$, for every fixed $l \in \mathbb{Z}$. In particular,*

$$\mathcal{M}^+_{\varepsilon,v}(l) \sim \mathcal{M}^+_{0,v} \sim H_+(v) = \mathbb{R}^{\operatorname{ind}_+(v)}. \tag{4.21}$$

Proof. The proof of this assertion is completely analogous to the proof of Theorem 3.1, so, we only indicate the main steps of this proof resting the details to the reader.

We first recall that, according to Theorem 2.3, we may fix ε_0 and δ such small that, for every $\varepsilon \le \varepsilon_0$ and every equilibrium $v \in \mathcal{R}_0$ (and for the corresponding $\bar{u}_{\varepsilon,v} \in \mathcal{R}_\varepsilon$), the associated local unstable sets

$$\bar{\mathcal{M}}_{\varepsilon,v}(l) := \mathcal{M}^{+,loc}_{\varepsilon,v}(l) \cap (\bar{u}_{\varepsilon,v}(l) + \mathcal{V}_+ \times H_-) \tag{4.22}$$

(for some neighborhood $\mathcal{V}_+ = \mathcal{V}_+(v) \subset H_+(v)$) are the C^1-submanifolds diffeomorphed to $H_+(v) \sim \mathbb{R}^{ind_+(v)}$, for every $l \in \mathbb{Z}$. Moreover, thanks to estimate (2.27), we may assume, without loss of generality that the constants ε_0 and δ and the neighborhood $\mathcal{V}_+$ are chosen in such way that

$$\bar{\mathcal{M}}_{\varepsilon,v}(l) \subset U_\varepsilon(l, l-1)\bar{\mathcal{M}}_{\varepsilon,v}(l-1), \ \forall l \in \mathbb{Z} \tag{4.23}$$

and, consequently, in order to introduce a structure of a C^1-manifold on the global unstable set $\mathcal{M}^+_{\varepsilon,v}(l)$, it is sufficient to use the following obvious decomposition:

$$\mathcal{M}^+_{\varepsilon,v}(l) = \cup_{n=1}^\infty U_\varepsilon(l, l-n)\bar{\mathcal{M}}_{\varepsilon,v}(l-n) \tag{4.24}$$

Finally, in order to verify that the topology induced on $\mathcal{M}^+_{\varepsilon,v}(l)$ by this structure coincides with the usual one induced by the embedding $\mathcal{M}^+_{\varepsilon,v}(l) \subset H$, it is sufficient to use Lemma 4.1 instead of Lemma 3.1 and the homeomorphisms

$$U_\varepsilon(l-n, l) := [U_\varepsilon(l, l-n)]^{-1} : \mathcal{A}_\varepsilon(l) \to \mathcal{A}_\varepsilon(l-n), \quad n \in \mathbb{N} \tag{4.25}$$

instead of the homeomorphisms $S_{-n} : \mathcal{A} \to \mathcal{A}$ (we recall that, due to assumption IV c) the maps $U_\varepsilon(l, l-n)$ are injective and, due to assumption V, the sets $\mathcal{A}_\varepsilon(l)$ are compact for all $\varepsilon \le \varepsilon_0$ and $l \in \mathbb{Z}$, so (4.25) are indeed the homeomorphisms). Thus, Theorem 4.2 is proven.

Corollary 4.2. *Let the assumptions of Theorem 4.2 hold. Then, for every $\varepsilon \le \varepsilon_0$ and every $l \in \mathbb{Z}$ the attractor $\mathcal{A}_\varepsilon(l)$ consists of the finite union of the finite-dimensional (unstable) C^1-submanifolds of H. Thus, the regular structure of the attractor preserves under small nonautonomous perturbations.*

Indeed, this assertion is an immediate corollary of (4.20) and Theorem 4.2.

Recall that, up to the moment, we know nothing about the attraction properties of the 'attractor' $\mathcal{A}_\varepsilon(l)$. The next theorem shows that the rate of attraction remains exponential for sufficiently small positive ε and, thus, the attractors $\mathcal{A}_\varepsilon(l)$ are indeed natural generalization of the concept of regular attractor to the nonautonomous case.

Theorem 4.3. *Let the assumptions I–V hold. Then, there exists positive ε_0 and α and a monotonic function Q such that, for every $\varepsilon \le \varepsilon_0$ and bounded subset B of H and every $l \in \mathbb{Z}$, the following estimate holds:*

$$\operatorname{dist}_H(U_\varepsilon(l+n, l)B, \mathcal{A}_\varepsilon(n+l)) \le Q(\|B\|_H)e^{-\alpha n}. \tag{4.26}$$

We emphasize that the constant α and the function Q are independent of ε, n and l.

Proof. We first note that, as in Theorem 3.2, it is sufficient to verify (4.26) for one fixed sufficiently small neighborhood B_0 of the attracting set $\mathcal{B}$. The rest of the proof of estimate (4.26) is completely analogous to the proof of Theorem 3.2, so we below only formulate the analogues of Lemmata 3.3–3.5 for the nonautonomous case resting the details to the reader.

Lemma 4.3. *Let the above assumptions hold. Then, for every sufficiently small $\delta > 0$, there exist $\varepsilon_0 = \varepsilon_0(\delta)$ and $T = T(\delta) > 0$ such that, for every $\varepsilon \le \varepsilon_0$ and every solution $u(n)$ of equation (4.1) with the initial data $u(l)$ belonging to B_0, one can find two sequences of numbers $\{T_-^i\}_{i=0}^M$ and $\{T_+^i\}_{i=1}^M$, where $M = M_u \le N := \#\mathcal{R}_0$ and a sequence of different equilibria $\{v_i\}_{i=1}^M \subset \mathcal{R}_0$ such that*

$$T_-^0 = l, \quad T_+^i - T_-^{i-1} \le T, \quad T_-^M = +\infty, \quad i = 1, \cdots, M \tag{4.27}$$

and

$$u(n) \in \mathcal{O}_\delta(v_i), \quad \forall n \in \{T_+^i, T_-^i\}, \quad i = 1, \cdots, M. \tag{4.28}$$

Indeed, this assertion is a standard corollary of Lemma 4.1 and 4.2 (see the proof of Lemma 3.3 for the details).

Lemma 4.4. *Let the above assumptions hold. Then, there exists positive constants C and K such that for every solution $u(n)$ starting from B_0 and every $k \ge l$,*

$$\operatorname{dist}_H(u(n+k), \mathcal{A}_\varepsilon(n+k)) \le Ce^{Kn} \operatorname{dist}_H(u(k), \mathcal{A}_\varepsilon(k)), \quad n \ge 0 \tag{4.29}$$

where the constants C and K are independent of n, ε, l and u.

Indeed, this assertion is an immediate corollary of the fact that the operators $S(\cdot) + P_\varepsilon(n, \cdot)$ are uniformly Lipschitz continuous on bounded subsets of H (due to assumption IV a)) (see the proof of Lemma 3.4 for the details).

Lemma 4.5. *Let the above assumptions hold and let the constants δ and ε_0 are chosen in such way that the assumptions of Theorem 2.4 be satisfied (in the δ-neighborhood) for every equilibrium $v \in \mathcal{R}_0$. Then, for every $\varepsilon \le \varepsilon_0$ and every solution $u(n)$ of equation (4.1) starting from B_0 at $n = l$ and satisfying*

$$u(n) \in \mathcal{O}_\delta(v), \quad n \in \{l', l' + T_0\}, \quad T_0 \in \{0, \infty\}, \quad l' \ge l, \tag{4.30}$$

the following estimate is valid:

$$\operatorname{dist}_H(v(l'+k), \mathcal{A}_\varepsilon(l'+k)) \le Ce^{-\gamma k} \left(\operatorname{dist}_H(u(l'), \mathcal{A}_\varepsilon(l'))\right)^\theta \tag{4.31}$$

where the positive constants C, γ and $0 < \theta < 1$ are independent of ε, l, l', k, T_0, v and u.

Indeed, this assertion is a standard corollary of Lemma 4.4 and Corollary 2.2 (see the proof of Lemma 3.5 for the details).

Finally, having Lemmata 4.3–4.5, we can verify (4.26) exactly as at the end of the proof of Theorem 3.2. Thus, Theorem 4.3 is proven.

The next corollary gives the estimate for the symmetric distance between the attractors $\mathcal{A}_\varepsilon(n)$ and the limit autonomous regular attractor $\mathcal{A}$.

Corollary 4.3. *Let the assumptions I–V hold. Then, for every $\varepsilon \le \varepsilon_0$, the symmetric distance between $\mathcal{A}_\varepsilon(l)$ and $\mathcal{A}$ possesses the following estimate:*

$$\operatorname{dist}_H^{symm}(\mathcal{A}_\varepsilon(l), \mathcal{A}) \le C\varepsilon^\kappa, \quad l \in \mathbb{Z}, \tag{4.32}$$

where the positive C and $0 < \kappa < 1$ are independent of ε and l.

Proof. Let $u_l \in \mathcal{A}_\varepsilon(l)$ be an arbitrary point of $\mathcal{A}_\varepsilon(l)$. Then, for every $k \in \mathbb{N}$, there exists $u_{l-k} \in \mathcal{A}_\varepsilon(l-k)$ such that $U_\varepsilon(l, l-k)u_{l-k} = u_l$. Let us consider a solution $\bar{u}(n)$, $n \geq l-k$ of the limit autonomous equation (3.1) such that $\bar{u}(l-k) = u_{l-k}$. Then, on the one hand, since the operator S is uniformly Lipschitz continuous on $\mathcal{B}$ and the operator P_ε satisfies (4.2), we have

$$\|\bar{u}(l) - u_l\| \leq C\varepsilon e^{Lk}, \tag{4.33}$$

where the positive constants C and L are independent of ε, u_l and k. On the other hand, since the attractor $\mathcal{A}$ is exponential, then, according to (3.40), we have

$$\operatorname{dist}_H(\bar{u}(l), \mathcal{A}) \leq Ce^{-\alpha k} \tag{4.34}$$

where the positive C and α are also independent of k and u_l. Combining (4.33) and (4.34) and using that $u_l \in \mathcal{A}_\varepsilon(l)$ and k are arbitrary, we deduce that

$$\operatorname{dist}_H(\mathcal{A}_\varepsilon(l), \mathcal{A}) \leq C \min_{k\in\mathbb{N}} \left(\varepsilon e^{Lk} + e^{-\alpha k}\right). \tag{4.35}$$

Fixing the parameter k in the right-hand side of (4.35) in an optimal way (i.e., such that $\varepsilon e^{Lk} \sim e^{-\alpha k}$), we have

$$\operatorname{dist}_H(\mathcal{A}_\varepsilon(l), \mathcal{A}) \leq C\varepsilon^\kappa \tag{4.36}$$

where $\kappa = \frac{\alpha}{L+\alpha}$. Thus, the first part of estimate (4.32) is proven. The second part

$$\operatorname{dist}_H(\mathcal{A}, \mathcal{A}_\varepsilon(l)) \leq C\varepsilon^\kappa \tag{4.37}$$

can be proven analogously, only instead of (3.30), we now need to use (4.26). Corollary 4.3 is proven.

Remark 4.1. Analogously to the autonomous case, estimates (4.26) can be slightly improved using the concept of N-composed trajectories introduced in Definition 3.2. Namely, for every bounded B, $l \in \mathbb{Z}$ and every trajectory $u(n) := U_\varepsilon(n, l)u_l$, $n \geq l$, there exists an N-composed trajectory $v_{u,l}(n)$ of the process $U_\varepsilon(n, l)$ such that $N \leq \#\mathcal{R}_0$ and

$$\|u(n) - v_{u,l}(n)\|_H \leq C\|u(l) - v_{u,l}(l)\|_H e^{-\alpha(n-l)},$$

where the positive constants C and α depend on B, but are independent of $\varepsilon \leq \varepsilon_0$, $l \in \mathbb{Z}$ and $u_l \in B$. The proof of this estimate can be obtained analogously to (3.55), see [3] and [8-9] for the details.

Moreover, estimate (4.32) also can be improved using the N-composed trajectories. Namely, for every complete trajectory $u_\varepsilon \in l^\infty(\mathbb{Z})$ of the perturbed process $U_\varepsilon(t, \tau)$, there exists an N-composed trajectory $u_0 \in l^\infty(\mathbb{Z})$ of the limit semigroup S_t belonging to the attractor $\mathcal{A}$ such that

$$\|u_\varepsilon(n) - u_0(n)\|_H \leq C\varepsilon^\kappa, \quad n \in \mathbb{Z},$$

where the positive constants C and κ are independent of ε and $\varepsilon \in [0, \varepsilon_0]$. This estimate can be obtained in a standard way based on Lemma 4.3 and the local analysis of the perturbed dynamics in the small neighborhood of the equilibria $z_0 \in \mathcal{R}_0$ (analogous to Theorems 2.2-2.4, see [3] and [8-9]).

In conclusion of this section, we clarify the relations between the nonautonomous regular attractor $\mathcal{A}_\varepsilon(l)$ obtained above and the known generalizations of the concept of a global attractor to the nonautonomous case. We recall that, up to the moment, there exist two main possibilities to generalize the global attractor to nonautonomous dynamical systems (see [4-5], [10], [13] and the references therein). The first one is the so-called *pullback* attractor's approach which treats the attractor for a nonautonomous equation as a time dependent set as well and the second one is the so-called *uniform* attractor's approach where the attractor for a nonautonomous equation remains time independent. For the convenience of the reader, we below recall the definitions of the attractors mentioned above.

Definition 4.3. Let $U_\varepsilon(n,l)$ be a dynamical (semi)process associated with equation (4.1). Then, a time dependent set $l \to \mathcal{A}_\varepsilon^{pb}(l)$, $l \in \mathbb{Z}$, is a *pullback* attractor if the following assumptions are satisfied:

1) For every $l \in \mathbb{Z}$, the set $\mathcal{A}_\varepsilon^{pb}(l)$ is compact in H;

2) The sets $\mathcal{A}_\varepsilon^{pb}(l)$ are strictly invariant, i.e., $U_\varepsilon(n,l)\mathcal{A}_\varepsilon^{pb}(l) = \mathcal{A}_\varepsilon^{pb}(n)$;

3) The sets $\mathcal{A}_\varepsilon^{pb}(l)$ possess the following *pullback* attraction property: for every bounded subset $B \subset H$, every $l \in \mathbb{Z}$ and every neighborhood $\mathcal{O}(\mathcal{A}_\varepsilon(l))$ there exists $T = T(l, B, \mathcal{O})$ such that

$$U_\varepsilon(l, l-n)B \subset \mathcal{O}(\mathcal{A}_\varepsilon^{pb}(l)), \quad \forall n \geq T. \tag{4.38}$$

Definition 4.4. Let $U_\varepsilon(n,l)$ be a dynamical (semi)process associated with equation (4.1). Then, a (time independent) set $\mathcal{A}_\varepsilon^{un} \subset H$ is a *uniform* attractor if the following assumptions are satisfied:

1) The set $\mathcal{A}_\varepsilon^{un}$ is compact in H;

2) The set $\mathcal{A}_\varepsilon^{un}$ possesses a uniform attraction property: for every bounded set $B \subset H$ and every neighborhood $\mathcal{O}(\mathcal{A}_\varepsilon^{un})$ of $\mathcal{A}_\varepsilon^{un}$ there exists $T = T(B)$ such that, for every $l \in \mathbb{Z}$,

$$U_\varepsilon(l+n, l)B \subset \mathcal{O}(\mathcal{A}_\varepsilon^{un}), \quad \forall n \geq T. \tag{4.39}$$

3) The set $\mathcal{A}_\varepsilon^{un}$ is a minimal set which satisfies assumptions 1) and 2).

We emphasize that, in contrast to (4.39), the attractors $\mathcal{A}_\varepsilon^{pb}(l)$ possess only the pullback attraction property (4.38) and, in general, the set $U_\varepsilon(l+k, l)B$ may not converge to $\mathcal{A}_\varepsilon^{pb}(l+k)$ as $k \to +\infty$. The next corollary shows that it is not the case for the nonautonomous perturbations of regular attractors.

Corollary 4.4. *Let the assumptions I–V hold. Then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \leq \varepsilon_0$ the following assertions are valid:*

1) The pullback attractor $\mathcal{A}_\varepsilon^{pb}(l)$ of the dynamical process associated with equation (4.1) *coincides with the nonautonomous regular attractor $\mathcal{A}_\varepsilon(l)$ obtained above:*

$$\mathcal{A}_\varepsilon^{pb}(l) = \mathcal{A}_\varepsilon(l), \quad l \in \mathbb{Z} \tag{4.40}$$

and, consequently, the pullback attractor $\mathcal{A}_\varepsilon^{pb}(l)$ possesses a uniform exponential attraction property (4.26) *and representation* (4.20) *as a finite union of the finite-dimensional C^1-submanifolds of H.*

2) The uniform attractor $\mathcal{A}_\varepsilon^{un}$ of the dynamical process associated with problem (4.1) *can be described as follows:*

$$\mathcal{A}_\varepsilon^{un} = \big[\cup_{l\in\mathbb{Z}}\mathcal{A}_\varepsilon(l)\big]_H = \big[\cup_{l\in\mathbb{Z}}\mathcal{A}_\varepsilon^{pb}(l)\big]_H, \tag{4.41}$$

where $[\,\cdot\,]_H$ stands for the closure in the space H.

Proof. Indeed, equality (4.40) is an immediate corollary of Definitions 4.1 and 4.2 and estimate (4.26). Let us verify equality (4.41). To this end, we note that the first assumption of Definition 4.3 holds, since $\mathcal{A}_\varepsilon(l)\subset\mathcal{B}$ for all $l\in\mathbb{Z}$ and the attracting set is compact in H. The second assumption is also satisfied due to estimate (4.26) and, finally, the third assumption is satisfied, since all the sets which satisfy 1) and 2) should contain $\mathcal{A}_\varepsilon(l)$ for all $l\in\mathbb{Z}$. Corollary 4.4 is proven.

§5 Nonautonomous regular attractors: the case of continuous time.

In this section, we extend the results obtained above to the case of continuous time. To be more precise, we consider a family $\{U_\varepsilon(t,\tau),\ t,\tau\in\mathbb{R},\ t\ge\tau\}$ of dynamical processes (with continuous time $t\in\mathbb{R}$) in the space H depending on the parameter $\varepsilon\in[0,1]$. It is now worth to recall that, by the definition of the dynamical process, the operators $U_\varepsilon(t,\tau): H\to H$ should satisfy the following generalization of the semigroup identity:

$$U_\varepsilon(t,\tau) = U_\varepsilon(t,s)\circ U_\varepsilon(s,\tau),\quad \forall t\ge s\ge\tau, \tag{5.1}$$

which is naturally satisfied for solving operators of *nonautonomous* evolution problems, see [4] and [10] and Section 6 below.

As in the previous section, we assume that the processes $U_\varepsilon(t,\tau)$ tend as $\varepsilon\to 0$ to the limit autonomous semigroup $S_{t-\tau}$. That is the reason why we split the operators $U_\varepsilon(t,\tau)$ as follows:

$$U_\varepsilon(t,\tau) = S_{t-\tau} + \bar P_\varepsilon(t,\tau), \tag{5.2}$$

where $\{S_t, t\ge 0\}$ is the limit autonomous semigroup and $\bar P_\varepsilon(t,\tau): H\to H$ is a perturbation.

Then, in order to handle the case of continuous time, we modify the assumptions I–V as follows.

VI. We assume that the operators $S(v):=S_1v$ and $P_\varepsilon(n,v):=\bar P_\varepsilon(n,n-1)v$, $n\in\mathbb{Z}$, satisfy the assumptions I–V of Sections 3 and 4 and, in addition,

a) Every equilibrium $z_0\in\mathcal{R}_0$ of the map $S(v):=S_1v$ is also an equilibrium of the continuous semigroup S_t (i.e., $S_tz_0=z_0$, for all $t\in\mathbb{R}_+$);

b) The operators $U_\varepsilon(t,\tau)\in C^1(H,H)$ for all $t\ge\tau$ and, for every $\varepsilon\in[0,1]$, $\tau\in\mathbb{R}$ and $s\in[0,1]$, we have

$$\|U_\varepsilon(\tau+s,\tau)v\|_H + \|U'_\varepsilon(\tau+s,\tau)\|_{\mathcal{L}(H,H)} \le Q_0(\|v\|_H) \tag{5.3}$$

where the monotonic function Q_0 is independent of ε, v, τ and s;

c) The operator $U_\varepsilon(\tau+s,\tau)$ is injective for every fixed $\varepsilon\in[0,1]$ $\tau\in\mathbb{R}$ and $s\in[0,1]$ and the kernel of its derivative $U'_\varepsilon(\tau+s,\tau)(v)$ is trivial for all $v\in H$;

d) The operators $\bar{P}_\varepsilon(t,\tau)$ are uniformly small in the following sense:

$$\|\bar{P}_\varepsilon(\tau+s,\tau)v\|_H + \|\bar{P}'_\varepsilon(\tau+s,\tau)(v)\|_{\mathcal{L}(H,H)} \le C\varepsilon,\ \varepsilon, s\in[0,1],\ \tau\in\mathbb{R}, \tag{5.4}$$

where the constant C depends (monotonically) on $\|v\|_H$, but is independent of ε, v, τ and s.

We are going to apply the theory developed in the previous sections for investigating the processes $\{U_\varepsilon(t,\tau),\ \tau\in\mathbb{R},\ t\ge\tau\}$ with continuous time. To this end, we first consider the restriction of these processes to discrete time, namely, $\{U_\varepsilon(m,n),\ m,n\in\mathbb{Z}, m\ge n\}$. We then note that, thanks to identity (5.1), every trajectory $u(n)=U_\varepsilon(n,l)u_l$, $u_l\in H$, $n\ge l$ of that processes satisfies the following difference equation:

$$u(n) = U_\varepsilon(n,n-1)u(n-1) \equiv S(u(n-1)) + P_\varepsilon(n,u(n-1)),\quad u(l)=u_l, \tag{5.5}$$

where $S(v):=S_1v$ and $P_\varepsilon(n,v):=\bar{P}_\varepsilon(n,n-1)v$ and, vise versa, every solution of (5.5) is a trajectory of $U_\varepsilon(n,l)$. Thus, studying of the trajectories of $U_\varepsilon(n,l)$ is equivalent to the study of solutions of the difference equation (5.5).

We now note that equation (5.5) has the form of (4.1) and (due to VI) the operators $S(\cdot)$ and $P_\varepsilon(n,\cdot)$ satisfy assumptions I–V of Sections 3 and 4. Thus, according to Section 3, the limit autonomous semigroup $\{S_n, n\in\mathbb{N}\}$ possesses the regular attractor $\mathcal{A}_0$ and there exists $\varepsilon_0>0$ such that, for every $\varepsilon\le\varepsilon_0$, the corresponding process $U_\varepsilon(m,n)$ possesses the nonautonomous regular attractors $n\to\mathcal{A}_\varepsilon(n)$. In particular, for every equilibrium $z_0\in\mathcal{R}_0$ of the map S, we have a unique trajectory $\bar{u}_{\varepsilon,z_0}\in l^\infty(\mathbb{Z})$ of the process $U_\varepsilon(m,n)$ belonging to the small neighborhood of z_0, the unstable sets $n\to\mathcal{M}^+_{\varepsilon,z_0}(n)$ of $\bar{u}_{\varepsilon,z_0}$ are the C^1-submanifolds in H and the attractor $\mathcal{A}_\varepsilon(n)$ is a finite union of that unstable manifolds, see (4.20) and (4.21). Moreover, the attractors $\mathcal{A}_\varepsilon(n)$ satisfy estimates (4.26) and (4.29).

Our task now is to extend this result to the case of continuous time. To this end, for every $t=n+s$, $n\in\mathbb{Z}$ and $0\le s\le 1$, we set

$$\begin{aligned}&\bar{u}_{\varepsilon,z_0}(t) := U_\varepsilon(n+s,n)\bar{u}_{\varepsilon,z_0}(n),\\ &\qquad\mathcal{M}^+_{\varepsilon,z_0}(t) := U_\varepsilon(n+s,n)\mathcal{M}^+_{\varepsilon,z_0}(n),\ \mathcal{A}_\varepsilon(t):=U_\varepsilon(n+s,n)\mathcal{A}_\varepsilon(n).\end{aligned} \tag{5.6}$$

Obviously, $U_\varepsilon(t,\tau)\bar{u}_{\varepsilon,z_0}(\tau)=\bar{u}_{\varepsilon,z_0}(t)$ and analogous formulae hold for the sets $\mathcal{M}^+_{\varepsilon,z_0}(t)$ and $\mathcal{A}^+_\varepsilon(t)$. The following theorem shows that the sets $t\to\mathcal{A}_\varepsilon(t)$ thus defined are indeed the nonautonomous regular attractors for the processes $U_\varepsilon(t,\tau)$ with continuous time.

Theorem 5.1. *Let assumption VI be satisfied. Then, for every $\varepsilon\in[0,\varepsilon_0]$, the following conditions hold:*

The second assumption is an immediate corollary of the analogous fact

1) The sets $t\to\mathcal{A}_\varepsilon(t)$ defined by (5.6) are the nonautonomous regular attractors for the process $U_\varepsilon(t,\tau)$ with continuous time and

$$\mathcal{A}_\varepsilon(t) = \cup_{z_0\in\mathcal{R}_0}\mathcal{M}^+_{\varepsilon,z_0}(t),\quad t\in\mathbb{R} \tag{5.7}$$

where the sets $\mathcal{M}^+_{\varepsilon,z_0}(t)$ are the unstable manifolds of continuous trajectory $\bar{u}_{\varepsilon,z_0}(t)$.

2) Every trajectory $u(t)$, $t\ge\tau$ of that process stabilizes as $t\to+\infty$ to one of the trajectories belonging to $\mathcal{R}_\varepsilon:=\{\bar{u}_{\varepsilon,z_0}(t),\ \ z_0\in\mathcal{R}_0\}$ and every complete bounded trajectory is a heteroclinic orbit between the trajectories belonging to $\mathcal{R}_\varepsilon$.

3) The attractors $t \to \mathcal{A}_\varepsilon(t)$ are uniformly exponential in the following sense: for every bounded subset B of H and every $\tau \in \mathbb{R}$, we have

$$(5.8) \qquad \operatorname{dist}_H(U_\varepsilon(\tau+t,\tau)B, \mathcal{A}_\varepsilon(\tau+t)) \le Q(\|B\|_H)e^{-\alpha t}, \quad t \in \mathbb{R}_+$$

where the positive constant α and the monotonic function Q are independent of ε, τ, t and B.

4) The attractors $t \to \mathcal{A}_\varepsilon(t)$ tend as $\varepsilon \to 0$ to the limit attractor $\mathcal{A}_0$, i.e.,

$$(5.9) \qquad \operatorname{dist}_H^{sym}(\mathcal{A}_\varepsilon(t), \mathcal{A}_0) \le C\varepsilon^\kappa, \quad t \in \mathbb{R}$$

where the positive constants C and κ are independent of ε and t.

Proof. We first note that, due to the uniform boundedness of the operators $U_\varepsilon(\tau+s,\tau)$ (see (5.3)), every bounded trajectory $u(n)$, $n \in \mathbb{Z}$ of $U_\varepsilon(m,n)$ (with discrete time) generates a bounded trajectory of $U_\varepsilon(t,\tau)$ (with continuous time) via

$$(5.10) \qquad u(n+s) := U_\varepsilon(n+s,n)u(n), \quad n \in \mathbb{Z}, \quad s \in (0,1)$$

and, vise versa, every bounded continuos trajectory $u(t)$, $t \in \mathbb{R}$, obviously generates a bounded discrete trajectory $u(n)$, $n \in \mathbb{Z}$. Moreover, according to Definition 4.1, the discrete attractors $\mathcal{A}_\varepsilon(n)$ are generated by all bounded discrete trajectories of $U_\varepsilon(m,n)$. Then, due to (4.18) and (5.6), the sets $t \to \mathcal{A}_\varepsilon(t)$, $t \in \mathbb{R}$, are generated by all bounded trajectories of $U_\varepsilon(t,\tau)$ (with continuous time), i.e.,

$$(5.11) \qquad \mathcal{A}_\varepsilon(\tau) = \mathcal{K}_\varepsilon^{cont}\big|_{t=\tau}$$

where $\mathcal{K}_\varepsilon^{cont}$ is a set of all bounded trajectories of $U_\varepsilon(t,\tau)$ (with continuous time). Thus, the sets $t \to \mathcal{A}_\varepsilon(t)$ defined by (5.6) satisfy the continuos analogue of Definition 4.1.

Analogously, due to the uniform boundedness of the derivatives $U'_\varepsilon(\tau+s,\tau)$, we have

$$(5.12) \qquad \|u_1(n+s) - u_2(n+s)\|_H \le C\|u_1(n) - u_2(n)\|_H, \quad n \in \mathbb{Z}, \quad s \in [0,1]$$

for every two trajectories u_1 and u_2 of $U_\varepsilon(t,\tau)$. Consequently, $u(t)$ tends to $\bar u_{\varepsilon,z_0}(t)$ as $t \to -\infty$ (continuous time, $t \in \mathbb{R}$) if and only if $u(n)$ tends to $\bar u_{\varepsilon,z_0}(n)$ (discrete time, $n \in \mathbb{Z}$). This shows that the sets $t \to \mathcal{M}^+_{\varepsilon,z_0}(t)$, $t \in \mathbb{R}$, defined by (5.6) are indeed the unstable sets of the continuous trajectory $\bar u_{\varepsilon,z_0}(t)$. We now recall that, due to Theorem 4.2, the sets $\mathcal{M}^+_{\varepsilon,z_0}(n)$ (with integer n) are finite-dimensional C^1-submanifolds of H. Then, the injectivity of $U_\varepsilon(\tau+s,\tau)$, the fact that the kernel $\ker U'_\varepsilon(\tau+s,\tau)(v) = \{0\}$ for all $v \in H$ and the continuation formulae (5.6) imply in a standard way that the sets $\mathcal{M}^+_{\varepsilon,z_0}(t)$ are C^1-submanifolds of H diffeomorphed to $\mathcal{M}^+_{0,z_0}$ for the noninteger t as well. Finally, formula (5.7) is an immediate corollary of the analogous representation of the attractors for discrete time and the continuation formulae (5.6). Thus, we have verified that $t \to \mathcal{A}_\varepsilon(t)$ are indeed the nonautonomous regular attractors for $U_\varepsilon(t,\tau)$ (with continuous time) and assertion 1) of Theorem 5.1 is verified.

The second assumption is an immediate corollary of the analogous fact for discrete time (see Theorem 4.1) and estimate (5.12). The exponential attraction property (5.8) also follows from the analogous estimate (4.26) for discrete time, uniform boundedness (5.3) and estimate (5.12).

Thus, it only remains to verify estimate (5.9). To this end, we note that, if $u_\varepsilon(t)$ and $u_0(t)$ be the trajectories of $U_\varepsilon(t,\tau)$ and $S_{t-\tau}$ respectively, then (thanks to estimates (5.3) and (5.4)) we have

$$\|u_\varepsilon(l+s)-u_0(l+s)\|_H \le C\|u_\varepsilon(l)-u_0(l)\|_H + C\varepsilon \tag{5.13}$$

where the constant C is independent of the trajectories u_ε and u_0 (belonging to some bounded subset of H) and of $l \in \mathbb{Z}$. Estimate (5.9) is now an immediate corollary of the analogous estimate (4.32) for discrete time and inequality (5.13). Thus, Theorem 5.1 is proven.

Remark 5.1. In particular, fixing $\varepsilon = 0$ in Theorem 5.1, we obtain that $\mathcal{A}_0$ is a regular attractor for the semigroup S_t with continuous time. The only difficulty here is that the continuation $z_0(t) := S_t z_0$ of the equilibrium $z_0 \in \mathcal{R}_0$ of the map $S = S_1$ is a priori a 1-periodic trajectory (and not necessarily an equilibrium of the semigroup S_t with continuous time). In order to exclude this case we impose the additional assumption VIa). We also note that this assumption is automatically satisfied if S_t possesses a global Lyapunov function (or if all of the trajectories of S_t are continuous with respect to t).

Remark 5.2. Let us assume that we are given a family of processes $U_{\varepsilon,\xi}(t,\tau) := S_{t-\tau} + \bar{P}_{\varepsilon,\xi}(t,\tau)$ depending on the additional parameter $\xi \in \Lambda$ (or, more general, $\xi \in \Lambda_\varepsilon$ where the sets Λ_ε can be different for different ε) such that all of the assumptions I-VI are satisfied *uniformly* with respect to ξ (i.e., the set attraction set $\mathcal{B}$ from condition V is also uniform with respect to ξ (i.e., (4.5) holds uniformly with respect to $\xi \in \Lambda$) and estimates (5.3) and (5.4) are also uniform with respect to $\xi \in \Lambda$. Then, it is not difficult to verify analyzing the proofs given above that estimates (5.8) and (5.9) for the corresponding regular attractors hold uniformly with respect to ξ. This simple observation allows to apply the above theory for constructing the regular attractors for the skew-product systems, see [6], (15) and [17].

§6 An example: reaction diffusion system.

In this concluding section, we illustrate the abstract results obtained above on a simplest model example of a reaction-diffusion system in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \partial_t u = a\Delta_x u - f(u) + g_0 + \varepsilon g_\varepsilon(t), \\ u\big|_{t=\tau} = u_\tau, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{6.1}$$

(more complicated applications will be considered in the forthcoming papers [15] and [17]). Here $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to the variables $x = (x^1, x^2, x^3)$, a is a given diffusion matrix which satisfies $a = a^* > 0$, $\varepsilon \ge 0$ is a small parameter and $f(u)$ and $g_0 + \varepsilon g_\varepsilon(t)$ are given nonlinear interaction function and the external forces respectively.

It is assumed that the nonlinearity $f \in C^2(\mathbb{R}^k, \mathbb{R}^k)$ has a gradient structure:

$$f(v) = \nabla_v F(v), \quad F \in C^3(\mathbb{R}^k, \mathbb{R}) \tag{6.2}$$

and satisfies the following standard dissipativity and growth restrictions:

$$1.\ f(v).v \ge -C, \quad 2.\ f'(v) \ge -K, \quad 3.\ |f''(v)| \le C(1+|v|^p), \quad p < 3. \tag{6.3}$$

Here and below $u.v$ denotes the standard inner product in $\mathbb{R}^k$ and $f'(v) \geq -K$ means that $f'(v)\theta.\theta \geq -K\|\theta\|^2$ for all $\theta \in \mathbb{R}^k$.

We also assume that the autonomous external forces $g_0 \in [L^2(\Omega)]^k$ and the nonautonomous external forces $g_\varepsilon(t)$ belong to the space $L^\infty(\mathbb{R}, L^2(\Omega))$ and are uniformly bounded (with respect to ε) in this space, i.e.,

$$\|g_\varepsilon\|_{L^\infty(\mathbb{R},L^2(\Omega))} \leq C, \tag{6.4}$$

where C is independent of ε.

Let us first consider the limit autonomous case $\varepsilon = 0$. It is well known (see [3]) that, under above assumptions, equation (6.1) generates a dissipative semigroup $\{S_t, t \geq 0\}$ in the phase space $H := [W_2^1(\Omega)]^k$ via

$$S_t u_0 := u(t), \quad u(t) \text{ is a unique solution of (6.1) with } u(0) = u_0 \tag{6.5}$$

which possesses a compact global attractor $\mathcal{A} = \mathcal{A}_0$ in H. Moreover, according to [3, Chapter VII], this semigroup belongs to the class $C^{1+\alpha}(H, H)$ for all $0 < \alpha < 1$ and

$$\|S_t(v)\|_{C^{1+\alpha}} \leq Q(\|v\|_H)e^{Kt} \tag{6.6}$$

where the positive constant K and the monotonic function Q are independent of t.

Let us verify the map $S = S_1$ satisfies the assumptions of Section 3. Indeed, since $S \in C^{1+\alpha}(H, H)$ then the Frechet derivative S' is uniformly bounded on bounded subsets of H. Thus, assumption Ia) and assumption (5.3) (with $\varepsilon = 0$) are satisfied. The injectivity assumptions Ib) and VIc) (with $\varepsilon = 0$) are standard corollaries of the backward uniqueness theory for the solutions of parabolic equations, see e.g., [1] and [3]. Thus assumption I is verified.

In contrast to this, the hyperbolicity assumption II is not automatically satisfied for all external forces g_0. Nevertheless, it is known (see [3]) that II is satisfied for *generic* external forces g_0 in $L^2(\Omega)$ (i.e., the set of g_0s for which II is satisfied is open and dense in $L^2(\Omega)$). Thus, we assume from now on that g_0 is chosen in such way that II is satisfied.

Let us verify assumption III. Indeed, IIIa) holds since the semigroup S_t possesses the global attractor $\mathcal{A}$ in H. Moreover, since the nonlinearity f is gradient, then the functional

$$L(u) := \int_\Omega a\nabla u(x).\nabla u(x) + 2F(u(x)) - 2g_0.u(x)\,dx$$

gives the global Lyapunov function for the semigroup S_t, see [3]. Thus, assumptions IIIb) and IIIc) are also verified.

Applying the abstract result of Section 3 to the semigroup S_n generated by equation (6.1) and noting that the continuation assumptions VI (with $\varepsilon = 0$) are also satisfied for the limit semigroup S_t, we obtain the following result of [3].

Theorem 6.1. *Let the above assumptions hold. Then, for generic external forces g_0, the attractor $\mathcal{A}_0$ of the semigroup S_t associated with problem* (6.1) *is regular (i.e., consists of a finite union of the finite dimensional unstable manifolds of the* *equilibria $z_0 \in \mathcal{R}_0$) and exponential (i.e., the image of every bounded set is attracted exponentially to the attractor $\mathcal{A}_0$, see Section 3).*

We now return to the nonautonomous problem (6.1). Then, as proven, e.g., in [4], for every $u_\tau \in H$ and every $\tau \in \mathbb{R}$ equation (6.1) possesses a unique solution $u(t)$ which satisfies

$$\|u(t)\|_H \le Q(\|u_\tau\|_H)e^{-\alpha(t-\tau)} + C_0 \tag{6.7}$$

where the positive constants C_0 and α are independent of $\varepsilon \in [0,1]$, $\tau \in \mathbb{R}$ and $t \ge \tau$. Consequently, the solving operators $u_\tau \to U_\varepsilon(t,\tau)u_0$ are well defined via

$$U_\varepsilon(t,\tau)u_\tau := u(t), \quad \text{where } u(t) \text{ is a unique solution of (6.1).} \tag{6.8}$$

Moreover, as known (see e.g., [4]), the operators $U_\varepsilon(t,\tau)$ satisfy identity (5.1) for every fixed $\varepsilon \in [0,1]$ and, thus, generate indeed a dynamical process in H (for every fixed ε). Let us verify that these processes satisfy the assumptions of Sections 4 and 5.

Indeed, arguing exactly as in the autonomous case, see [3, Chapter VII], we deduce that $U_\varepsilon(t,\tau) \in C^{1+\alpha}(H,H)$ for every $\tau \in \mathbb{R}_+$ and $t \ge \tau$ and, analogously to (6.6),

$$\|U_\varepsilon(\tau+s,\tau)(v)\|_{C^{1+\alpha}} \le Q(\|v\|_H)e^{Ks} \tag{6.9}$$

where the positive constant C and the monotonic function Q are independent of $\varepsilon \in [0,1]$, $\tau \in \mathbb{R}$ and $s \in \mathbb{R}_+$. Thus, the regularity assumption IVa) and estimate (5.3) are satisfied.

Moreover, arguing in a standard way, see e.g. [3] and [4], we can check that

$$\|U_\varepsilon(\tau+s,\tau)v - S_s v\|_H + \|U'_\varepsilon(\tau+s,\tau)(v) - S'_s(v)\|_{\mathcal{L}(H,H)} \le \varepsilon Q(\|v\|_H)e^{Kt} \tag{6.10}$$

where the positive constant K and the monotonic function Q are independent of ε, s and τ. Thus, assumptions IVb) and (5.4) is also satisfied. As in the autonomous case, the injectivity assumptions IVc) and VIc) are corollaries of the standard backward uniqueness theory for the parabolic equations, see [1] and [3-4]. Thus, assumption IV is verified.

Let us verify assumption V of Section 4. To this end, we note that, thanks to (6.9) the R-ball $\mathcal{B}_R$ of H centered at zero will be a *uniformly* (with respect to ε and τ) absorbing set for the family of processes $U_\varepsilon(t,\tau)$ if R is large enough, but this set is not compact in H. In order to construct the compact one, we recall that, due to the standard smoothing property for parabolic equations, for every $0 < \delta < 1$ and every solution $u(t)$ of (6.1), we have

$$\|u(\tau+1)\|_{H^{1+\delta}(\Omega)} \le \bar{Q}_\delta(\|u(\tau)\|_H) \tag{6.11}$$

where the monotonic function Q_δ depends on δ, but is independent of $\varepsilon \in [0,1]$, $\tau \in \mathbb{R}$ and $u(\tau) \in H$, see [3], [4] and [7] for the details. Thus, the $\bar{Q}_\delta(R)$-ball of $H^{1+\delta}(\Omega)$ (for some fixed positive δ) centered at zero will be the desired compact uniformly absorbing set for the family $U_\varepsilon(t,\tau)$ and assumption V is verified.

It remains to note that the continuation assumption VI is automatically satisfied due to (6.9) and (6.10). Thus, all of the assumptions of Sections 3–5 are verified for problem (6.1) and, consequently, the following result holds.

Theorem 6.2. *Let the above assumptions hold and let the external forces g_0 is chosen in such way that the hyperbolicity assumption II is satisfied (and, consequently, the limit autonomous equation* (6.1) *possesses a regular attractor). Then, there exists $\varepsilon_0 = \varepsilon_0(f, g_0, \Omega) > 0$ such that, for every $\varepsilon \le \varepsilon_0$ the dynamical process $U_\varepsilon(t, \tau)$ possesses the nonautonomous regular attractors $t \to \mathcal{A}_\varepsilon(t)$, $t \in \mathbb{R}$. Thus,*

1) For every equilibrium $z_0 \in \mathcal{R}_0$ of the limit autonomous equation (6.1) *there exist a unique solution $\bar{u}_{\varepsilon,z_0}(t)$ of the nonautonomous problem* (6.1) *belonging to small neighborhood of z_0.*

2) The unstable sets $t \to \mathcal{M}^+_{\varepsilon,z_0}(t)$ of every such solution are the C^1-submanifolds of H diffeomorphed to $\mathbb{R}^{\kappa_{z_0}}$ where κ_{z_0} is the instability index of the equilibrium z_0 and the attractors $\mathcal{A}_\varepsilon(t)$ possess the following description:

$$\mathcal{A}_\varepsilon(t) = \cup_{z_0 \in \mathcal{R}_0} \mathcal{M}^+_{\varepsilon,z_0}(t), \quad t \in \mathbb{R}. \tag{6.12}$$

3) The attractors $t \to \mathcal{A}_\varepsilon(t)$ are uniformly exponential in the following sense: for every bounded subset B of H

$$\operatorname{dist}_H(U_\varepsilon(\tau + t, \tau)B, \mathcal{A}_\varepsilon(t + \tau)) \le Q(\|B\|_H)e^{-\alpha t}, \tag{6.13}$$

where the positive constant α and the monotonic function Q are independent of ε, t and τ.

4) The attractors $t \to \mathcal{A}_\varepsilon(t)$ are close to the limit autonomous attractor $\mathcal{A}_0$:

$$\operatorname{dist}^{sym}_H(\mathcal{A}_\varepsilon(t), \mathcal{A}_0) \le C\varepsilon^\kappa, \tag{6.14}$$

where the positive constants C and κ are also independent of ε, ξ, t and τ.

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