DISSIPATIVE DYNAMICAL SYSTEMS OF INFINITE DIMENSIONS

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Introduction.

A dynamical system (DS) is a system which evolves with respect to the time. To be more precise, a DS $(S(t), \Phi)$ is determined by a phase space Φ which consists of all possible values of the parameters describing the state of the system and an evolution map $S(t) : \Phi \to \Phi$ which allows to find the state of the system at time $t > 0$ if the initial state at $t = 0$ is known. Very often, in mechanics and physics, the evolution of the system is governed by systems of differential equations. If the system is described by ordinary differential equations (ODE),

$$\frac{d}{dt}y(t) = F(t, y(t)), \quad y(0) = y_0, \quad y(t) := (y_1(t), \cdots, y_N(t)), \tag{0.1}$$

for some nonlinear function $F : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, we have a so-called *finite-dimensional* DS. In that case, the phase space Φ is some (invariant) subset of $\mathbb{R}^N$ and the evolution operator $S(t)$ is defined by

$$S(t)y_0 := y(t), \quad y(t) \text{ solves (0.1).} \tag{0.2}$$

We also recall that, in the case where equation (0.1) is *autonomous* (i.e. does not depend explicitly on the time), the evolution operators $S(t)$ generate a semigroup on the phase space Φ, i.e.

$$S(t_1 + t_2) = S(t_1) \circ S(t_2), \quad t_1, t_2 \in \mathbb{R}_+. \tag{0.3}$$

Now, in the case of a distributed system whose initial state is described by functions $u_0 = u_0(x)$ depending on the *spatial* variable x, the evolution is usually governed by *partial* differential equations (PDE) and the corresponding phase space Φ is some infinite-dimensional function space (e.g., $\Phi := L^2(\Omega)$ or $\Phi := L^\infty(\Omega)$ for some domain $\Omega \subset \mathbb{R}^N$). Such DS are usually called *infinite-dimensional.*

The qualitative study of DS of finite dimensions goes back from the beginning of the 20th century with the pioneering works of Poincaré on the N-body problem (one should also acknowledge the contributions of Lyapunov on stability and of Birkhoff on minimal sets and the ergodic theorem). One of the most surprising and significant fact discovered at the very beginning of the theory is that even relatively simple equations can generate very complicated chaotic behaviors. Moreover, these types of systems are extremely sensitive to initial conditions (the trajectories with close but different initial data diverge exponentially). Thus, in spite of the deterministic nature of the system (we recall that it is generated by a system of ODE, for which we usually have the unique solvability theorem), its temporal evolution is *unpredictable* on time scales larger than some critical time T_0 (which depends obviously on the error of approximation and on the rate of divergence of close trajectories) and can show typical stochastic behaviors. To the best of our knowledge, one of the first ODE for which such types of behaviors were established is the physical pendulum parametrically perturbed by time periodic external forces,

$$y''(t) + \sin(y(t))(1 + \varepsilon \sin(\omega t)) = 0, \tag{0.4}$$

where ω and $\varepsilon > 0$ are physical parameters. We also mention the more recent (and more relevant for our topic) famous example of the Lorenz system which is defined by the following system of ODE in $\mathbb{R}^3$:

$$\begin{cases} x' = \sigma(y - x), \\ y' = -xy + rx - y, \\ z' = xy - bz, \end{cases} \tag{0.5}$$

where σ, r and b are some parameters. These equations are obtained by truncation of the Navier-Stokes equations and give an approximate description of a horizontal fluid layer heated from below. The warmer fluid formed at the bottom tends to rise, creating convection currents. This is similar to what happens in the earth's atmosphere. For a sufficiently intense heating, the time evolution has a sensitive dependence on the initial conditions, thus representing a very irregular and chaotic convection. This fact was used by Lorenz to justify the so-called "butterfly effect", a metaphor for the imprecision of weather forecast.

The theory of DS in finite dimensions has been extensively developed during the 20th century, due to the efforts of many famous mathematicians (such as Anosov, Arnold, LaSalle, Sinai, Smale, etc.) and, nowadays, very much is known on the chaotic behaviors in such systems, at least in low dimensions. In particular, it is known that, very often, the trajectories of a chaotic system are localized, up to a transient process, in some subset of the phase space having a very complicated fractal geometric structure (e.g., locally homeomorphic to the cartesian product of R^m and some Cantor set) which, thus, accumulates the nontrivial dynamics of the system (the so-called strange attractor). The chaotic dynamics on such sets are usually described by *symbolic* dynamics generated by Bernoulli shifts on the space of sequences. We also note that, nowadays, a mathematician has a large amount of different concepts and methods for the extensive study of concrete chaotic DS in finite dimensions. In particular, we mention here different types of bifurcation theories (including the KAM theory and the homoclinic bifurcation theory with related Shilnikov chaos), the theory of hyperbolic sets, stochastic description of deterministic processes, Lyapunov exponents and entropy theory, dynamical analysis of time series, etc.

We now turn to infinite-dimensional DS generated by PDE. A first important difficulty which arises here is related to the fact that the analytic structure of a PDE is *essentially* more complicated than that of an ODE and, in particular, we do not have in general the unique solvability theorem as for ODE, so that even finding the proper phase space and the rigorous construction of the associated DS can be a highly nontrivial problem. In order to indicate the level of difficulties arising here, it suffices to recall that, for the three-dimensional Navier-Stokes system (which is one of the most important equation of mathematical physics), the required associated DS has not been constructed yet. Nevertheless, there exists a large number of equations for which the problem of the global existence and uniqueness of a solution has been solved. Thus, the question of extending the highly developed finite-dimensional DS theory to infinite dimensions arises naturally.

One of the first and most significant result in that direction was the development of the theory of integrable Hamiltonian systems in infinite dimensions and the explicit resolution (by inverse scattering methods) of several important *conservative* equations of mathematical physics (such as the Korteweg-de Vries (and the generalized Kadomtsev-Petiashvilli hierarchy), the Sine-Gordon, the nonlinear Schrödinger, etc., equations). Nevertheless, it is worth noting that integrability is a very rare phenomena, even among ODE, and this theory is clearly *insufficient* to understand the dynamics arising in PDE. In particular, there exist many important equations which are essentially out of reach of this theory.

One of the most important class of such equations consists of the so-called *dissipative* PDE which are the main subject of our study. As hinted by this denomination, these systems exhibit some energy dissipation process (in contrast to conservative systems for which the energy is preserved) and, of course, in order to have nontrivial dynamics, these models should also account for the energy income. Roughly speaking, the complicated chaotic behaviors in such systems usually arise from the interaction of the following mechanisms:

1) energy dissipation in the higher part of the Fourier spectrum;

2) external energy income in its lower part;

3) energy flux from lower to higher Fourier modes provided by the nonlinear terms of the equation.

We chose not to give a rigorous definition of a dissipative system here (although the concepts of energy dissipation and of related dissipative systems are more or less obvious from the physical point of view, they seem too general to have an adequate mathematical definition). Instead, we only indicate several basic classes of equations of mathematical physics which usually exhibit the above behaviors.

The first example is, of course, the Navier-Stokes system which describes the motion of a viscous incompressible fluid in a bounded domain Ω (we will only consider here the two-dimensional case $\Omega \subset \mathbb{R}^2$, since the adequate formulation in 3D is still an open problem):

$$\begin{cases} \partial_t u - (u, \nabla_x)u = \nu\Delta_x u + \nabla_x p + g(x), \\ \operatorname{div} u = 0, \ u\big|_{t=0} = u_0, \quad u\big|_{\partial\Omega} = 0. \end{cases} \tag{0.6}$$

Here, $u(t,x) = (u_1(t,x), u_2(t,x))$ is the unknown velocity vector, $p = p(t,x)$ is the unknown pressure, Δ_x is the Laplacian with respect to x, $\nu > 0$ and g are given kinematic viscosity and external forces respectively and $(u, \nabla_x)u$ is the inertial term ($[(u, \nabla_x)u]_i = \sum_{j=1}^2 u_j \partial_{x_j} u_i$, $i = 1, 2$). The unique global solvability of (0.6) has been proven by Ladyzhenskaya. Thus, this equation generates an infinite-dimensional DS in the phase space Φ of divergence free square integrable vector fields.

The second example is the damped nonlinear wave equation in $\Omega \subset \mathbb{R}^n$:

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u + f(u) = 0, \ u\big|_{\partial\Omega} = 0, \ u\big|_{t=0} = u_0, \ \partial_t u\big|_{t=0} = u_0', \tag{0.7}$$

which models, e.g., the dynamics of a Josephson junction driven by a current source (Sine-Gordon equation). It is known that, under natural sign and growth assumptions on the nonlinear interaction function f, this equation generates a DS in the energy phase space E of pairs of functions $(u, \partial_t u)$ such that $\partial_t u$ and $\nabla_x u$ are square integrable.

The last class of equations that we will consider consists of reaction-diffusion systems in a domain $\Omega \subset \mathbb{R}^n$:

$$\partial_t u = a\Delta_x u - f(u), \quad u\big|_{t=0} = u_0 \tag{0.8}$$

(endowed with Dirichlet $u\big|_{\partial\Omega} = 0$ or Neumann $\partial_n u\big|_{\partial\Omega} = 0$ boundary conditions), which describes some chemical reaction in Ω. Here, $u = (u^1, ..., u^N)$ is an unknown vector-valued function which describes the concentrations of the reactants, $f(u)$ is a given interaction function and a is a diffusion matrix. It is known that, under natural assumptions on f and a, these equations also generate an infinite-dimensional DS, e.g., in the phase space $\Phi := [L^\infty(\Omega)]^n$.

We emphasize once more that the phase spaces Φ in all these examples are appropriate *infinite-dimensional* function spaces. Nevertheless, it was observed in experiments that, up to a transient process, the trajectories of the DS considered are localized inside of "very thin" invariant subset of the phase space having a complicated geometric structure which, thus, accumulates all the nontrivial dynamics of the system. It was conjectured a little later that these invariant sets are, in some proper sense, *finite-dimensional* and that the dynamics restricted to these sets can be effectively described by a finite number of parameters. Thus (when this conjecture is true), in spite of the infinite-dimensional initial phase space, the effective dynamics (reduced to this invariant set) is finite-dimensional and can be studied by using the algorithms and concepts of the classical finite-dimensional DS theory. In particular, this means that the infinite-dimensionality plays here only the role of (possibly essential) technical difficulties which cannot, however, produce any new dynamical phenomena which are not observed in the finite-dimensional theory.

The above finite-dimensional reduction principle of dissipative PDE in *bounded* domains has been given solid mathematical grounds (based on the concept of the so-called *global attractor*) over the last three decades, starting from the pioneering papers of Ladyzhenskaya. We discuss this theory in more details in the first section.

We now discuss the *limitations* of the finite-dimensional reduction theory. Of course, the first and most obvious restriction of this principle is the effective *dimension* of the reduced finite-dimensional DS. Indeed, it is known that, typically, this dimension grows at least linearly with respect to the volume $\mathrm{vol}(\Omega)$ of the spatial domain Ω of the DS considered (and the growth of the size of Ω is the same (up to a rescaling) as the decay of the viscosity coefficient ν or the diffusion matrix a, see equations (0.6)–(0.8)). So, for sufficiently large domains Ω, the reduced DS can be too large for reasonable investigations.

The next, less obvious, but much more essential, restriction is the growing *spatial* complexity of the DS. Indeed, as shown by Babin-Buinimovich, the spatial complexity of the system (e.g., the number of topologically different equilibria) grows *exponentially* with respect to $\mathrm{vol}(\Omega)$. Thus, even in the case of relatively small dimensions, the reduced system can be out of reasonable investigations, due to its extremely complicated structure.

Therefore, the approach based on the finite-dimensional reduction does not look so attractive for large domains. It seems instead more natural, at least from the physical point of view, to replace large bounded domains by their limit unbounded ones (e.g., $\Omega = \mathbb{R}^n$ or cylindrical domains). Of course, this approach requires a systematic study of dissipative DS associated with PDE in *unbounded* domains.

The dynamical study of PDE in unbounded domains started from the pioneering paper of Kolmogorov-Petrovskij-Piskunov, in which the travelling wave solutions of reaction-diffusion equations in a strip were constructed and the convergence of the trajectories (for specific initial data) to this travelling wave solutions were established. Starting from this, a large amount of results on the dynamics of PDE in unbounded domains have been obtained. However, for a long period, the general features of such dynamics remained completely unclear. The main problems arising here are:

1) the essential infinite-dimensionality of the DS considered (absence of any finite-

dimensional reduction) which leads to essentially new dynamical effects which are not observed in finite-dimensional theories;

2) the additional spatial "unbounded" directions lead to the so-called *spatial* chaos and the interaction between spatial and temporal chaotic modes generates the *spatio-temporal* chaos which has also no analogue in finite dimensions.

Nevertheless, we mention below several ideas which (from our point of view) were the most important for the further development of these topics. The first one is the pioneering paper of Kirchgässner, in which dynamical methods were applied to the study of the *spatial* structure of solutions of elliptic equations in cylinders (which can be considered as equilibria equations for evolution PDE in unbounded cylindrical domains). The second one is the Sinai-Buinimovich model of space-time chaos in discrete lattice DS. Finally, the third one is the adaptation of the concept of a global attractor to unbounded domains by Abergel and Babin-Vishik.

We note however that the situation on the understanding of the general features of the dynamics in unbounded domains seems to have changed in the last several years, due to the works of Collet-Eckmann and Zelik. That is the reason why we devote the second section of this review to a more detailed discussion on this topic.

Other important questions are the object of current studies and we only briefly mention some of them. We mention for instance the study of attractors for nonautonomous systems (i.e. systems in which the time appears explicitly). This situation is much more delicate and is not completely understood; notions of attractors for such systems have been proposed by Chepyzhov-Vishik, Haraux and Kloeden-Schmalfuss. We also mention that theories of (global) attractors for non well-posed problems have been proposed by Babin-Vishik, Ball, Chepyzhov-Vishik, Melnik-Valero and Sell.

§1. Global attractors and finite-dimensional reduction.

1.1. Global attractors: the abstract setting. As already mentioned, one of the main concepts of the modern theory of DS in infinite dimensions is that of the *global attractor*. We give below its definition for an abstract semigroup $S(t)$ acting on a metric space Φ, although, without loss of generality, the reader may think that $(S(t),\Phi)$ is just a DS associated with one of the PDE (0.6)–(0.8) described in the introduction.

To this end, we first recall that a subset K of the phase space Φ is an attracting set of the semigroup $S(t)$ if it attracts the images of all the *bounded* subsets of Φ, i.e., for every bounded set B and every $\varepsilon>0$, there exists a time T (depending in general on B and ε) such that the image $S(t)B$ belongs to the ε-neighborhood of K if $t\geq T$. This property can be rewritten in the equivalent form

$$\lim_{t\to\infty}\operatorname{dist}_H(S(t)B,K)=0, \tag{1.1}$$

where $\operatorname{dist}_H(X,Y):=\sup_{x\in\mathbb{X}}\inf_{y\in\mathbb{Y}}d(x,y)$ is the nonsymmetric Hausdorff distance between subsets of Φ.

We now give the definition of a global attractor, following Babin-Vishik.

Definition 1.1. A set $\mathcal{A} \subset \Phi$ is a global attractor for the semigroup $S(t)$ if
1) $\mathcal{A}$ is *compact* in Φ;
2) $\mathcal{A}$ is *strictly invariant*: $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;
3) $\mathcal{A}$ is an *attracting* set for the semigroup $S(t)$.

Thus, the second and third properties guarantee that a global attractor, if it exists, is unique and that the DS reduced to the attractor contains all the nontrivial dynamics of the initial system. Furthermore, the first property indicates that the reduced phase space $\mathcal{A}$ is indeed "thinner" than the initial phase space Φ (we recall that, in infinite dimensions, a compact set cannot contain, e.g., balls and should thus be nowhere dense).

In most applications, one can use the following attractor's existence theorem.

Theorem 1.1. *Let a DS $(S(t), \Phi)$ possess a compact attracting set and the operators $S(t) : \Phi \to \Phi$ be continuous for every fixed t. Then, this system possesses the global attractor $\mathcal{A}$ which is generated by all the trajectories of $S(t)$ which are defined for all $t \in \mathbb{R}$ and are globally bounded.*

The strategy for applying this theorem to concrete equations of mathematical physics is the following. In a first step, one verifies a so-called *dissipative* estimate which has usually the form

$$\|S(t)u_0\|_\Phi \leq Q(\|u_0\|_\Phi)e^{-\alpha t} + C_*, \quad u_0 \in \Phi, \tag{1.2}$$

where $\|\cdot\|_\Phi$ is a norm in the function space Φ and the positive constants α and C_* and the monotonic function Q are independent of t and $u_0 \in \Phi$ (usually, this estimate follows from energy estimates and is sometimes even used in order to "define" a dissipative system). This estimate obviously gives the existence of an attracting set for $S(t)$ (e.g., the ball of radius $2C_*$ in Φ), which is however *noncompact* in Φ. In order to overcome this problem, one usually derives, in a second step, a *smoothing* property for the solutions, which can be formulated as follows:

$$\|S(1)u_0\|_{\Phi_1} \leq Q_1(\|u_0\|_\Phi), \quad u_0 \in \Phi, \tag{1.3}$$

where Φ_1 is another function space which is *compactly* embedded into Φ. In applications, Φ is usually the space $L^2(\Omega)$ of square integrable functions, Φ_1 is the Sobolev space $H^1(\Omega)$ of the functions u such that u and $\nabla_x u$ belong to $L^2(\Omega)$ and estimate (1.3) is a classical smoothing property for solutions of parabolic equations (for hyperbolic equations, a slightly more complicated *asymptotic* smoothing property should be used instead of (1.3)).

Since the continuity of the operators $S(t)$ usually arises no difficulty (if the uniqueness is proven), then the above scheme gives indeed the existence of the global attractor for most of the PDE of mathematical physics in bounded domains.

1.2. Dimension of the global attractor. In this subsection, we start by discussing one of the basic questions of the theory: in which sense is the dynamics on the global attractor finite-dimensional? As already mentioned, the global attractor is usually not a manifold, but has a rather complicated geometric structure. So, it is natural to use the definitions of dimensions adopted for the study of fractal sets here. We restrict ourselves to the so-called fractal (or box-counting, entropy) dimension, although other dimensions (e.g., Hausdorff, Lyapunov, etc.) are also used in the attractors' theory.

In order to define the fractal dimension, we first recall the concept of Kolmogorov's ε-entropy which comes from the information theory and plays a fundamental role in the theory of DS in unbounded domains considered in the next section.

Definition 1.2. Let $\mathcal{A}$ be a compact subset of a metric space Φ. For every $\varepsilon > 0$, we define $N_\varepsilon(K)$ as the minimal number of ε-balls which are necessary to cover $\mathcal{A}$. Then, Kolmogorov's ε-entropy $\mathcal{H}_\varepsilon(\mathcal{A}) = \mathcal{H}_\varepsilon(\mathcal{A}, \Phi)$ of $\mathcal{A}$ is the digital logarithm of this number, $\mathcal{H}_\varepsilon(\mathcal{A}) := \log_2 N_\varepsilon(\mathcal{A})$. We recall that $\mathcal{H}_\varepsilon(\mathcal{A})$ is finite for every $\varepsilon > 0$, due to the Hausdorff criterium. The fractal dimension $d_f(\mathcal{A}) \in [0, \infty]$ of $\mathcal{A}$ is then defined by

$$d_f(\mathcal{A}) := \limsup_{\varepsilon \to 0} \mathcal{H}_\varepsilon(\mathcal{A}) \big/ \log_2 1/\varepsilon. \tag{1.4}$$

We also recall that, although this dimension coincides with the usual dimension of the manifold for Lipschitz manifolds, it can be noninteger for more complicated sets. For instance, the fractal dimension of the standard ternary Cantor set in $[0, 1]$ is $\ln 2 \big/ \ln 3$.

The so-called Mané theorem (which can be considered as a generalization of the classical Yitni embedding theorem for fractal sets) plays an important role in the finite-dimensional reduction theory.

Theorem 1.2. *Let Φ be a Banach space and $\mathcal{A}$ be a compact set such that $d_f(\mathcal{A}) < N$ for some $N \in \mathbb{N}$. Then, for "almost all" $2N+1$-dimensional planes L in Φ, the corresponding projector $\Pi_L : \Phi \to L$ restricted to the set $\mathcal{A}$ is a Hölder continuous homeomorphism.*

Thus, if the finite fractal dimensionality of the attractor is established, then, fixing a hyperplane L satisfying the assumptions of the Mané theorem and projecting the attractor $\mathcal{A}$ and the DS $S(t)$ restricted to $\mathcal{A}$ onto this hyperplane ($\bar{\mathcal{A}} := \Pi_L \mathcal{A}$ and $\bar{S}(t) := \Pi_L \circ S(t) \circ \Pi_L^{-1}$), we obtain indeed a reduced DS $(\bar{S}(t), \bar{\mathcal{A}})$ which is defined on a finite-dimensional set $\bar{\mathcal{A}} \subset L \sim \mathbb{R}^{2N+1}$. Moreover, this DS will be Hölder continuous with respect to the initial data.

1.3. Estimates on the fractal dimension. Obviously, good estimates on the dimension of the attractors in terms of the physical parameters are crucial for the finite-dimensional reduction described above and (consequently) there exists a highly developed machinery for obtaining such estimates. The best known upper estimates are usually obtained by the so-called volume contraction method which is based on the study of the evolution of infinitesimal k-dimensional volumes in the neighborhood of the attractor (and, if the DS considered contracts the k-dimensional volumes, then the fractal dimension of the attractor is less than k). Lower bounds on the dimension are usually based on the observation that the global attractor always contains the unstable manifolds of the (hyperbolic) equilibria. Thus, the instability index of a properly constructed equilibrium gives a lower bound on the dimension of the attractor.

We formulate below several estimates for the classes of equations given in the introduction. We start with the most studied case of the reaction-diffusion system (0.8). For this system, sharp upper and lower bounds are known, namely

$$C_1 \operatorname{vol}(\Omega) \le d_f(\mathcal{A}) \le C_2 \operatorname{vol}(\Omega), \tag{1.5}$$

where the constants C_1 and C_2 depend on a and f (and, possibly, on the shape of Ω), but are independent of its size. The same types of estimates also hold for the hyperbolic equation (0.7). Concerning the Navier-Stokes system (0.6) in general two-dimensional domains Ω, the asymptotics of the fractal dimension as $\nu \to 0$ is not known. The best known upper bound has the form $d_f(\mathcal{A}) \le C\nu^{-2}$ and was obtained by Foias-Temam by using the so-called Lieb-Thirring inequalities. Nevertheless, for *periodic* boundary conditions, Constantin-Foias-Temam and Liu obtained upper and lower bounds of the same order (up to a logarithmic correction):

$$C_1\nu^{-4/3} \le d_f(\mathcal{A}) \le C_2\nu^{-4/3}(1+\ln(\nu^{-1}))^{\frac{1}{3}}. \tag{1.6}$$

1.4. Global Lyapunov functions and the structure of global attractors. Although the global attractor has usually a very complicated geometric structure, there exists one exceptional class of DS for which the global attractor has a relatively simple structure which is completely understood, namely the DS having a global Lyapunov function. We recall that a continuous function $\mathcal{L} : \Phi \to \mathbb{R}$ is a global Lypanov function if

1) $\mathcal{L}$ is nonincreasing along the trajectories, i.e. $\mathcal{L}(S(t)u_0) \le \mathcal{L}(u_0)$, for all $t \ge 0$;

2) $\mathcal{L}$ is *strictly* decreasing along all nonequilibrium solutions, i.e. $\mathcal{L}(S(t)u_0) = \mathcal{L}(u_0)$ for some $t > 0$ and u_0 implies that u_0 is an equilibrium of $S(t)$.

For instance, in the scalar case $N = 1$, the reaction-diffusion equations (0.8) possess the global Lyapunov function $\mathcal{L}(u_0) := \int_\Omega [a|\nabla_x u_0(x)|^2 + F(u_0(x))]\,dx$, where $F(v) := \int_0^v f(u)\,du$. Indeed, multiplying equation (0.8) by $\partial_t u$ and integrating over Ω, we have

$$\frac{d}{dt}\mathcal{L}(u(t)) = -2\|\partial_t u(t)\|^2_{L^2(\Omega)} \le 0. \tag{1.7}$$

Analogously, in the scalar case $N = 1$, multiplying the hyperbolic equation (0.7) by $\partial_t u(t)$ and integrating over Ω, we obtain the standard global Lyapunov function for this equation.

It is well known that, if a DS posseses a global Lyapunov function, then, at least under the generic assumption that the set $\mathcal{R}$ of equilibria is finite, every trajectory $u(t)$ *stabilizes* to one of these equilibria as $t \to +\infty$. Moreover, every complete bounded trajectory $u(t)$, $t \in \mathbb{R}$, belonging to the attractor is a heteroclinic orbit joining two equilibria. Thus, the global attractor $\mathcal{A}$ can be described as follows:

$$\mathcal{A} = \cup_{u_0 \in \mathcal{R}} \mathcal{M}^+(u_0), \tag{1.8}$$

where $\mathcal{M}^+(u_0)$ is the so-called unstable set of the equilibrium u_0 (which is generated by all heteroclinic orbits of the DS which start from the given equilibrium $u_0 \in \mathcal{A}$). It is also known that, if the equilibrium u_0 is *hyperbolic* (generic assumption), then the set $\mathcal{M}^+(u_0)$ is a κ-dimensional submanifold of Φ, where κ is the instability index of u_0. Thus, under the generic hyperbolicity assumption on the equilibria, the attractor $\mathcal{A}$ of a DS having a global Lyapunov function is a finite union of smooth finite-dimensional submanifolds of the phase space Φ. These attractors are called *regular* (following Babin-Vishik).

It is also worth emphasizing that, in contrast to general global attractors, regular attractors are robust under perturbations. Moreover, in some cases, it is also possible to verify the so-called *transversality* conditions (for the intersection of stable and unstable manifolds of the equilibria) and, thus, verify that the DS considered is a Morse-Smale system. In particular, this means that the dynamics restricted to the regular attractor $\mathcal{A}$ is also preserved (up to homeomorphisms) under perturbations.

A disadvantage of the regular attractor's approach is the fact that, except for scalar parabolic equations in one space dimension, it is usually extremely difficult to verify the "generic" hyperbolicity and transversality assumptions for concrete values of the physical parameters and the associated hyperbolicity constants, as a rule, cannot be expressed in terms of these parameters.

1.5. Inertial manifolds. We can note that the scheme for the finite-dimensional reduction described above has essential drawbacks. Indeed, the reduced system $(\bar{S}(t), \bar{\mathcal{A}})$ is only Hölder continuous and, consequently, cannot be realized as a DS generated by a system of ODE (and reasonable conditions on the attractor $\mathcal{A}$ which guarantee the Lipschitz continuity of the Mané projections are not known). On the other hand, the complicated geometric structure of the attractor $\mathcal{A}$ (or $\bar{\mathcal{A}}$) makes the use of this finite-dimensional reduction in computations hazardous (in fact, only the heuristic information on the number of unknowns which are necessary to capture all the dynamical effects in approximations can be extracted).

In order to overcome these problems, the concept of an inertial manifold (which allows to embed the global attractor into a smooth manifold) has been suggested by Foias-Sell-Temam. To be more precise, a Lipschitz finite-dimensional manifold $\mathbb{M} \subset \Phi$ is an inertial manifold for the DS $(S(t), \Phi)$ if

1) $\mathbb{M}$ is semiinvariant, i.e. $S(t)\mathbb{M} \subset \mathbb{M}$, for all $t \geq 0$;

2) $\mathbb{M}$ satisfies the following asymptotic completeness property: for every $u_0 \in \Phi$, there exists $v_0 \in \mathbb{M}$ such that

$$\|S(t)u_0 - S(t)v_0\|_\Phi \leq Q(\|u_0\|_\Phi)e^{-\alpha t}, \tag{1.9}$$

where the positive constant α and the monotonic function Q are independent of u_0.

We can see that an inertial manifold, if it exists, confirms in a *perfect* way the heuristic conjecture on the finite-dimensionality formulated in the introduction. Indeed, the dynamics of $S(t)$ *restricted* to an inertial manifold can be, obviously, described by a system of ODE (which is called the *inertial form* of the initial PDE). On the other hand, the asymptotic completeness gives (in a very strong form) the equivalence of the initial DS $(S(t), \Phi)$ with its inertial form $(S(t), \mathbb{M})$. Moreover, in turbulence, the existence of an inertial manifold would yield an exact interaction law between the small and large structures of the flow.

Unfortunately, all the known constructions of inertial manifolds are based on a very restrictive condition, the so-called spectral gap condition, which requires arbitrarily large gaps in the spectrum of the linearization of the initial PDE and which can usually be verified only in one space dimension. So, the existence of an inertial manifold is still an open problem for many important equations of mathematical physics (including in particular the 2D Navier-Stokes equations; some nonexistence results have also been proven by Mallet-Paret).

1.6. Exponential attractors. We first recall that definition 1.1 of a global attractor only guarantees that the images $S(t)B$ of all the bounded subsets converge to the attractor, without saying anything on the rate of convergence (in contrast to inertial manifolds for which this rate of convergence can be controlled). Furthermore, as elementary examples show, this convergence can be arbitrarily slow, so that, until now, we have no effective way for estimating this rate of convergence in terms of the physical parameters of the system (an exception is given by the regular attractors described in subsection 1.4 for which the rate of convergence can be estimated in terms of the hyperbolicity constants of the equilibria. However, even in this situation, it is usually very difficult to estimate these constants for concrete equations). Furthermore, there exists a large number of physically relevant systems (e.g., the so-called slightly dissipative gradient systems) which have trivial global attractors, but very rich and physically relevant transient dynamics which are automatically forgotten under the global attractor's approach. Another important problem is the robustness of the global attractor under perturbations. In fact, global attractors are usually only upper semicontinuous under perturbations (which means that they cannot explode) and the lower semicontinuity (which means that they cannot also implode) is much more delicate to prove and requires some hyperbolicity assumptions (which are usually impossible to verify for concrete equations).

In order to overcome these difficulties, Eden-Foias-Nicolaenko-Temam have introduced an intermediate object (between inertial manifolds and global attractors), namely an exponential attractor (also called an inertial set).

Definition 1.3. A compact set $\mathcal{M} \subset \Phi$ is an exponential attractor for the DS $(S(t), \Phi)$ if

1) $\mathcal{M}$ has finite fractal dimension: $d_f(\mathcal{M}) < \infty$;
2) $\mathcal{M}$ is semiinvariant: $S(t)\mathcal{M} \subset \mathcal{M}$, for all $t \geq 0$;

3) $\mathcal{M}$ attracts *exponentially* the images of all the bounded sets $B \subset \phi$:

$$\text{dist}_H(S(t)B, \mathcal{M}) \leq Q(\|B\|_\Phi)e^{-\alpha t}, \tag{1.10}$$

where the positive constant α and the monotonic function Q are independent of B.

Thus, on the one hand, an exponential attractor remains finite-dimensional (like the global attractor) and, on the other hand, estimate (1.10) allows to control the rate of attraction (like an inertial manifold). We note however that the relaxation of *strict invariance* to *semiinvariance* allows this object to be nonunique. So, we have here the problem of the "best choice" of the exponential attractor. We also mention that an exponential attractor, if it exists, always contains the global attractor.

Although the initial construction of exponential attractors is based on the so-called squeezing property (and requires Zorn's lemma), we formulate below a simpler construction, due to Efendiev-Miranville-Zelik, which is similar to the method proposed by Ladyzhenskaya to verify the finite-dimensionality of global attractors. We formulate it for discrete times and for a DS generated by iterations of some map $S : \Phi \to \Phi$, since the passage from discrete to continuous times usually arises no difficulty (without loss of generality, the reader may think that $S = S(1)$ and $(S(t), \Phi)$ is one of the DS mentioned in the introduction).

Theorem 1.3. *Let the phase space Φ_0 be a closed bounded subset of some Banach space H and let H_1 be another Banach space compactly embedded into H. Assume also that the map $S : \Phi_0 \to \Phi_0$ satisfies the following "smoothing" property:*

$$\|Su_1 - Su_2\|_{H_1} \leq K\|u_1 - u_2\|_H, \quad u_1, u_2 \in \Phi_0, \tag{1.11}$$

for some constant K independent of u_i. Then, the DS (S, Φ_0) possesses an exponential attractor.

In applications, Φ_0 is usually a bounded absorbing/attracting set whose existence is guaranteed by the dissipative estimate (1.2), $H := L^2(\Omega)$ and $H_1 := H^1(\Omega)$. Furthermore, estimate (1.11) simply follows from the classical parabolic smoothing property, but now applied to the equation of variations (as in (1.3), hyperbolic equations require a slightly more complicated analogue of (1.11)). These simple arguments show that exponential attractors are as general as global attractors and, to the best of our knowledge, exponential attractors exist indeed for all the equations of mathematical physics for which the finite-dimensionality of the global attractor can be established. Moreover, since $\mathcal{A} \subset \mathcal{M}$, this scheme can also be used to prove the finite-dimensionality of global attractors.

It is finally worth emphasizing that the control on the rate of convergence provided by (1.10) makes exponential attractors much more robust than global attractors. In particular, they are upper and lower semicontinuous under perturbations (of course, up to the "best choice", since they are not unique), as shown by Efendiev-Miranville-Zelik.

§2. Essentially infinite-dimensional dynamical systems – the case of unbounded domains.

As already mentioned in the introduction, the theory of dissipative DS in unbounded domains is developing only now and the results given here are not as complete as for bounded domains. Nevertheless, we indicate below several of the most interesting (from our point of view) results concerning the general description of the dynamics generated by such problems by considering a system of reaction-diffusion equations (0.8) in $\mathbb{R}^n$ with phase space $\Phi = L^\infty(\mathbb{R}^n)$ as a model example (although all the results formulated below are general and depend weakly on the choice of equation).

2.1. Generalization of the global attractor and Kolmogorov's ε-entropy. We first note that definition 1.1 of a global attractor is too strong for equations in unbounded domains. Indeed, as seen in subsection 1.1, the compactness of the attractor is usually based on the compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$, which does not hold in unbounded domains. Furthermore, an attractor, in the sense of definition 1.1, *does not exist* for most of the interesting examples of equations (0.7) in $\mathbb{R}^n$.

It is natural to use instead the concept of the so-called *locally compact* global attractor which is well adapted to unbounded domains. This attractor $\mathcal{A}$ is only bounded in the phase space $\Phi = L^\infty(\mathbb{R}^n)$, but its restrictions $\mathcal{A}\big|_\Omega$ to all *bounded* domains Ω are compact in $L^\infty(\Omega)$. Moreover, the attraction property should also be understood in the sense of a *local* topology in $L^\infty(\mathbb{R}^n)$. It is known that this generalized global attractor $\mathcal{A}$ exists indeed for problem (0.8) in $\mathbb{R}^n$ (of course, under some "natural" assumptions on the nonlinearity f and the diffusion matrix a). As for bounded domains, its existence is based on the dissipative estimate (1.2), the smoothing property (1.3) and the compactness of the embedding $H^1_{loc}(\mathbb{R}^n) \subset L^2_{loc}(\mathbb{R}^n)$ (we need to use the local topology only to have this compactness).

The next natural question which arises here is how to control the "size" of the attractor $\mathcal{A}$ if its fractal dimension is infinite (which is usually the case in unbounded domains). One of the most natural way to handle this problem (which was first suggested by Chepyzhov-Vishik in the different context of *uniform* attractors associated with nonautonomous equations in *bounded* domains and appears as extremely fruitful for the theory of dissipative PDE in unbounded domains) is to study the asymptotics of Kolmogorov's ε-entropy of the attractor. Actually, since the attractor $\mathcal{A}$ is compact *only* in a local topology, it is natural to study the entropy of its restrictions, say, to balls $B^R_{x_0}$ of $\mathbb{R}^n$ of radius R centered at x_0 with respect to the three parameters R, x_0 and ε. A more or less complete answer to this question is given by the following estimate:

$$\mathcal{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}) \le C(R + \log_2 1/\varepsilon)^n \log_2 1/\varepsilon, \tag{2.1}$$

where the constant C is independent of $\varepsilon \le 1$, R and x_0. Moreover, it can be shown that this estimate is sharp for all R and ε under the very weak additional assumption that equation (0.8) possesses at least one exponentially unstable spatially homogeneous equilibrium.

Thus, formula (2.1) (whose proof is also based on a smoothing property for the equation of variations) can be interpreted as a natural generalization of the heuristic principle of finite-dimensionality of global attractors to unbounded domains. It is also worth recalling that the entropy of the embedding of a ball $\mathcal{B}_k$ of the space $C^k(B^R_{x_0})$ into $C(B^R_{x_0})$ has the asymptotic $\mathcal{H}_\varepsilon(\mathcal{B}) \sim C_R(1/\varepsilon)^{n/k}$, which is essentially worse than (2.1). So, (2.1) is not based on the smoothness of the attractor $\mathcal{A}$ and, therefore, reflects deeper properties of the equation.

2.2. Spatial dynamics and spatial chaos. The next main difference with bounded domains is the existence of unbounded spatial directions which can generate the so-called spatial chaos (in addition to the "usual" temporal chaos arising under the evolution). In order to describe this phenomenon, it is natural to consider the group $\{T_h, h \in \mathbb{R}^n\}$ of spatial translations acting on the attractor $\mathcal{A}$:

$$(T_h u_0)(x) := u_0(x+h), \quad T_h : \mathcal{A} \to \mathcal{A}, \tag{2.2}$$

as a DS (with multidimensional "times" if $n > 1$) acting on the phase space $\mathcal{A}$ and to study its dynamical properties.

In particular, it is worth noting that the lower bounds on the ε-entropy that one can derive imply that the topological entropy of this spatial DS is infinite and, consequently, the classical symbolic dynamics with a *finite* number of symbols is not adequate to clarify the nature of chaos in (2.2). In order to overcome this difficulty, it was suggested by Zelik to use Bernoulli shifts with an *infinite* number of symbols, belonging to the whole interval $\omega \in [0,1]$. To be more precise, let us consider the cartesian product $\mathbb{M}_n := [0,1]^{\mathbb{Z}^n}$ endowed with the Tikhonov topology. Then, this set can be interpreted as the space of all the functions $v : \mathbb{Z}^n \to [0,1]$, endowed with the standard local topology. We define a dynamical system $\{\mathcal{T}_l, l \in \mathbb{Z}^n\}$ on $\mathbb{M}_n$ by

$$(\mathcal{T}_l v)(m) := v(m+l), \quad v \in \mathbb{M}_n, \quad l, m \in \mathbb{Z}^n. \tag{2.3}$$

Based on this model, the following description of spatial chaos was obtained.

Theorem 2.1. *Let equation* (0.8) *in* $\Omega = \mathbb{R}^n$ *possess at least one exponentially unstable spatially homogeneous equilibrium. Then, there exist* $\alpha > 0$ *and a homeomorphic embedding* $\tau : \mathbb{M}_n \to \mathcal{A}$ *such that*

$$T_{\alpha l} \circ \tau(v) = \tau \circ \mathcal{T}_l(v), \quad \forall l \in \mathbb{Z}^n, \quad v \in \mathbb{M}_n. \tag{2.4}$$

Thus, the spatial dynamics, restricted to the set $\tau(\mathbb{M}_n)$, is conjugated to the symbolic dynamics on $\mathbb{M}_n$. Moreover, there exists a dynamical invariant (the so-called mean toplogical dimension) which is always finite for the spatial DS (2.3) and strictly positive for the Bernoulli scheme $\mathbb{M}_n$. So, the embedding (2.4) clarifies indeed the nature of chaos arising in the spatial DS (2.2).

2.3. Spatio-temporal chaos. To conclude, we briefly discuss an extension of Theorem 2.1 which takes into account the temporal modes and, thus, gives a description of the spatio-temporal chaos. In order to do so, we first note that the spatial DS (2.2) commutes obviously with the temporal evolution operators $S(t)$ and, consequently, an extended $(n+1)$-parametric semigroup $\{\mathbb{S}(t,h), (t,h) \in \mathbb{R}_+ \times \mathbb{R}^n\}$ acts on the attractor:

$$\mathbb{S}(t,h) := S(t) \circ T_h, \quad \mathbb{S}(t,h) : \mathcal{A} \to \mathcal{A}, \quad t \in \mathbb{R}_+, \quad h \in \mathbb{R}^n. \tag{2.5}$$

Then, this semigroup (interpreted as a DS with multidimensional times) is responsible for all the spatio-temporal dynamical phenomena in the initial PDE (0.8) and, consequently, the question of finding adequate dynamical characteristics is of a great interest. Moreover, it is also natural to consider the subsemigroups $\mathbb{S}_{V_k}(t,h)$ associated with the k-dimensional planes V_k of the space-time $\mathbb{R}_+ \times \mathbb{R}^n$, $k < n+1$.

Although finding an adequate description of the dynamics of (2.5) seems to be an extremely difficult task, some particular results in this direction have already been obtained. Thus, it has been proven by Zelik that the semigroup (2.5) has finite topological entropy and the entropy of its subsemigroups $\mathbb{S}_{V_k}(t,h)$ is usually infinite if $k < n+1$. Moreover (adding a natural transport term of the form $(L, \nabla_x)u$ to equation (0.8)), it was proven that the analogue of Theorem 2.1 holds for the subsemigroups $\mathbb{S}_{V_n}(t,h)$ associated with the n-dimensional hyperplanes V_n of the space-time. Thus, the infinite-dimensional Bernoulli shifts introduced in the previous subsection can be used to describe the temporal evolution in unbounded domains as well.

In particular, as a consequence of this embedding, the topological entropy of the initial purely temporal evolution semigroup $S(t)$ is also infinite, which indicates that (even without considering the spatial directions) we have indeed here essential new levels of dynamical complexity which are not observed in the classical DS theory of ODE.

See also

Chaos and attractors. Discrete dynamical systems, chaos and strange attractors. Dynamical systems and thermodynamics. Dynamical systems in fluid mechanics. Dynamical systems in mathematical physics. Lyapunov exponents and strange attractors.

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