THE ATTRACTOR FOR A NONLINEAR REACTION–DIFFUSION SYSTEM IN THE UNBOUNDED DOMAIN AND KOLMOGOROV'S ε-ENTROPY.

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July 10, 1999

Contents

Introduction

In this paper the quasilinear second order parabolic equations and systems of a reaction-diffusion type

$$\begin{cases} \partial_t u - \Delta_x u + f(u) + \lambda_0 u = g(t);\ x \in \Omega \\ u\big|_{t=0} = u_0\,, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{0.1}$$

are considered.

1991 *Mathematics Subject Classification.* 35B40, 35B45.

Key words and phrases. Nonlinear reaction-diffusion systems, unbounded domains, nonautonomous attractors, Kolmogorov's entropy.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Here $\Omega \subset \mathbb{R}^n$ is an unbounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary (see §1), $u = (u^1, \cdots, u^k)$ is unknown vector-valued function, Δ_x is a Laplacian with respect to $x = (x_1, \cdots, x_n)$, f and g are given functions and λ_0 is fixed positive constant.

It is assumed also that the nonlinear term $f(u)$ satisfies the conditions

$$\begin{cases} 1.\, f \in C^1(\mathbb{R}^k, \mathbb{R}^k) \\ 2.\, f(u).u \geq -C \end{cases} \tag{0.2}$$

Here and below we denote by $u.v$ the inner product in the space R^k.

It is well known that in many cases the longtime behavior of dynamical systems, generated by evolutionary equations of mathematical physics can be naturally described in terms of attractors of the corresponding semigroups (see [2], [11], [20]). In bounded domains the existence of the attractor is established for a large class of equations such as reaction-diffusion equations, nonlinear wave equations, 2D Navier–Stokes system, etc. Under some natural assumptions it is proved that in the autonomous case for all equations mentioned above the attractor has the finite Hausdorff and fractal dimension (see [11], [20]).

The equations of mathematical physics which can depend explicitly on t in bounded domains Ω are considered in [4], [5]. Recall that according to the construction of the uniform attractor, suggested there, one should consider not only the initial nonautonomous problem but simultaneously the whole family of problems which are obtained from the initial one by all positive shifts along the t-axis and their closure in the appropriate topology. Moreover, if this family is in a certain sense "infinite dimensional" (for example the right–hand side is almost-periodic by t with the infinite number of independent frequences) then the uniform attractor of the corresponding equation naturally has infinite Hausdorff and fractal dimension.

Thus, in contrast to the autonomous case in the nonautonomous one the fractal dimension is not a convenient quantitative characteristic of the "size" of attractors and consequently the problem of finding another characteristics arises.

One of possible approaches to handle this problem, which is suggested in [6], is to estimate Kolmogorov's ε-entropy of the attractor. Recall, that by definition Kolmogorov's ε-entropy $\mathbb{H}_\varepsilon(\mathcal{A})$ of the attractor $\mathcal{A}$ is the logarithm from the minimal number $N_\varepsilon(\mathcal{A})$ of ε-balls in the appropriate phase space which cover the attractor:

$$\mathbb{H}_\varepsilon(\mathcal{A}) = \ln N_\varepsilon(\mathcal{A}) \tag{0.3}$$

Note that since $\mathcal{A}$ is compact then (0.3) is well defined and finite for every $\varepsilon > 0$.

For unbounded domains Ω the behavior of solutions for (0.1) becomes much more complicated. In this case even the problem of finding the appropriate phase space for (0.1) becomes nontrivial. For instance, in [1], [3], [9] this equation has been studied in weighted Sobolev spaces $W^{l,p}_\phi(\Omega)$ with $\phi(x) = \phi_\alpha(x) = (1 + |x|^2)^{\alpha/2}$. The case of general weights ϕ is considered in [10].

In this paper we assume that the solution $u(t,x)$ is bounded with respect to $|x| \to \infty$. To be more precise it is assumed that for every fixed $t \geq 0$

$$u(t) \in W^{l,p}_b(\Omega) \equiv \{v : \|v\|_{W^{l,p}_b} = \sup_{x_0 \in \Omega} \|v\|_{W^{l,p}(\Omega \cap B^1_{x_0})} < \infty\} \tag{0.4}$$

with the appropriate exponents l and p. (Here and below we denote by $B^R_{x_0}$ the R-ball in $\mathbb{R}^k$ centered in x_0.)

In the autonomous case $g = g(x)$ reaction-diffusion equations and systems of the type (0.1) under the assumptions (0.4) are considered in [8], [17], [18], [19].

Recall that under the above assumptions the attractor $\mathcal{A}$ of the equation (0.1) may have (and has in general) infinite Hausdorff and fractal dimension even in the autonomous case (see [8], [10]). Thus, in contrast to the case of bounded domains where the infinite dimensional attractor can appear only in the nonautonomous case and only due to the "infinite dimensional" external time-depended forces, in the case where Ω is unbounded the infinite dimensionality appears even in the autonomous case and has consequently the internal nature.

The main aim of this paper is to give a systematical study of Kolmogorov's ε-entropy of attractors for autonomous and nonautonomous reaction-diffusion equations of the type (0.1) in unbounded domains Ω.

It is known (see Remark 5.1) that in general the attractor $\mathcal{A}$ of the problem (0.1) is not compact in the uniform topology of the space (0.4) but only in a local topology of the space $W^{l,p}_{loc}(\Omega)$, that is why we consider the entropy of restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ and study the dependence of $\mathbb{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}})$ on three parameters ε, R and x_0.

In the autonomous case $g = g(x)$ we prove that the entropy of the attractor $\mathcal{A}$ possesses the following estimate:

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}) \le C\,\mathrm{vol}_{x_0,\Omega}(R + K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} \tag{0.5}$$

where $\mathrm{vol}_{x_0,\Omega}(r) = \mathrm{vol}(\Omega\cap B^r_{x_0})$, $\mathrm{vol}(\cdot)$ means the n-dimensional volume, and constants C, K are independent of R, ε and x_0. Particularly, if $\Omega = \mathbb{R}^n$, then (0.5) implies that

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}) \le C(R + K\ln\frac{1}{\varepsilon})^n\ln\frac{1}{\varepsilon} \tag{0.6}$$

We verify also the sharpness of the estimate (0.6). To this end Chafee-Infante equation in $\mathbb{R}^n$

$$\partial_t u = \Delta_x u + \alpha u - u^3, \quad \alpha > 0 \tag{0.7}$$

is considered. It is proved that the entropy of the attractor of (0.7) possesses the following estimate

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}) \ge C_1 R^n \ln\frac{1}{\varepsilon} \tag{0.8}$$

for $R \ge R_0$ and $\varepsilon < \varepsilon_0$ and consequently the estimate (0.6) is sharp if $R \sim \ln\frac{1}{\varepsilon}$ or $R \gg \ln\frac{1}{\varepsilon}$. For the case where $R \ll \ln\frac{1}{\varepsilon}$ (particularly for $R = 1$) we obtain that for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^1_{x_0}}) \ge C_\delta \left(\ln\frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{0.9}$$

Moreover, the upper estimates of the ε-entropy in the nonautonomous case are also obtained. In this case the term with the appropriate entropy of the right–hand side g should be added to the right–hand side of (0.5) (see §8).

This paper consists of three Parts.

Part 1 is devoted to study the analytical properties of the problem (0.1) such as different a priori estimates, existence of solutions, uniqueness, smoothing property, etc. Note that the technique employed in this Part is further development of methods, suggested in [10].

The existence of attractors in the autonomous and nonautonomous case and related problems are considered in Part 2.

The ε-entropy for the attractors obtained above is studied in Part 3. Moreover, we recall here some definitions related with the abstract concept of ε-entropy and give some examples of it for the typical sets in different functional spaces.

Part 1. A priori estimates, existence of solutions, uniqueness.

This part is devoted to study the analytical properties of solutions (such as a priori estimates, existence, uniqueness, etc.) of (0.1) in unbounded domains.

In Section 1 we introduce a wide class of weights and the corresponding weighted Sobolev spaces and formulate a number of useful auxiliary results.

The linear equation (with $f(u) \equiv 0$) is considered in Section 2.

A priori estimates for the solutions of the nonlinear equation (0.1) are obtained in Section 3. Moreover, using these estimates and Leray-Schauder principle we obtain then the existence of a solution u for (0.1).

Section 4 is devoted to study the problems related with the uniqueness of solutions and smoothing properties for a difference between two solutions of (0.1). The estimates, obtained here will be essentially used in Part 3 for the entropy estimates.

§1 Functional spaces.

In this Section we introduce and study some functional spaces which will be used throughout of the paper.

Definition 1.1. *A function $\phi \in L^\infty_{loc}(\mathbb{R}^n)$ is called a weight function with the rate of growth $\mu \geq 0$ if the condition*

$$\phi(x+y) \leq C_\phi e^{\mu|x|}\phi(y),\ \ \phi(x) > 0 \tag{1.1}$$

is satisfied for every $x, y \in \mathbb{R}^n$.

Remark 1.1. *It is not difficult to deduce from* (1.1) *that*

$$\phi(x+y) \geq C_\phi^{-1} e^{-\mu|x|}\phi(y) \tag{1.2}$$

is also satisfied for every $x, y \in \mathbb{R}^n$.

Proposition 1.1. *Let ϕ_1 and ϕ_2 be weight functions with the rates of growth μ_1 and μ_2 correspondingly. Then,*

1. $\alpha\phi_1 + \beta\phi_2$, $\max\{\phi_1, \phi_2\}$, and $\min\{\phi_1, \phi_2\}$ are the weight functions with the rate of growth $\max\{\mu_1, \mu_2\}$ for every $\alpha, \beta > 0$.

2. $\phi_1 \cdot \phi_2$ *and* $\phi_1 \cdot (\phi_2)^{-1}$ *are the weight functions with the rate of growth* $\mu_1 + \mu_2$.
3. $(\phi_1)^\alpha$ *is the weight function with the rate of growth* $|\alpha|\mu_1$.

The assertions of this proposition are immediate corollaries of (1.1) and (1.2).

The following example of weight functions are of fundamental significance for our purposes:

$$\phi_{\{\varepsilon\},x_0}(x) = e^{-\varepsilon|x-x_0|},\ \varepsilon \in \mathbb{R},\ x_0 \in \mathbb{R}^n$$

(Evidently this weight has the rate of growth $|\varepsilon|$.)

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be some (unbounded) domain in $\mathbb{R}^n$ and let ϕ be a weight function with the rate of growth μ. Define the space*

$$L^p_\phi(\Omega) = \left\{ u \in D'(\Omega)\ :\ \|u, \Omega, \|_{\phi,0,p} \equiv \int_\Omega \phi(x)|u(x)|^p\, dx < \infty \right\}$$

Analogously the weighted Sobolev space $H^{l,p}_\phi(\Omega)$, $l \in \mathbb{N}$ is defined as the space of distributions whose derivatives up to the order l inclusively belong to $L^p_\phi(\Omega)$.

For the simplicity of notations we will right throughout of the paper $W^{s,p}_{\{\varepsilon\}}$ instead of $W^{s,p}_{e^{-\varepsilon|x|}}$.

We define also another class of weighted Sobolev spaces

$$W^{l,p}_{b,\phi}(\Omega) = \left\{ u \in D'(\Omega)\ : \|u, \Omega\|^p_{b,\phi,l,p} = \sup_{x_0 \in \mathbb{R}^n} \phi(x_0)\|u, \Omega \cap B^1_{x_0}\|^p_{l,p} < \infty \right\}$$

Here and below we denote by $B^R_{x_0}$ the ball in R^n of radius R, centered in x_0, and $\|u, V\|_{l,p}$ means $\|u\|_{W^{l,p}(V)}$.

We will write $W^{l,p}_b$ instead of $W^{l,p}_{b,1}$.

Theorem 1.1. *Let $u \in L^p_\phi(\Omega)$, where ϕ is a weight function with the rate of growth μ. Then for any $1 \leq q \leq \infty$ the following estimate is valid*

$$\left(\int_\Omega \phi(x_0)^q \left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\, dx \right)^q dx_0 \right)^{1/q} \leq C \int_\Omega \phi(x)|u(x)|^p\, dx \tag{1.3}$$

for every $\varepsilon > \mu$, where the constant C depends only on ε, μ and C_ϕ from (1.1) *(and independent of Ω).*

Proof. Let $q = 1$. Then, according to (1.1),

$$\int_\Omega \int_\Omega \phi(x_0) e^{-\varepsilon|x-x_0|}|u(x)|^p\, dx\, dx_0 \leq$$
$$\leq C_\phi \int_{\Omega^2} e^{\mu|x-x_0|} e^{-\varepsilon|x-x_0|}\phi(x)|u(x)|^p\, dx\, dx_0 \leq$$
$$\leq C_\phi \left(\int_{\mathbb{R}^n} e^{-(\varepsilon-\mu)|y|}\, dy \right) \left(\int_\Omega \phi(x)|u(x)|^p\, dx \right) \leq C_1 \int_\Omega \phi(x)|u(x)|^p\, dx$$

Let now $q = \infty$. Then applying (1.1) again we obtain

$$\sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right\} \le$$
$$\le C_\phi \int_{x\in\Omega} \sup_{x_0\in\Omega}\left\{e^{\mu|x-x_0|}e^{-\varepsilon|x-x_0|}\right\}\phi(x)|u(x)|^p\,dx \le$$
$$\le C_\phi \int_{x\in\Omega}\phi(x)|u(x)|^p\,dx$$

Thus, we proved the inequality (1.3) for $q = 1$ and $q = \infty$. For $1 < q < \infty$ it follows now from the interpolation inequality

$$\|\cdot\|_{L^q} \le \|\cdot\|^{\theta}_{L^1}\|\cdot\|^{1-\theta}_{L^\infty},\ \theta = 1/q$$

Theorem 1.1 is proved. $\square$

Corollary 1.1. *Let $u \in L^\infty_\phi(\Omega)$, where ϕ is the same as in Theorem 1.1. Then the following analogue of the estimate* (1.3) *is valid*

$$\sup_{x_0\in\Omega}\left\{\phi(x_0)\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\} \le C\sup_{x\in\Omega}\{\phi(x)|u(x)|\} \tag{1.4}$$

The proof of this Corollary is the same as the proof of Theorem 1.1 for $q = \infty$.

For the more detailed study of weighted Sobolev spaces defined above we need some regularity assumptions on the domain $\Omega \subset \mathbb{R}^n$ which are assumed to be valid throughout of the paper.

We suppose that there exists a positive number $R_0 > 0$ such that for every point $x_0 \in \Omega$ there exists a smooth domain $V_{x_0} \subset \Omega$ such that

$$B^{R_0}_{x_0} \cap \Omega \subset V_{x_0} \subset B^{R_0+1}_{x_0} \cap \Omega \tag{1.5}$$

Moreover it is assumed also that there exists a diffeomorphizm $\theta_{x_0} : B^1_0 \to V_{x_0}$ such that $\theta_{x_0}(x) = x_0 + p_{x_0}(x)$ and

$$\|p_{x_0}\|_{C^N} + \|p^{-1}_{x_0}\|_{C^N} \le K \tag{1.6}$$

where the constant K is assumed to be independent of $x_0 \in \Omega$ and N is large enough. For simplicity we suppose below that (1.5) and (1.6) hold for $R_0 = 2$.

Note that in the case when Ω is bounded the conditions (1.5) and (1.6) are equivalent to the condition: the boundary $\partial\Omega$ is a smooth manifold, but for unbounded domains the only smoothness of the boundary is not sufficient to obtain the regular structure of Ω when $|x| \to \infty$ since some uniform with respect to $x_0 \in \Omega$ smoothness conditions are required. It is the most convenient for us to formulate these conditions in the form (1.5) and (1.6).

Theorem 1.2. *Let the domain Ω satisfy the conditions* (1.5) *and* (1.6)*, the weight function – the condition* (1.1) *and let R be some positive number. Then the following estimates are valid*

$$(1.7) \quad C_2 \int_\Omega \phi(x)|u(x)|^p\,dx \le \int_\Omega \phi(x_0) \int_{\Omega\cap B^R_{x_0}} |u(x)|^p\,dx\,dx_0 \le \le C_1 \int_\Omega \phi(x)|u(x)|^p\,dx$$

Proof. Let us change the order of integration in the middle part of (1.7)

$$(1.8) \quad \int_\Omega \phi(x_0) \int_{\Omega\cap B^R_{x_0}} |u(x)|^p\,dx\,dx_0 = = \int_\Omega |u(x)|^p \left(\int_\Omega \chi_{\Omega\cap B^R_x}(x_0)\phi(x_0)\,dx_0\right) dx$$

Here $\chi_{\Omega\cap B^R_x}$ is the characteristic function of the set $\Omega\cap B^R_x$.

It follows from the inequalities (1.1) and (1.2) that

$$(1.9) \quad C_1\phi(x) \le \inf_{x_0\in B^R_x}\phi(x_0) \le \sup_{x_0\in B^R_x}\phi(x_0) \le C_2\phi(x)$$

and the assumptions (1.5) and (1.6) imply that

$$(1.10) \quad 0 < C_1 \le \operatorname{mes}(\Omega\cap B^R_x) \le C_2$$

uniformly with respect to $x\in\Omega$.

The estimate (1.7) is an immediate corollary of the estimates (1.8)–(1.10). Theorem 1.2 is proved. □

Corollary 1.2. *Let* (1.5) *and* (1.6) *be valid. Then the equivalent norm in weighted Sobolev space $W^{l,p}_\phi(\Omega)$ is given by the following expression:*

$$(1.11) \quad \|u,\Omega\|_{\phi,l,p} = \left(\int_\Omega \phi(x_0)\|u,\Omega\cap B^R_{x_0}\|^p_{l,p}\,dx_0\right)^{1/p}$$

Particularly, the norms (1.11) *are equivalent for different $R\in\mathbb{R}_+$.*

To study the equation (0.1) we need also weighted Sobolev spaces with fractional derivatives $s\in\mathbb{R}_+$ (not only $s\in\mathbb{Z}$). For the first we recall (see [21] for details) that if V is a bounded domain the norm in the space $W^{s,p}(V)$, $s=[s]+l$, $0<l<1$, $[s]\in\mathbb{Z}_+$ can be given by the following expression

$$(1.12) \quad \|u,V\|^p_{s,p} = \|u,V\|^p_{[s],p} + \sum_{|\alpha|=[s]} \int_{x\in V}\int_{y\in V} \frac{|D^\alpha u(x)-D^\alpha u(y)|^p}{|x-y|^{n+lp}}\,dx\,dy$$

It is not difficult to prove arguing as in Theorem 1.2 and using this representation that for any bounded domain V with a sufficiently smooth boundary

$$(1.13) \quad \|u,V\|^p_{s,p} \le C_1 \int_{x_0\in V} \|u,V\cap B^R_{x_0}\|^p_{s,p}\,dx_0 \le C_2\|u,V\|^p_{s,p}$$

This justifies the following definition.

Definition 1.3. *Define the space $W^{s,p}_{\phi}(\Omega)$ for any $s \in \mathbb{R}_+$ by the norm* (1.11).

It is not difficult to check that these norms are also equivalent for different $R > 0$. Note in conclusion of this Section that the weight functions

$$\phi_{\{\varepsilon\},x_0} = e^{-\varepsilon|x-x_0|} \tag{1.14}$$

satisfy the conditions (1.1) *uniformly* with respect to $x_0 \in \mathbb{R}^n$, consequently all estimates obtained above for the arbitrary weights will be valid for the family (1.14) with constants, independent of $x_0 \in \mathbb{R}^n$. Since these estimates are of fundamental significance for us we write it explicitly in a number of corollaries formulated below.

Corollary 1.3. *Let $u \in L^p_{\{\delta\}}(\Omega)$ for $0 < \delta < \varepsilon$. Then the following estimate holds uniformly with respect to $y \in \mathbb{R}^n$*

$$\left(\int_\Omega e^{-q\delta|x_0-y|}\left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \le \\ \le C_{\varepsilon,q}\int_\Omega e^{-\delta|x-y|}|u(x)|^p\,dx \tag{1.15}$$

Moreover if $u \in L^\infty_{\{\delta\}}(\Omega)$, $\delta < \varepsilon$ then

$$\sup_{x_0\in\Omega}\left\{e^{-\delta|x_0-y|}\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\} \le C_{\varepsilon,\delta}\sup_{x\in\Omega}\{e^{-\delta|x-y|}|u(x)|\} \tag{1.16}$$

Corollary 1.4. *Let $u \in W^{l,p}_{b,\phi}(\Omega)$ and ϕ be a weight function with the rate of growth $\mu < \varepsilon$. Then*

$$C_1\|u,\Omega\|^p_{b,\phi,l,p} \le \\ \le \sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx\right\} \le C_2\|u,\Omega\|^p_{b,\phi,l,p} \tag{1.17}$$

Proof. Indeed,

$$\sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx\right\} \le \\ \le C\sup_{x\in\Omega}\left\{\phi(x)\|u,\Omega\cap B^1_x\|^p_{l,p}\right\}\sup_{x_0\in\Omega}\left\{\int_{x\in\mathbb{R}^n}\phi(x_0)\phi(x)^{-1}e^{-\varepsilon|x-x_0|}\,dx\right\} \le \\ \le C_1\sup_{x\in\Omega}\left\{\phi(x)\|u,\Omega\cap B^1_x\|^p_{l,p}\right\}\sup_{x_0\in\Omega}\left\{\int_{x\in\mathbb{R}^n}e^{\mu|x-x_0|}e^{-\varepsilon|x-x_0|}\,dx\right\} \le \\ \le C_2\|u,\Omega\|^p_{b,\phi,l,p}$$

Conversely, using the evident inequality

$$\|u,\Omega\cap B^1_{x_0}\|^p_{l,p} \le C\int_\Omega e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx \tag{1.18}$$

we obtain that

$$\|u,\Omega\|^p_{b,\phi,l,p} = \sup_{x_0\in\Omega}\left\{\phi(x_0)\|u,\Omega\cap B^1_{x_0}\|^p_{l,p}\right\} \le \\ \le C\sup_{x_0\in\Omega}\left\{\phi(x_0)\int_\Omega e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx\right\}$$

Corollary 1.4 is proved.

This Section is devoted to study the linear problem of the type (0.1)

$$\begin{cases} \partial_t u - \Delta_x u + \lambda_0 u = g(t) \\ u\big|_{t=0} = u_0 \ ; \ u\big|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

in the unbounded domain Ω which is assumed to satisfy the conditions (1.5) and (1.6), formulated in previous Section. To this end we will use weighted Sobolev spaces introduced above.

Theorem 2.1. *Let $g \in L^2([0,T], L^2_{\{\varepsilon_1\}}(\Omega))$ for some $\varepsilon_1 \geq 0$ and let $u_0 \in L^2_{\{\varepsilon_1\}}(\Omega)$. Then there exists the unique solution of the problem (2.1), such that*

$$u \in L_2([0,T], W^{1,2}_{\{\varepsilon_1\}}(\Omega)) \cap W^{1,2}([0,T], W^{-1,2}_{\{\varepsilon_1\}}(\Omega)) \tag{2.2}$$

and for any $\varepsilon > \varepsilon_1$, the following estimate is valid uniformly with respect to $x_0 \in \Omega$:

$$\|u(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_{T-1}^{T} \|u(t), \Omega \cap B^1_{x_0}\|^2_{1,2} + \|\partial_t u(t), \Omega \cap B^1_{x_0}\|^2_{-1,2}\, dt \leq \\ \leq C(|v_0|^2, e^{-\varepsilon|x-x_0|})e^{-(\lambda_0-\varepsilon^2)T} + \int_0^T e^{(\lambda_0-\varepsilon^2)(t-T)}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\, dt \tag{2.3}$$

Here and below $\int_{T-1}^T$ means $\int_0^T$ if $T < 1$.

Proof. We deduce only a priori estimate (2.3) for the solutions of the problem (2.1). The existence of solutions can be obtained from this estimate in a standard way.

Let us multiply the equation (2.1) by $u(t)e^{-\varepsilon|x-x_0|}$ and integrate over $x \in \Omega$

$$\partial_t(|u(t)|^2, e^{-\varepsilon|x-x_0|}) + 2\lambda_0(|u(t)|^2, e^{-\varepsilon|x-x_0|}) + 2(|\nabla_x u(t)|^2, e^{-\varepsilon|x-x_0|}) = \\ = 2(g, ue^{-\varepsilon|x-x_0|}) - 2(\nabla_x u, u\nabla_x e^{-\varepsilon|x-x_0|})$$

Applying Holder inequality to the right–hand side of the last formula and using the evident estimate $|\nabla_x e^{-\varepsilon|x-x_0|}| \leq \varepsilon e^{-\varepsilon|x-x_0|}$ we obtain

$$\partial_t(|u(t)|^2, e^{-\varepsilon|x-x_0|}) + (\lambda_0 - \varepsilon^2)(|u(t)|^2, e^{-\varepsilon|x-x_0|}) + \\ + (|\nabla_x u(t)|^2, e^{-\varepsilon|x-x_0|}) \leq C(|g|^2, e^{-\varepsilon|x-x_0|}) \tag{2.4}$$

Applying Gronewal inequality to the estimate (2.4) we obtain that

$$(|u(T)|^2, e^{-\varepsilon|x-x_0|}) + \int_{T-1}^{T} (|\nabla_x u(t)|^2, e^{-\varepsilon|x-x_0|})\, dt \leq \\ \leq (|u(0)|^2, e^{-\varepsilon|x-x_0|})e^{-(\lambda_0-\varepsilon^2)T} + C\int_0^T e^{(\lambda_0-\varepsilon^2)(t-T)}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\, dt \tag{2.5}$$

Taking into account that $e^{-\varepsilon|x-x_0|} \geq C$ if $x \in B^1_{x_0}$ we derive from (2.5) that

$$
\begin{aligned}
(2.6)\quad & \|u(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_{T-1}^{T} \|\nabla_x u(t), \Omega \cap B^1_{x_0}\|^2_{0,2}\, dt \leq \\
& \leq C(|u(0)|^2, e^{-\varepsilon|x-x_0|})e^{-(\lambda_0-\varepsilon^2)T} + C\int_0^T e^{(\lambda_0-\varepsilon^2)(t-T)}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\, dt
\end{aligned}
$$

The estimate of $\partial_t u$ follows now from (2.6) and from the equation (2.1). The estimate (2.3) is proved.

Note that (2.2) is a corollary of (2.3). Indeed, take in (2.3) $\varepsilon > \varepsilon_1$, multiply it by $e^{-\varepsilon_1|x_0|}$ and integrate over $x_0 \in \Omega$. Then after using the estimates (1.3) and (1.7) we obtain that

$$
\begin{aligned}
& \|u(T), \Omega\|^2_{\{\varepsilon_1\},0,2} + \int_{T-1}^{T} \|u(t), \Omega\|^2_{\{\varepsilon_1\},1,2} + \|\partial_t u(t), \Omega\|^2_{\{\varepsilon_1\},-1,2}\, dt \leq \\
& \qquad \leq \|u(0), \Omega\|^2_{\{\varepsilon_1\},0,2} e^{-(\lambda_0-\varepsilon^2)T} + \int_0^T e^{(\lambda_0-\varepsilon^2)(t-T)}\|g(t), \Omega\|^2_{\{\varepsilon_1\},0,2}\, dt
\end{aligned}
$$

Theorem 2.1 is proved. $\square$

Theorem 2.2. *Let u be a solution of* (2.1) *satisfied* (2.2), *$u_0 \in W^{1,2}_{\{\varepsilon_1\}}(\Omega)$, ε and ε_1 be small enough, $\varepsilon > \varepsilon_1 \geq 0$, and g be the same as in previous Theorem. Then*

$$
(2.7)\qquad u \in L^2([0,T], W^{2,2}_{\{\varepsilon_1\}}(\Omega)) \cap W^{1,2}([0,T], L^2_{\{\varepsilon_1\}}(\Omega))
$$

and the following estimate is valid uniformly with respect to $x_0 \in \Omega$

$$
\begin{aligned}
(2.8)\quad & \|u(T), \Omega \cap B^1_{x_0}\|^2_{1,2} + \int_{T-1}^{T} \|u(t), \Omega \cap B^1_{x_0}\|^2_{2,2} + \|\partial_t u(t), \Omega \cap B^1_{x_0}\|^2_{0,2}\, dt \leq \\
& \leq C\left(\|\nabla_x u(0), \Omega \cap B^3_{x_0}\|^2 + (|u_0|^2, e^{-\varepsilon|x-x_0|})\right) e^{-\gamma T} + \\
& \qquad + C\int_0^T e^{\gamma(t-T)}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\, dt
\end{aligned}
$$

for $\gamma = \lambda_0 - \varepsilon^2 > 0$ for ε small enough.

Proof. Recall firstly that we assume that the domain Ω satisfy the conditions (1.5) and (1.6) and the constant $R_0 = 2$.

Let us consider the cut–off function $\psi_{x_0}(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_{x_0} = 1$ if $x \in B^1_{x_0}$ and $\psi_{x_0} = 0$ if $x \notin B^2_{x_0}$ and let $v_{x_0} = \psi_{x_0} u$. It follows from the equation (2.1) and from the condition (1.5) that v_{x_0} is the solution of the following equation

$$
(2.9)\qquad \begin{cases} \partial_t v_{x_0} - \Delta_x v_{x_0} + \lambda_0 v_{x_0} = \psi_{x_0} g - 2\nabla_x \psi_{x_0} \nabla_x v_{x_0} - \Delta_x \psi_{x_0} v_{x_0} \equiv h_{x_0}(t) \\ v_{x_0}\big|_{\partial V_{x_0}} = 0\ ;\ v_{x_0}\big|_{t=0} = \psi_{x_0} u(0) \end{cases}
$$

where the domains V_{x_0} were defined in (1.5) and (1.6).

Multiplying the equality (2.9) by $\Delta_x v_{x_0}$ and integrating over $x \in V_{x_0}$ we obtain after simple computation involving the integrating by parts and Gronewal inequality that the following estimate holds uniformly with respect to $x_0 \in \Omega$:

$$\begin{aligned}(2.10)\quad &\|v_{x_0}(T), V_{x_0}\|_{1,2}^2 + \int_{T-1}^T \|\Delta_x v_{x_0}(t), V_{x_0}\|_{0,2}^2 + \|\partial_t v_{x_0}(t), V_{x_0}\|_{0,2}^2\, dt \le \\ &\le C_1 \|v_{x_0}(0), V_{x_0}\|_{1,2}^2 e^{-\lambda_0 T} + C_1 \int_0^T e^{\lambda_0(t-T)} \|h_{x_0}(t), V_{x_0}\|_{0,2}^2\, dt \le \\ &\le C_2 \|u(0), \Omega \cap B_{x_0}^3\|_{1,2}^2 e^{-\lambda_0 T} + C_2 \int_0^T e^{\lambda_0(t-T)} \|h_{x_0}(t), \Omega \cap B_{x_0}^3\|_{0,2}^2\, dt\end{aligned}$$

Taking into the account the assumptions (1.5) and (1.6) for the domains V_{x_0} we obtain from the elliptic regularity theorem (see [21]) that

$$\|u(t), \Omega \cap B_{x_0}^1\|_{2,2} \le C\|v_{x_0}(t), V_{x_0}\|_{2,2} \le C_1 \|\Delta_x v_{x_0}(t), V_{x_0}\|_{0,2}$$

Estimating

$$(2.11)\qquad \|h_{x_0}(t), \Omega \cap B_{x_0}^3\|_{0,2}^2 \le C(\|g(t), \Omega \cap B_{x_0}^3\|_{0,2}^2 + \|u(t), \Omega \cap B_{x_0}^3\|_{1,2}^2)$$

and using the estimate (2.3) we obtain now the estimate (2.8). The assertion (2.7) can be reduced from (2.8) in the same way as (2.2) from (2.3). Theorem 2.2 is proved. □

Theorem 2.3. *Let u be a solution of the problem* (2.1) *which satisfies* (2.2), $\varepsilon > \varepsilon_1 \ge 0$ *such as in Theorem 2.2,* $u_0 \in W^{2(1-1/p),p}_{\{\varepsilon_1\}}(\Omega)$ *and* $g \in L^p([0,T], L^p_{\{\varepsilon_1\}}(\Omega))$, $2 \le p < \infty$. *Then*

$$(2.12)\quad u \in C([0,T], W^{2(1-1/p),p}_{\{\varepsilon_1\}}(\Omega)) \cap W^{1,p}([0,T], L^p_{\{\varepsilon_1\}}(\Omega)) \cap L^p([0,T], W^{2,p}_{\{\varepsilon_1\}}(\Omega))$$

and the following estimate is valid uniformly with respect to $x_0 \in \Omega$*:*

(2.13)

$$\begin{aligned}&\|u(T), \Omega \cap B_{x_0}^1\|_{2(1-1/p),p}^p + \int_{T-1}^T \|u(t), \Omega \cap B_{x_0}^1\|_{2,p}^p + \|\partial_t u(t), \Omega \cap B_{x_0}^1\|_{0,p}^p dt \le \\ &\le C e^{-\gamma T} \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} \|u(0), \Omega \cap B_x^1\|_{2(1-1/p),p}^p dx + \\ &\qquad + \int_0^T e^{\gamma(t-T)} (|g(t)|^p, e^{-\varepsilon|x-x_0|})\, dt\end{aligned}$$

for some positive constant γ.

The proof of this Theorem is based on the following result for the auxiliary problem (2.9)

Lemma 2.1. *Let v_{x_0} be a solution of the problem (2.9) and let the initial condition $v_{x_0}(0)\in W^{2(1-1/p),p}\cap W_0^{1,p}(V_{x_0})$ and $h_{x_0}\in L^p([0,T],L^p(V_{x_0}))$. Then*
(2.14)

$$v_{x_0}\in C([0,T],W^{2(1-1/p),p}(V_{x_0}))\cap W^{1,p}([0,T],L^p(V_{x_0}))\cap L^p([0,T],W^{2,p}(V_{x_0}))$$

and the following estimate is valid uniformly with respect to $x_0\in\Omega$

$$\|v_{x_0}(T),V_{x_0}\|^p_{2(1-1/p),p}+\int_{T-1}^T\|v_{x_0}(t),V_{x_0}\|^p_{2,p}+\|\partial_t v_{x_0}(t),V_{x_0}\|^p_{0,p}dt\le$$
$$\le C\|v_{x_0}(0),V_{x_0}\|^p_{2(1-1/p),p}e^{-\lambda_0 T}+C_1\int_0^T e^{\lambda_0(t-T)}\|h_{x_0}(t),V_{x_0}\|^p_{0,p}\,dt \tag{2.15}$$

Proof. The estimate (2.15) is a corollary of well known L^p-regularity theorem for parabolic equations (see [16]). Since the domains V_{x_0} satisfy the conditions (1.6) uniformly with respect to $x_0\in\Omega$ then the constants C,C_1 in (2.15) are also independent of x_0.

The proof of Theorem 2.3. Without loss of generality we may assume that $p>2$. (The case $p=2$ is considered in Theorem 2.2.)

It follows from Lemma 2.1 that

(2.16)

$$\int_{T-1}^T\|u(t),\Omega\cap B^1_{x_0}\|^p_{2,p}+\|\partial_t u(t),\Omega\cap B^1_{x_0}\|^p_{0,p}dt+\|u(T),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}\le$$
$$C\|v_{x_0}(0),V_{x_0}\|^p_{2(1-1/p),p}e^{-\lambda_0 T}+C\int_{T-1}^T\|v_{x_0}(t),V_{x_0}\|^p_{2,p}+\|\partial_t v_{x_0}(t),V_{x_0}\|^p_{0,p}dt\le$$
$$\le C_1\|v_{x_0}(0),V_{x_0}\|^p_{2(1-1/p),p}e^{-\lambda_0 T}+C_1\int_0^T e^{\lambda_0(t-T)}\|h_{x_0}(t),V_{x_0}\|^p_{0,p}$$

We used here the notations of Theorem 2.2.

Estimating the function h_{x_0} using Galiardo-Nirenberg inequality and the estimate (2.8) we obtain

$$\|h_{x_0}(t),V_{x_0}\|^p_{0,p}\le C(\|u(t),V_{x_0}\|^p_{1,p}+\|g(t),V_{x_0}\|^p_{0,p})\le \tag{2.17}$$
$$C_1(\|u(t),V_{x_0}\|^{p\theta}_{2(1-1/p),p}\|u(t),V_{x_0}\|^{p(1-\theta)}_{1,2}+\|g(t),V_{x_0}\|^p_{0,p})\le\mu\|u(t),V_{x_0}\|^p_{2(1-1/p),p}$$
$$+C_\mu\left(\|u(t),V_{x_0}\|^p_{1,2}+\|g(t),V_{x_0}\|^p_{0,p}\right)\le\mu\|u(t),V_{x_0}\|^p_{2(1-1/p),p}+$$
$$+C_\mu\|g(t),V_{x_0}\|^p_{0,p}+C_\mu e^{-\gamma t}\left(\int_\Omega e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_x\|^2_{1,2}dx\right)^{p/2}+$$
$$+C_\mu\left(\int_0^t e^{\gamma(s-t)}\left(|g(s)|^2,e^{-\varepsilon|x-x_0|}\right)ds\right)^{p/2}\le$$
$$\le\mu\|u(t),V_{x_0}\|^p_{2(1-1/p),p}+C_\mu\|g(t),V_{x_0}\|^p_{0,p}+$$
$$C_\mu e^{-\gamma t}\int_\Omega e^{-\varepsilon|x-x_0|}\|u(0),B^1_x\cap\Omega\|^p_{1,p}dx+C_\mu\int_0^t e^{\gamma(s-t)}\left(|g(s)|^p,e^{-\varepsilon|x-x_0|}\right)ds$$

Inserting the estimate (2.17) into the inequality (2.16) we obtain after simple calculations that

$$
\begin{aligned}
(2.18)\quad & \|u(T), \Omega \cap B^1_{x_0}\|^p_{2(1-1/p),p} \le \\
& \le C_\mu e^{-\gamma T}\left(\|u(0), V_{x_0}\|^p_{2(1-1/p),p} + \int_\Omega e^{-\varepsilon|x-x_0|}\|u(0), B^1_x \cap \Omega\|^p_{1,p}dx\right) + \\
& \qquad + \mu \int_0^T e^{\gamma(t-T)}\|u(t), V_{x_0}\|^p_{2(1-1/p),p}dt + \\
& \qquad\qquad + C_\mu \int_0^T e^{\gamma(t-T)}\left(|g(t)|^p, e^{-\varepsilon|x-x_0|}\right) dt
\end{aligned}
$$

Multiplying the inequality by $e^{-\beta|z-x_0|}$, $0<\beta<\varepsilon$ and integrating over $x_0 \in \Omega$ we derive, using the estimates (1.7) and (1.15) that

$$
\begin{aligned}
(2.19)\quad & \int_{x_0\in\Omega} e^{-\beta|z-x_0|}\|u(T), B^1_{x_0} \cap \Omega\|^p_{2(1-1/p),p}dx_0 \le \\
& \qquad \le Ce^{-\gamma T}\int_{x_0\in\Omega} e^{-\beta|z-x_0|}\|u(0), V_{x_0}\|^p_{2(1-1/p),p}dx_0 + \\
& + \mu \sup\nolimits_{t\in[0,T]}\left\{e^{\gamma(t-T)/2}\int_{x_0\in\Omega} e^{-\beta|z-x_0|}\|u(t), V_{x_0}\|^p_{2(1-1/p),p}dx_0\right\} + \\
& \qquad\qquad + C_\mu \int_0^T e^{\gamma(t-T)}\left(|g(t)|^p, e^{-\beta|z-x_0|}\right) dt
\end{aligned}
$$

Recall now that the norms (1.11) are equivalent for different R, since (2.19) implies that

$$
\begin{aligned}
(2.20)\quad & U_z(T) \le C_\mu e^{-\gamma T}\int_{x_0\in\Omega} e^{-\beta|z-x_0|}\|u(0), \Omega \cap B^1_{x_0}\|^p_{2(1-1/p),p}dx_0 + \\
& + C_\mu \int_0^T e^{\gamma(t-T)}\left(|g(t)|^p, e^{-\beta|z-x_0|}\right) dt + \mu \sup\nolimits_{t\in[0,T]} e^{\gamma(t-T)/2}U_z(t)
\end{aligned}
$$

where

$$
U_z(t) = \int_{x_0\in\Omega} e^{-\beta|z-x_0|}\|u(t), \Omega \cap B^1_{x_0}\|^p_{2(1-1/p),p}dx_0
$$

and $\mu > 0$ can be chosen arbitrary small.

To complete the proof of Theorem 2.3 we need one more lemma.

Lemma 2.2. *Let a function $U_z(t)$ be a solution of the following inequality*

$$
(2.21)\qquad U_z(T) \le C_1 e^{-\beta_1 T} + C_2 + \mu \sup_{t\in[0,T]}\{e^{\beta_1(t-T)}U_z(t)\}
$$

and let $\mu \le 1/2$ and $\beta_1 > 0$. Then

$$
(2.22)\qquad U_z(T) \le 2C_1 e^{-\beta_1 T} + 2C_2
$$

Proof. Multiplying the inequality (2.21) by $e^{\beta_1 T}$ and applying $\sup_{T\in[0,s]}$ to the both sides of the obtained inequality we get after simple calculations

$$\sup_{T\in[0,s]}\{e^{\beta_1 T}U_z(T)\} \le C_1 + C_2 e^{\beta_1 s} + \mu \sup_{T\in[0,s]}\{e^{\beta_1 T}U_z(T)\}$$

Taking into account that $\mu \le 1/2$ we derive that

$$\sup_{t\in[0,T]}\{e^{\beta_1 t}U_z(t)\} \le 2C_1 + 2C_2 e^{\beta_1 T}$$

Replacing the last term in (2.21) by this estimate we obtain (2.22).

The end of the proof of Theorem 2.3. Lemma 2.2, applied to the inequality (2.20) with $\mu > 0$ small enough, implies that

$$U_z(T) \le Ce^{-\gamma T/2}\int_{x_0\in\Omega} e^{-\beta|z-x_0|}\|u(0),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}dx_0 + \\ + C\int_0^T e^{\gamma(t-T)}\left(|g(t)|^p, e^{-\beta|z-x_0|}\right)dt \tag{2.23}$$

The estimate (2.23) together with the evident inequality $\|u(T),\Omega\cap B^1_z\|^p_{2(1-1/p),p} \le CU_z(T)$ complete the proof of Theorem 2.3. $\square$

Corollary 2.1. *Let the assumptions of previous Theorem hold and let* $p > \frac{n+2}{2}$*. Then* $u \in C([0,T]\times\Omega)$ *and the following estimate is valid:*

$$|u(T,x_0)|^p \le Ce^{-\gamma T}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}dx + \\ + C\int_0^T e^{\gamma(t-T)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(t,x)|^p dxdt \tag{2.24}$$

Indeed, the assertion of this Corollary follows immediately from the estimate (2.13) and from the embedding theorem (see [16])

$$L^p([T,T+1],W^{2,p}(V_{x_0}))\cap W^{1,p}([T,T+1],L^p(V_{x_0})) \subset C([T,T+1]\times V_{x_0}) \tag{2.25}$$

for $p > \frac{n+2}{2}$.

We finish this Section by some version of comparison principle for the parabolic equations in weighted Sobolev spaces.

Theorem 2.5. *Let a function* u *satisfy* (2.7) *for a certain* $\varepsilon_1 > 0$, $u(0) = 0$ *and let the following inequality be valid almost everywhere in* $[0,T]\times\Omega$*:*

$$\partial_t u - \Delta_x u + \lambda_0 u \ge 0 \tag{2.26}$$

Then almost everywhere in $[0,T]\times\Omega$ $\quad u(t,x) \ge 0$.

Proof. Let us consider the functions $u_+(t,x) = \max\{0, u(t,x)\}$, $u_-(t,x) = u_+(t,x) - u(t,x)$. Using the technique of [23] it is not difficult to prove that

$$u_-, u_+ \in W^{1,2}([0,T], L^2_{\{\varepsilon\}}(\Omega)) \cap L^2([0,T], W^{1,2}_{\{\varepsilon\}}(\Omega)) \tag{2.27}$$

for $\varepsilon \geq \varepsilon_1$ and the following equalities are valid almost everywhere

$$(2.28) \qquad (\partial_t u_+(t), u_-(t))_{\{\varepsilon\}} = (\partial_t u_-(t), u_+(t))_{\{\varepsilon\}} = (\nabla_x u_+(t), \nabla_x u_-(t))_{\{\varepsilon\}} = 0$$

Let us multiply (2.26) by u_- and integrate over Ω. Then due to (2.28) we obtain

$$(2.29) \qquad -1/2\partial_t\|u_-(t),\Omega\|^2_{\{\varepsilon\},0,2} - \lambda_0\|u_-(t),\Omega\|^2_{\{\varepsilon\},0,2} + \varepsilon^2\|u_-(t),\Omega\|^2_{\{\varepsilon\},0,2} \geq 0$$

Applying Gronewal inequality to the estimate (2.29) and taking into account that $u_-(0) = 0$ we obtain that $\|u_-(t),\Omega\|_{\{\varepsilon\},0,2} = 0$ almost everywhere, i.e. $u \geq 0$ almost everywhere. Theorem 2.5 is proved.

§3 The nonlinear equation: a priori estimates, existence of solutions .

In this Section we consider the parabolic boundary problem:

$$(3.1) \qquad \begin{cases} \partial_t u - \Delta_x u + f(u) + \lambda_0 u = g(t) \\ u\big|_{t=0} = u_0; \quad u\big|_{\partial\Omega} = 0 \end{cases}$$

in the unbounded domain Ω which is assumed to to satisfy the conditions (1.5) and (1.6) formulated in Section 1.

Recall that $u = (u^1, \cdots, u^k)$, $\lambda_0 > 0$ – is some positive number, $f = (f^1, \cdots, f^k)$, $g = (g^1, \cdots, g^k)$ and the nonlinear term f satisfies the following conditions

$$(3.2) \qquad \begin{cases} 1. \quad f(u).u \geq -C \\ 2. \quad f \in C^1(\mathbb{R}^k, \mathbb{R}^k) \end{cases}$$

We suppose in this Section that the right–hand side $g = g(t)$ is from the space $L^p_b(\mathbb{R}_+ \times \Omega)$ for some $p > \max\{2, \frac{n+2}{2}\}$ and the initial date u_0 – from the space $W_b^{2(1-1/p),p}(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}$

A solution of the equation (3.1) is defined to be a function u from the space

$$(3.3) \qquad \cap_{\varepsilon>0}\left\{L^p([0,T], W^{2,p}_{\{\varepsilon\}}(\Omega)) \cap W^{1,p}([0,T], L^p_{\{\varepsilon\}}(\Omega))\right\}$$

which satisfies the equation (3.1) in the sense of distributions.

The main aim of this Section is to prove a number of a priori estimates for the solutions of (3.1) and to derive the existence of solutions for this equation.

Theorem 3.1. *Let u be a solution of* (3.1). *Then the following estimate is valid:*

$$(3.4) \quad |u(T,x_0)|^p \leq Ce^{-\gamma T}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_{x_0}\|^p_{0,p}dx + \\ + C\left(1 + \int_0^T e^{\gamma(t-T)}\left(|g(t)|^p, e^{-\varepsilon|x-x_0|}\right)dt\right)$$

for some positive $\gamma > 0$ *and sufficiently small* $\varepsilon > 0$.

Proof. Let us consider the function $w(t,x) = u(t,x).u(t,x)$. Then due to the equation (3.1)

$$\partial_t w - \Delta_x w + 2\lambda_0 w = -2\nabla_x u.\nabla_x u - 2f(u).u + 2g.u \le C + 2g.u \equiv h_u(t) \tag{3.5}$$

We consider also the auxiliary linear problem

$$\begin{cases} \partial_t v - \Delta_x v + 2\lambda_0 v = h_u(t) \\ v\big|_{t=0} = w\big|_{t=0} = u_0.u_0 \end{cases} \tag{3.6}$$

Due to the comparison principle (Theorem 2.4)

$$w(t,x) \le v(t,x), \quad (t,x) \in [0,T] \times \Omega \tag{3.7}$$

Applying Corollary 2.1 to the linear equation (3.6) we obtain that

$$\begin{aligned} |w(T,x_0)|^p \le |v(T,x_0)|^p \le Ce^{-2\gamma T} \int_{x\in\Omega} e^{-2\varepsilon|x-x_0|} \|v(0), \Omega \cap B^1_x\|^p_{2(1-1/p),p} dx + \\ + C\left(1 + \int_0^T e^{2\gamma(t-T)} \int_{x\in\Omega} e^{-2\varepsilon|x-x_0|} |g(t,x)|^p |u(t,x)|^p dx dt\right) \end{aligned} \tag{3.8}$$

Using the estimates (1.15) and (1.3) and the fact that the space $W^{2(1-1/p),1/p}$ is an algebra if $p > \frac{n+2}{2}$ we derive that

$$\begin{aligned} \int_{x\in\Omega} e^{-2\varepsilon|x-x_0|} \|v(0), \Omega \cap B^1_x\|^p_{2(1-1/p),p} dx \le \\ \le C \int_{x\in\Omega} e^{-2\varepsilon|x-x_0|} \|u(0), \Omega \cap B^1_x\|^{2p}_{2(1-1/p),p} dx \le \\ \le C_1 \int_{x\in\Omega} e^{-2\varepsilon|x-x_0|} \left(\int_{y\in\Omega} e^{-\beta|y-x|} \|u(0), \Omega \cap B^1_y\|^p_{2(1-1/p),p} dy\right)^2 dx \le \\ \le C_2 \left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|} \|u(0), \Omega \cap B^1_x\|^p_{2(1-1/p),p} dx\right)^2 \end{aligned} \tag{3.9}$$

Estimating the last integral in (3.8) by Holder inequality we will have

$$\begin{aligned} \int_0^T e^{2\gamma(t-T)} \int_{x\in\Omega} e^{-2\varepsilon|x-x_0|} |g(t,x)|^p |u(t,x)|^p dx dt \le \\ \le \int_0^T e^{2\gamma(t-T)} \sup_{x\in\Omega} \left\{e^{-\varepsilon|x-x_0|} |u(t,x)|^p\right\} \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} |g(t)|^p dx dt \le \\ \le \mu \left(\sup_{t\in[0,T]} e^{\gamma(t-T)} \sup_{x\in\Omega} e^{-\varepsilon|x-x_0|} |u(t,x)|^p\right)^2 + \\ + C_\mu \left(\int_0^T e^{\gamma(t-T)} \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} |g(t)|^p dx dt\right)^2 \end{aligned} \tag{3.10}$$

Inserting these estimates into the inequality (3.8) we obtain that

$$
\begin{aligned}
(3.11)\quad |u(T,x_0)|^p \le Ce^{-\gamma T}\int_{x\in\Omega} & e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_x\|^p_{2(1-1/p),p}dx+\\
&+\mu\sup_{t\in[0,T]} e^{\gamma(t-T)}\sup_{x\in\Omega} e^{-\varepsilon|x-x_0|}|u(t,x)|^p+\\
&\qquad + C_\mu\left(1+\int_0^T e^{\gamma(t-T)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(t)|^p dxdt\right)
\end{aligned}
$$

Applying the operator $\sup_{x_0\in\Omega} e^{-\delta|z-x_0|}$, $\delta<\varepsilon$ to the both sides of the inequality (3.11) and using the estimates (1.3) and (1.4) we will have

$$
\begin{aligned}
(3.12)\quad U_z(T)\le C^{-\gamma T}\int_{x\in\Omega} & e^{-\delta|z-x|}\|u(0),\Omega\cap B^1_x\|^p_{2(1-1/p),p}dx+\\
&+\mu\sup_{t\in[0,T]} e^{\gamma(t-T)}U_z(t)+\\
&\qquad + C_\mu\left(1+\int_0^T e^{\gamma(t-T)}\int_{x\in\Omega} e^{-\delta|z-x|}|g(t)|^p dxdt\right)
\end{aligned}
$$

with $U_z(t)=\sup_{x_0\in\Omega}\{e^{-\delta|z-x_0|}|u(t,x_0)|^p\}$.

Lemma 2.2 implies now (if μ is small enough) that

$$
\begin{aligned}
(3.13)\quad U_z(T)\le C^{-\gamma T}\int_{x\in\Omega} & e^{-\delta|z-x|}\|u(0),\Omega\cap B^1_x\|^p_{2(1-1/p),p}dx+\\
&\qquad + C\left(1+\int_0^T e^{\gamma(t-T)}\int_{x\in\Omega} e^{-\delta|z-x|}|g(t)|^p dxdt\right)
\end{aligned}
$$

Theorem 3.1 is proved.

Corollary 3.1. *Under the assumptions of previous Theorem the following estimate is valid*

$$
(3.14)\qquad \|u(T)\|_{L^\infty(\Omega)}\le Ce^{-\gamma T}\|u(0),\Omega\|_{b,2(1-1/p),p}+C_1(1+\|g\|_{L^p_b(\mathbb{R}_+\times\Omega)})
$$

Indeed, the estimate (3.14) is an immediate corollary of (3.4) and (1.17).

Theorem 3.2. *Let u be a solution of the problem* (3.1) *and let the above assumptions be valid. Then the following estimate is valid:*

(3.15)

$$
\begin{aligned}
\|u(T),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}+\int_{T-1}^T \|u(t),\Omega\cap B^1_{x_0}\|^p_{2,p}+\|\partial_t u(t),\Omega\cap B^1_{x_0}\|^p_{0,p}dt\le\\
\le Q\left(\|u(0),\Omega\|_{b,2(1-1/p),p}\right)e^{-\gamma T}+Q\left(\|g\|_{L^p_b(\mathbb{R}_+\times\Omega)}\right)
\end{aligned}
$$

Here $\gamma>0$ and Q is a certain monotonic function independent of $x_0\in\Omega$.

Proof. Let us rewrite the equation (3.1) in the form of linear one

$$
(3.16)\qquad \partial_t u=\Delta_x u-\lambda_0 u+h_f(t)
$$

where $h_f(t) \equiv g(t) - f(u(t))$. Applying the estimate (2.13) to the equation (3.16) and using the inequalities (1.17) we derive that

(3.17)

$$\|u(T), \Omega \cap B^1_{x_0}\|^p_{2(1-1/p),p} + \int_{T-1}^{T} \|u(t), \Omega \cap B^1_{x_0}\|^p_{2,p} + \|\partial_t u(t), \Omega \cap B^1_{x_0}\|^p_{0,p} dt \le$$
$$\le Ce^{-\gamma T}\|u(0), \Omega\|^p_{b,2(1-1/p),p} + C\|g\|^p_{L^p_b(\mathbb{R}_+\times\Omega)} +$$
$$+ C\int_0^T e^{\gamma(t-T)}\|f(u(t))\|^p_{L^\infty(\Omega)} dt$$

To complete the proof of Theorem 3.2 we need the following Lemma.

Lemma 3.1. *Let the function f be continuous and the function u satisfy the estimate* (3.14)*. Then*

$$\|f(u(t))\|_{L^\infty(\Omega)} \le Q(\|u(0)\|_{b,2(1-1/p),p})e^{-\gamma T} + Q(\|g\|_{L^p_b(\mathbb{R}_+\times\Omega)}) \tag{3.18}$$

For a certain monotonic function Q.

The proof of Lemma 3.1 is given in [22].

Inserting the estimate (3.18) into the inequality (3.17) we immediately obtain the assertion of the theorem. □

Corollary 3.2. *Let the above assumptions be valid. Then*

$$\|u(T), \Omega\|_{b,2(1-1/p),p} \le Q(\|u(0), \Omega\|_{b,2(1-1/p),p})e^{-\gamma T} + Q(\|g\|_{L^p_b(\mathbb{R}_+\times\omega)}) \tag{3.19}$$

Remark 3.1. *Note that all estimates derived above depend only on the constant K, R_0 which are defined in the assumptions* (1.5) *and* (1.6)*. Thus if we consider a sequence Ω_N which satisfy these assumptions uniformly with respect to $N \in \mathbb{N}$ then the function Q in* (3.20) *can be chosen independently of Ω_N.*

Now we are in position to prove the existence of solutions for the problem (3.1). To this end we prove firstly this existence in the case when the domain Ω is bounded.

Theorem 3.3. *Let the above assumptions be valid and let Ω be bounded. Then the problem* (3.1) *has at least one solution in the space*

$$W_\Omega([0,T]) = L^p([0,T], W^{2,p}(\Omega)) \cap W^{1,p}([0,T], L^p(\Omega)) \tag{3.20}$$

and the following estimate is valid:

$$\|u\|_{W_\Omega([0,T])} \le Q(\|u(0), \Omega\|_{b,2(1-1/p),p}) + Q(\|g\|_{L^p_b([0,T]\times\Omega)}) \tag{3.21}$$

Proof. A priori estimate (3.21) is an immediate corollary of the estimate (3.19) and the existence of solutions for the equation (3.1) can be deduced from this estimate in a standard way involving for instance Leray-Schauder principle (see [24]). Theorem 3.3 is proved.

Theorem 3.4. *Let the above assumptions hold and let Ω be an arbitrary unbounded domain which satisfies the assumptions* (1.5) *and* (1.6)*. Then the problem* (3.1) *has at least one solution from the class* (3.3)*.*

Proof. Let Ω_N, $N = 1, 2, \cdots$ be the sequence of smooth bounded domains, which satisfy the conditions (1.5) and (1.6) uniformly with respect to $N \in \mathbb{N}$, such that

$$\begin{cases} \Omega_N \subset \Omega_{N+1} \subset \Omega \ ; \ \Omega = \cup_{N=1}^{\infty} \Omega_N \\ \Omega \cap B_0^N \subset \Omega_N \subset \Omega \cap B_0^{N+1} \end{cases} \tag{3.22}$$

It is not difficult to check that such sequence exists.

Let us introduce the sequence of the cut–off functions $\psi_N(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_N(x) = 1$ if $x \in B_0^{N-1}$, $\psi_N(x) = 0$ if $x \notin B_0^N$ and $\|\psi_N\|_{C^2} \le C$.

Let u_N be a solution of the following problem

$$\begin{cases} \partial_t u_N - \Delta_x u_N + \lambda_0 u_N + f(u_N) = g(t) \\ u_N\big|_{\partial\Omega_N} = 0 \ ; \ u_N\big|_{t=0} = \psi_N u_0 \end{cases} \tag{3.23}$$

According to Remark 3.1 the estimates (3.21) with u replaced by u_N are valid uniformly with respect to $N \in \mathbb{N}$. Thus, for every $M \in \mathbb{N}$ the sequence $u_N\big|_{\Omega\cap B_0^M}$, $N \ge M$ is bounded in the space $W_{\Omega\cap B_0^M}([0,T])$. Note that the space $W_{\Omega\cap B_0^M}([0,T])$ is reflexive since using Cantor's diagonal procedure we can extract from the sequence u_N a subsequence (which we denote by u_N also for simplicity) converging weakly to u in $W_{\Omega\cap B_0^M}([0,T])$ for every $M \in \mathbb{N}$. Using the compactness of the embedding $W_{\Omega\cap B_0^M}([0,T]) \subset C([0,T] \times \Omega \cap B_0^M)$ we obtain that $u_N \to u$ strongly in the space $C_{loc}([0,T] \times \Omega)$.

Thus, $f(u_N) \to f(u)$ strongly in $C_{loc}([0,T] \times \Omega)$ and consequently u satisfies the equation (3.1) in the sense of distributions.

Fixing $x_0 \in \Omega$ in the estimate (3.15) for the solutions u_N and passing to the limit $N \to \infty$ we obtain that the function u is also satisfies this estimate $uniformly$ with respect to $x_0 \in \Omega$. The embedding (3.3) is a trivial corollary of this estimate and consequently u is a solution of the problem (3.1). Theorem 3.4 is proved. $\square$

§4 The nonlinear equation: uniqueness of solutions.

In this Section we prove the uniqueness of the solution for the equation (3.1) and derive a number of estimates for the difference between two solutions of this equation (for different initial conditions and different right–hand sides) in weighted Sobolev spaces which are of fundamental significance in our study of the attractor.

Theorem 4.1. *Let the assumptions of previous Section be valid and let $u_1(t)$ and $u_2(t)$ be two solutions of the problem* (3.1) *with the right–hand sides g_1 and g_2 correspondingly. Then the following estimate holds:*

$$\|u_1(T) - u_2(T), \Omega \cap B_{x_0}^1\|_{0,2}^2 + \int_{T-1}^T \|u_1(t) - u_2(t), \Omega \cap B_{x_0}^1\|_{1,2}^2 dt \le \\ \le Ce^{KT} \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} |u_1(0) - u_2(0)|^2 dx + \\ + C \int_0^T e^{K(T-t)} \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} |g_1(t) - g_2(t)|^2 dx dt \tag{4.1}$$

Here $\gamma, \varepsilon > 0$ – some positive constants, independent of $u_i(0)$ and g_i and constants C, K depend on $\|u_i(0)\|_{b,2(1-1/p),p}$ and $\|g_i\|_{L^p_b}$

Proof. Since u_1, u_2 are two solutions of (3.1) then according to Corollary 3.1

$$\|u_i, \mathbb{R}_+ \times \Omega\|_{0,\infty} \le Q(\|u_i(0), \Omega\|_{b,2(1-1/p),p}) + Q(\|g_i, \mathbb{R}_+ \times \Omega\|_{b,0,p}) \tag{4.2}$$

for i=1,2. Let $v(t) = u_2(t) - u_1(t)$, $h(t) = g_2(t) - g_1(t)$. Then

$$\begin{cases} \partial_t v - \Delta_x v + \lambda_0 v = -\widehat{L}(t,x)v + h(t) \\ v\big|_{t=0} = u_2(0) - u_1(0) \end{cases} \tag{4.3}$$

Here

$$\widehat{L}(t,x) = \int_0^1 f'_u(u_1 + \theta v)\,d\theta \tag{4.4}$$

It follows from the condition (3.2) that

$$\begin{aligned} (4.5)\quad & \|\widehat{L}(t)\|_{0,\infty} \le K_1 = \\ &= K_1(\|u_1,\Omega\|_{b,2(1-1/p),p}, \|u_2,\Omega\|_{b,2(1-1/p),p}, \|g_1, \mathbb{R}_+ \times \Omega\|_{b,0,p}, \|g_2, \mathbb{R}_+ \times \Omega\|_{b,0,p}) \end{aligned}$$

After multiplying the equation (4.3) by $ve^{-\varepsilon|x-x_0|}$ and integrating over Ω we obtain that

$$\begin{aligned} (4.6)\quad \partial_t \left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) &+ 2\left(|\nabla_x v(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \\ &+ 2\lambda_0 \left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) + 2\left(\widehat{L}_1 v.v, e^{-\varepsilon|x-x_0|}\right) = \\ &= 2\left(h, ve^{-\varepsilon|x-x_0|}\right) - 2\left(ve^{-\varepsilon|x-x_0|}, \nabla_x v.\nabla_x e^{-\varepsilon|x-x_0|}\right) \end{aligned}$$

Fixing $\varepsilon > 0$ small enough and applying Holder inequality (as in in the proof of Theorem 2.1) we derive from (4.5) and (4.6) that

$$\begin{aligned} (4.7)\quad \partial_t \left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) &+ \left(|\nabla_x v(t)|^2, e^{-\varepsilon|x-x_0|}\right) - \\ &- K\left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) \le C\left(|h(t)|^2, e^{-\varepsilon|x-x_0|}\right) \end{aligned}$$

The estimate (4.1) is an immediate corollary of this inequality and Gronewal's lemma. Theorem 4.1 is proved. $\square$

Corollary 4.1. *Let the assumptions of Section 3 be valid. Then the problem* (3.1) *has the unique solutions in the class* (3.3).

Corollary 4.2. *Let the assumptions of Theorem 4.1 hold. Then the following estimate is valid uniformly with respect to* $x_0 \in \Omega$

$$
\begin{aligned}
(4.8)\quad \|u_2(T)-u_1(T),\Omega\cap B^1_{x_0}\|^2_{1,2} \le \\
\le Ce^{KT}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{1,2}dx+ \\
+C\int_0^T e^{K(T-t)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g_1(t)-g_2(t)|^2dxdt
\end{aligned}
$$

Indeed, applying the inequality (2.8) to the equation (4.3) we will have

$$
\begin{aligned}
(4.9)\quad \|v(T),\Omega\cap B^1_{x_0}\|^2_{1,2} \le Ce^{-\gamma T}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|v(0),\Omega\cap B^1_x\|^2_{1,2}dx+ \\
+C\int_0^T e^{\gamma(t-T)}e^{-\varepsilon|x-x_0|}|h(t)|^2dxdt+ \\
+C\int_0^T e^{\gamma(T-t)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|\widehat{L}(t)v(t)|^2dxdt
\end{aligned}
$$

Estimating the last integral in (4.9) by (4.1) and (4.5) we obtain the estimate (4.8).

We formulate now some smoothing properties for the difference of two solutions of (3.1) which will be used below for the entropy estimates.

Proposition 4.1. *Let the assumptions of Theorem 4.1 hold. Then*

$$
\begin{aligned}
(4.10)\quad \|u_2(T)-u_1(T),\Omega\cap B^1_{x_0}\|^2_{1,2} \le C\frac{e^{KT}}{T^2}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|u_1(0)-u_2(0)|^2dx+ \\
+C\int_0^T e^{K(T-t)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g_1(t)-g_2(t)|^2dxdt
\end{aligned}
$$

for certain K, C *which depend on* $\|u_i(0)\|_{b,2(1-1/p),p}$ *and* $\|g_i\|_{L^2_b}$.

Proof. Indeed, let us decompose the function v into a sum of two functions $v(t) = v_1(t)+v_2(t)$, where $v_1(t)$ is a solution of the equation (4.3) with $v_1(0) = v(0)$ and $h \equiv 0$ and $v_2(t)$ satisfies (4.3) with $v_2(0) = 0$.

Then, Corollary 4.2 implies

$$
(4.11)\quad \|v_2(T),\Omega\cap B^1_{x_0}\|^2_{1,2} \le C\int_0^T e^{K(T-t)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g_1(t)-g_2(t)|^2dxdt
$$

and it follows from Theorem 4.1 that

$$
(4.12)\quad \|v_1(T),\Omega\cap B^1_{x_0}\|^2_{0,2} \le Ce^{KT}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|v(0)|^2dx
$$

Consider now the function $w(t) = tv_1(t)$ which satisfies the equation

$$
(4.13)\quad \partial_t w = \Delta_x w - \lambda_0 w - \widehat{L}(t)w + v_1 \quad , \quad w(0) = 0
$$

The estimate (4.8) applied to (4.13) implies that

$$T^2\|v_1(T),\Omega\cap B^1_{x_0}\|^2_{1,2}\le C\int_0^T e^{K(T-t)}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}|v_1(t)|^2dxdt \tag{4.14}$$

Estimating the integral in the right–hand side of (4.14) by (4.12) we obtain after simple calculations

$$T^2\|v_1(T),\Omega\cap B^1_{x_0}\|^2_{1,2}\le Ce^{KT}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}|u_1(0)-u_2(0)|^2dx \tag{4.15}$$

The estimate (4.15) together with (4.11) implies (4.10). Proposition 4.1 is proved.

Proposition 4.2. *Let the assumptions of previous Theorem be valid. Then*

$$\begin{aligned}\|u_2(T)-u_1(T),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}\le\\ \le C\frac{e^{KT}}{T^{p^2/2}}\left(\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}|u_1(0)-u_2(0)|^2dx\right)^{p/2}+\\ +C\int_0^T e^{K(T-t)}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}|g_1(t)-g_2(t)|^pdxdt\end{aligned} \tag{4.16}$$

Proof. To deduce the estimate (4.16) we need the following lemma which is the analogue of Theorem 2.3 for the equation (4.3).

Lemma 4.1. *Let v be a solution of the equation* (4.3) *and let* (4.5) *be valid. Then*

$$\begin{aligned}\|v(T),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}\le Ce^{KT}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(0),\Omega\cap B^1_x\|^p_{2(1-1/p),p}dx+\\ +\int_0^T e^{K(T-t)}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}|h(t)|^pdxdt\end{aligned} \tag{4.17}$$

The proof of this lemma is analogous to the proof of Theorem 2.3 and we left it to a pedant reader.

Let the functions $v_1(t)$ and $v_2(t)$ be the same as in the proof of Proposition 4.1. Then, according to Lemma 4.1,

$$\begin{aligned}\|v_2(T),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}\le\\ \le C\int_0^T e^{K(T-t)}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}|g_1(t)-g_2(t)|^pdxdt\end{aligned} \tag{4.18}$$

Consider now the function $w(t)=t^{p/2}v_1(t)$. Then

$$\partial_t w=\Delta_x w-\lambda_0 w-\widehat{L}(t)w+p/2t^{p/2-1}v_1\quad,\quad w(0)=0 \tag{4.19}$$

Lemma 4.1 implies that

(4.20)

$$\|w(T),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}\le C\int_0^T e^{K(T-t)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}t^{(p-2)p/2}|v_1(t)|^p dxdt\le$$
$$\le C\int_0^T e^{K(T-t)}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|w(t)|^{p-2}|v_1(t)|^2 dxdt$$

Estimating the last integral into the right–hand side of (4.20) by Holder inequality and by (4.12) and using the embedding $W^{2(1-1/p),p}\subset C$ we will have

$$(4.21)\quad \int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|w(t)|^{p-2}|v_1(t)|^2dx\le$$
$$\le C\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\||w(t)|^{p-2}|v_1(t)|^2,\Omega\cap B^1_x\|_{0,1}dx\le$$
$$\le C_1\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|w(t),\Omega\cap B^1_x\|^{p-2}_{0,\infty}\|v_1(t),\Omega\cap B^1_x\|^2_{0,2}dx\le$$
$$\le C_2\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|w(t),\Omega\cap B^1_x\|^p_{2(1-1/p),p}dx+$$
$$+C_3e^{K_1t}\left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|v(0)|^2dx\right)^{p/2}$$

Inserting this estimate into the inequality (4.20), multiplying the obtained inequality by $e^{-\delta|z-x_0|}$ and integrating over $x_0\in\Omega$ we obtain after simple calculations that

$$(4.22)\qquad U_z(T)\le Ce^{K_1T}\int_0^T U_z(t)+Ce^{K_1T}\left(\int_{x\in\Omega} e^{-\delta|x-z|/2}|v(0)|^2dx\right)^{p/2}$$

where $U_z(t)=\int_{x\in\Omega} e^{-\delta|x-z|}\|u,\Omega\cap B^1_x\|^p_{2(1-1/p),p}dx$

Applying Gronewal inequality to the estimate (4.22) we will have

$$(4.23)\quad \|w(T),\Omega\cap B^1_{x_0}\|^p_{2(1-1/p),p}\le CU_{x_0}(T)\le$$
$$\le C_1e^{K_2T}\left(\int_{x\in\Omega} e^{-\delta|x-x_0|/2}|v(0)|^2dx\right)^{p/2}$$

The estimate (4.23) together with the estimate (4.18) prove Proposition 4.2.

We conclude this Section by deriving the estimates for a difference between two solution of the equation (3.1) in weighted Sobolev spaces, introduced in Section 1.

Corollary 4.2. *Let $\phi(x)\in L^1(\mathbb{R}^n)$ be a weight function with a sufficiently small rate of growth (see Section 1) and let the assumptions of Corollary 4.1 hold. Then*

$$(4.24)\quad \|u_1(1)-u_2(1),\Omega\|_{b,\phi,1,2}\le C\|u_1(0)-u_2(0),\Omega\|_{b,\phi,0,2}+$$
$$+C\|g_1-g_2,[0,1]\times\Omega\|_{b,\phi,0,2}$$

where the constants C and K depend only on $\|u_i(0),\Omega\|_{b,2(1-1/p),p}$ and $\|g_i,\mathbb{R}_+\times\Omega\|_{b,0,p}$, $i=1,2$.

Indeed, multiplying the estimate (4.10) by $\phi(x_0)$, taking $\sup_{x_0\in\Omega}$ to the both sides of the obtained inequality and using (1.17) we obtain (4.24).

Corollary 4.3. *Let the assumptions of previous corollary be valid. Then*

$$\|u_1(1)-u_2(1),\Omega\|_{b,\phi^{p/2},2(1-1/p),p}\le C\|u_1(0)-u_2(0),\Omega\|_{b,\phi,0,2}+ \\ + C\|g_1-g_2,[0,1]\times\Omega\|_{b,\phi^{p/2},0,p} \tag{4.25}$$

Indeed, the estimate (4.25) is an immediate corollary of (4.16), (1.3) and (1.17).

Part 2. The attractors.

This part of the paper is devoted to study the longtime behavior of solutions of (0.1) in the spaces of functions which are bounded with respect to $|x|\to\infty$.

In Section 5 we construct the attractor $\mathcal{A}$ for the equation (0.1) in the particular case where the right–hand side g is independent of t.

The general case where g depends on t and translation-compact in the appropriate space is considered in Section 6. Moreover, the class of admitable right–hand sides g (for which we can construct the attractor) is also studied.

§5 The autonomous attractor.

In this Section we prove the existence of the attractor for the equation (3.1) under the additional assumption

$$g(t)\equiv g\in L^p_b(\Omega) \tag{5.1}$$

The general case will be considered in the next Section.

The assumption (5.1) together with Corollary 4.1 imply that under the conditions of Section 3 the equation (3.1) generates a semigroup $S_t:\Phi_b(\Omega)\to\Phi_b(\Omega)$, where $\Phi_b(\Omega)=W^{2(1-1/p),p}_b(\Omega)\cap\{u_0\big|_{\partial\Omega}=0\}$, by formula

$$S_tu(0)=u(t)\ \text{ where } u(t) \text{ is a solution of (3.1)} \tag{5.2}$$

Moreover, it follows from Corollary 3.2 that this semigroup possesses a bounded absorbing set K in the space $\Phi_b(\Omega)$, i.e. for any other bounded subset $B\subset\Phi_b(\Omega)$ there exists $T=T(B)$ such that

$$S_tB\subset K \text{ if } t\ge T$$

It seems natural to consider the attractor of (5.2) in the 'uniform' topology of the space $\Phi_b(\Omega)$ but, as it shown in Remark 5.1, in contrast to the case of bounded domains Ω in our situation the existence of a compact attractor for (5.2) in the 'uniform' topology of $\Phi_b(\Omega)$ is very restrictive assumption which violates even in the simplest examples. That's why we will construct below the attractor $\mathcal{A}$ of the semigroup (5.2) which attracts bounded subsets of $\Phi_b(\Omega)$ only in a local topology of the space $\Phi_{loc}=W^{2(1-1/p),p}_{loc}(\Omega)$ (i.e., $\mathcal{A}$ is the (Φ_b,Φ_{loc})-attractor of (5.2) in notations of [2]).

Recall that the space $\Phi_{loc}(\Omega)$ is reflexive metrizable F-space which defines by seminorms $\|\cdot,\Omega\cap B^1_{x_0}\|_{2(1-1/p),p}$, $x_0\in\Omega$.

Definition 5.1. *The set $\mathcal{A} \subset \Phi_b(\Omega)$ is defined to be the attractor of the semigroup S_t if the following assumptions hold:*

1. The set $\mathcal{A}$ is compact in $\Phi_{loc}(\Omega)$.

2 The set $\mathcal{A}$ is strictly invariant with respect to S_t, i.e.

$$S_t\mathcal{A} = \mathcal{A} \text{ for } t \geq 0$$

3. The set $\mathcal{A}$ is the attracting set for S_t in local topology, i.e. for every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of the space $\Phi_{loc}(\Omega)$ and for every bounded in uniform topology subset $B \subset \Phi_b(\Omega)$ there exists $T = T(\mathcal{O}, B)$ such that

$$S_tB \subset \mathcal{O}(\mathcal{A}) \text{ if } t \geq T$$

Recall that the first condition means that the restriction $\mathcal{A}\big|_{\Omega_1}$ is compact in $W^{2(1-1/p),p}(\Omega_1)$ for every *bounded* $\Omega_1 \subset \Omega$.

Analogously, the third condition means that for every *bounded* $\Omega_1 \subset \Omega$, every bounded B in $\Phi_b(\Omega)$ and every $W^{2(1-1/p),p}(\Omega_1)$-neighborhood $\mathcal{O}(\mathcal{A}\big|_{\Omega_1})$ of the restriction $\mathcal{A}\big|_{\Omega_1}$ there exists $T = T(\Omega_1, \mathcal{O}, B)$ such that

$$(S_tB)\big|_{\Omega_1} \subset \mathcal{O}(\mathcal{A}\big|_{\Omega_1}) \text{ if } t \geq T$$

Theorem 5.1. *Let the above assumptions be valid. Then the semigroup S_t, defined by* (5.2), *possesses an attractor $\mathcal{A}$ in the sense of Definition 5.1 which has the following structure:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0} \tag{5.3}$$

where we denote by $\mathcal{K}$ the set of all solutions of (3.1), *defined and bounded for all $t \in \mathbb{R}$ ($\sup_{t\in\mathbb{R}} \|u(t)\|_{\Phi_b(\Omega)} < \infty$).*

Proof. According to the attractor's existence theorem for abstract semigroups (see [2]), it is sufficient to verify the following conditions:

1. The operators S_t is Φ_{loc}-continuous on every Φ_b-bounded set and for every fixed $t \geq 0$.

2. The semigroup S_t possesses the precompact attracting set in Φ_{loc}-topology.

The continuity of S_t is an immediate corollary of Lemma 4.1. Since it remains only to construct the compact attracting set.

According to Corollary 3.2, the set $B_R = \{u_0 \in \Phi_b : \|u_0\|_{\Phi_b} \leq R\}$ is the absorbing set for the semigroup S_t if R is large enough. Hence the set $K = S_1B_R$ is the absorbing set also. So it remains to prove that K is precompact in Φ_{loc}.

According to Cantor's diagonal procedure it is sufficient to prove that the restriction $K\big|_{\Omega_1}$ is precompact for every bounded $\Omega_1 \subset \Omega$. To this end we fix an arbitrary bounded subdomain $\Omega_1 \subset \Omega$ and consider an arbitrary sequence $u_n(1)$, $n \in \mathbb{N}$, $u_n(0) \in B_R$.

Since Ω_1 is bounded that $\Omega_1 \subset B_0^M$ for a sufficiently large M. Let $\psi(x)$ be the cut-off function, such that $\psi(x) = 1$ if $|x| \leq M$ and $\psi(x) = 0$ if $|x| > M + 1$. Then

$\psi\big|_{\Omega_1} \equiv 1$. Let us consider now a sequence $w_n(t) = t\psi(x)u_n(t)$ which evidently satisfy the equations

$$
(5.4)\qquad \begin{cases} \partial_t w_n(t) - \Delta_x w_n(t) + \lambda_0 w_n(t) = -t\psi(x)f(u_n(t)) + \\ +t\psi(x)g - t\Delta_x\psi(x)u_n(t) - 2t\nabla_x\psi(x).\nabla_x u_n(t) + \psi(x)u_n(t) \equiv h_n(t) \\ w_n(0) = 0; \quad w_n\big|_{\tilde\Omega_M} = 0 \end{cases}
$$

where $\tilde\Omega_M \subset \Omega$ is bounded domain with smooth boundary such that $\Omega \cap B_0^{M+1} \subset \tilde\Omega_M$.

According to Corollary 3.2, the sequence $u_n(t)$ is bounded in

$$
(5.5)\qquad W_M = L^p([0,1], W^{2,p}(\tilde\Omega_M)) \cup W^{1,p}([0,1], L^p(\tilde\Omega_M))
$$

hence without loss of generality we may assume that $u_n \to u$ weakly in this space. Using the compactness of embedding $W_M \subset L^p([0,1], W^{1,p}(\tilde\Omega_M)) \cap C([0,1] \times \tilde\Omega_M)$ one can easily derive that $h_n \to h$ strongly in $L^p([0,1] \times \tilde\Omega_M)$. The parabolic regularity theorem, applied to the equation (5.4), implies now that $u_n(1) \to u(1)$ in $W^{2(1-1/p),p}(\Omega_1)$. Theorem 5.1 is proved.

Let us discuss now the problem of the attractor's existence in a 'uniform' topology of the space $\Phi_b(\Omega)$.

Lemma 5.1. *Let $\Omega = \mathbb{R}^n$ and the right–hand side g be independent of x. Let us suppose also that the attractor $\mathcal{A}$ of the problem* (3.1) *is compact in the space $W_b^{2(1-1/p),p}(\Omega)$. Then*

$$
(5.6)\qquad \mathcal{A} \subset AP(\mathbb{R}^n)
$$

Here we denote by $AP(\mathbb{R}^n)$ the space of almost-periodic functions in $\mathbb{R}^n$ (see [12]*).*

Proof. Let $u_0 \in \mathcal{A}$. Then by definition, we should verify that the hull

$$
(5.7)\qquad H(u_0) = \{T_h u_0, h \in \mathbb{R}^n\}_{C_b(\mathbb{R}^n)}, \text{ where } (T_h u_0)(x) = u_0(x+h)
$$

is compact in $C_b(\mathbb{R}^n)$. (Here and below we denote by $\{\cdot\}_X$ the closure in the topology of the space X.)

Note that our equation is invariant with respect to T_h since $u_0 \in \mathcal{A}$ implies $H(u_0) \subset \mathcal{A}$. But according to our assumptions $\mathcal{A}$ is compact in $\Phi_b(\Omega)$ and consequently (since $\Phi_b(\Omega)$ is continuously embedded in $C_b(\mathbb{R}^n)$) the hull $H(u_0)$ is compact in $C_b(\mathbb{R}^n)$. Lemma 5.1 is proved.

Remark 5.1. It is worth to emphasize now that the obtained embedding $\mathcal{A} \subset AP(\mathbb{R}^n)$ is not natural. Indeed, consider the Chafee-Infante equation in $\mathbb{R}^n$

$$
(5.8)\qquad \partial_t u = \Delta_x u - u^3 + \alpha^2 u
$$

Then, the equilibria point

$$
u_0(x) = \alpha \tanh\left(\frac{\alpha}{2^{1/2}} x_1\right)
$$

evidently belongs to the attractor $\mathcal{A}$ but not almost periodic. Thus, the attractor $\mathcal{A}$ of the equation (5.8) which exists according to Theorem 5.1 is not compact in $\Phi_b(\mathbb{R}^n)$.

Note in conclusion of this Section that we construct the attractor $\mathcal{A}$, which attracts bounded subsets of $\Phi_b(\Omega)$ only in a local topology of F-space $\Phi_{loc}(\Omega)$ which is not a B-space. The following Lemma allows us to avoid this difficulty.

Lemma 5.2. *Let $\phi \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ be a weight function in the sense of Definition 1.1 and let B be a bounded subset of $\Phi_b(\Omega)$. Then the topologies induced on B by the embeddings $B \subset W^{2(1-1/p),p}_{loc}(\Omega)$ and $B \subset W^{2(1-1/p),p}_{\phi}(\Omega)$ are coincide.*

The assertion of this Lemma is more or less evident and based on the fact that for every $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that

$$\|B, \Omega \cap \{|x| > R\}\|_{\phi,2(1-1/p),p} \le \varepsilon$$

if $\phi \in L^1(\mathbb{R}^n)$.

Corollary 5.1. *Let the assumptions of Theorem 5.1 be valid and let the weight ϕ be the same as in Lemma 5.2. Then the attractor $\mathcal{A}$ ((Φ_b, Φ_{loc})-attractor) of the equation* (3.1), *constructed in Theorem 5.1, is simultaneously (Φ_b, Φ_ϕ)-attractor, where $\Phi_\phi = W^{2(1-1/p),p}_{\phi}(\Omega)$.*

§6 The nonautonomous attractor.

In this Section we consider the general case of the equation (3.1) where the right–hand side $g = g(t)$ depends on t. In order to construct the attractor of the nonautonomous equation we will use the approach, developed in [4], [5].

Together with our initial equation we will consider simultaneously a family of equation of the type (3.1), obtained from the initial one by positive shifting along the t axis and by taking a closure in the corresponding topology.

To be more precise we consider the family of problems of type (3.1)

$$\partial_t u = \Delta_x u - \lambda_0 u - f(u) + \xi(t), \quad \xi \in \mathcal{H}^+(g) \tag{6.1}$$

where in contrast to the case of almost-periodic functions (see (5.7)) we define the hull by taking the closure of the set $\{T_h g, h \in \mathbb{R}_+\}$ ($(T_h)g(t) = (T^t_h g) = g(t+h)$) in a *local* topology of $L^p_{loc}(\mathbb{R}_+ \times \Omega)$:

$$\mathcal{H}^+(g) = \{T_h g, h \in \mathbb{R}_+\}_{L^p_{loc}(\mathbb{R}_+ \times \Omega)} \tag{6.2}$$

The main requirement to the right–hand side g of the initial equation (3.1) is: the hull $\mathcal{H}^+(g)$ is compact in the space $L^p_{loc}(\mathbb{R}_+ \times \Omega)$. The functions, which satisfy this assumption, is called translation-compact in L^p_{loc} (following to [4]). Some discussion of this concept will be given in the end of this Section.

Define a semigroup $\{\mathbb{S}_h, h \ge 0\}$, acting on the extended phase space $\Phi_b \times \mathcal{H}^+(g)$, by formula

$$\mathbb{S}_h(u_0, \xi) = (u_\xi(h), T_h \xi) \tag{6.3}$$

where $u_\xi(t)$ is the solution of the problem (3.1) with the right–hand $\xi \in \mathcal{H}^+(g)$ and $u(0) = u_0 \in \Phi_b(\Omega)$.

Theorem 6.1. *Let the assumptions of Section 3 hold and let the right–hand side g be translation-compact in $L^p_{loc}(\mathbb{R}_+\times\Omega)$. Then the semigroup $\mathbb{S}_h$ possesses the $(\Phi_b\times\mathcal{H}^+(g),\Phi_{loc}\times\mathcal{H}^+(g))$-attractor $\mathbb{A}$ (see Definition 5.1).*

Proof. As in the proof of Theorem 5.1 we will check the conditions of the attractor's existence for the abstract semigroups (see [2]).

1. The continuity of $\mathbb{S}_h$ is an immediate corollary of Lemma 4.1. Thus it remains to verify the existence of precompact absorbing set.

The estimate (3.19) together with the evident inequality

$$\|\xi\|_{L^p_b(\mathbb{R}_+\times\Omega)}\le\|g\|_{L^p_b(\mathbb{R}_+\times\Omega)}\quad\text{for }\xi\in\mathcal{H}^+(g)\tag{6.4}$$

imply that the set $\mathbb{B}_R=B_R\times\mathcal{H}^+(g)$, where B_R is the R-ball in Φ_b, is the absorbing set for the semigroup (6.3) for R large enough. Let us define a set

$$\mathbb{K}=\mathbb{S}_1\mathbb{B}_R$$

Then, arguing as in the proof of Theorem 5.1, we deduce that $\mathbb{K}$ is precompact in $\Phi_{loc}\times\mathcal{H}^+(g)$. Theorem 6.1 is proved.

Definition 6.1. *The projection $\mathcal{A}=\Pi_1\mathbb{A}$ to the first component of the attractor $\mathbb{A}$ is called the (uniform) attractor of the family* (6.1) *or the (nonautonomous) attractor of the equation* (3.1).

Corollary 6.1. *Let the assumptions of Theorem 6.1 hold. Then the equation* (3.1) *possesses the attractor $\mathcal{A}$.*

Remark 6.1. There exists the internal definition of the attractor $\mathcal{A}$ without using the corresponding semigroup in the extended phase space. Namely, the set $\mathcal{A}$ is called the uniform attractor of the family (6.1) if the following conditions hold:

1. $\mathcal{A}\subset\Phi_b$ is compact in Φ_{loc}.
2. For every bounded $B\subset\Phi_b$ and every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of Φ_{loc} there exists $T=T(\mathcal{O},B)$, such that

$$u(t)\in\mathcal{O}(\mathcal{A})$$

for every solution $u(t)$ of the equation (6.1) with $u_0\in B$, the right–hand side $h\in\mathcal{H}^+(g)$ and $t\ge T$.

3. The set $\mathcal{A}$ is minimal set which satisfy the condition 1 and 2.

It is proved in [4] that the attractor thus defined coincides with the attractor, defined above.

We study now the structure of the obtained attractor $\mathcal{A}$. To this end we need the next definitions

Definition 6.2. *Let $\omega(g)$ be the attractor (ω-limit set) of the semigroup $\{T_h,h\in\mathbb{R}_+\}$, acting in the compact metric space $\mathcal{H}^+(g)$, i.e. (see* [2], [11], [20]*)*

$$\omega(g)=\cap_{h\ge0}\left\{\cup_{s\ge h}T_s\mathcal{H}^+(g)\right\}_{L^p_{loc}(\mathbb{R}_+\times\Omega)}\tag{6.5}$$

Definition 6.3. *Let us denote by $Z(g)$ the set of functions $\widehat{\xi} \in L^p_b(\mathbb{R}\times\Omega)$ which satisfy the condition:*

$$\Pi_+(T_h\widehat{\xi}) \subset \omega(g) \text{ for every } h \in \mathbb{R} \tag{6.6}$$

where Π_+ is the restriction operator to the semiaxis $t \in \mathbb{R}_+$.

It is known (see [4], [7]) that the sets $\omega(g)$ and $Z(g)$ is not empty and compact in the spaces $L^p_{loc}(\mathbb{R}_+\times\omega)$ and $L^p_{loc}(\mathbb{R}\times\Omega)$ correspondingly. Moreover, for every $\xi \in \omega(g)$ there exists $\widehat{\xi} \in Z(g)$ such that $\Pi_+\widehat{\xi} = \xi$, i.e

$$\Pi_+ Z(g) = \omega(g) \tag{6.7}$$

Theorem 6.2. *Let the assumptions of Theorem 6.1 hold. Then the attractor $\mathcal{A}$ of the equation* (3.1) *has the following structure*

$$\mathcal{A} = \Pi_0 \cup_{\widehat{\xi}\in Z(g)} \mathcal{K}_{\widehat{\xi}} \tag{6.8}$$

where $\mathcal{K}_{\widehat{\xi}}$ is the union of all solutions $\widehat{u}$ of the equation (3.1) *with the right–hand side $\widehat{\xi} \in Z(g)$ which are defined for every $t \in \mathbb{R}$ and bounded with respect to $t \in \mathbb{R}$ (as usual $\Pi_0 u \equiv u(0)$).*

Theorem 6.2 is a corollary of general theorem which describes the structure of nonautonomous attractors (see [4], [5]).

Thus, we constructed the (uniform) attractor for every right–hand side $g(t)$ which is translation-compact in the space $L^p_{loc}(\mathbb{R}_+\times\Omega)$. The rest part of this Section is devoted to study the class of translation-compact functions. The following evident proposition reduce this problem to the case where Ω is bounded.

Proposition 6.1. *The right–hand side $g \in L^p_b(\mathbb{R}_+\times\Omega)$ is translation-compact in L^p_{loc} if and only if the restriction $g\big|_{\Omega_1}$ is translation-compact in $L^p_{loc}(\mathbb{R}_+, L^p(\Omega_1))$ for every bounded subdomain $\Omega_1 \subset \Omega$.*

Indeed, by definition $\mathcal{H}^+(g)$ is compact in $L^p_{loc}(\mathbb{R}_+\times\Omega)$ if and only if $\mathcal{H}^+(g\big|_{\Omega_1}) = \mathcal{H}^+(g)\big|_{\Omega_1}$ is compact in $L^p_{loc}(\mathbb{R}_+, L^p(\Omega_1))$ for every bounded $\Omega_1 \subset \Omega$.

Now we formulate some necessary and sufficient conditions for the translation compactness in the case where Ω_1 is bounded.

Note for the first that by definition a function $g \in L^p_b(\mathbb{R}_+\times\Omega_1)$ is translation-compact in $L^p_{loc}(\mathbb{R}_+, L^p(\Omega_1))$ if and only if the set

$$\{(T_h g)\big|_{t\in[0,1]}, h \in \mathbb{R}_+\} \subset\subset L^p([0,1]\times\Omega_1) \tag{6.9}$$

(is precompact in $L^p([0,1]\times\Omega_1)$). The following Proposition gives these conditions in the spirit of Arcela-Ascoli theorem.

Proposition 6.2. *Let $\Omega_1 \subset \Omega$ be a bounded domain. Then a function $g \in L^p_b(\mathbb{R}_+\times\Omega_1)$ is translation-compact in $L^p_{loc}(\mathbb{R}_+, L^p(\Omega_1))$ if and only if the following conditions hold:*

(a) for any fixed $t > 0$ the set $\{\int_s^{t+s} g(z)dz,\ s \in \mathbb{R}_+\}$ is precompact in the space $L^p(\Omega_1)$;

(b) there exists a function $\beta(s)$, $s \geq 0$, $\beta(s) \to 0$ as $s \to 0$, such that

$$\int_t^{t+1} \|g(z) - g(z+l)\|^p_{L^p(\Omega_1)} dz \leq \beta(|l|) \ \ \forall t \in \mathbb{R}_+,\ t+l \in \mathbb{R}_+$$

The proof of Proposition 6.2 is given in [7].

Corollary 6.2. *Let Ω_1 be a bounded domain in $\mathbb{R}^n$ and let*

$$g \in W_b^{\alpha,p}(\mathbb{R}_+, W^{\alpha,p}(\Omega_1)), \ \alpha > 0 \tag{6.10}$$

then g is translation-compact in $L^p_{loc}(\mathbb{R}_+, L^p(\Omega_1))$.

Corollary 6.3. *Let Ω be arbitrary, $\phi \in L^1(\mathbb{R}^n)$ be the weight function in the sense of Section 1 and let*

$$g \in W_b^{\alpha,p}(\mathbb{R}_+, W_\phi^{\alpha,p}(\Omega)), \ \alpha > 0 \tag{6.11}$$

then g is translation-compact in $L^p_{loc}(\mathbb{R}_+ \times \Omega)$.

Remark 6.2. Note that the conditions (6.10) or (6.11) is not necessary for the translation-compactness. Indeed, any periodic, quasiperiodic, or almost periodic function is evidently translation-compact. Moreover, if g is translation-compact in $L^p_{loc}(\mathbb{R}_+ \times \Omega)$ and $g_1 \in L^p_{loc}(\mathbb{R}_+ \times \Omega)$ satisfy the condition $\operatorname{supp} g \subset [0,T] \times \Omega$ then $g + g_1$ is also translation-compact. Thus, in contrast to the concept of almost periodicity, the translation-compactness is some kind of regularity condition when $t \to \infty$.

The following Proposition shows the relations between the translation- compactness and smoothness.

Proposition 6.3. *Let Ω_1 be bounded domain. Let $TC_p(\mathbb{R}_+ \times \Omega_1)$ be the closure of the set $C^1_b(\mathbb{R}_+ \times \Omega_1)$ in the space $L^p_b(\mathbb{R}_+ \times \Omega_1)$*

$$TC_p(\mathbb{R}_+ \times \Omega_1) = \left\{C^1_b(\mathbb{R}_+ \times \Omega_1)\right\}_{L^p_b(\mathbb{R}_+ \times \Omega_1)} \tag{6.12}$$

Then g is translation-compact in $L^p_{loc}(\mathbb{R}_+ \times \Omega_1)$ if and only if $g \in TC_p(\mathbb{R}_+ \times \Omega_1)$.

Proof. Let $g \in TC_p$. Then for every $\varepsilon > 0$ there exists $g_\varepsilon \in C^1_b$, such that

$$\|g - g_\varepsilon, \mathbb{R}_+ \times \Omega_1\|_{b,0,p} \le \varepsilon \tag{6.13}$$

According to Corollary 6.2, g_ε is translation-compact, consequently $K_\varepsilon = \mathcal{H}^+(g_\varepsilon)$ is compact in $L^p_{loc}(\mathbb{R}_+, L^p(\Omega_1))$.

The assumption (6.13) implies that for every $h \in \mathcal{H}^+(g)$ there exists $h_\varepsilon \in \mathcal{H}^+(g_\varepsilon)$ such that

$$\|h - h_\varepsilon, \mathbb{R}_+ \times \Omega_1\|_{b,0,p} \le \varepsilon \tag{6.13}$$

and consequently, for every $\varepsilon > 0$ there exists a compact K_ε such that

$$\mathcal{H}^+(g) \subset \mathcal{O}_\varepsilon(K_\varepsilon) \tag{6.14}$$

where $\mathcal{O}_\varepsilon$ is ε-neighborhood in L^p_b.

The assertion (6.14) (together with Hausdorff criteria) implies that $\mathcal{H}^+(g)$ is compact in $L^p_{loc}(\mathbb{R}_+, L^p(\Omega_1))$.

Assume now that g is translation-compact. Let us consider the usual averaging operator $\mathbb{T}_\varepsilon$ for $(t,x) \in \mathbb{R}^{n+1}$

$$(\mathbb{T}_\varepsilon \phi)(t,x) = \int_{\mathbb{R}^{n+1}} F_\varepsilon(|(t,x) - (t_1,x_1)|)\phi(t_1,x_1)dx_1dt_1$$

where $F_\varepsilon(z) = \frac{1}{\varepsilon^{n+1}}F(z/\varepsilon)$, $\int_{\mathbb{R}^{n+1}} F(|z|)dz = 1$ and $\operatorname{supp} F \subset [-1,1]$,

It is well known that $\mathbb{T}_\varepsilon\phi \in C^\infty(\mathbb{R}^{n+1})$,

$$\|\mathbb{T}_\varepsilon\phi - \phi\|_{L^p([0,1]\times\Omega_1)} \to 0\ , \quad \text{when } \varepsilon \to 0 \tag{6.15}$$

for every $\phi \in L^p([0,1]\times\Omega_1)$ and this convergence is uniform with respect to ϕ on compact sets in $L^p([0,1]\times\Omega_1)$.

Note that $\mathbb{T}_\varepsilon g \in C_b^1(\mathbb{R}_+\times\Omega_1)$ and (6.9) together with (6.15) imply that $\mathbb{T}_\varepsilon g \to g$ in $L_b^p(\mathbb{R}_+\times\Omega)$. Proposition 6.3 is proved.

Part 3. Kolmogorov's ε-entropy and attractors.

This part is devoted to study Kolmogorov's ε-entropy of the attractors constructed in previous part.

For the reader convenience we recall firstly (in Section 7) the definition of entropy and give some examples of asymptotics for the typical sets in the spaces of functions.

The upper bounds of entropy for the attractor $\mathcal{A}$ of the equation (3.1) are established in Section 8.

The examples of equations of the type (3.1) which show the sharpness of the estimates obtained above are considered in Section 9.

In Section 10 using the infinite dimensional unstable manifolds we extend the lower estimates of entropy derived in previous Section to a wide class of natural equations of mathematical physics such as Chafee-Infanter equation, complex Ginzburg-Landay equation, etc.

§7 Definitions and typical examples.

In this Section we recall briefly the definition of ε-entropy and give the upper and lower estimates of it when $\varepsilon \to 0$ for the typical sets in functional spaces. For the detailed study of this concept see [15], [21].

Definition 7.1. *Let $\mathbb{M}$ be a metric space and let K be precompact subset of it. For a given $\varepsilon > 0$ let $N_\varepsilon(K) = N_\varepsilon(K,\mathbb{M})$ be the minimal number of ε-balls in $\mathbb{M}$ which cover the set K (this number is evidently finite by Hausdorff criteria). By definition, Kolmogorov's ε-entropy of K in $\mathbb{M}$ is the following number*

$$\mathbb{H}_\varepsilon(K) = \mathbb{H}_\varepsilon(K,\mathbb{M}) \equiv \ln N_\varepsilon(K) \tag{7.1}$$

Example 7.1. Let K be compact n-dimensional Lipschitz manifold in $\mathbb{M}$. Then the evident estimates imply that

$$C_1\left(\frac{1}{\varepsilon}\right)^n \le N_\varepsilon(K) \le C_2\left(\frac{1}{\varepsilon}\right)^n \tag{7.2}$$

and consequently

$$\mathbb{H}_\varepsilon(K) = (n + \overline{\overline{o}}(1))\ln\frac{1}{\varepsilon} \tag{7.3}$$

when $\varepsilon \to 0$.

This example justifies the following definition

Definition 7.2. *The fractal (box-counting) dimension of the set $K \subset\subset \mathbb{M}$ is defined to be the following number:*

$$\dim_F(K) = \dim_F(K, \mathbb{M}) = \limsup_{\varepsilon \to 0} \frac{\mathbb{H}_\varepsilon(K)}{\ln \frac{1}{\varepsilon}} \tag{7.4}$$

Note that the fractal dimension $\dim_F(K) \in [0, \infty]$ is defined for any compact set in $\mathbb{M}$ but may be not integer if K is not a manifold.

Example 7.2. Let $\mathbb{M} = [0, 1]$ and let K be the ternary Cantor set in $\mathbb{M}$. Then it is not difficult to obtain that

$$C_1 \left(\frac{1}{\varepsilon}\right)^d \le N_\varepsilon(K) \le C_2 \left(\frac{1}{\varepsilon}\right)^d \quad , \quad d = \frac{\ln 2}{\ln 3} \tag{7.5}$$

and consequently $\dim_F(K) = d = \frac{\ln 2}{\ln 3}$.

Consider now the examples of infinite dimensional sets (i.e. $\dim_F(K) = \infty$).

The following two examples give the typical asymptotics for the entropy in the spaces of analitycal functions.

Example 7.3. Let K be the set of all analytic functions f in a ball $B(R)$ of radius $R > 1$ in $\mathbb{C}^n$ such that $\|f\|_{L^\infty(B(R))} \le 1$ and let $\mathbb{M}$ be the space $C(B^{Re})$, where $B^{Re} = \{z \in \mathbb{C}^n \ : \operatorname{Im} z_i = 0 \, , |z| \le 1\}$. Thus, K consists of all functions from $C(B^{Re})$ which can be extended holomorphically to the ball $B(R) \subset \mathbb{C}^n$ and the C-norm of this extension is not greater then one. Then

$$C_1 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \le \mathbb{H}_\varepsilon(K, \mathbb{M}) \le C_2 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \tag{7.6}$$

For the proof of this estimate see [15].

Example 7.4. Let $\mathbb{M}$ be the same as in previous example and let K be the set of all functions f in $\mathbb{M}$ which can be extended to the entire function $\widehat{f}$ in $\mathbb{C}^n$ which satisfy the estimate

$$|\widehat{f}(z)| \le K_1 e^{K_2 |z|}, \quad z \in \mathbb{C}^n \tag{7.7}$$

Then, as proved in [15],

$$C_1 \frac{\left(\ln \frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln \ln \frac{1}{\varepsilon}\right)^n} \le \mathbb{H}_\varepsilon(K) \le C_2 \frac{\left(\ln \frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln \ln \frac{1}{\varepsilon}\right)^n} \tag{7.8}$$

The next example gives the typical asymptotics for the entropy in the class of Sobolev spaces in bounded domains.

Example 7.5. Let Ω be smooth bounded domain in $\mathbb{R}^n$ and

$$W^{l_1, p_1}(\Omega) \subset\subset W^{l_2, p_2}(\Omega) \ , \ 0 \le l_i < \infty, \ 1 < p_i < \infty, \ l_1 > l_2$$

i.e., according to the embedding theorem $\frac{l_1}{n} - \frac{1}{p_1} > \frac{l_2}{n} - \frac{1}{p_2}$.

Let now $\mathbb{M} = W^{l_2, p_2}(\Omega)$ and K be the unitary ball in $W^{l_1, p_1}(\Omega)$. Then

$$C_1 \left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1 - l_2}} \le \mathbb{H}_\varepsilon(K) \le C_2 \left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1 - l_2}} \tag{7.9}$$

The proof of this estimate can be found in [21].

The following proposition, which will be essentially used in the next Section gives the dependence of the constants C_i in (7.9) on the 'size' of Ω in the particular case $p_1 = p_2 = 2$.

Proposition 7.1. *Let Ω be a bounded domain, which satisfies the conditions* (1.5) *and* (1.6), $\mathbb{M} = L^2_b(\Omega)$ *and let* $\mathbb{K}$ *be the unitary ball in* $W^{1,2}_b(\Omega)$. *Then*

$$C_1 \operatorname{vol}(\Omega) \left(\frac{1}{\varepsilon}\right)^n \le \mathbb{H}_\varepsilon(\mathbb{K}) \le C_2 \operatorname{vol}(\Omega) \left(\frac{1}{\varepsilon}\right)^n \quad \textit{for } \varepsilon < \varepsilon_0 \tag{7.10}$$

where $\operatorname{vol}(\Omega)$ *is n-dimensional volume of* Ω. *Moreover constants C_i and ε_0 in* (7.10) *depends only on K and R_0 from the assumptions* (1.5) *and* (1.6).

Sketch of the proof. We prove only the right-part of the inequality (7.10). The left one can be proved analogously.

It is not difficult to prove, using the conditions (1.5) and (1.6) that there exists the covering of Ω by cubes $\mathcal{C}_j = x_j + [-r,r]^n$, $j = 1, \cdots, N$, such that $N \le C_r \operatorname{vol}(\Omega)$. Moreover, the constants r and C_r depends only on the conditions (1.5) and (1.6) and are independent of the size of Ω. Using now the evident estimate

$$\begin{aligned}\mathbb{H}_{C'_r\varepsilon}\left(B(0,1,W^{1,2}_b(\Omega)), L^2_b(\Omega)\right) \le \sum_{j=1}^N \mathbb{H}_\varepsilon\left(B(0,1,W^{1,2}_b(\mathcal{C}_j)), L^2_b(\mathcal{C}_j)\right) = \\ = N\mathbb{H}_\varepsilon\left(B(0,1,W^{1,2}_b([-r,r]^n)), L^2_b([-r,r]^n)\right)\end{aligned}$$

(where $B(0,1,W^{1,2}(V))$ means the unitary ball in the space $W^{1,2}_b(V)$, and C'_r depends only on n and r) and the estimate (7.9) we obtain the right part of the inequality (7.10).

§8 The entropy of the attractor: the upper bounds.

In this Section we obtain the upper estimates of ε-entropy for the attractor $\mathcal{A}$ of the equation (3.1). Recall that we construct the attractor $\mathcal{A}$ which is compact only in F-space Φ_{loc} but not in the uniform topology of $\Phi_b(\Omega)$. That's why we will estimate the entropy of the restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$.

The main result of this Section is the following theorem.

Theorem 8.1. *Let the assumptions of Theorem 6.1 be valid and let*

$$\operatorname{vol}_{\Omega,x_0}(R) = \operatorname{vol}(\Omega \cap B^R_{x_0}) \tag{8.1}$$

Then for every $R \in \mathbb{R}_+$, $x_0 \in \Omega$

$$\begin{aligned}\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, W^{2(1-1/p),p}_b(\Omega\cap B^R_{x_0})\right) \le C \operatorname{vol}_{\Omega,x_0}(R + K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} + \\ + 2\mathbb{H}_{\varepsilon/L}\left(\omega(g)\big|_{[0,K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0}}, L^p_b([0,K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0})\right)\end{aligned} \tag{8.2}$$

where the constants C, K and L are independent of R and $x_0 \in \Omega$.

Proof. Define a family of weight functions with the rate of growth 1 by the following formula

$$\phi_{R,x_0}(x) = \begin{cases} e^{R-|x-x_0|} & \text{if } |x-x_0| \ge R \\ 1 & \text{if } |x-x_0| \le R \end{cases} \tag{8.3}$$

Note that we have defined these weight functions in such a way that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, W_b^{2(1-1/p),p}(\Omega\cap B^R_{x_0})\right)\le \mathbb{H}_\varepsilon\left(\mathcal{A}, W^{2(1-1/p),p}_{b,\phi_{R,x_0}}(\Omega)\right) \tag{8.4}$$

Hence, instead of estimating the entropy of the restriction $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ we will estimate below the entropy of the attractor in weighted Sobolev spaces $W^{2(1-1/p),p}_{b,\phi_{R,x_0}}(\Omega)$.

Let $u_1(t)$ and $u_2(t)$ be two solutions of the family (6.1) with the right–hand sides g_1 and g_2 which lie on the attractor $\mathcal{A}$. Then, according to the estimates (4.25)

$$\|u_1(1)-u_2(1)\|_{W^{2(1-1/p),p}_{b,\phi^{p/2}_{R,x_0}}(\Omega)}\le C\|u_1(0)-u_2(0)\|_{L^2_{b,\phi_{R,x_0}}(\Omega)}+ \\ + C\|g_1-g_2\|_{L^p_{b,\phi^{p/2}_{R,x_0}}([0,1]\times\Omega)} \tag{8.5}$$

Here the constant C in (8.5) is independent of $u_1,u_2\in\mathcal{A}$. Moreover, since

$$\phi_{R,x_0}(x+y)\le e^{|x|}\phi_{R,x_0}(y)$$

then this constant is independent of R and x_0 also.

The estimate (8.5) together with the description (6.8) of the attractor $\mathcal{A}$ implies immediately that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, W^{2(1-1/p),p}_{b,\phi^{p/2}_{R,x_0}}(\Omega)\right)\le \mathbb{H}_{\varepsilon/(2C)}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}(\Omega)\right)+ \\ + \mathbb{H}_{\varepsilon/(2C)}\left(\omega(g), L^p_{b,\phi^{p/2}_{R,x_0}}(\Omega)\right) \tag{8.6}$$

Since $\|w(g)\|_{L^p_b(\mathbb{R}_+\times\Omega)}\le C_1<\infty$ and $\phi^{1/2}_{R,x_0}(x)\le\varepsilon/K_1$ if $|x-x_0|>R+K\ln\frac{1}{\varepsilon}$ then the estimates (8.4) and (8.6) imply

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, W_b^{2(1-1/p),p}(\Omega\cap B^R_{x_0})\right)\le \mathbb{H}_{\varepsilon/(2C)}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}(\Omega)\right)+ \\ + \mathbb{H}_{\varepsilon/L}\left(\omega(g), L^p_b([0,1]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0})\right) \tag{8.7}$$

Moreover, constants C, K, K_1, L are independent of ε, x_0 and R.

The estimate (8.7) reduces our problem to estimating the entropy of the attractor in the space $L^2_{b,\phi_{R,x_0}}(\Omega)$.

The following corollary of the estimate (4.24) is of fundamental significance for this estimation:

Let u_1 and u_2 be arbitrary two solutions of the family (6.1) with the right–hand sides g_1 and g_2 correspondingly which belong to the attractor. Then the following estimate is valid

$$\|u_1(1)-u_2(1)\|_{W^{1,2}_{b,\phi_{R,x_0}}(\Omega)}\le C\|u_1(0)-u_2(0)\|_{L^2_{b,\phi_{R,x_0}}(\Omega)}+ \\ + C\|g_1-g_2\|_{L^2_{b,\phi_{R,x_0}}([0,1]\times\Omega)} \tag{8.8}$$

Where the constant C depends only on the equation.

Lemma 8.1. *The following recurrent inequality is valid*

$$\mathbb{H}_{\varepsilon/2^k}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) \le \mathbb{H}_{\varepsilon}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) + \\ + k\ln M_k(\varepsilon) + \mathbb{H}_{\varepsilon/2^{k-1}}\left(\omega(g), L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)\right) \tag{8.9}$$

where

$$\ln M_k(\varepsilon) \le C\,\mathrm{vol}_{\Omega,x_0}(R + L\ln\frac{2^k}{\varepsilon}) \tag{8.10}$$

Moreover, the constants C and L is independent of k, R, ε and x_0.

Proof. Let $\{B(u_0^i,\varepsilon,L^2)\}$, $i=1,\cdots,N_0(\varepsilon)$ be the initial ε-covering of $\mathcal{A}$ (as before $B(u_0^i,\varepsilon,L^2)$ means ε-ball in $L^2_{b,\phi_{R,x_0}}(\Omega)$, centered in u_0). We will call this system of balls by the ε-system of 0th order. Let us fix also the $\varepsilon/2^{k-1}$-covering of the set $w(g)\big|_{[0,k]\times\Omega}$. Let h_j, $j=1,\cdots,N_g(\varepsilon)$ be the centers of this covering. Having the ε-system of 0th order we construct now the $\varepsilon/2$-system of 1st order. To this end we construct firstly the system $B(v_1^{i,j}, C(\varepsilon+\varepsilon/2^{k-1}), W^{1,2}_{b,\phi_{R,x_0}})$, $i=1,\cdots,N_0(\varepsilon)$, $j=1,\cdots,N_g(\varepsilon)$ of $C(\varepsilon+\varepsilon/2^{k-1})$-balls in the space $W^{1,2}_{b,\phi_{R,x_0}}(\Omega)$ with the constant C the same as in the estimate (8.8) by taking

$$v_1^{i,j} = U_{h_j}(1,0)u_0^i \tag{8.11}$$

Here and below $U_h(t,\tau)u_0$ means the solution $u(t)$ of the equation (3.1) with the right–hand side h and $u(\tau)=u_0$.

Recall now that the set $B(v_1^{i,j}, C(\varepsilon+\varepsilon/2^{k-1}), W^{1,2})\cap\mathcal{A}$ is compact in $L^2_{b,\phi}$ and consequently it can be covered by the finite number of $\varepsilon/2$-balls. Let

$$M_1(\varepsilon) = \max\nolimits_{i,j} N_{\varepsilon/2}\left(\mathcal{A}\cap B(v_1^{i,j}, C(\varepsilon+\varepsilon/2^{k-1}), W^{1,2}), L^2_{b,\phi_{R,\varepsilon}}\right) \tag{8.12}$$

Let us cover now every ball in the $W^{1,2}$-covering by $\le M_1(\varepsilon)$ $\varepsilon/2$-balls in $L^2_{b,\phi_{R,x_0}}$. Thus, we constructed the $\varepsilon/2$-system $B(u_1^{i,j},\varepsilon/2,L^2)$, $i=1,\cdots,N_1'(\varepsilon/2)$, $j=1,\cdots,N_g(\varepsilon)$ of the 1st order with $N_1'(\varepsilon/2)\le M_1(\varepsilon)N_0(\varepsilon)$ and consequently the number of balls in this system not exceed

$$N_1(\varepsilon/2) \le M_1(\varepsilon)N_0(\varepsilon)N_g(\varepsilon) \tag{8.13}$$

Note that by definition $B(u_1^{i,j},\varepsilon/2,L^2)$ belong to the covering of $B(v_1^{i_1,j}, C(\varepsilon+\varepsilon/2^{k-1}), W^{1,2})$ (with the same j!).

Having the $\varepsilon/2$-system of 1st order, we construct the $\varepsilon/4$-system of 2nd order. To this end we consider the system $B(v_2^{i,j}, C(\varepsilon/2+\varepsilon/2^{k-1}), W^{1,2})$ of $W^{1,2}$-balls of radius $C(\varepsilon/2+\varepsilon/2^{k-1})$ centered in $v_2^{i,j}=U_{h_j}(2,1)u_1^{i,j}$. Note that in contrast to the first step we will not change the function h_j any more, i.e. if $B(u_1^{i,j},\varepsilon/2,L^2)$ belongs to the covering of $B(v^{i_1,j}, C(\varepsilon+\varepsilon/2^{k-1}), W^{1,2})$ with $v_1^{i_1,j}=U_{h_j}(1,0)u_0^{i_1}$ then we apply the operator $U_h(2,1)$ to $u_1^{i,j}$ *only*(!) with $h=h_j$. Covering now every $C(\varepsilon/2+\varepsilon/2^{k-1})$-ball in $W^{1,2}$ buy the finite number of $\varepsilon/4$-balls, we obtain the $\varepsilon/4$-system $B(u_2^{i,j},\varepsilon/4,L^2)$ of 2nd order. Analogously to (8.12) we define

$$M_2(\varepsilon)=\max_{i,j} N_{\varepsilon/4}\left(\mathcal{A}\cap B(v_2^{i,j},C(\varepsilon/2+\varepsilon/2^{k-1}),W^{1,2}),L^2_{b,\phi_{R,\varepsilon}}\right)$$

Then the number of $\varepsilon/4$-balls in the covering of 2nd order not exceed

$$N_2(\varepsilon/4)\le M_2(\varepsilon)M_1(\varepsilon)N_0(\varepsilon)N_g(\varepsilon)$$

Iterating the above procedure we obtain finally the $\varepsilon/2^k$-system $B(u_k^{i,j},\varepsilon/2^k,L^2)$ of kth order and the number of balls in this system not exceed

$$N_k(\varepsilon/2^k)\le M_1(\varepsilon)\cdots M_k(\varepsilon)N_0(\varepsilon)N_g(\varepsilon) \tag{8.14}$$

where

$$M_l(\varepsilon)=\max_{i,j} N_{\varepsilon/2^l}\left(\mathcal{A}\cap B(v_l^{i,j},C(\varepsilon/2^{l-1}+\varepsilon/2^{k-1}),W^{1,2}),L^2_{b,\phi_{R,\varepsilon}}\right) \tag{8.15}$$

We claim that $\varepsilon/2^k$-system of kth order covers $\mathcal{A}$. Indeed, let $u\in\mathcal{A}$. Then due to (6.8) there exists $u_0\in\mathcal{A}$ and $h\in\omega(g)$ such that $u=U_h(k,0)u_0$. Let us find the indexes i and j such that

$$\|u_0-u_0^i\|_{L^2_{b,\phi_{R,x_0}}}\le\varepsilon\quad,\quad \|h-h_j\|_{L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)}\le\varepsilon/2^{k-1} \tag{8.16}$$

It is possible to do due to our assumptions. Let $u_l=U_h(l,0)u_0$, $l=1,\cdots,k$. Then, according to the estimates (8.8) and (8.16), $u_1\in B(v_1^{i,j},C(\varepsilon+\varepsilon/2^{k-1}),W^{1,2})$ and consequently, there exists i_1, such that $u_1\in B(u_1^{i_1,j},\varepsilon/2,L^2)$. Applying the estimate (8.8) again we obtain that $u_2\in B(v^{i_1,j},C(\varepsilon/2+\varepsilon/2^{k-1}),W^{1,2})$ and consequently there exists i_2 such that $u_2\in B(u_2^{i_2,j},\varepsilon/4,L^2)$. Arguing analogously, we obtain finally that $u=u_k\in B(u_k^{i_k,j},\varepsilon/2^k,L^2)$. Since $u\in\mathcal{A}$ is arbitrary then the $\varepsilon/2^k$-system of kth order covers $\mathcal{A}$.

Thus, the estimate (8.14) implies now that

$$\begin{aligned}\mathbb{H}_{\varepsilon/2^k}\left(\mathcal{A},L^2_{b,\phi_{R,x_0}}\right)\le\sum_{i=1}^k\ln M_i(\varepsilon)+\\ +\mathbb{H}_\varepsilon\left(\mathcal{A},L^2_{b,\phi_{R,x_0}}\right)+\mathbb{H}_{\varepsilon/2^{k-1}}\left(w(g),L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)\right)\end{aligned} \tag{8.17}$$

To complete the proof of the lemma it remains to estimate the numbers $M_i(\varepsilon)$.

It follows from Corollary 3.2 that

$$\|\mathcal{A},\Omega\|_{b,1,2}\le K_1 \tag{8.18}$$

Consequently, according to the estimate $\phi_{R,x_0}(x)^{1/2}\le\frac{\varepsilon}{C_1 2^{l+1}K_1}$ if $|x-x_0|>R+L\ln\frac{2^{l+1}}{\varepsilon}\le R+L_1\ln\frac{2^k}{\varepsilon}\equiv R_k(\varepsilon)$, we obtain that

$$\begin{aligned}M_l(\varepsilon)\le\max_{i,j} N_{\varepsilon/2^l}\left(\mathcal{A}\cap B(v_l^{i,j},C\varepsilon/2^{l-2},W^{1,2}),L^2_{b,\phi_{R,x_0}}\right)\le\\ \max_{i,j} N_{\varepsilon/2^{l+1}}\left(\mathcal{A}\cap B(v_l^{i,j},C\varepsilon/2^{l-2},W^{1,2}_{b,\phi_{R,x_0}}(\Omega\cap B_{x_0}^{R_k(\varepsilon)})),L^2_{b,\phi_{R,x_0}}(\Omega\cap B_{x_0}^{R_k(\varepsilon)})\right)\\ \le N_{\varepsilon/2^{l+1}}\left(B(0,C\varepsilon/2^{l-2},W^{1,2}_{b,\phi_{R,x_0}}(\Omega\cap B_{x_0}^{R_k(\varepsilon)})),L^2_{b,\phi_{R,x_0}}(\Omega\cap B_{x_0}^{R_k(\varepsilon)})\right)\le\\ \le N_{1/(8C)}\left(B(0,1,W^{1,2}_{b,\phi_{R,x_0}}(\Omega\cap B_{x_0}^{R_k(\varepsilon)})),L^2_{b,\phi_{R,x_0}}(\Omega\cap B_{x_0}^{R_k(\varepsilon)})\right)\end{aligned} \tag{8.19}$$

Thus, it remains to estimate the entropy of the unitary $W^{1,2}_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})$-ball in the space $L^2_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})$. To this end we need the following proposition.

Proposition 8.1. *Let $F: u\to \phi^{1/2}_{R,x_0}u$. Then F realizes the linear isomorphizm between L^2_b and $L^2_{b,\phi_{R,x_0}}$ and also between $W^{1,2}_b$ and $W^{1,2}_{b,\phi_{R,x_0}}$. Moreover*
(8.20)

$$C_1\|u,\Omega\cap B^{R_k(\varepsilon)}_{x_0}\|_{b,\phi_{R,x_0},i,2}\le \|Fu,\Omega\cap B^{R_k(\varepsilon)}_{x_0}\|_{b,i,2}\le C_2\|u,\Omega\cap B^{R_k(\varepsilon)}_{x_0}\|_{b,\phi_{R,x_0},i,2}$$

where $i=0,1$ and constants C_1 and C_2 are independent of R_k, R, and x_0.

Proof. Let us consider only the case $i=1$ (the case $i=0$ can be considered analogously). Let $\psi=\phi^{1/2}_{R,x_0}$. Then, using the evident estimate $|\nabla_x\psi(x)|\le\psi(x)$, we obtain that

$$\begin{aligned}(8.21)\quad &|\psi(x)u(x)|^2+|\nabla_x(\psi(x)u(x)|^2=\\ &=\psi^2(x)\left((1+|\frac{\nabla_x\psi(x)}{\psi(x)}|^2)|u(x)|^2+|\nabla_xu(x)|^2+2u(x)\frac{\nabla_x\psi(x)}{\psi(x)}\nabla_xu(x)\right)\le\\ &\qquad\qquad\le 3\psi^2(x)\left(|u(x)|^2+|\nabla_xu(x)|^2\right)\end{aligned}$$

Analogously,

$$(8.22)\qquad |\psi(x)u(x)|^2+|\nabla_x(\psi(x)u(x)|^2\ge 1/2\psi^2(x)\left(|u(x)|^2+|\nabla_xu(x)|^2\right)$$

According to estimates (1.9) we obtain now that

$$\begin{aligned}(8.23)\quad &C_1\phi_{R,x_0}(z)\int_{\Omega\cap B^1_z}|u(x)|^2+|\nabla_xu(x)|^2dx\le\\ &\le\int_{\Omega\cap B^1_z}|Fu(x)|^2+|\nabla_xFu(x)|^2dx\le\\ &\le C_2\phi_{R,x_0}(z)\int_{\Omega\cap B^1_z}|u(x)|^2+|\nabla_xu(x)|^2dx\end{aligned}$$

The estimate (8.20) is an immediate corollary of (8.23). Proposition 8.1 is proved.

According to Proposition 8.1 instead of estimating the entropy in weighted Sobolev spaces $L^2_{b,\phi_{R,x_0}}$ it is sufficient to estimate it in the spaces L^2_b, i.e.

$$\begin{aligned}(8.24)\quad M_l(\varepsilon)&\le N_{1/(8C)}\left(B(0,1,W^{1,2}_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})),L^2_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})\right)\le\\ &\le N_{1/(C_1)}\left(B(0,1,W^{1,2}_b(\Omega\cap B^{R_k(\varepsilon)}_{x_0})),L^2_b(\Omega\cap B^{R_k(\varepsilon)}_{x_0})\right)\end{aligned}$$

Applying the estimate (7.10) to the right–hand side of the estimate (8.24) we obtain finally

$$(8.25)\qquad \ln M_l(\varepsilon)\le C_3\,\mathrm{vol}_{\Omega,x_0}\left(R+L\ln\frac{2^k}{\varepsilon}\right)$$

Lemma 8.1 is proved.

Now we are in position to complete the proof of Theorem 8.1. Let us fix ε_0 large enough that $\mathcal{A} \subset B(0, \varepsilon_0, L^2_b(\Omega))$. Then

$$\mathbb{H}_{\varepsilon_0}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}(\Omega)\right) = 0$$

for any R and x_0. Let us apply now the recurrent estimate (8.9) with $\varepsilon = \varepsilon_0$. Then we will have

$$\begin{aligned}(8.26)\quad \mathbb{H}_{\varepsilon_0/2^k}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) \le Ck\,\mathrm{vol}_{\Omega,x_0}\left(R + L\ln\frac{2^k}{\varepsilon_0}\right) + \\ + \mathbb{H}_{\varepsilon_0/2^{k-1}}\left(\omega(g), L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)\right)\end{aligned}$$

Let us fix an arbitrary $\beta < \varepsilon_0$ and take $k = k(\beta)$ such that

$$(8.27)\qquad \frac{\varepsilon_0}{2^{k-1}} \ge \beta \ge \frac{\varepsilon_0}{2^k} \text{ and consequently } 2^k \le \frac{2\varepsilon_0}{\beta}$$

Then (8.26) and (8.27) imply that

$$\begin{aligned}(8.28)\quad \mathbb{H}_\beta\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) \le \mathbb{H}_{\varepsilon_0/2^k}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) \le \\ \le Ck\,\mathrm{vol}_{\Omega,x_0}\left(R + L\ln\frac{2^k}{\varepsilon_0}\right) + \mathbb{H}_{\varepsilon_0/2^{k-1}}\left(\omega(g), L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)\right) \le \\ \le C_1\,\mathrm{vol}_{\Omega,x_0}\left(R + L\ln\frac{2}{\beta}\right)\ln\frac{1}{\beta} + \mathbb{H}_\beta\left(\omega(g), L^2_{b,\phi_{R,x_0}}([0, L_1\ln\frac{1}{\beta}]\times\Omega)\right)\end{aligned}$$

It follows immediately from the estimate (8.28) that

$$\begin{aligned}(8.29)\quad \mathbb{H}_\varepsilon\left(\mathcal{A}, L^2_b(\Omega\cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}\left(R + K\ln\frac{1}{\varepsilon}\right)\ln\frac{1}{\varepsilon} + \\ + \mathbb{H}_\varepsilon\left(\omega(g), L^2_b([0, K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0})\right)\end{aligned}$$

The estimates (8.7) and (8.29) complete the proof of Theorem 8.1.

Remark 8.1. A little more precise using of the recurrent scheme from Lemma 8.1 admits to improve the estimate (8.2) in the following way

$$\begin{aligned}(8.30)\quad \mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, W^{2(1-1/p),p}_b(\Omega\cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}(R + K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} + \\ + \mathbb{H}_{\varepsilon/L}\left(\omega(g), L^p_b([0, K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0})\right)\end{aligned}$$

without the multiplier 2 before the entropy of $\omega(g)$.

We consider now a number of corollaries of the main Theorem 8.1.

Corollary 8.1. Since $C \subset W_b^{2(1-1/p),p}$ then

$$
\text{(8.31)} \quad \mathbb{H}_\varepsilon\left(\mathcal{A}, C(\Omega \cap B_{x_0}^R)\right) \le C\,\mathrm{vol}_{\Omega,x_0}(R + K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} + \\
+ \mathbb{H}_{\varepsilon/L}\left(\omega(g), L_b^p([0, K\ln\frac{1}{\varepsilon}] \times \Omega \cap B_{x_0}^{R+K\ln\frac{1}{\varepsilon}})\right)
$$

Remark 8.2. Since the embedding $W_b^{2(1-1/p),p} \subset C$ is compact then a little more accurate using the recurrent scheme, introduced in Lemma 8.1, allows to obtain (8.31) with an arbitrary $L > 0$ particularly with $L = 1$.

Corollary 8.2. Let the equation (3.1) be autonomous ($g = g(x)$). Then

$$
\text{(8.32)} \qquad \mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2(1-1/p),p}(\Omega \cap B_{x_0}^R)\right) \le C\,\mathrm{vol}_{\Omega,x_0}\left(R + K\ln\frac{1}{\varepsilon}\right)\ln\frac{1}{\varepsilon}
$$

Particularly, if $\Omega = R^n$ then $\mathrm{vol}_{\Omega,x_0}(r) = cr^n$ and consequently

$$
\text{(8.33)} \qquad \mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2(1-1/p),p}(B_{x_0}^R)\right) \le C\left(R + K\ln\frac{1}{\varepsilon}\right)^n \ln\frac{1}{\varepsilon}
$$

Taking $R = \ln\frac{1}{\varepsilon}$ we obtain that

$$
\text{(8.34)} \qquad \mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2(1-1/p),p}(B_{x_0}^{\ln\frac{1}{\varepsilon}})\right) \le C_1\left(\ln\frac{1}{\varepsilon}\right)^{n+1}
$$

Note that the estimate (8.33) gives the same type of upper bounds for $R = 1$ and $R = \ln\frac{1}{\varepsilon}$. We return to this surprising fact in the next Section.

Corollary 8.3. Let Ω be a bounded domain. Then Theorem 8.1 implies the estimate

$$
\text{(8.35)} \quad \mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2(1-1/p),p}(\Omega)\right) \le C\,\mathrm{vol}(\Omega)\ln\frac{1}{\varepsilon} + \\
+ \mathbb{H}_{\varepsilon/L}\left(\omega(g), L_b^p([0, K\ln\frac{1}{\varepsilon}] \times \Omega)\right)
$$

which improves slightly the estimate, obtained in [6]. Particularly if the equation (3.1) is autonomous ($g = g(x)$), then the estimate (8.35) reflects the well-known fact that in this case the attractor $\mathcal{A}$ has the finite fractal dimension.

Corollary 8.4. Let $\Omega = \mathbb{R}^k \times \omega^{n-k}$ be a cylindrical domain where ω is bounded. Then the estimate (8.32) gives the following bound of the ε-entropy of the autonomous attractor

$$
\text{(8.36)} \qquad \mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2(1-1/p),p}(\Omega \cap B_{x_0}^R)\right) \le C\left(R + K\ln\frac{1}{\varepsilon}\right)^k \ln\frac{1}{\varepsilon}
$$

Definition 8.1. *Let $\mathcal{A} \subset \Phi_b(\Omega)$ be a compact set in the space $\Phi_{loc}(\Omega)$. Then the ε-entropy per unit volume is defined to be the following number*

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) = \limsup_{R\to\infty} \frac{\mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2(1-1/p),p}(\Omega \cap B_0^R)\right)}{\mathrm{vol}_{\Omega,0}(R)} \tag{8.37}$$

Corollary 8.5. Let the equation (3.1) be autonomous. Then

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C \ln\frac{1}{\varepsilon} \tag{8.38}$$

Indeed, the estimate (8.38) is an immediate corollary of the estimate (8.32) and trivial assertion

$$\lim{}_{R\to\infty} \frac{\mathrm{vol}_{\Omega,x_0}(R + C_1)}{\mathrm{vol}_{\Omega,x_0}(R)} = 1 \tag{8.39}$$

Remark 8.3. For the case $\Omega = \mathbb{R}^n$ and for the complex Ginzburg-Landay equation the estimate (8.38) has been obtained in [8].

To formulate the result for the entropy per unit volume for the nonautonomous case we need the following definition

Definition 8.2. *Let the entropy per unit volume of the right–hand side be the following number*

$$\overline{\mathbb{H}}_\varepsilon(g) = \limsup_{R\to\infty} \frac{\mathbb{H}_\varepsilon\left(\omega(g), L_b^p([0, K\ln\frac{1}{\varepsilon}] \times \Omega \cap B_0^{R+K\ln\frac{1}{\varepsilon}})\right)}{\mathrm{vol}_{\Omega,0}(R)} \tag{8.40}$$

Corollary 8.6. Let $\overline{\mathbb{H}}_\varepsilon(g) < \infty$. Then

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C \ln\frac{1}{\varepsilon} + \overline{\mathbb{H}}_{\varepsilon/L}(g) \tag{8.41}$$

Corollary 8.7. Note that if $g(t,x) = \phi(t)g_0(x)$ where $g_0 \in L_b^p(\Omega)$ and ϕ is translation-compact in $L^\infty_{loc}(\mathbb{R}_+)$ then $\overline{\mathbb{H}}_\varepsilon(g) \equiv 0$ and consequently

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C \ln\frac{1}{\varepsilon} \tag{8.42}$$

Remark 8.4. Let the right–hand side g be (t,x)-almost-periodic in $C(\mathbb{R} \times \mathbb{R}^n)$. Then, it not difficult to verify that $\overline{\mathbb{H}}_\varepsilon(g) = 0$ and consequently the assertion of Corollary 8.7 remains valid for such right–hand sides.

Definition 8.3. *Let $h_{sp}(\mathcal{A})$ be the following number*

$$h_{sp}(\mathcal{A}) = \limsup_{\varepsilon\to 0} \frac{\overline{\mathbb{H}}_\varepsilon(\mathcal{A})}{\ln\frac{1}{\varepsilon}} \tag{8.43}$$

Corollary 8.8. Let the assumptions of Corollary 8.5 or 8.7 or Remark 8.4 hold. Then

$$h_{sp}(\mathcal{A}) < \infty \tag{8.44}$$

Remark 8.5. The number $h_{sp}(\mathcal{A})$ can be interpreted as some quantitative characteristic of the phenomena of *space* chaotisity of the dynamical system, generated by the equation (3.1). In order to understand this relationship it is worth to compare the definition of h_{sp} with the definition of the topological (time) entropy h_{top} of the dynamical system (see [14]). For the reader convenience we recall shortly this definition. Let M be compact metric space and let $S_t : M \to M$ be a dynamical system (semigroup) on it. For a given $T > 0$ we consider the set $M(0,T) \subset L^\infty([0,T], M)$ of all trajectories $u(t) = S_t u_0$, $t \in [0,T]$ with $u_0 \in M$. Then by definition

$$h_{top} = \limsup_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\mathbb{H}_\varepsilon\left(M(0,T), L^\infty([0,T],M)\right)}{T} \tag{8.45}$$

§9 The entropy of the attractor: the examples of lower bounds.

In this Section we show that the estimates, obtained in previous Section are sharp. For simplicity we restrict ourselves by considering only the case $\Omega = \mathbb{R}^n$, $g \equiv 0$. We construct below the nonlinearity f in such a way that the entropy of the attractor $\mathcal{A}$ for this nonlinearity possesses the lower estimates with the same type of asymptotics as in the estimate (8.33). To this end we need firstly to define the special class of functions

Definition 9.1. *Let us denote by $\mathbb{B}_\sigma(\mathbb{R}^n)$ the subspace of $L^\infty(\mathbb{R}^n)$ which consists of all functions ϕ with the Fourier transform $\widehat{\phi}$ satisfying the condition*

$$\operatorname{supp}\widehat{\phi} \subset [-\sigma,\sigma]^n \tag{9.1}$$

It is well-known that every function $\phi \in \mathbb{B}_\sigma$ can be extended to entire function $\tilde{\phi}(z) \in A(\mathbb{C}^n)$ which satisfy the estimate

$$\sup_{x\in\mathbb{R}^n}|\tilde{\phi}(x+iy)| \le C\|\phi,\mathbb{R}^n\|_{0,\infty}e^{\sigma\sum_{i=1}^n|y_i|} \tag{9.2}$$

Moreover, every function $\phi \in L^\infty$, which possesses the entire extension $\tilde{\phi}$ which satisfies (9.2) belongs in fact to the space $\mathbb{B}_\sigma$.

The applications of this class to our problem are based on the following simple proposition.

Proposition 9.1. *For every $u_0 \in \mathbb{B}_\sigma(\mathbb{R}^n)$ the backward parabolic problem*

$$\partial_t u = \Delta_x u + (2n+1)\sigma^2 u,\ \ u\big|_{t=0} = u_0,\ \ t \le 0 \tag{9.3}$$

has a unique solution $u \in C_b(\mathbb{R}_- \times \mathbb{R}^n)$. Moreover the following estimate is valid

$$\sup_{t\in\mathbb{R}_-} e^{-t\sigma^2}\|u(t),\mathbb{R}^n\|_{0,\infty} \le C\|u_0,\mathbb{R}^n\|_{0,\infty} \tag{9.4}$$

The assertion of this Proposition can be verified directly by applying Fourier transform to the problem (9.3) and using (9.1).

Proposition 9.1 allows to construct the nonlinearity f in such a way that the attractor $\mathcal{A}$ of the problem (3.1) thus obtained contains the unitary ball of the space $\mathbb{B}_\sigma(\mathbb{R}^n)$.

Proposition 9.2. *For a given $\sigma > 0$ let*

$$f(u) = -(2n+1)\sigma^2 u\theta(u) + u^3(1-\theta(u)) \tag{9.5}$$

where $\theta(u)$ is a cut-off function, which equals 1 if $|u| \le C$ and 0 if $|u| > 2C$ and C is the same as in (9.4)*. Let us assume also that the right–hand side $g = 0$. Then the equation* (3.1) *possesses an attractor $\mathcal{A}$ which contains the unitary ball of the space $\mathbb{B}_\sigma(\mathbb{R}^n)$*

$$B(0,1,\mathbb{B}_\sigma) \subset \mathcal{A} \tag{9.6}$$

Proof. Indeed, the existence of the attractor $\mathcal{A}$ is an immediate corollary of Theorem 5.1. So it remains to verify (9.6). Let $u_0 \in B(0,1,\mathbb{B}_\sigma)$. Then according to Proposition 7.1 there exists a backward solution $u(t)$, $t \le 0$ of the parabolic problem (9.3). Since $\|u(t),\mathbb{R}^n\|_{0,\infty} \le C\|u_0,\mathbb{R}^n\|_{0,\infty} \le C$ then, by the definition of f, $u(t)$ is simultaneously a backward solution of the nonlinear problem (3.1). According to Theorem 3.4, there exists the solution $u(t)$, $t \ge 0$ of the problem (3.1) with $u(0) = u_0$. Thus, we have constructed the complete bounded solution $u(t)$, $t \in \mathbb{R}$ of the problem (3.1), such that

$$\sup_{t\in\mathbb{R}} \|u(t),\mathbb{R}^n\|_{0,\infty} \le C_1 < \infty \tag{9.7}$$

Using the smoothing property for the equation (3.1) we deduce immediately from (9.7) that $\sup_{t\in\mathbb{R}} \|u(t)\|_{\Phi_b} < \infty$ and consequently (by Theorem 5.1) $u_0 \in \mathcal{A}$. Proposition 9.2 is proved.

According to the embedding (9.6) it is sufficient now to estimate the entropy of the unitary ball of $\mathbb{B}_\sigma$ in $C_b(\mathbb{R}^n)$.

Theorem 9.1. *The following estimate is valid for $R \ge R_0$ and $\varepsilon < \varepsilon_0$*

$$\mathbb{H}_\varepsilon\left(B(0,1,\mathbb{B}_\sigma), C_b(B_0^R)\right) \ge CR^n \ln\frac{1}{\varepsilon} \tag{9.8}$$

where the constant C is independent of R and ε.

Proof. The ε-entropy in the spaces $\mathbb{B}_\sigma$ were estimated in [15]. For the reader convenience we give below the sketch of this proof, adopted to our situation.

For every $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ we define a function

$$\phi_k(z) = \Pi_{i=1}^n \frac{\sin^2\left(\frac{\sigma z_i}{2} - \pi k_i\right)}{\left(\frac{\sigma z_i}{2} - \pi k_i\right)^2} \tag{9.9}$$

Direct calculations reveal now that $\phi_k \in \mathbb{B}_\sigma$.

Let L_N be the subspace of $\mathbb{B}_\sigma$ spanned by $\phi_k(z)$, $k \in [-N,N]^n \equiv K_N$, i.e. L_N consists of all functions

$$u(z) = \sum_{k\in K_N} a_k\phi_k(z), \quad a_k \in R \tag{9.10}$$

The proof of Theorem 9.1 is based on the following two Lemmata.

Lemma 9.1. *The following estimate holds*

$$\sup_{k\in K_N}|a_k| \le \|u,\mathbb{R}^n\|_{0,\infty} \le C\sup_{k\in K_N}|a_k|, \quad u\in L_N \tag{9.11}$$

Moreover, the constant C is independent of N.

Indeed, the left inequality is an immediate corollary of the equality $u(\frac{2\pi k}{\sigma}) = a_k$ and the right one can be deduced easily from the assertion

$$\sum_{k\in\mathbb{Z}^n}\frac{1}{\Pi_{i=1}^n(\pi k_i - \sigma z_i/2)^2} < \infty$$

for every $\frac{\sigma z}{2\pi}\notin\mathbb{Z}^n$ and from the evident inequality $|\phi_k(z)|\le 1$.

Lemma 9.2. *Let $N\ge N_0$ and $R\ge bN$ for a some fixed $b\ge 1$. Then*

$$\|u,B_0^R\|_{0,\infty} \ge \frac{1}{2}\|u,\mathbb{R}^n\|_{0,\infty} \tag{9.12}$$

for every $u\in L_N$.

Proof. We give below the proof of (9.12) only for the case $n=1$. The estimate (9.12) in general case can be obtained analogously. Let $u\in L_N$. Then, according to Lemma 9.1

$$\|u,|z|>R\|_{0,\infty} \le \sup_{k\in K_N}\left\{|a_k|\right\}\cdot\sum_{k\in K_N}\frac{1}{(\sigma R/2-\pi N)^2} \le$$
$$\le C\|u,\mathbb{R}^1\|_{0,\infty}\frac{2N}{(R-lN)^2} \le \frac{1}{2}\|u,\mathbb{R}^1\|_{0,\infty}$$

if $R\ge 2lN$, $l=\frac{2\pi}{\sigma}$, and N is large enough. Lemma 9.2 is proved.

Now we are in position to complete the proof of Theorem 9.1. Let us fix R is large enough and ε is small enough, define $N=\left[\frac{R}{b}\right]+1$ where b is the same as in Lemma 9.2 and consider the subspace L_N. We divide the segment $[-\frac{1}{C},\frac{1}{C}]$, where C is defined in Lemma 9.1, by points $a_j = 4\varepsilon j$, $j=-[\frac{1}{4C\varepsilon}],\cdots,[\frac{1}{4C\varepsilon}]$ and define a number of functions

$$\phi_J(z) = \sum_{k\in K_N} a_{J(k)}\phi_k(z) \tag{9.13}$$

where $J:[-N,\cdots,N]^n\to -[\frac{1}{4C\varepsilon}],\cdots,[\frac{1}{4C\varepsilon}]$ an arbitrary integer map. It follows from Lemma 9.1 that $\phi_J(z)\in B(0,1,\mathbb{B}_\sigma)$, and Lemma 9.2 implies that

$$\|\phi_{J_1}-\phi_{J_2},B_0^R\|_{0,\infty}\ge 2\varepsilon$$

if $J_1\ne J_2$. Thus, finally

$$N_\varepsilon(B(0,1,\mathbb{B}_\sigma)) \ge \left(2[\frac{1}{4C\varepsilon}]+1\right)^{(2N+1)^n} \tag{9.14}$$

and consequently

$$\mathbb{H}_\varepsilon\left(B(0,1,\mathbb{B}_\sigma),C(B_0^R)\right) \ge (2N+1)^n\ln\frac{C}{2\varepsilon} \ge C_1R^n\ln\frac{1}{\varepsilon}$$

if ε is small enough. Theorem 9.1 is proved. $\square$

Corollary 9.1. *Let $\mathcal{A}$ be the attractor, constructed in Proposition 9.2 and let $R \geq \ln \frac{1}{\varepsilon}$, $\varepsilon < \varepsilon_0$. Then*

$$C_1 R^n \ln \frac{1}{\varepsilon} \leq \mathbb{H}_\varepsilon \left(\mathcal{A}, C_b(B_0^R)\right) \leq C_2 R^n \ln \frac{1}{\varepsilon} \tag{9.15}$$

Particularly, for every fixed $K > 0$

$$C_1 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \leq \mathbb{H}_\varepsilon \left(\mathcal{A}, C_b(B_0^{K \ln \frac{1}{\varepsilon}})\right) \leq C_2 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \tag{9.16}$$

Indeed, the left inequality is an immediate corollary of Theorem 9.1 and Proposition 9.2 and the right-one follows from the estimate (8.33).

Remark 9.1. Note, that (9.16) implies particularly that

$$C_1 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \leq \mathbb{H}_\varepsilon \left(B(0,1,\mathbb{B}_\sigma), C_b(B_0^{K \ln \frac{1}{\varepsilon}})\right) \leq C_2 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \tag{9.17}$$

Corollary 9.2. *Let $\mathcal{A}$ be the attractor of the equation, constructed in Proposition 9.2. Then*

$$C_1 \ln \frac{1}{\varepsilon} \leq \overline{\mathbb{H}}_\varepsilon \left(\mathcal{A}\right) \leq C_2 \ln \frac{1}{\varepsilon} \tag{9.18}$$

and consequently $0 < C_1 \leq h_{sp}\left(\mathcal{A}\right) \leq C_2$.

Thus, we have proved that the estimate (8.33) is sharp when $R \geq \ln \frac{1}{\varepsilon}$. Note that the lower estimate (9.8) is far from optimal if $R \ll \ln \frac{1}{\varepsilon}$. The following Theorem shows that the estimate (8.33) remains sharp even in the case when $R = 1$.

Theorem 9.2. *For every $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\mathbb{H}_\varepsilon \left(B(0,1,\mathbb{B}_\sigma), C(B_0^1)\right) \geq C_\delta \left(\ln \frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{9.19}$$

Proof. The proof of this Theorem is based on the estimate (9.8) and the following Lemma

Lemma 9.3. *Let $D_n^R = \{z \in \mathbb{C}^n : |z_i| \leq R\}$. Then the following estimate holds for every entire function $u \in A(\mathbb{C}^n)$ and for sufficiently large r and R, $4 < r < R$:*

$$\|u, D_n^{r/4}\|_{0,\infty} \leq \|u, [-1,1]^n\|_{0,\infty}^\beta \|u, D_n^R\|_{0,\infty}^{1-\beta} \tag{9.20}$$

where $\beta = \left(1 - \frac{\ln r}{\ln R}\right)^n$

Proof. We deduce the estimate (9.20) only for the case $n = 1$ (The general case can be easily reduced to $n = 1$ by induction.)

Recall firstly that according to Hadamard lemma

$$\|u, \{|z| = r\}\|_{0,\infty} \leq \|u, \{|z| = 1\}\|_{0,\infty}^{1-\beta} \|u, \{|z| = R\}\|_{0,\infty}^\beta \tag{9.21}$$

for every $u \in A(\overline{D_1^R \backslash D_1^1})$. Applying Zukovskij conformal transformation $w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ to the ring $\{1 \le |z| \le R\}$ we deduce from (9.21) that

$$\|u, E(r)\|_{0,\infty} \le \|u, [-1,1]\|_{0,\infty}^{\beta} \|u, E(R)\|_{0,\infty}^{1-\beta} \tag{9.22}$$

holds for every $u \in A(\mathbb{C})$ and $E(r) = \{z = x + iy \in \mathbb{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, $a = \frac{1}{2}(r + \frac{1}{r})$, $b = \frac{1}{2}(r - \frac{1}{r})$. The estimate (9.22) together with the evident embeddings

$$D_1^{r/4} \subset E(r) \subset D_1^r$$

imply the estimate (9.20) for $n = 1$. Lemma 9.3 is proved.

Now we are in position to complete the proof of Theorem 9.2. To this end we fix an arbitrary $\theta > 1$ and set $R = r^\theta$ in (9.20). Then using (9.2) we obtain that

$$\|u, B_0^r\|_{0,\infty} \le e^{C_\theta r^\theta} \|u, [-1,1]^n\|_{0,\infty}^{(1-1/\theta)^n} \tag{9.23}$$

for every $u \in B(0, 2, \mathbb{B}_\sigma)$.

Let now $\{u_i\}_{i=1}^N$ be the ε-covering of the set $B(0,1,\mathbb{B}_\sigma)$ in $C([-1,1]^n)$. Then the estimate (9.23) with $r = \left(m \ln \frac{1}{\varepsilon}\right)^{1/\theta}$ implies that this set is simultaneously is $\varepsilon^{\delta(\theta)}$-covering of the set $B(0,1,\mathbb{B}_\sigma)$ in $C(B_0^r)$, where $\delta(\theta) = (1 - 1/\theta)^n - mC_\theta > 0$ if m is small enough. Consequently,

$$\mathbb{H}_{\varepsilon^{\delta(\theta)}} \left(B(0,1,\mathbb{B}_\sigma), C(B_0^r)\right) \le \mathbb{H}_\varepsilon \left(B(0,1,\mathbb{B}_\sigma), C([-1,1]^n)\right) \tag{9.24}$$

Estimating now the left-hand side of (9.24) by (9.8) we obtain that

$$\mathbb{H}_\varepsilon \left(B(0,1,\mathbb{B}_\sigma), C([-1,1]^n)\right) \ge C'_\theta \left(\ln \frac{1}{\varepsilon}\right)^{\frac{n}{\theta}+1} \tag{9.25}$$

Since $\theta > 1$ is arbitrary then (9.25) proves Theorem 9.2.

Corollary 9.3. *Let $\mathcal{A}$ be the attractor of the equation, constructed in Proposition 9.2. Then for every $\delta > 0$ there exists C_δ such that*

$$\mathbb{H}_\varepsilon \left(\mathcal{A}, C(B_0^1)\right) \ge C_\delta \left(\ln \frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{9.26}$$

Remark 9.2. Note that the assertion of Theorem 9.2 is not valid for $\delta = 0$. Indeed, according to (9.2) the set $B(0,1,\mathbb{B}_\sigma)$ is a subset of the class of functions, considered in Example 7.4. Thus, according to (7.8),

$$\mathbb{H}_\varepsilon \left(B(0,1,\mathbb{B}_\sigma), C(B_0^1)\right) \le C \frac{\left(\ln \frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln \ln \frac{1}{\varepsilon}\right)^n} \tag{9.27}$$

Remark 9.3. Let $n = 1$. Then a little more accurate using Hadamard lemma implies the estimate

$$\mathbb{H}_\varepsilon \left(B(0,1,\mathbb{B}_\sigma), C(B_0^1)\right) \ge C_1 \left(\frac{\ln \frac{1}{\varepsilon}}{\ln \ln \frac{1}{\varepsilon}}\right)^2 \tag{9.28}$$

§10 Unstable manifolds and lower bounds of ε-entropy.

In previous Section the examples of equations which possess sharp lower bounds of Kolmogorov's entropy have been constructed. Note, however that these examples seemed artificial. In this Section, using the technique of infinite dimensional unstable manifolds, developed in [10], we obtain the same estimate for rather wide class of natural equations, including Chafee-Infante equation, complex Ginzburg-Landay equation, etc.

We assume in this Section that $\Omega = \mathbb{R}^n$ and the equation (3.1) has the form

$$\partial_t u = \Delta_x u + \alpha^2 u - f(u), \quad f(0) = 0, \quad f'(0) = 0, \quad \alpha > 0 \tag{10.1}$$

and the nonlinear term $f \in C^2$ satisfies the condition

$$f(u).u \geq -C + \beta |u|^2, \quad \beta > \alpha^2 \tag{10.2}$$

To construct the unstable manifold of the equation (10.1) near the equilibria point $u \equiv 0$ we study for the first the linear ($f \equiv 0$) nonhomogeneous equation (10.1) with the right–hand side $h(t)$.

Definition 10.1. *Let $\beta > 0$. Then we define the space $\mathbb{L}_\beta$ by the following expression*

$$\mathbb{L}_\beta = \{u \in L^\infty(\mathbb{R}_- \times \Omega) : \|u\|_{\mathbb{L}_\beta} = \sup_{t \leq 0} e^{-\beta t} \|u(t), \mathbb{R}^n\|_{0,\infty} < \infty\} \tag{10.3}$$

Proposition 10.1. *Let $\beta > \alpha^2$ and $h \in \mathbb{L}_\beta$. Then the equation*

$$\partial_t v = \Delta_x v + \alpha^2 v + h(t), \quad t \leq 0 \tag{10.4}$$

possesses the unique solution $v \in \mathbb{L}_\beta$ and consequently defines the linear operator $\mathbb{T}_\beta : \mathbb{L}_\beta \to \mathbb{L}_\beta$, $v(t) = (\mathbb{T}_\beta h)(t)$.

The proof of Proposition can be derived in a standard way (see [10], for instance).

Proposition 10.2. *Let $\sigma < \frac{\alpha}{2n^{1/2}}$ then for every $u_0 \in \mathbb{B}_\sigma$ the backward parabolic problem*

$$\begin{cases} \partial_t v = \Delta_x v + \alpha^2 v, \quad t \leq 0 \\ v\big|_{t=0} = u_0 \end{cases} \tag{10.5}$$

possesses the unique solution $v \in \mathbb{L}_{\beta_0}$ with $\beta_0 = \frac{3\alpha^2}{4}$ and consequently the linear operator $\mathbb{P}_{\beta_0} : \mathbb{B}_\sigma \to \mathbb{L}_{\beta_0}$, $v(t) = (\mathbb{P}_{\beta_0} u_0)(t)$ is defined.

The proof of this Proposition can be derived in a standard way applying the Fourier transform to the problem (10.5).

Now we are in position to study the neighborhood of zero equilibria point for the nonlinear equation.

Theorem 10.1. *Let the nonlinearity f satisfy* (10.2) *and $\mathcal{A}$ be the attractor of the equation* (10.1) *which exists according to Theorem 5.1. Then there exist $\sigma = \sigma(\alpha)$, $\delta_0 = \delta_0(f, \alpha)$ and C^1-map*

$$\mathcal{U}_0 : B(0, \delta_0, \mathbb{B}_\sigma) \to \mathcal{A} \tag{10.6}$$

Moreover,

$$\|\mathcal{U}_0(u_0) - u_0, \mathbb{R}^n\|_{0,\infty} \leq C\|u_0, \mathbb{R}^n\|^2_{0,\infty} \tag{10.7}$$

for every $u_0 \in B(0, \delta_0, \mathbb{B}_\sigma)$

Proof. The proof of this Theorem is based on the implicit function theorem and on the following lemma.

Lemma 10.1. *Let $f \in C^2$ such that $f(0) = 0$ and $f'(0) = 0$. Then for every $\beta > 0$ the Nemitskij operator $Fu = f(u)$ belongs to the space $C^1(\mathbb{L}_\beta, \mathbb{L}_{2\beta})$.*

Indeed, since $f(0) = f'(0) = 0$ then $f(u) = u^2\phi(u)$ and consequently the Nemitskij operator $F : \mathbb{L}_\beta \to \mathbb{L}_{2\beta}$. The differentiability of this map follows from the following estimate with $v(t) = u_1(t) - u_2(t)$

$$\begin{aligned} e^{-2\beta t}|f(u_1(t) - f(u_2(t)) - f'(u_1(t))(u_1(t) - u_2(t))| \leq \\ \leq e^{-2\beta t}|v(t)| \int_0^1 |f'(u_1(t) - \theta_1 v(t)) - f'(u_1(t))| d\theta_1 \leq \\ \leq e^{-2\beta t}|v(t)|^2 \int_0^1 \int_0^1 \theta_1 |f''(u_1(t) - \theta_2\theta_1 v(t))| d\theta_1 d\theta_2 \leq C\|v\|^2_{\mathbb{L}_\beta} \end{aligned}$$

Let us fix $\beta = \frac{3\alpha^2}{4}$ and rewrite the equation (10.1) near the equilibria point $u = 0$ in the following form

$$u + \mathbb{T}_{2\beta} F u = \mathbb{P}_\beta u_0, \quad u \in \mathbb{L}_\beta \tag{10.8}$$

where $u_0 \in \mathbb{B}_\sigma$ and σ satisfies the conditions of Proposition 10.2. Note that every solution of (10.8) is simultaneously a solution of the equation (10.1) hence it is sufficient to solve (10.8) in $\mathbb{L}_\beta$.

We will solve the equation (10.8) using the implicit function theorem. To this end we introduce a function $\mathcal{F} : \mathbb{L}_\beta \times \mathbb{B}_\sigma \to \mathbb{L}_\beta$ by formula

$$\mathcal{F}(u, u_0) = u + \mathbb{T}_{2\beta} F u - \mathbb{P}_\beta u_0 \tag{10.9}$$

It follows from Propositions 10.1, 10.2 and from Lemma 10.1 that $\mathcal{F} \in C^1(\mathbb{L}_\beta \times \mathbb{B}_\sigma, \mathbb{L}_\beta)$ and $D_u\mathcal{F}(0, 0) = Id$. Hence due to the implicit function theorem (see [24] for instance) there exists a neighborhood $B(0, \delta_0, \mathbb{B}_\sigma)$ and a C^1-function

$$\mathcal{U} : B(0, \delta_0, \mathbb{B}_\sigma) \to \mathbb{L}_\beta \tag{10.10}$$

such that $\mathcal{F}(\mathcal{U}(u_0), u_0) \equiv 0$ and consequently $\mathcal{U}(u_0)(t)$ is a backward solution of the problem (10.1). The equation (10.8) implies now that

$$\|\mathcal{U}(u_0) - \mathcal{P}_\beta u_0\|_{\mathbb{L}_\beta} \leq C\|f(\mathcal{U}(u_0))\|_{\mathbb{L}_{2\beta}} \leq C_1\|\mathcal{U}(u_0)\|^2_{\mathbb{L}_\beta} \leq C_2\|u_0\|^2_{\mathbb{B}_\sigma} \tag{10.11}$$

Let us define now $\mathcal{U}_0(u_0) = \mathcal{U}(u_0)\big|_{t=0}$. Then (10.11) together with the evident assertion $(\mathbb{P}_\beta u_0)(0) = u_0$ imply the estimate (10.7). The assertion $\mathcal{U}_0(B(0, \delta_0, \mathbb{B}_\sigma)) \subset \mathcal{A}$ can be derived as in the proof of Proposition 9.2. Theorem 10.1 is proved.

Corollary 10.1. *Let $u_0^1, u_0^2 \in B(0,\delta,\mathbb{B}_\sigma)$ and $\delta \le \delta_0$. Then for every $R > 0$*

$$\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{L^\infty(B_0^R)} \ge \|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} - C\delta^2 \tag{10.12}$$

with C independent of R.

Indeed,

$$\begin{aligned}
&\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{L^\infty(B_0^R)} \ge \\
&\quad \ge \|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} - \|\mathcal{U}_0(u_0^1) - u_0^1\|_{L^\infty(\mathbb{R}^n)} + \|\mathcal{U}_0(u_0^2) - u_0^2\|_{L^\infty(\mathbb{R}^n)} \ge \\
&\quad \ge \|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} - C_1(\|u_0^1\|_{\mathbb{B}_\sigma}^2 + \|u_0^2\|_{\mathbb{B}_\sigma}^2) \ge \|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} - 2C_1\delta^2
\end{aligned}$$

The estimate (10.12) admits to obtain the analogues of (9.15),(9.16) and (9.26) for the equation (10.1). Indeed, let $\varepsilon > 0$ be small enough, $\delta = \left(\frac{\varepsilon}{2C}\right)^{1/2}$ and $u_0^1, u_2^0 \in B(0,\delta,\mathbb{B}_\sigma)$ such that

$$\|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} \ge \varepsilon \tag{10.13}$$

Then it follows from (10.12) that

$$\|\mathcal{U}(u_0^1) - \mathcal{U}(u_0^2)\|_{L^\infty(B_0^R)} \ge \varepsilon/2 \tag{10.14}$$

The estimates (10.13),(10.14) together with the assertion (10.6) imply that

$$\begin{aligned}
&\mathbb{H}_{\varepsilon/4}\left(\mathcal{A}, C(B_0^R)\right) \ge \\
&\quad \ge \mathbb{H}_\varepsilon\left(B(0,\left(\frac{\varepsilon}{2C}\right)^{1/2},\mathbb{B}_\sigma), C(B_0^R)\right) = \mathbb{H}_{(2C\varepsilon)^{1/2}}\left(B(0,1,\mathbb{B}_\sigma), C(B_0^R)\right)
\end{aligned} \tag{10.15}$$

The last estimate together with (9.8) and (9.19) imply that

Corollary 10.2. *Let $\mathcal{A}$ be the attractor of the equation* (10.1) *and let $\varepsilon < \varepsilon_0$ and $R \ge \ln\frac{1}{\varepsilon}$. Then*

$$C_1 R^n \ln\frac{1}{\varepsilon} \le \mathbb{H}_\varepsilon\left(\mathcal{A}, C(B_0^R)\right) \le C_2 R^n \ln\frac{1}{\varepsilon} \tag{10.16}$$

Particularly, $0 < C_1 \ln\frac{1}{\varepsilon} \le \overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C_2 \ln\frac{1}{\varepsilon}$.

Corollary 10.3. *Let $\mathcal{A}$ be the attractor of the equation* (10.1). *Then for every $\delta > 0$ there exists C_δ such that*

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, C(B_0^1)\right) \ge C_\delta\left(\ln\frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{10.17}$$

Example 10.1. The simplest example of the equation (10.1) for which our estimates are valid is Chafee-Infante equation in $\mathbb{R}^n$

$$\partial_t u - \Delta_x u = \alpha^2 u - u^3, \quad \alpha > 0 \tag{10.18}$$

Thus, the estimates (10.16) and (10.17) are valid for the attractor $\mathcal{A}$ of the equation (10.18)

Acknowledgements. The research was partially supported by the DFG – Schwerpunktprogram "Dynamische Systeme" under Mi 459 | 2–3.

The author finished this manuscript while visiting the University of Stuttgart in 1999. He is grateful for discussions with Prof. A.Mielke and M.I.Vishik.

References

1. F. Abergel, *Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains.*, J. Diff.Equ. **83** (1990), 85–108.
2. A. V. Babin, M. I. Vishik, *Attractors of evolutionary equations*, Nauka, Moscow, 1989; English transl. Stud. Math. Appl., 25, North Holland, Amsterdam, 1992.
3. A.V.Babin and M.I.Vishik, *Attractors of partial differential evolution equations in an unbounded domain*, New directions in differential equations and dynamical systems. Published by the Royal Society of Edinburg (1991), 39–63.
4. V.V.Chepyzhov and M.I.Vishik, *Attractors of nonautonomous dynamical systems and their dimension*, J. Math. Pures Appl. **73** (1994), 279–333.
5. V.V.Chepyzhov and M.I.Vishik, *Nonautonomous evolutionary equations with translation-compact symbols and their attractors*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), 153–158.
6. V.V.Chepyzhov and M.I.Vishik, *Kolmogorov's ε-entropy for the attractor of reaction-diffusion equation*, Math. Sbornik **189(2)** (1998), 81–110.
7. V.V.Chepyzhov and M.I.Vishik, *Evolution equations and their trajectory attractors*, J. Math. Pures Appl. **76** (1997), 913–964.
8. P.Collet and J.Eckmann, *Extensive properties of the complex Ginzburg–Landay equation.*, Commun. Math. Phys. **200** (1999), 699-722.
9. M.A.Efendiev and A.Miranville, *Finite dimensional attractors for RDE in R^n with a strong nonlinearity*, Discrete and Continuous Dynamical Systems, **15, (2)** (1999), 399—424.
10. M.A.Efendiev and S.V.Zelik, *The attractor for a nonlinear reaction–diffusion system in the unbounded domain*, preprint (1999).
11. J. K. Hale, *Asymptotic behavior of dissipative systems*, Math. Surveys and Mon., 25, Amer. Math. Soc., Providence, RI (1987).
12. E.Hewitt and K.Ross, *Abstract harmonic analysis. Volume I. Structure of topological groups. Integration theory. Group representations.*, Springer-Verlag, 1963.
13. W. Hutson and J. S. Pym, *Applications of Functional Analysis and Operator Theory*, Math. Sci. Engrg., vol. 146, Academic Press, London, 1980.
14. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1998.
15. A.N. Kolmogorov and V.M. Tikhomirov ε-entropy and ε-capacity of sets in functional spaces, In: Selected works of A.N. Kolmogorov **Vol III**, ed., Dordrecht: Kluver (1993).
16. Ladizanskia O.A., Solonnikov V.A., Uraltseva N.N., *Linear and quasilinear equations of parabolic type*, M.:Nauka, 1967.
17. S.Merino, *On the existence of the global attractor for semilinear RDE on R^n*, Journal Diff. Equation **132** (1996), 87–106.
18. A.Mielke, *The complex Ginzburg–Landay equation on large and unbounded domains: sharper bounds and attractors*, Nonlinearity **10** (1997), 199–222.
19. A.Mielke, *Bounds for the solutions of the complex Ginzburg–Landay equation in terms of the dispersion parameters*, Physica D **117** (1998), 106–116.
20. R.Temam, *Infinite–dimensional dynamical systems in mechanics and physics*, Springer–Verlag, New-York, 1988.
21. H. Triebel, *Interpolation theory, function space, differential operators*, North-Holland, Amsterdam–New York, 1978.
22. M. I. Vishik and S. V. Zelik, *The trajectory attractor for a nonlinear elliptic system in a cylindrical domain*, Russian Acad. Sci. Sb. Math. **187** (1996), 1755–1789.
23. A. I. Volpert and S. I. Khudyaev, *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics*, Nijhoff, Dordrecht, 1985.
24. E. Zeidler, *Nonlinear Functional Analysis and its applications*, Springer-Verlag.

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