

# EXPONENTIAL ATTRACTORS FOR EVOLUTION EQUATIONS

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**Abstract :** Our aim in this article is to review recent results in the theory of exponential attractors, both in the autonomous and the nonautonomous cases.

## 1. Introduction.

Our aim in this article is to discuss the behavior as time goes to infinity of partial differential equations of the form

$$\frac{\partial u}{\partial t} = F(t, u), \quad (1.1)$$

in a Banach space  $H$ .

When the system is autonomous, i.e., when the time does not appear explicitly in (1.1) ( $F(t, u) \equiv F(u)$ ), then, very often, the long time behavior of the system can be described in terms of the global attractor  $\mathcal{A}$ . More precisely, assuming that the system is well-posed, we can define the family of solving operators

$$S(t) : u_0 \mapsto u(t), \quad t \geq 0,$$

acting on  $H$ , which maps the initial datum  $u_0$  onto the solution at time  $t$ . This family of operators satisfies

$$S(0) = I,$$

$$S(t + s) = S(t) \circ S(s), \quad \forall t, s \geq 0,$$

$I$  denoting the identity operator, and we say that it forms a semigroup on the phase space  $H$ . Then, we say that a set  $\mathcal{A}$  is the global attractor for  $S(t)$  in  $H$  if

- (i) It is compact in  $H$ .
- (ii) It is invariant, i.e.,  $S(t)\mathcal{A} = \mathcal{A}$ ,  $\forall t \geq 0$ .

(iii) It attracts (uniformly) the bounded sets of initial data in the following sense :

$$\forall B \subset H \text{ bounded, } \lim_{t \rightarrow +\infty} \text{dist}(S(t)B, \mathcal{A}) = 0,$$

where  $\text{dist}$  denotes the Hausdorff semidistance between sets, defined by

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_H.$$

This is equivalent to the following :  $\forall B \subset H$  bounded,  $\forall \epsilon > 0$ ,  $\exists t_0 = t_0(B, \epsilon)$  such that  $t \geq t_0$  implies  $S(t)B \subset \mathcal{U}_\epsilon$ , where  $\mathcal{U}_\epsilon$  denotes the  $\epsilon$ -neighborhood of  $\mathcal{A}$ .

We note that it follows from (ii) and (iii) that the global attractor, if it exists, is unique. Furthermore, it follows from (i) that it is essentially thinner than the original phase space  $H$  ; indeed, here, in general,  $H$  is an infinite-dimensional function space and, in infinite dimensions, a compact set cannot contain a ball and is nowhere dense. It is not difficult to prove that the global attractor is the smallest (for the inclusion) closed set enjoying the attraction property (iii) ; it is also the largest bounded invariant set. Finally, in most (if not all) cases, one can prove that the global attractor has finite dimension (in the sense of covering dimensions, such as the Hausdorff and the fractal dimensions ; the global attractor is not a smooth manifold in general, but it can have a very complicated geometric structure), so that, even though the initial phase space is infinite-dimensional, the dynamics, reduced to the global attractor, is, in some proper sense, finite-dimensional and can be described by a finite number of parameters. It thus follows that the global attractor appears as a suitable object in view of the study of the long time behavior of the system. We refer the reader to [BV], [CheV2], [H], [L2], [R] and [Te] for extensive reviews on this subject.

Now, the global attractor may present some defaults. Indeed, it may attract the trajectories slowly (see, e.g., [Ko]). Furthermore, in general, it is very difficult, if not impossible, to express the convergence rate in terms of the physical parameters of the problem. A second drawback, which can also be seen as a consequence of the first one, is that the global attractor may be sensitive to perturbations ; a given system is only an approximation of reality and it is thus essential that the objects that we study be robust under small perturbations. Actually, in general, the global attractor is upper semicontinuous with respect to perturbations, i.e.,

$$\text{dist}(\mathcal{A}_\epsilon, \mathcal{A}_0) \rightarrow 0 \text{ as } \epsilon \rightarrow 0,$$

where  $\mathcal{A}_0$  is the global attractor associated with the nonperturbed system and  $\mathcal{A}_\epsilon$  that associated with the perturbed one,  $\epsilon > 0$  being the perturbation parameter. Very roughly speaking, this property means that the global attractor cannot explode under small perturbations. Now, the lower semicontinuity, i.e.,

$$\text{dist}(\mathcal{A}_0, \mathcal{A}_\epsilon) \rightarrow 0 \text{ as } \epsilon \rightarrow 0,$$

which, roughly speaking, means that the global attractor cannot implode, is much more difficult to prove (see, e.g., [R]). Furthermore, this property may not hold. This is in particular the case when the perturbed and nonperturbed problems do not have the same equilibria (stationary solutions). This can already be seen in finite dimensions by considering the following ordinary differential equation (see [R]) :

$$x' = (1 - x^2)(1 - \lambda^2), \quad \lambda \in [-1, 1].$$

Then, when  $\lambda = 0$ ,  $\mathcal{A}_\lambda = [0, 1]$ , whereas  $\mathcal{A}_\lambda = \{1\}$  for  $\lambda < 0$  and  $\mathcal{A}_\lambda = [-\sqrt{\lambda}, 1]$  for  $\lambda > 0$ . Thus, there is a bifurcation phenomenon at  $\lambda = 0$  and the global attractor is not lower semicontinuous at  $\lambda = 0$ . It thus follows that the global attractor may change drastically under small perturbations. Furthermore, in many situations, the global attractor may not be observable in experiments or in numerical simulations. This can be due to the fact that it has a very complicated geometric structure, but not necessarily. Indeed, we can consider for instance the following Chafee-Infante equation in one space dimension :

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0, \quad x \in [0, 1], \quad \nu > 0,$$

$$u(0, t) = u(1, t) = -1, \quad t \geq 0.$$

Then, due to the boundary conditions,  $\mathcal{A} = \{-1\}$ . Now, this problem possesses many metastable “almost stationary” equilibria which live up to a time  $t_\star \sim e^{\nu^{-\frac{1}{2}}}$ . Thus, for  $\nu$  small, one will not see the global attractor in numerical simulations. Finally, in some situations, the global attractor may fail to capture important transient behaviors. This can be observed, e.g., on some models of one-dimensional Burgers equations with a weak dissipation term (see [Bi]). In that case, the global attractor is trivial, it is reduced to one exponentially attracting point, but the system presents very rich and important transient behaviors, which resemble some modified version of the Kolmogorov law. We can also mention models of pattern formation equations in chemotaxis for which one observes important transient behaviors which are not contained in the global attractor (see [TTY]).

So, it follows from the above considerations that it should be useful to have a (possibly) larger object which contains the global attractor, attracts the trajectories at a fast rate, is still finite-dimensional and is more robust under perturbations.

Our purpose in what follows is to study such an object, namely, an exponential attractor, proposed by A. Eden, C. Foias, B. Nicolaenko and R. Temam in [EFNT].

We can mention that another possible object is an inertial manifold (see [FoST]). An inertial manifold  $\mathcal{M}$  is a smooth (at least Lipschitz) finite-dimensional manifold which satisfies the following asymptotic completeness property :  $\forall u \in H, \exists v \in \mathcal{M}$  such that

$$\|S(t)u - S(t)v\|_H \leq Q(\|u\|_H)e^{-\alpha t}, \quad t \geq 0,$$

where the monotonic function  $Q$  and the constant  $\alpha > 0$  are independent of  $u$  and  $v$ . Actually, an inertial manifold would be a perfect object in view of the remarks made above. Indeed, since it is a smooth manifold and owing to the asymptotic completeness, the dynamics, reduced to  $\mathcal{M}$ , is equivalent, in a strong way, to that of the initial system and can be described by a finite system of ordinary differential equations (called the inertial form). By comparison, the finite dimensionality of the global attractor (and, more precisely, the finite fractal dimensionality ; the same will hold for exponential attractors) only gives an estimate of the number of unknowns which are necessary to capture all the dynamics in, say, numerical simulations, even though one has a partial reduction principle via the so-called Mané theorem

(see [EFNT]). Unfortunately, all the known constructions of inertial manifolds are based on a very restrictive condition, namely, the so-called spectral gap condition (see, e.g., [FoST] and [T]). Consequently, the existence of an inertial manifold is not known in general. In particular, it is not known for the two-dimensional Navier-Stokes equations and for reaction-diffusion and weakly damped wave equations in arbitrary domains in three space dimensions ; one has even nonexistence results for reaction-diffusion equations in higher space dimensions (see [MaS]).

## 2. Exponential attractors for autonomous systems.

Let  $S(t)$ ,  $t \geq 0$ , be the semigroup associated with the problem

$$\frac{\partial u}{\partial t} = F(u), \quad (2.1)$$

$$u|_{t=0} = u_0, \quad (2.2)$$

in a Banach space  $H$  (in particular, we assume that (2.1)-(2.2) is well-posed),  $u_0 \in H$ . We have the following definition.

**Definition 2.1 :** A set  $\mathcal{M}$  is an exponential attractor for  $S(t)$  in  $H$  if

- (i) It is compact in  $H$ .
- (ii) It is positively invariant, i.e.,  $S(t)\mathcal{M} \subset \mathcal{M}$ ,  $\forall t \geq 0$ .
- (iii) It has finite fractal dimension.
- (iv) It attracts exponentially fast the bounded sets of initial data in the following sense : There exists a monotonic function  $Q$  and a constant  $\alpha > 0$  such that

$$\forall B \subset H \text{ bounded, } \text{dist}(S(t)B, \mathcal{M}) \leq Q(\|B\|_H)e^{-\alpha t}, \quad t \geq 0.$$

It follows from this definition that an exponential attractor always contains the global attractor (actually, it follows from the definition that, if  $S(t)$  possesses an exponential attractor  $\mathcal{M}$ , then it also possesses the global attractor  $\mathcal{A} \subset \mathcal{M}$  ; indeed,  $\mathcal{M}$  is a compact attracting set (see, e.g., [BV] ; the continuity of  $S(t)$ ,  $\forall t \geq 0$ , generally holds)). Thus, an exponential attractor is still finite-dimensional, like the global attractor, and, moreover, one now has an explicit (exponential) control on the rate of attraction of the trajectories.

### Remark 2.1 :

- (i) Actually, proving the existence of an exponential attractor is also one way of proving the finite (fractal) dimensionality of the global attractor. In general, in order to prove the finite dimensionality of the global attractor, one uses the so-called volume contraction method, based on the study of the evolution of infinitesimal  $k$ -dimensional volumes in a neighborhood of the global attractor : if the semigroup contracts the  $k$ -dimensional volumes, then the fractal dimension of the global attractor is less than  $k$ . This method usually gives the best estimates on the dimension in terms of the physical parameters of the problem, see, e.g., [T] ; see also [EfM2] for the derivation of sharp estimates on the dimension of exponential attractors for reaction-diffusion systems. However, the volume contraction method requires some differentiability of the associated semigroup. As we shall see below, the construction of exponential attractors requires weaker assumptions, namely,

some Lipschitz or Hölder property, which can be useful when the differentiability is not known or difficult to prove, see, e.g., [MiZ2].

(ii) The choice of the fractal dimension over other dimensions, e.g., the Hausdorff dimension, in Definition 2.1 is related, as mentioned above, with the Mané theorem which gives some indications on the existence of a reduced finite-dimensional system which is Hölder continuous (but, unfortunately, not Lipschitz continuous) with respect to the initial data, see [EFNT].

Now, the main drawback of exponential attractors is that an exponential attractor, if it exists, is not unique. Therefore, the question of the best choice, if it makes sense, of an exponential attractor is a crucial one. In order to overcome this drawback, one possibility consists in finding a “simple” algorithm  $S \mapsto \mathcal{M}(S)$  which maps a given mapping or semigroup  $S$  onto an exponential attractor  $\mathcal{M}(S)$ ; by simple, we have in particular in mind the numerical realization of such an algorithm.

The first construction of exponential attractors was due to A. Eden, C. Foias, B. Nicolaenko and R. Temam [EFNT]. This construction is based on the so-called squeezing property which, roughly speaking, says that either the higher modes are dominated by the lower ones or that the flow is contracted exponentially. It is non-constructible (indeed, Zorn’s lemma is used in order to construct the appropriate exponential attractor) and is only valid in Hilbert spaces (since it makes an essential use of orthogonal projectors with finite rank). Furthermore, based on this construction, it is possible to prove the lower semicontinuity of proper exponential attractors under perturbations, but only up to some time shift, so that, essentially, one only proves that

$$\text{dist}(\mathcal{A}_0, \mathcal{M}_\epsilon) \rightarrow 0 \text{ as } \epsilon \rightarrow 0,$$

where  $\mathcal{A}_0$  is the global attractor associated with the nonperturbed system and  $\mathcal{M}_\epsilon$  an exponential attractor associated with the perturbed one, which is not satisfactory.

In [EfMZ1], we proposed a second construction, valid in Banach spaces also (see also [DN] for another construction of exponential attractors valid in Banach spaces; this second construction consists in adapting that of [EFNT] to a Banach setting and has thus some of the drawbacks mentioned above). The key point in this construction is a smoothing property on the difference of two solutions (actually, such a smoothing property was already considered by O.A. Ladyzhenskaya in order to prove the finite dimensionality of the global attractor, see, e.g., [L1]) of the form

$$\|S(t_\star)x_1 - S(t_\star)x_2\|_{H_1} \leq c\|x_1 - x_2\|_H, \quad (2.3)$$

where  $H_1$  is a second Banach space which is compactly embedded into  $H$ , which has to hold for some  $t_\star > 0$  and on some bounded positively invariant subset of  $H$ . We can note that, in a Hilbert setting, i.e., when  $H$  and  $H_1$  are Hilbert spaces, then (2.3) implies the squeezing property, see [EfM2]. Furthermore, based on this construction, it is possible to construct robust (i.e., lower and upper semicontinuous with respect to perturbations) families of exponential attractors (see [EfMZ3], [EfY], [FGMZ] and [GaGMP2]) which satisfy in particular an estimate of the form

$$\text{dist}_{\text{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leq c\epsilon^\kappa, \quad c > 0, \quad \kappa \in (0, 1), \quad (2.4)$$

where the constants  $c$  and  $\kappa$  are independent of  $\epsilon$  and can be computed explicitly in terms of the physical parameters of the problem and where  $\text{dist}_{\text{sym}}$  denotes the symmetric Hausdorff distance between (closed) sets :  $\text{dist}_{\text{sym}}(A, B) = \max(\text{dist}(A, B), \text{dist}(B, A))$ . Of course, such constructions are obtained having in mind the nonuniqueness problem.

**Remark 2.2 :**

(i) It is in general very difficult, if not impossible, to prove an estimate of the form (2.4) for global attractors. This is possible, for instance, when the stationary solutions enjoy some hyperbolicity assumption. In that case, the global attractor (the so-called regular attractor in the terminology of A.V. Babin and M.I. Vishik [BV]) is regular and exponential and one has an estimate of the form (2.4). However, even in that case, one cannot compute in general the constants  $c$  and  $\kappa$  in terms of the physical parameters of the problem.

(ii) We refer the reader to the bibliography for many references on constructions of exponential attractors and of robust families of exponential attractors in various situations. In particular, the existence of exponential attractors (and of families of robust exponential attractors) is as general as that of global attractors, in the sense that we shall have exponential attractors (and, in some sense, robust families of exponential attractors) as soon as we have the existence of the global attractor.

(iii) We also refer to [AY] for results on the stability of exponential attractors under numerical approximations and [EfMZ5] for the stability of approximations of exponential attractors for equations in unbounded domains by exponential attractors for the corresponding equation in approximating bounded domains.

In the constructions of exponential attractors mentioned above, the exponential attractors  $\mathcal{M}_\epsilon$ ,  $\epsilon > 0$ , depend on the perturbed semigroup  $S_\epsilon(t)$ , but also on the exponential attractor  $\mathcal{M}_0$  for the nonperturbed semigroup  $S_0(t)$ . Consequently, the continuity only holds at one point, namely, at  $\epsilon = 0$ .

We improved this construction in [EfMZ6], allowing now to have a mapping  $S \mapsto \mathcal{M}(S)$  which is Hölder continuous. To state this result, we first define the class of mappings that we shall consider (of course, the smoothing property (2.3) will play a fundamental role).

We consider two Banach spaces  $H$  and  $H_1$  such that  $H_1$  is compactly embedded into  $H$  and a bounded subset  $B$  of  $H_1$ .

**Definition 2.2 :** A mapping  $S : H_1 \rightarrow H_1$  belongs to the class  $\mathcal{S}_{\delta, K}(B)$ ,  $\delta, K$  being two positive constants, if

- (i) The mapping  $S$  maps a  $\delta$ -neighborhood (for the topology of  $H_1$ )  $\mathcal{O}_\delta(B)$  of  $B$  onto  $B$ .
- (ii)  $\forall h_1, h_2 \in \mathcal{O}_\delta(B)$ ,

$$\|Sh_1 - Sh_2\|_{H_1} \leq K\|h_1 - h_2\|_H.$$

We then have the following result (see [EfMZ6]).

**Theorem 2.1 :** *For every  $S \in \mathcal{S}_{\delta, K}(B)$ , the discrete dynamical system generated by the iterations of  $S$  (i.e., the discrete semigroup  $S(n) = S \circ \dots \circ S$  ( $n$  times),  $n \in \mathbb{N}$ ) possesses an exponential attractor  $\mathcal{M}(S)$  which satisfies the following properties :*

- (i) *It is compact in  $H_1$ .*

- (ii) It is positively invariant, i.e.,  $SM(S) \subset \mathcal{M}(S)$ .
- (iii) It has finite fractal dimension.
- (iv) It enjoys the following exponential attraction property :

$$\text{dist}_{H_1}(S^n B, \mathcal{M}(S)) \leq c_1 e^{-\alpha n}, \quad n \in \mathbb{N}, \quad \alpha > 0.$$

- (v) The map  $S \mapsto \mathcal{M}(S)$  is Hölder continuous in the following sense :  $\forall S_1, S_2 \in \mathcal{S}_{\delta, K}(B)$ ,

$$\text{dist}_{\text{sym}, H_1}(\mathcal{M}(S_1), \mathcal{M}(S_2)) \leq c_2 \|S_1 - S_2\|_{\mathcal{S}}^{\kappa}, \quad \kappa > 0,$$

where  $\|S_1 - S_2\| = \sup_{h \in \mathcal{O}_{\delta}(B)} \|S_1 h - S_2 h\|_{H_1}$ .

Furthermore, all the constants only depend on  $B, H, H_1, \delta$  and  $K$ , but they are independent of the concrete choice of  $S \in \mathcal{S}_{\delta, K}(B)$  ; they can be computed explicitly in terms of the physical parameters of the problem.

In order to construct an exponential attractor in the continuous case, i.e., for a semigroup  $S(t)$ ,  $t \geq 0$ , acting on  $H_1$ , one proceeds as follows (typically, we consider here the semigroup  $S(t)$  associated with a parabolic system ; however, the construction can be extended to allow to construct such exponential attractors to, e.g., weakly damped wave equations, see [EfMZ6]).

First, one considers a bounded absorbing set  $B$  of  $H_1$  for which one can prove, e.g., that there exists  $T > 0$  such that  $S(T)\mathcal{O}_1(B) \subset B$  (typically, this will follow from proper a priori estimates) and such that  $S(T) \in \mathcal{S}_{1, K}(B)$ , for some proper constant  $K > 0$ . Then, one considers the discrete dynamical system generated by the map  $S_T = S(T)$ . It thus follows from Theorem 2.1 that this dynamical system possesses an exponential attractor  $\mathcal{M}_T$  which satisfies all the assertions of Theorem 2.1. Finally, one sets

$$\mathcal{M}(S) = \cup_{t \in [0, T]} S(t) \mathcal{M}_T. \quad (2.5)$$

We now assume that the mapping  $(t, x) \mapsto S(t)x$  is Lipschitz with respect to the  $x$ -variable and Hölder with respect to the time, with exponent  $\kappa_1$  (this is what we usually have in applications).

It is then standard to prove the

**Theorem 2.2 :** *The set  $\mathcal{M}(S)$  is an exponential attractor for  $S(t)$  in  $H_1$  which satisfies the following properties :*

- (i) It is compact in  $H_1$ .
- (ii) It is positively invariant.
- (iii) It has finite fractal dimension.
- (iv) It enjoys the following exponential attraction property :

$$\text{dist}_{H_1}(S(t)B, \mathcal{M}(S)) \leq c'_1 e^{-\alpha' t}, \quad t \geq 0, \quad \alpha' > 0.$$

- (v) If  $S_1(t)$  and  $S_2(t)$  are two semigroups such that  $S_i(T) \in \mathcal{S}_{1, K}(B)$ ,  $i = 1, 2$ , then

$$\text{dist}_{\text{sym}, H_1}(\mathcal{M}(S_1), \mathcal{M}(S_2)) \leq c'_2 \|S_1 - S_2\|_{\mathcal{S}}^{\kappa'}, \quad \kappa' > 0,$$

where  $\|S_1 - S_2\|_{\mathcal{S}} = \sup_{t \in [0, T], h \in \mathcal{O}_1(B)} \|S_1(t)h - S_2(t)h\|_{H_1}$ .

Furthermore, all the constants only depend on  $B$ ,  $H$ ,  $H_1$ ,  $K$  and  $\kappa_1$  and can be computed explicitly.

**Remark 2.4 :**

(i) Noting that  $B$  is a bounded absorbing set, one can prove that the exponential attractor  $\mathcal{M}(S)$  actually attracts exponentially fast (and uniformly) all the bounded sets of initial data in  $H_1$ .

(ii) It follows from the Hölder property (v) that one can now construct a robust family of exponential attractors which are continuous at every point (and not only at 0), thus improving the previous constructions.

**3. Exponential attractors for nonautonomous systems.**

We now assume that the time appears explicitly in the equation (typically, in the forcing terms) and we consider the problem

$$\frac{\partial u}{\partial t} = F(t, u), \quad (3.1)$$

$$u|_{t=\tau} = u_\tau, \quad \tau \in R, \quad (3.2)$$

in a Banach space  $H$ ,  $u_\tau \in H$ . Assuming that the problem is well-posed, we have the family of solving operators

$$\begin{aligned} U(t, \tau) &: H \rightarrow H \\ u_\tau &\mapsto u(t), \quad t \geq \tau, \quad \tau \in R, \end{aligned}$$

such that

$$U(\tau, \tau) = I, \quad \forall \tau \in R,$$

$$U(t, s) \circ U(s, \tau) = U(t, \tau), \quad t \geq s \geq \tau, \quad \tau \in R.$$

We say that this family of operators forms a process on  $H$ .

The theory of attractors for nonautonomous systems is less understood than that for autonomous systems. We have essentially two approaches.

The first one, initiated by A. Haraux (see [Ha]) and further studied and developed by V.V. Chepyzhov and M.I. Vishik (see, e.g., [CheV1] and [CheV2]), is based on the notion of a uniform attractor. Actually, in order to construct the uniform attractor, one considers, together with (3.1)-(3.2), a whole family of equations. Then, one proves the existence of the global attractor for a proper semigroup on an extended phase space, and, finally, projecting this global attractor onto the first component, one obtains the uniform attractor. The major drawback of this approach is that the extended dynamical system is essentially more complicated than the initial one, which leads, for general (translation-compact, see [CheV1]) time dependences, to an artificial infinite dimensionality of the uniform attractor. This can already be seen for the following simple linear equation :

$$\frac{\partial u}{\partial t} - \Delta u = h(t),$$



$$u = 0 \text{ on } \partial\Omega,$$

in a bounded smooth domain  $\Omega$ , whose dynamics is simple, namely, one has one exponentially attracting trajectory. However, the uniform attractor has infinite dimension and infinite topological entropy (see [CheV2]). However, for periodic and quasiperiodic time dependences, one has in general finite-dimensional uniform attractors (i.e., if the same is true for the corresponding autonomous system). Furthermore, one can derive sharp upper and lower bounds on the dimension of the uniform attractor, so that this approach is quite relevant in that case. We can note that, as in the autonomous case, an exponential attractor in this setting always contains the uniform attractor and, again, one has, for general time dependences, an artificial infinite dimensionality. We shall not develop this approach here and we refer the reader to [EfMZ2], [FM], [Mi1] and [Mi3] for more details.

The second approach is based on the notion of a pullback attractor (see, e.g., [CrF], [K] and [S]). In that case, one has a time dependent attractor  $\{\mathcal{A}(t), t \in R\}$ , contrary to the uniform attractor which is time independent. More precisely, a family  $\{\mathcal{A}(t), t \in R\}$  is a pullback attractor for the process  $U(t, \tau)$  in  $H$  if

- (i) The set  $\mathcal{A}(t)$  is compact in  $H, \forall t \in R$ .
- (ii) It is invariant, i.e.,  $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t), \forall t \geq \tau, \tau \in R$ .
- (iii) It satisfies the following pullback attraction property :

$$\forall B \subset H \text{ bounded, } \lim_{s \rightarrow +\infty} \text{dist}(U(t, t-s)B, \mathcal{A}(t)) = 0.$$

One can prove that, in general,  $\mathcal{A}(t)$  has finite dimension,  $\forall t \in R$ , see, e.g., [CLV] and [LaS]. We also note that it follows from the above definition that the pullback attractor, if it exists, is unique. Furthermore, if the system is autonomous, then one recovers the global attractor. Now, the attraction property essentially means that, at time  $t$ , the attractor  $\mathcal{A}(t)$  attracts the bounded sets of initial data coming from the past (i.e., from  $-\infty$ ). However, in (iii), the rate of attraction is not uniform in  $t$ , so that the forward convergence is not true in general (see nevertheless [CKR], [ChKS] and [LaR] for cases where the forward convergence can be proven). We can illustrate this on the following nonautonomous ordinary differential equation :

$$y' = f(t, y),$$

where  $f(t, y) = -y$  if  $t \leq 0$ ,  $(-1 + 2t)y - ty^2$  if  $t \in [0, 1]$  and  $y - y^2$  if  $t \geq 1$ . Then, one has the existence of the pullback attractor  $\{\mathcal{A}(t), t \in R\}$ , where  $\mathcal{A}(t) = \{0\}, \forall t \in R$ . However, for  $t \geq 0$ , every trajectory, different from  $\{0\}$ , starting from a small neighborhood of 0, will leave this neighborhood, never to enter it again. This clearly contradicts our intuitive understanding of attractors. We shall see that this essential drawback will be removed when constructing exponential attractors in this setting (indeed, in that case, one has a control (and, more precisely, an exponential control) of the rate of attraction).

We first consider a discrete process  $U(l, m), l, m \in Z, l \geq m$ , acting on the phase space  $H$ . We set  $\mathcal{U}(n) = U(n+1, n), n \in Z$ . Then, the process  $U$  is uniquely defined by the family  $\{\mathcal{U}(n), n \in Z\}$ . Indeed, one has

$$U(n+k, n) = \mathcal{U}(n+k-1) \circ \mathcal{U}(n+k-2) \circ \dots \circ \mathcal{U}(n), n \in Z, k \in N.$$

We then consider a second Banach space  $H_1$  such that  $H_1$  is compactly embedded into  $H$  and a bounded subset  $B$  of  $H_1$ .

We have the following result (see [EfMZ6]).

**Theorem 3.1 :** *We assume that,  $\forall n \in Z$ ,  $\mathcal{U}(n) \in \mathcal{S}_{\delta,K}(B)$ , where  $\delta$  and  $K$  are independent of  $n$ . Then, the process  $U$  possesses an exponential attractor  $\{\mathcal{M}_U(n), n \in Z\}$  which satisfies the following properties :*

- (i)  $\forall n \in N$ ,  $\mathcal{M}_U(n) \subset B$  and is compact in  $H_1$ .
- (ii) It is positively invariant, i.e.,  $U(k, m)\mathcal{M}_U(m) \subset \mathcal{M}_U(k)$ ,  $k \geq m$ ,  $m \in Z$ .
- (iii)  $\forall n \in N$ ,  $\mathcal{M}_U(n)$  has finite fractal dimension, bounded independently of  $n$ .
- (iv) It enjoys the following exponential attraction property :

$$\text{dist}_{H_1}(U(n+k, n)B, \mathcal{M}_U(n+k)) \leq c_1 e^{-\alpha k}, \quad n \in Z, \quad k \in N, \quad (3.3)$$

where  $c_1$  and  $\alpha > 0$  are independent of  $n$  and  $k$ .

- (v) The map  $U \mapsto \mathcal{M}_U(n)$  is uniformly Hölder continuous in the following sense :  $\forall U_1, U_2$  processes such that  $\mathcal{U}_i(n) \in \mathcal{S}_{\delta,K}(B)$ ,  $n \in Z$ ,  $i = 1, 2$ ,

$$\text{dist}_{\text{sym}, H_1}(\mathcal{M}_{U_1}(n), \mathcal{M}_{U_2}(n)) \leq c_2 \sup_{l \in (-\infty, n)} \{e^{-\beta(n-l)} \|\mathcal{U}_1(l) - \mathcal{U}_2(l)\|_{\mathcal{S}}^{\kappa}\}, \quad (3.4)$$

$\beta > 0$ ,  $\kappa > 0$ .

Furthermore, all the constants only depend on  $B$ ,  $H$ ,  $H_1$ ,  $\delta$  and  $K$ , but they are independent of  $n$  and of the concrete choice of the  $U_i$ ,  $i = 1, 2$  ; they can be computed explicitly in terms of the physical parameters of the problem.

**Remark 3.1 :**

(i) It follows from (3.3) that one has an exponentially fast pullback attraction. Now, this estimate also shows that the forward convergence holds, so that the main drawback of pullback attractors is solved. Thus, this is the first general result on the fact that the asymptotic behavior of nonautonomous dynamical systems is finite-dimensional, as in the autonomous case.

(ii) It follows from the Hölder property (3.4) that the attractor  $\mathcal{M}_U(n)$  is independent of  $\mathcal{U}(k)$ ,  $k \geq n$ , so that the causal principle which, roughly speaking, says that the state of a system is uniquely determined by its past, is satisfied. Furthermore, (3.4) shows that the influence of the past decays exponentially, which is in agreement with our physical intuition.

(iii) We can also construct the exponential attractor  $\{\mathcal{M}_U(n), n \in Z\}$  such that the following cocycle identity holds :

$$\mathcal{M}_U(n+k) = \mathcal{M}_{T_k U}(n), \quad n, \quad k \in Z,$$

where  $T_k U(l, m) = U(l+k, m+k)$ ,  $k, l, m \in Z$ ,  $l \geq m$ .

(iv) In the autonomous case, i.e., when  $\mathcal{U}(n) \equiv S$ ,  $\forall n \in Z$ , and we consider the dynamical system generated by the iterations of  $S$ , then,  $\mathcal{M}_U(n) \equiv \mathcal{M}(S)$ ,  $n \in Z$ , is independent of  $n$  and we recover the (autonomous) exponential attractor constructed in the previous section.

(v) If the dependence of  $\mathcal{U}(n)$  on  $n$  is periodic, quasiperiodic or almost periodic, then, the same is true for the dependence of  $\mathcal{M}_U(n)$  on  $n$ .

We now consider a continuous process  $U(t, \tau)$ ,  $t \geq \tau$ ,  $\tau \in R$ , acting on  $H$ . In order to construct an exponential attractor for this process, we shall typically follow the following steps (again, typically, we consider here a process associated with a nonautonomous parabolic system ; however, we can extend the construction to, e.g., nonautonomous weakly damped wave equations).

First, we consider a uniform (with respect to  $\tau$ ) bounded absorbing set  $B \subset H_1$  (i.e.,  $\forall B_0 \subset H_1$  bounded,  $\exists t_0 = t_0(B_0)$  such that  $t \geq t_0$  implies  $U(t + \tau, \tau)B_0 \subset B$ ,  $\forall \tau \in R$ ) such that  $U(T + \tau, \tau)\mathcal{O}_1(B) \subset B$ ,  $\forall \tau \in R$ , for some  $T > 0$ , and  $U(T + \tau, \tau) \in \mathcal{S}_{1,K}(B)$ ,  $\forall \tau \in R$ , for some positive constant  $K$  independent of  $\tau$ . Then, for every  $\tau \in R$ , we consider the discrete process

$$U^\tau(m, l) = U(\tau + mT, \tau + lT), \quad m, l \in Z, \quad m \geq l.$$

Thanks to Theorem 3.1, we can construct,  $\forall \tau \in R$ , a discrete exponential attractor  $\{\mathcal{M}_U(l, \tau), l \in Z\}$  which satisfies all the assertions of the theorem. It also satisfies the following properties :

$$\mathcal{M}_U(l, \tau) = \mathcal{M}_U(0, lT + \tau), \quad l \in Z, \quad \tau \in R ;$$

$$\mathcal{M}_{T_s U}(l, \tau) = \mathcal{M}_U(l, \tau + s), \quad l \in Z, \quad s, \tau \in R,$$

where  $T_s U(t, \tau) = U(t + s, \tau + s)$ ,  $t \geq \tau$ ,  $\tau, s \in R$ . We finally set

$$\mathcal{M}_U(t) = \cup_{s \in [0, T]} U(t, t - T - s)\mathcal{M}_U(0, t - T - s), \quad t \in R.$$

Then, assuming that  $L : (t, \tau, x) \mapsto U(t, \tau)x$  is Lipschitz with respect to the  $x$ -variable and satisfies proper Hölder type properties with respect to the time (i.e.,  $t$  and  $\tau$ ), see [EfMZ6], we can prove the

**Theorem 3.2 :** *The set  $\{\mathcal{M}_U(t), t \in R\}$  is an exponential attractor for the process  $U$  in  $H_1$  which satisfies the following properties :*

- (i)  $\forall t \in R$ ,  $\mathcal{M}_U(t)$  is compact in  $H_1$ .
- (ii) It is positively invariant, i.e.,  $U(t, \tau)\mathcal{M}_U(\tau) \subset \mathcal{M}_U(t)$ ,  $t \geq \tau$ ,  $\tau \in R$ .
- (iii) It satisfies the following cocycle identity :

$$\mathcal{M}_{T_s U}(t) = \mathcal{M}_U(t + s), \quad \forall t, s \in R.$$

(iv)  $\forall t \in R$ ,  $\mathcal{M}_U(t)$  has finite fractal dimension, bounded independently of  $t$ .

(v) It enjoys the following exponential attraction property :

$$\text{dist}_{H_1}(U(t + \tau, \tau)B, \mathcal{M}_U(t + \tau)) \leq c'_1 e^{-\alpha' t}, \quad t \geq \tau, \quad \tau \in R, \quad \alpha' > 0,$$

where the constants  $c'_1$  and  $\alpha'$  are independent of  $t$  and  $\tau$ .

(vi) It satisfies the following Hölder continuity property : If  $U_1(t, \tau)$ ,  $U_2(t, \tau)$ ,  $t \geq \tau$ ,  $\tau \in R$ , are two processes such that  $U_i(t + T, t) \in \mathcal{S}_{1,K}(B)$ ,  $\forall t \in R$ ,  $i = 1, 2$ , then

$$\text{dist}_{\text{sym}, H_1}(\mathcal{M}_{U_1}(t), \mathcal{M}_{U_2}(t)) \leq c'_2 \sup_{s \in R^+} \{e^{-\beta' s} \|U_1(t, t - s) - U_2(t, t - s)\|_{\mathcal{S}}^{\kappa'}\},$$

$\beta' > 0$ ,  $\kappa' > 0$ .

Furthermore, all the constants only depend on  $B$ ,  $H$ ,  $H_1$  and the Hölder exponents of the mapping  $L$  with respect to the time ; they can be computed explicitly in terms of the physical parameters of the problem.

**Remark 3.2 :**

- (i) We again have properties that are similar to those listed in Remark 3.1.
- (ii) We can also prove some Hölder property of the mapping  $t \mapsto \mathcal{M}_U(t)$  (see [EfMZ6]).

#### §4 Applications and generalizations

As we have already mentioned, some smoothing property for the differences of trajectories of the considered dynamical system plays a crucial role for the above exponential attractor theory. For simplicity, we have considered above only this smoothing property of the form (2.3) although this concrete form is not unique and, up to the moment, there exists a number of interesting and significant generalizations of that property which increase greatly the area of applications of the above exponential attractors theory. The main task of this concluding section is to give a brief exposition of that topic.

We first recall that our choice of the smoothing property of the form (2.3) is well adapted to the study of parabolic evolutionary problems in bounded domains where we usually have the smoothing effect in a finite time. Indeed, the model problem of that type is the following semilinear reaction-diffusion system:

$$(4.1) \quad \frac{\partial u}{\partial t} = a\Delta_x u - f(x, u), \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$ ,  $u = (u^1, \dots, u^k)$  is an unknown vector-valued function and  $a$  and  $f$  are given diffusion matrix and the nonlinear interaction function respectively.

It is well-known (see e.g., [BV], [Te]) that, under some dissipativity and growth restrictions on the nonlinearity  $f$ , equation (4.1) generates a dissipative semigroup in the phase space  $H = L^2(\Omega)$  and the ball  $B_R$  of a sufficiently large radius  $R$  in the Sobolev space  $H_1 = H^1(\Omega)$  is an absorbing set for the associated semigroup  $S(t)$ . This ball naturally plays the role of the set  $B$  introduced in Definition 2.2. Moreover, the difference  $v(t) := u_1(t) - u_2(t)$  between two solutions of (4.1) obviously satisfies the linear parabolic equation

$$(4.2) \quad \frac{\partial v}{\partial t} = a\Delta_x v - l(t, x)v, \quad l(t, x) := \int_0^1 f'_u(x, su_1(t, x) + (1-s)u_2(t, x)) ds.$$

According to the classical theory of linear parabolic equations, this equation possesses a smoothing property of the following form:

$$(4.3) \quad \|v(t)\|_{H^1(\Omega)} \leq Ct^{-1/2}e^{Kt}\|v(0)\|_{L^2(\Omega)}, \quad t > 0$$

where  $C$  and  $K$  are the appropriate positive constants depending only on  $f$  and on the norms  $\|u_i(0)\|_{H^1(\Omega)}$  (in order to verify (4.3) it is usually sufficient to multiply this equation by  $t\Delta_x v$  and integrate over  $[0, t] \times \Omega$ , see e.g. [BV]). Therefore, the operators  $S(t)$  satisfy (2.3) for every  $t > 0$ . Moreover, using that  $B = B_R$  is an

absorbing set, it is not difficult to verify that  $S(t_*) \in \mathcal{S}_{\delta, K}(B)$  for some  $t_*$ ,  $\delta$  and  $K$  and, consequently, the assertions of Theorem 2.1 hold for problem (4.1) (see also [EfMZ6]) for the non-autonomous case). Thus, the dissipative estimate for the solutions of (4.1) together with the classical smoothing property (4.3) for linear parabolic equations guarantee (due to Theorem 2.1) the finite-dimensionality of the associated global attractor and the existence of an exponential attractor.

We however note that for the evolutionary problems which are far from being parabolic (e.g., for the nonlinear damped wave equation) and even for parabolic problems in unbounded domains with finite global attractors the smoothing property of the form (2.3) is too strong and should be weakened. Indeed, in contrast to the parabolic equation (4.2), the linear hyperbolic equation can possess only the asymptotic smoothing property and (4.3) can never hold. Instead of that, very often the difference  $u_1(t) - u_2(t)$  can be splitted into a contraction and compact parts. This gives the following generalization of (2.3):

$$(4.4) \quad S(t_*)u_1 - S(t_*)u_2 = \mathcal{C}_1(u_1, u_2) + \mathcal{C}_2(u_1, u_2)$$

where the operators  $\mathcal{C}_i$  satisfy

$$(4.5) \quad \|\mathcal{C}_1(u_1, u_2)\|_H \leq \alpha \|u_1 - u_2\|_H, \quad \|\mathcal{C}_2(u_1, u_2)\|_{H^1} \leq K \|u_1 - u_2\|_H$$

where  $\alpha < 1$  and  $K > 0$  are some numbers independent of  $u_1$  and  $u_2$ . The usage of the generalized smoothing property of the form (4.4), (4.5) allows indeed to extend the above "parabolic" exponential attractors theory to wide class of damped hyperbolic and other partially dissipative problems, see [EfMZ1], [EfMZ4], [FGMZ], [MiZ1-MiZ3].

Another form of the smoothing property which is also useful for hyperbolic equation and is very close (but nonequivalent) to (4.4), (4.5) is the following one:

$$(4.6) \quad \|S(t_*)u_1 - S(t_*)u_2\|_{H^1} \leq \alpha \|u_1 - u_2\|_{H^1} + K \|u_1 - u_2\|_H$$

where as before  $H_1 \subset \subset H$ ,  $\alpha < 1$  and  $C$  are the appropriate positive constants, see [EfMZ6].

We now mention also the so-called  $l$ -trajectory method of verifying the finite-dimensionality of global attractors and constructing exponential attractors, see [MN], [Z1], [MP] where the smoothing property for differences is formulated not in the phase space, but in the appropriate functional spaces related with the pieces of trajectories of the considered dynamical system of length  $l$ . This method allows, in particular, this method allows to extend the exponential attractors theory to the class of equations of non-Newtonian fluid mechanics ([MN]), hyperbolic equations with nonlinear damping ([Pr1]), parabolic equations with supercritical nonlinearities ([Z1]) and degenerate doubly nonlinear parabolic equations ([EfZ3]). The application of this method naturally leads to the following form of the smoothing property:

$$(4.7) \quad \|S(l_*)u_1 - S(l_*)u_2\|_{H^1} \leq \alpha \|u_1 - u_2\|_{H^1} + K(\|u_1 - u_2\|_H + \|S(l_*)u_1 - S(l_*)u_2\|_H)$$

where  $S(l_*)$  is the extension of the evolution semigroup to the space of  $l_*$ -trajectories,  $H_1 \subset \subset H$  and  $\alpha < 1$  and  $K > 0$  are some positive numbers. The smoothing property (4.7) together with the Lipschitz continuity

$$(4.8) \quad \|S(l_*)u_1 - S(l_*)u_2\|_{H^1} \leq L \|u_1 - u_2\|_{H^1}$$

are sufficient in order to verify the existence of a finite-dimensional exponential attractor, see [MP].

The next relaxation (see [MiZ2]) of smoothing property (2.3) uses the possibility for this property to be satisfied not on the whole absorbing set  $B$ , but only on some subset  $\mathcal{B}$  of it which possesses the appropriate recurrence properties. Namely, it is assumed that there exists  $\varepsilon_0 > 0$  and  $N \in \mathbb{N}$  such that, for every  $\varepsilon_0$ -ball  $B(\varepsilon_0, u_0)$  of space  $H$  centered at  $u_0$  there exists  $k = k(u_0) \in \mathbb{N}$ ,  $k \leq N$  such that

$$(4.9) \quad S(kt_*)(B(\varepsilon_0, u_0) \cap B) \subset \mathcal{B}$$

If (4.9) is satisfied and if, in addition,  $S(t_*)$  is globally Lipschitz continuous on the whole invariant set  $B$  in  $H$ -metric and possesses the uniform smoothing property (2.3) for every  $u_1, u_2 \in \mathcal{B}$ , then, as proved in [MiZ2],  $S(t_*)$  possesses an exponential attractor on  $B$ .

This scheme has been applied in [MiZ2] for the study of the 3D Cahn-Hilliard problem

$$(4.10) \quad \frac{\partial u}{\partial t} = -\Delta_x(\Delta_x u - f_{\log}(u)), \quad \partial_n u|_{\partial\Omega} = \partial_n \Delta_x u|_{\partial\Omega} = 0$$

in a bounded domain  $\Omega \subset \mathbb{R}^3$  with logarithmic nonlinearity

$$(4.11) \quad f_{\log}(u) := \alpha u - \beta \log \frac{1+u}{1-u}, \quad \alpha, \beta > 0$$

We note that the case of logarithmic nonlinearity is much more difficult than the case of regular potentials due to the presence of singular points  $u = \pm 1$  in the nonlinearity which does not allow to prove neither the differentiability of the associated semigroup with respect to the initial data nor the smoothing property (2.3) for the whole absorbing set and, thus, the classical theory is not applicable here. Nevertheless, the smoothing property (2.3) on a smaller set

$$(4.12) \quad \mathcal{B} := \{u_0 \in C(\Omega), \quad \|u_0\|_{C(\Omega)} \leq 1 - \delta\}, \quad \delta > 0$$

can be easily verified. Moreover, the recurrence assumption (4.9) also can be verified using the maximum principle for the second order elliptic equations and some regularity reasons, see [MiZ2]. This allows, in turns, to prove the finite-dimensionality of the global attractor for (4.10) and to construct the associated exponential attractors.

Next generalization of smoothing property (2.3) is related with singular perturbed problems whose model example is the following damped hyperbolic equation:

$$(4.13) \quad \varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta_x u + f(u) = g(x), \quad u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}|_{t=0} = u'_0$$

where  $0 \leq \varepsilon \ll 1$  is a small parameter and  $g$  is a given external force and  $f$  is a given nonlinearity which satisfies some dissipativity and growth restrictions. In this case, for  $\varepsilon > 0$  we have a hyperbolic problem with two initial data (for  $u(0)$  and  $\frac{\partial u}{\partial t}(0)$ ) and for the limit case  $\varepsilon = 0$  we have a parabolic problem which requires

only one initial data (for  $u(0)$ ). Thus, the associated family of semigroups  $S_\varepsilon(t)$  act in *different* functional spaces and, moreover, the natural energy norms

$$(4.14) \quad \|(u, \frac{\partial u}{\partial t})\|_{\mathcal{E}^\alpha(\varepsilon)}^2 := \varepsilon \|\frac{\partial u}{\partial t}\|_{H^\alpha(\Omega)}^2 + \|\frac{\partial u}{\partial t}\|_{H^{\alpha-1}(\Omega)}^2 + \|u\|_{H^{\alpha+1}(\Omega)}^2$$

in which we can verify the smoothing property also depend on  $\varepsilon$ . Therefore, in order to treat this problem, one needs a generalization of the above theory to the case where different semigroups  $S_\varepsilon(t)$  are defined on different spaces and the spaces  $H$  and  $H_1$  in (4.5) are also depend on  $\varepsilon$  ( $H = H(\varepsilon)$  and  $H^1 = H^1(\varepsilon)$ ). Such a generalization has been suggested in [FGMZ] in an abstract setting. We do not give here the precise formulation of that result and only mention the most important conditions: 1) estimates (4.5) should be satisfied with constants  $\alpha$  and  $K$  independent of  $\varepsilon$ . 2) The compact embedding  $H^1(\varepsilon) \subset\subset H(\varepsilon)$  also should be uniform with respect to  $\varepsilon$  (in the sense of Kolmogorov's  $\varepsilon$ -entropy, see [FGMZ]). 3) for small  $\varepsilon$ ,  $S_\varepsilon(t_*)$  should be in a sense close to  $S_0(t_*)$ . Then, there exists a robust (at  $\varepsilon = 0$ ) family of exponential attractors for the maps  $S_\varepsilon(t_*)$ .

This abstract theorem is occurred very useful not only for hyperbolic equation (4.13), but also for wide class of more complicated singular perturbed problems including including phase-field problems ([MiZ1], [MiZ3], [GrMPZ]), hyperbolic Cahn-Hilliard equation ([GaGMP3]), Cahn-Hilliard equations with vanishing microforces ([MiZ2]) and even for some class of evolution equations with memory ([GaGMP1], [GP2], [GP3]).

We also mention that the other generalization of (2.3), where the space  $H^1$  depends on the points  $u_1$  and  $u_2$  ( $H^1 = H^1(u_1, u_2)$ ) can be used in order to construct the finite dimensional attractors for some class of porous media equations, see [EfZ4].

We conclude our exposition by considering the evolution equations in unbounded domains with *infinite-dimensional* attractors whose model example is the following Chafee-Infante equation in  $\mathbb{R}^n$ :

$$(4.15) \quad \frac{\partial u}{\partial t} = \Delta_x u + u - u^3, \quad u|_{t=0} = u_0$$

It is well-known that this equation is uniquely solvable for every  $u_0 \in L^\infty(\mathbb{R}^n)$  and generated a dissipative semigroup  $S(t)$  in that space. Furthermore, this semigroup possesses the so-called *locally compact* global attractor  $\mathcal{A}$  which, in contrast to the usual one is only bounded in the phase space  $\Phi := L^\infty(\Omega)$ , compact in the local topology  $\Phi_{loc} := L^\infty_{loc}(\mathbb{R}^n)$  and the attraction property for such attractor is also understood in the sense of local topology, see e.g. [EfZ1], [Z2], [Z4]. It is easy to see that, in contrast to the case of bounded domains, the attractor  $\mathcal{A}$  has infinite dimension, therefore, one usually uses the concept of the so-called Kolmogorov's  $\varepsilon$ -entropy in order to obtain some information on the "size" of such attractors and their complexity.

We recall that, by definition, see [KT], Kolmogorov's  $\varepsilon$ -entropy  $\mathbb{H}(K, M)$  of a compact set  $K$  in a metric space  $M$  is a binary logarithm of the minimal number of  $\varepsilon$ -balls which is necessary for covering the set  $K$ :

$$(4.16) \quad \mathbb{H}_\varepsilon(K, M) := \log_2 N_\varepsilon(K, M)$$

Moreover, since the attractor  $\mathcal{A}$  is compact only in the local topology, it is natural to consider the  $\varepsilon$ -entropy of its restrictions  $\mathcal{A}|_{B_{x_0}^R}$  on the balls  $B_{x_0}^R$  in  $\mathbb{R}^n$  of radius  $R$  centered at  $x_0$  and study the dependence of that quantity on three parameters  $\varepsilon$ ,  $R$  and  $x_0$ .

As shown in [Z2] (see also [EfZ2], [Z3], [Z4]), this quantity has the following asymptotics

$$(4.17) \quad \mathbb{H}_\varepsilon(\mathcal{A}|_{B_{x_0}^R}, L^\infty) \sim (R + \log_2 \frac{1}{\varepsilon})^n \log \frac{1}{\varepsilon}$$

(up to the double logarithm of  $\varepsilon^{-1}$ ).

The exponential attractor theory for reaction-diffusion equations in unbounded domains has been developed in [EfMZ6]. Of course, since the exponential attractors always contain the global ones, the exponential attractor  $\mathcal{M}$  for problem (4.15) should be also infinite dimensional and it is also natural to use the Kolmogorov's  $\varepsilon$ -entropy for its construction. To be more precise, as shown in [EfMZ6], there exists an infinite-dimensional exponential attractor  $\mathcal{M} \subset \Phi$  for equation (4.15) which is semiinvariant with respect to the semiflow:  $S(t)\mathcal{M} \subset \mathcal{M}$ , possesses the same type of asymptotics for Kolmogorov's entropy (see (4.17)) as the global attractor and attracts exponentially all of the bounded subsets of  $\Phi$  in the uniform topology of  $\Phi$ , i.e., for every  $B \subset \Phi$  bounded,

$$(4.18) \quad \text{dist}_\Phi(S(t)B, \mathcal{M}) \leq Q(\|B\|)e^{-\alpha t}$$

where the function  $Q$  and positive constant  $\alpha$  are independent of  $B$ . We see that, in contrast to the global ones, this exponential attractor attracts the bounded subsets in the initial topology of the phase space  $\Phi$  and, moreover, as shown in [EfMZ6], it can be effectively approximated by the usual (finite-dimensional) exponential attractors associated with (4.15) in large bounded domains. We also note that this result is essentially based on the following weighted analogue of the smoothing property (2.3):

$$(4.19) \quad \|S(t_*)u_1 - S(t_*)u_2\|_{H^1_{e^{-|x-x_0|}}(\mathbb{R}^n)} \leq Kt^{-1/2}e^{Kt_*}\|u_1 - u_2\|_{L^2_{e^{-|x-x_0|}}(\mathbb{R}^n)}$$

where  $x_0 \in \mathbb{R}^n$  is a parameter and the constant  $K$  is independent of  $x_0 \in \mathbb{R}^n$ , see [Z2], [EfMZ6].

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