The dynamics of fast non-autonomous travelling waves and homogenization. *

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1 Introduction

The elliptic boundary value problem

$$a(\partial_t^2 u+\Delta_x u)-\frac{\gamma}{\varepsilon}\partial_t u-f(u)=g(t);\ u|_{\partial\Omega}=0;\ u|_{t=0}=u_0 \tag{1}$$

in a semicylinder $(t,x)\in\Omega_+ :=\mathbb{R}_+\times\omega$, where ω is a bounded smooth domain in $\mathbb{R}^n$, $u=(u^1,\cdots,u^k)$ is an unknown vector function, f, g and u_0 are given vector functions, a and γ are given constant $k\times k$-matrices such that $a+a^*>0$ and $\gamma=\gamma^*>0$, and ε is assumed to be a small positive parameter ($\varepsilon\ll 1$), is studied.

The problems of the type (1) arise studying the travelling wave solutions of the non-autonomous evolution equations in a cylindrical domains $\Omega:=\mathbb{R}\times\omega$. Indeed, consider the second order non-autonomous parabolic equation in $(t,x)\in\Omega$

$$\partial_\eta v=a(\partial_t^2 v+\Delta_x v)-f(v)-g(t-\frac{\gamma}{\varepsilon}\eta,x) \tag{2}$$

with the fast travelling wave external force $g(t-\frac{\gamma}{\varepsilon}\eta)$ (where $\gamma\backslash\varepsilon\gg 1$ is a wave speed and the variable η plays the role of time). Then the problem of finding the travelling wave solution (modulated by the external travelling wave g) $v(\eta,t,x):=v(t-\frac{\gamma}{\varepsilon}\eta,x)$ leads to the elliptic boundary problem (1) in a cylinder Ω. Applying the dynamical approach to studying the problem

*This research is partially supported by the Russian Foundation for Fundamental Research (grant# 99-01-00304).

in the full cylinder (see e.g. [1], [3], [8], [14]) we obtain the auxiliary problem of the type (1) in a semicylinder Ω_+. Note also that the problem (1) is of independent interest.

It is assumed that the nonlinear term $f(u)$ in (1) satisfies the following assumptions

$$1.\ f(v).v \geq -C;\quad 2.\ f'(v) \geq -K;\quad 3.\ |f(v)| \leq C(1+|v|^q);\quad \forall v \in \mathbb{R}^k \tag{3}$$

for the appropriate constants C and K and with the growth exponent $q < q_{max} = \frac{n+2}{n-2}$.

It is assumed also that the external force $g \in C_b(\mathbb{R}, L^2(\omega))$ is almost periodic with respect to t with values in $L^2(\omega)$. Recall (see e.g. [9]) that it means by definition that the hull

$$H(g) := [T_h g, h \in \mathbb{R}]_{C_b(\mathbb{R},L^2(\omega))},\quad (T_h g)(t) := g(t+h) \tag{4}$$

is compact in $C_b(\mathbb{R}, L^2(\omega))$ (Here we have denoted by $\{\cdot\}_V$ the closure in the space V).

The solution u of the equation (1) is defined to be a function, which belongs to $W^{2,2}(\Omega_T)$ for every $T \geq 0$ ($\Omega_T := [T, T+1] \times \omega$) and has the finite norm

$$\|u\|_{W_b^{2,2}(\mathbb{R}_+)} = \sup_{T\geq 0} \|u, \Omega_T\|_{2,2} < \infty \tag{5}$$

and therefore we restrict ourselves to consider only bounded with respect to $t \to \infty$ solutions of the problem (1).

Here and below $\|g, V\|_{l,p} := \|g\|_{W^{l,p}(V)}$ and as usual $W^{l,p}$ means the Sobolev space of distributions which derivatives up to the order l inclusively belonging to L^p. Moreover, we denote by $W_b^{2,2}(\Omega_+)$ the space of functions which have the finite norm (5). The spaces $W_b^{l,p}(\Omega)$ and $W_b^{l,p}(\Omega_+)$ can be defined analogously.

It is natural to assume also that the initial data u_0 belongs to the space $V_0 := W^{3/2,2}(\omega) \cap \{u_0|_{\partial\omega} = 0\}$ which in fact the space of traces of functions from $W_b^{2,2}(\Omega_+) \cap \{u|_{\partial\omega} = 0\}$ when $t = 0$ (see e.g. [13]).

The equations of the type (1) under the various assumptions on a, γ, f, g and ε have been studied in [1], [3], [10], [11], [14], [15].

It is known (see e.g. [3], [15]) that under our assumptions for every fixed $\varepsilon < \varepsilon_0 \ll 1$ the problem (1) possesses a unique (bounded with respect to $t \to \infty$) solution $u(t)$ which satisfies the estimate

$$\|u, \Omega_T\|_{2,2} \leq Q_\varepsilon(\|u_0\|_{V_0})e^{-\alpha T} + Q_\varepsilon(\|g\|_{L_b^2}) \tag{6}$$

with a certain monotonic function Q_ε depending only on f, a, γ and ε (and independent of u_0 and g) and positive α. Consequently, the problem (1) defines a non-autonomous dynamical system $U_g^\varepsilon(t,\tau)$ in the phase space V_0 (a process in V_0 using the terminology of [4]) by formula

$$U_g(t,\tau)u_\tau = u(t) \text{ where } u(t),\ t\geq\tau \text{ is a solution of (1) with } u(\tau)=u_\tau \quad (7)$$

Using the standard skew product technique (see e.g. [4], [6]) this process can be extended to a semigroup acting in a larger phase space. Indeed, consider with the initial problem (1) a family of problems of the type (1) with all right-hand sides $\xi(t)$ belonging to the hull (4) of the initial external force g:

$$a(\partial_t^2 u + \Delta_x u) - \frac{\gamma}{\varepsilon}\partial_t u - f(u) = \xi(t), \quad \xi\in H(g) \tag{8}$$

and the corresponding family of processes $\{U_\xi^\varepsilon(t,\tau), \xi\in H(g), t\geq\tau\}$ acting in V_0. Then the semigroup $\mathbb{S}_t^\varepsilon$ acting in the extended phase space $V_0\times H(g)$ can be defined in the following way (see e.g. [4]):

$$\mathbb{S}_t^\varepsilon(u_0,\xi) := (U_\xi(t,0)u_0, T_t\xi), \quad t\geq 0, \quad \xi\in H(g), \quad u_0\in V_0 \tag{9}$$

It is not difficult to prove (see [14]), using the dissipative estimate (6) and the fact that the hull $H(g)$ is compact in $C_b(\mathbb{R}, L^2(\omega))$ that the semigroup (9) possesses a (global) attractor $\mathbb{A}^\varepsilon$ in $V_0\times H(g)$. The projection of $\mathcal{A}^\varepsilon := \Pi_1\mathbb{A}^\varepsilon$ of this attractor to the first component (V_0) is defined to be the (uniform) attractor for the initial equation (1).

Note that the attractor $\mathcal{A}^\varepsilon$ is generated by all *bounded* solutions of the family of equations (8) which is defined in a full cylinder Ω:

$$\mathcal{A}^\varepsilon = \cup_{\xi\in H(g)} K_\xi^\varepsilon\big|_{t=0} \tag{10}$$

where K_ξ^ε is a set of all bounded in $W_b^{2,2}(\Omega)$ solutions of the equation (8) with the right-hand side ξ or which is the same the set of all travelling wave solutions of the evolution equation (2) (with g is replaced by ξ). This justifies the attractor's approach to study the travelling wave solutions.

The main aim of the paper is to study the behaviour of the attractors $\mathcal{A}^\varepsilon$ when $\varepsilon\to 0$. To this end we make the time rescaling $t\to\varepsilon t$ and write the problem in the following more convenient form:

$$a(\varepsilon^2\partial_t^2 u + \Delta_x u) - \gamma\partial_t u - f(u) = g_\varepsilon(t), \quad u\big|_{t=0} = u_0 \tag{11}$$

where $g_\varepsilon(t) = g_\varepsilon(t,x) := g(\frac{t}{\varepsilon}, x)$. Evidently the attractors of the equations (1) and (11) coincide. Note that we obtain the rapidly t-oscillating external force $g(\frac{t}{\varepsilon})$ in the right-hand side of (11) and consequently it is natural to introduce it's averaging (see [9])

$$\widehat{g}(x) := \lim_{T\to\infty} \frac{2}{T}\int_{-T}^{T} g(t,x)\,dt \tag{12}$$

and write the limit ($\varepsilon = 0$) equation in the following form:

$$\gamma\partial_t u = a\Delta_x u - f(u) - \widehat{g}, \quad u|_{t=0} = u_0 \tag{13}$$

The equation (13) has the form of autonomous dissipative reaction-diffusion equation and consequently (see e.g. [2], [12]) possesses a (global) attractor $\mathcal{A}^0$ in the phase space $L^2(\omega)$.

Theorem 0.1. *Let the above assumptions hold. Then the attractors* $\mathcal{A}^\varepsilon$ *converge to the attractor* $\mathcal{A}^0$ *in the spaces* $W^{1-\delta,2}(\omega)$ *for every* $\delta > 0$ *in the following sense:*

$$\lim_{\varepsilon\to 0} \operatorname{dist}_{W^{1-\delta,2}(\omega)}(\mathcal{A}^\varepsilon, \mathcal{A}^0) = 0 \tag{14}$$

where dist *means the non-symmetric Hausdorff distance between sets.*

Assume now that the limit attractor $\mathcal{A}^0$ is exponential, i.e.

$$\operatorname{dist}_{L^2(\omega)}(S_t B, \mathcal{A}^0) \le C(B)e^{-\nu t} \tag{15}$$

for every *bounded* subset $B \subset L^2(\omega)$. Here S_t is a semigroup generated by the autonomous equation (13), $\nu > 0$, and the constant $C(B)$ depends on $\|B\|_{L^2}$. It is known that (15) is true for generic $\widehat{g}$ at least if the equation (13) possesses a Lyapunov function, e.g. if $a = a^*$ and $f(u) = \nabla_u F(u)$ (see [2]).

Theorem 0.2. *Let the assumptions of previous theorem hold, let the limit attractor be exponential and let the almost-periodic function* $g(t) - \widehat{g}$ *have bounded primitive in* $L^2(\omega)$*, i.e.*

$$G(T) := \int_0^T (g(t) - \widehat{g})\,dt, \quad \|G(T)\|_{L^2(\omega)} \le C(g), \quad \forall T \ge 0 \tag{16}$$

Then the following estimate is valid:

$$\operatorname{dist}_{L^2(\omega)}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le C_g \varepsilon^\kappa \tag{17}$$

where $0 < \kappa < 1$ *and* C_g *can be calculated explicitly.*

Note that the assumption (16) is evidently valid for any *periodic* function g but may be not valid for more general almost periodic ones. Some sufficient conditions on g to satisfy this assumption are given at the end of Section 1.

Note also that the estimates (17) for differences between the regular attractors for semigroups which possesse global Lyapunov functions and depend regularly on a parameter ε have been obtained in [2].

These results have been recently extended in [5] to regular attractors of some autonomous reaction-diffusion equations with spatially oscillating coefficients (x/ε) and their homogenizations.

2 The estimates for solutions in a half-cylinder

In this Section we derive a number of estimates for the solutions of the auxiliary problem

$$a(\varepsilon^2\partial_t^2 u + \Delta_x u) - \gamma\partial_t u - f(u) = g_\varepsilon(t), \quad u|_{t=0} = u_0 \tag{18}$$

in a half-cylinder Ω_+ which are useful to study the behaviour of the attractors.

We start with the uniform with respect to $\varepsilon \to 0$ analogue of the estimate (6) for this equation.

Theorem 1.1 *Let the above assumptions hold. Then for every $\varepsilon < \varepsilon_0$ small enough the problem (18) possesses a unique solution which satisfies the following estimate:*

$$\|u, \Omega_T\|_{\Lambda^\varepsilon} \le Q(\|u_0\|_{V_0^\varepsilon})e^{-\alpha T} + Q(\|g\|_{L_b^2}) \tag{19}$$

where by definition

$$\|u, \Omega_T\|^2_{\Lambda^\varepsilon} := \varepsilon^4\|\partial_t^2 u, \Omega_T\|^2_{0,2} + \|\partial_t u, \Omega_T\|^2_{0,2} + \|u, \Omega_T\|^2_{2,2}, \tag{20}$$

$\|u_0\|^2_{V_0^\varepsilon} := \varepsilon\|u_0\|^2_{V_0} + \|u_0\|^2_{0,2}$ *and the monotonic function Q and the exponent α are independent of $\varepsilon < \varepsilon_0$.*

The uniform estimates of the form (19) are more or less known for the particular cases of equations (18) ([3], [15]) but the rigorous proof of (19) is rather technical so we omit it here (see [16] for details).

Remark 1.1 Note that if we put formally $g_0(t) := \widehat{g}$, where $\widehat{g}$ has been defined by (12) then taking formally $\varepsilon = 0$ in (20) we obtain well known estimate for the solutions of the limit parabolic equation (13). Note also that the norm in V_0^ε is a uniform with respect to ε norm in the trace space at $t = 0$ for $\Lambda_\varepsilon(\Omega_0)$, so the estimate (19) implies particularly that the solutions $u(t)$ are uniformly with respect to ε bounded in V_0^ε.

Our next task is to estimate the difference between the individual solutions of (18) and (13). These estimates are of fundamental significance for estimating the differences between the corresponding attractors which will be derived in the next Section.

Theorem 2.2 *Let the assumptions of previous Theorem hold. Assume in addition that the right-hand sides g satisfies (16)*

Let $u_\varepsilon(t)$ and $\widehat{u}(t)$ be the solutions of the problems (18) and (13) respectively such that $u_\varepsilon(0) = \widehat{u}(0) = u_0$. Then

$$\|u_\varepsilon(t) - \widehat{u}(t)\|_{0,2} \le C_1 \varepsilon^{1/2} e^{K_1 t} \tag{21}$$

where the constants C_1 and K_1 are independent of ε and uniform with respect to bounded in V_0^ε sets of u_0.

If the nonlinear term satisfies the additional regularity assumption

$$|f'(u)| \le C(1 + |u|^{4/(n-2)}) \tag{22}$$

and the primitive $G(T)$ is bounded not only in $L^2(\omega)$ but in $W_0^{1,2}(\omega)$, then the estimate (21) remains true with ε^1 instead of $\varepsilon^{1/2}$ in the right-hand side:

$$\|u_\varepsilon(t) - \widehat{u}(t)\|_{0,2} \le C\varepsilon e^{K_1 T} \tag{23}$$

The sketch of the proof. Denote $v(t) = u_\varepsilon(t) - \widehat{u}(t)$. Then this function evidently satisfies the equation

$$\gamma \partial_t v = a\Delta_x v - (f(u_\varepsilon) - f(\widehat{u})) - h_\varepsilon(t); \quad v|_{t=0} = 0 \tag{24}$$

where $h_\varepsilon(t) := \varepsilon^2 \partial_t^2 u_\varepsilon(t) + (g_\varepsilon(t) - \widehat{g})$. Multiplying this equation by $v(t)$ and integrating over $(t,x) \in [0,T] \times \omega$ we obtain using the monotonicity assumption $f' \ge -K$ and the positiveness of matrices γ and a that

$$\alpha(\|v(T)\|_{0,2}^2 + \int_0^T \|v(t)\|_{1,2}^2\, dt) \le K \int_0^T \|v(t)\|_{0,2}^2\, dt + \int_0^T (h_\varepsilon(t), v(t))dt \tag{25}$$

with the appropriate positive α. In order to apply the Gronewal inequality to (25) we should estimate only the last integral in the right-hand side of it. To this end we decompose it in a sum of two integrals:

$$I_1(T) := \varepsilon^2 \int_0^T (\partial_t^2 u_\varepsilon(t), v(t))dt; \quad I_2(T) := \int_0^T (g_\varepsilon(t) - \widehat{g}, v(t))dt$$

Let us introduce a function $G_\varepsilon(T) := \int_0^T (g_\varepsilon(t) - \widehat{g})dt$. Then, evidently $G_\varepsilon(T) = \varepsilon G(T/\varepsilon)$. Thus, (16) implies that

$$\|G_\varepsilon(T)\|_{0,2} \le C\varepsilon \tag{26}$$

and the assumptions of the second part of the theorem imply $\|G(T)\|_{1,2} \le C\varepsilon$.

Integrating by parts in I_1 and using the facts that $\partial_t u_\varepsilon$ and $\partial_t v$ is uniformly bounded with respect to ε in $L^2_b(\Omega_+)$ (according to Theorem 1.1) we derive that

$$I_1 = -\varepsilon^2 \int_0^T (\partial_t u_\varepsilon, \partial_t v)\, dt + \varepsilon^2 (\partial_t u_\varepsilon(T), v(T)) \le C_\mu T \varepsilon^2 + \mu \|v(T)\|^2_{0,2} \quad (27)$$

where $\mu > 0$ can be chosen arbitrary small. We have also used here the evident fact that $\varepsilon\|\partial_t u_\varepsilon(T)\|_{0,2}$ is uniformly bounded with respect to ε. Indeed, according to the standard interpolation inequality,

$$\varepsilon\|\partial_t u_\varepsilon(T)\|_{0,2} \le C\left(\varepsilon^2 \|\partial_t^2 u_\varepsilon\|_{L^2([T,T+1],L^2)}\right)^{1/2} \left(\|\partial_t u_\varepsilon\|_{L^2([T,T+1],L^2)}\right)^{1/2} \le C_1$$

Thus, it remains to estimate I_2. Integrating by parts again we obtain that

$$I_2 = -\int_0^T (G_\varepsilon(t), \partial_t v(t))\, dt + (G_\varepsilon(T), v(T)) \quad (28)$$

The second term can be easily estimated by Holder inequality and the assumption (26):

$$(G_\varepsilon(T), v(T)) \le C_\mu \varepsilon^2 + \mu\|v(T)\|^2_{0,2}$$

Estimating the first integral (we denote it by I_2^1) in the right-hand side of (28) by Holder inequality and using (26) and the fact that $\partial_t v$ is uniformly bounded in $L^2_b(\Omega_+)$ one can easlily derive the rough estimate $I_2^1 \le C\varepsilon T$ where in contrast to the previous estimates we have only ε^1 but not ε^2. This leads (after the inserting all obtained inequality to the right-hand side of (25) and applying the Gronewal inequality) to the rough estimate (21) with the rate of converging $\varepsilon^{1/2}$ instead of ε^1.

Our task now is to derive more sharp estimate for the integral I_2^1 under the assumptions of the second part of Theorem 1.2. To this end we express $\partial_t v$ from the equation (24) and insert it to I_2^1:

$$I_2^1 = -\int_0^T (G_\varepsilon(t), \gamma^{-1} a \Delta_x v(t)) dt + \int_0^T \left(G_\varepsilon(t), \gamma^{-1}(f(u_\varepsilon) - f(\widehat{u}))\right) dt +$$
$$+ \int_0^T (G_\varepsilon(t), \gamma^{-1} G'_\varepsilon(t))\, dt = J_1 + J_2 + J_3 \quad (29)$$

In order to estimate J_1 we integrate by part with respect to x and use the estimate $\|\nabla_x G_\varepsilon(t)\|_{0,2} \le C\varepsilon$ together with Holder inequality:

$$J_1 = \int_0^T (\nabla_x G_\varepsilon(t), \gamma^{-1} a \nabla_x v(t)) dt \le C_\mu \varepsilon^2 T + \mu \int_0^T \|v(t)\|^2_{1,2} dt \quad (30)$$

Here we essentially use the fact that G_ε equals zero on the boundary $\partial\omega$. (Without this assumption we would obtain the additional boundary terms after the integration by parts for which we cannot derive the good estimate).

In order to estimate J_2 we use the fact that, according to Theorem 1.1 the functions $u_\varepsilon(t)$ are uniformly with respect to ε bounded in $W^{1,2}(\omega)$, the growth assumption (22) and Sobolev embedding theorem. Indeed, since $f(u_\varepsilon) - f(\widehat{u}) = \int_0^1 f'(su_\varepsilon + (1-s)\widehat{u})\, dsv$ then applying Holder inequality with the exponents $p_1 = p_2 = 2n/(n-2)$ and $p_3 = n/2$ and the embedding $W^{1,2} \subset L^{p_1}$ we will have

$$J_2 \le C\int_0^T (1 + \|u_\varepsilon(t)\|_{0,p_1} + \|\widehat{u}(t)\|_{0,p_1})^{4/(n-2)}\|v(t)\|_{0,p_1}\|G_\varepsilon(t)\|_{0,p_1}\, dt \le$$

$$C_1\int_0^T (1 + \|u_\varepsilon(t)\|_{1,2} + \|\widehat{u}(t)\|_{1,2})^{4/(n-2)}\|v(t)\|_{1,2}\|G_\varepsilon(t)\|_{1,2}\, dt \le$$

$$\le C_2\varepsilon^2 T + \mu\int_0^T \|v(t)\|_{1,2}^2 dt \tag{31}$$

Note that the integral J_3 can be calculated explicitly:

$$J_3 = 1/2\left(G_\varepsilon(T), \gamma^{-1}G_\varepsilon(T)\right) \le C\varepsilon^2 \tag{32}$$

Inserting the estimates (29)–(32) in (28) we obtain that

$$I_2(T) \le C\varepsilon^2(1+T) + \mu\int_0^T \|v(t)\|_{1,2}^2\, dt + \mu\|v(T)\|_{0,2}^2 \tag{33}$$

Inserting the estimates (27) and (33) in (25), taking μ small enough and applying the Gronewal inequality we derive the estimate (23). Theorem 1.2 is proved.

Let us formulate now some sufficient conditions for almost periodic right-hand sides g to satisfy the assumptions of Theorem 1.2. To this end we recall (see e.g. [9] for details) that every almost periodic in $C_b(\mathbb{R}, L^2(\omega))$ function g possesses a Fourier expansion

$$g(t) = \sum_{k=-\infty}^{\infty} g_{\alpha_k}(x)e^{i\alpha_k t} \tag{34}$$

where $\{\alpha_k\} \subset \mathbb{R}$ is countable set of Fourier modes for f and the corresponding amplitudes $g_{\alpha_k} \in L^2(\omega)$ and satisfy the estimate

$$\sum_{k=-\infty}^{\infty} \|g_{\alpha_k}\|_{0,2}^2 < \infty \tag{35}$$

Moreover, $\widehat{g}(x) = g_0$ (Here we define $g_\alpha \equiv 0$ if $\alpha \notin \{\alpha_k\}$).

Proposition 1.1. *Let the Fourier amplitudes g_{α_k} of the almost periodic function g satisfy the assumption*

$$\sum_{\alpha_k \neq 0} \frac{1}{|\alpha_k|} \|g_{\alpha_k}\|_{0,2} < \infty \tag{36}$$

Then the function $G(T)$ satisfies the inequality (16). Analogously if

$$\sum_{\alpha_k \neq 0} \frac{1}{|\alpha_k|} \|g_{\alpha_k}\|_{1,2} < \infty \tag{37}$$

then $\|G(T)\|_{1,2} \leq C$.

Proof. Let us verify (16) using the Fourier expansion (34). Indeed, subtracting $\widehat{g} = g_0$ in (34) and integrating over t we derive that

$$G(t) = \sum_{\alpha_k \neq 0} g_{\alpha_k}(x) 1/(i\alpha_k) \left(e^{i\alpha_k t} - 1 \right) \tag{38}$$

Taking the L^2-norm from the both sides of (38) and using (36) we obtain (16). The second part of the proposition can be verified analogously. Proposition 1.1 is proved.

Corollary 1.1. *Let the assumptions of Theorem 1.1 hold and let the function g satisfies (36). Then the estimate (21) hold for the difference of the non-averaged $u_\varepsilon(t)$ and 'averaged' $\widehat{u}(t)$ solutions of (18). If in addition $G(t)$ belongs to $W_0^{1,2}(\omega)$ and the assumptions (22) and (37) hold then the improved estimate (23) is valid.*

Remark 1.2. Note that we have used the non-natural assumption that $G(t)|_{\partial\omega} = 0$ (which means that the oscillations decay near the boundary) in order to obtain the estimate (23) (with ε^1 instead $\varepsilon^{1/2}$). This supposition occured to be nonessential under the assumptions of Corollary 1.1 and can be removed (see [16]) so we use it only in order to simplify the proofs.

We conclude this Section by considering the case of so called quasiperiodic right-hand sides.

Example 1.1. Quasiperiodic functions. In this case (by definition) there exists a finite vector of frequencies $\beta = (\beta_1, \cdots, \beta_m) \in \mathbb{R}^m$, $m > 1$ such that $\alpha_k = (\beta, l(k)) := \sum_{i=1}^m \beta_i \cdot l(k)_i$ for the appropriate $l(k) \in \mathbb{Z}^m$ and β_i are rationally independent. Then (34) reads

$$g(t) = \sum_{l \in \mathbb{Z}^m} g_l e^{i(\beta,l)t} \tag{39}$$

Moreover, it is known that for every such $g \in C_b(\mathbb{R}, L^2(\omega))$ there exist a 2π-periodic with respect to every z_i, $i = 1, \cdots, m$ function $\Phi \in C_b(\mathbb{R}^m, L^2(\omega))$ such that

$$g(t) = \Phi(\beta_1 z_1, \cdots, \beta_m z_m); \quad \Phi(z, x) = \sum_{l \in \mathbb{Z}^m} g_l(x) e^{i(z,l)} \tag{40}$$

(In a fact (40) gives another equivalent definition of a quasiperiodic function).

In order to verify the condition (36) which reads in our case as $I := \sum_{l \in \mathbb{Z}^m, l \neq 0} \|g_l\|_{0,2} / |(\beta, l)| < \infty$ we recall that due to the theory of Diophantine approximations for every $\delta > 0$ and for almost every $\beta \in \mathbb{R}^m$ (with respect to Lebesgue measure) the following estimate is valid:

$$|(\beta, l)| \geq C_\beta |l|^{-m-\delta}, \quad l \neq 0 \tag{41}$$

Assume that β is chosen in such a way that (41) hold. Then the sum I can be estimated by

$$I \leq C \sum_{l \in \mathbb{Z}^m} |l|^{m+\delta} \|g_l\|_{0,2} \leq C(\sum_{l \neq 0} |l|^{2(m+\delta-\alpha)})^{1/2} (\sum_{l \neq 0} |l|^{2\alpha} \|g_l\|_{0,2}^2)^{1/2} \tag{42}$$

Note that the first integral in (42) is finite if $2(m + \delta - \alpha) < -m$, i.e. $\alpha > 3m/2 + \delta$ and the second one is finite for every g such that the function Φ from the representation (40) belonging to $C_b^\alpha(\mathbb{R}^m, L^2(\omega))$.

Thus, for every β satisfying (41) and every periodic $\Phi \in C_b^\alpha(\mathbb{R}^m, L^2(\omega))$ with $\alpha > \delta + 3m/2$ the function (40) satisfies the assumption (36).

3 The attractors.

In this Section we give a sketch of the proof of Theorems 0.1 and 0.2 which is based on the estimates obtained in the previous Section.

We start with the proof of Theorem 0.1. Indeed, it follows form the estimate (19) of Theorem 1.1 that the attractors $\mathcal{A}^\varepsilon$ are uniformly bounded in in the norms of V_0^ε and particularly in the norm of $W_0^{1,2}(\omega)$:

$$\|\mathcal{A}^\varepsilon\|_{1,2} \leq C, \quad \varepsilon < \varepsilon_0 \tag{43}$$

So in order to prove the upper semi-continuity (14) it is sufficient to verify that if $u_{\varepsilon_n} \in \mathcal{A}^{\varepsilon_n}$ and $u_{\varepsilon_n} \to u_0$ when $\varepsilon_n \to 0$ weakly in $W_0^{1,2}$ then $u_0 \in \mathcal{A}^0$. Then we will have the upper semicontinuity in a weak toplogy of $W^{1,2}(\omega)$

and consequently (due to the compactness of the embedding $W^{1-\delta,2} \subset W^{1,2}$) – the upper semicontinuity in the spaces $W^{1-\delta,2}(\omega)$.

According to the attractor's structure theorem (see (10)), there exists a sequence $\xi_n \in H(g)$ and a complete bounded solutions $u_{\varepsilon_n}(t)$ $t \in \mathbb{R}$ of the equations

$$a(\varepsilon_n^2 \partial_t^2 u_{\varepsilon_n} + \Delta_x u_{\varepsilon_n}) - \gamma \partial_t u_{\varepsilon_n} - f(u_{\varepsilon_n}) = \xi_n(t/\varepsilon_n) \tag{44}$$

such that $u_{\varepsilon_n} = u_{\varepsilon_n}(0)$. Our task now is to pass to the limit $n \to \infty$ in (44). To this end we recall, that according to Theorem 1.1, $u_{\varepsilon_n}(t)$ are uniformly bounded in $\Lambda_\varepsilon(\Omega_T)$ for every $T \in \mathbb{R}$. Thus, passing to a subsequence if necessary, we may assume that for every $T \in \mathbb{R}$,

$$\partial_t u_{\varepsilon_n}(t) \rightharpoonup \partial_t \widehat{u}(t) \ in \ L^2(\Omega_T), \ \ and \ \ u_{\varepsilon_n}(t) \rightharpoonup \widehat{u}(t) \ in \ W^{2,2}(\Omega_T) \tag{45}$$

and the limit function $\widehat{u}$ is bounded with respect to t, i.e $\partial_t \widehat{u}, \Delta_x \widehat{u} \in L_b^2(\Omega)$ and $\widehat{u}(0) = u_0$. So, it remains to prove that the function $\widehat{u}(t)$ satisfies the limit equation (13). Then (10) will imply that $u_0 \in \mathcal{A}^0$.

Note that the convergence (45) together with a growth restrictions (3) admits to pass to the limit in the left-hand side of (44) in a standard way (see e.g. [2]). The passing to the limit in the right hand side of this equation is based on the following lemma.

Lemma 2.1 Let $g \in C_b(\mathbb{R}, L^2(\omega))$ be almost periodic with a hull $H(g)$. Let also $\xi_n \in H(g)$ and $\varepsilon_n \to 0$ when $n \to \infty$. Then for every $T \in \mathbb{R}$

$$\xi_n(t/\varepsilon_n) \rightharpoonup \widehat{g} \tag{46}$$

weakly in $L^2(\Omega_T)$.

Indeed, since the almost periodic flow is strictly ergodic then it follows from the Birkghoff–Hinchin ergodic theorem (see e.g. [7]) that the L^2-limit

$$\widehat{g} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \xi(t)\, dt \tag{47}$$

is uniform with respect to $\xi \in H(g)$. The assertion of the lemma is a simple corollary of this fact (see [16] for details). Lemma 2.1 is proved. Theorem 0.1 is proved.

Proof of Theorem 0.2. In a fact the result of this Theorem is a simple corollary of Theorem 1.2. Indeed, assume that (16) is valid for the initial right-hand side g. Then it is not difficult to verify that it is valid uniformly with respect to $\xi \in H(g)$ and consequently the estimate (21) is also valid uniformly with respect to $\xi \in H(g)$. Namely, let $u_{\varepsilon,\xi}(t)$ be a solution of the equation (18) with the right hand side $\xi(t/\varepsilon)$, $\xi \in H(g)$ and let $\widehat{u}(t)$ be the corresponding solution ($u_{\varepsilon,\xi}(0) = \widehat{u}(0) = u_0$) of the limit problem (13). Then

$$\|u_{\varepsilon,\xi}(t) - \widehat{u}(t)\|_{0,2} \leq C\varepsilon^{1/2}e^{K_1 t} \tag{48}$$

uniformly with respect $\xi \in H(g)$ and bounded in V_0^ε sets of initial data u_0.

Assume now that $\phi \in \mathcal{A}^\varepsilon$. According to the attractor's structure theorem there exists a complete bounded trajectory $u_\varepsilon(t)$, $t \in \mathbb{R}$ of the equation (18) with the right-hand side $\xi \in H(g)$. Let us fix an arbitrary $T > 0$ and consider the trajectory $\widehat{u}(t)$ of the limit equation such that $\widehat{u}(0) = u_\varepsilon(-T)$. Then (since $\mathcal{A}^\varepsilon$ are uniformly bounded in V_0^ε) (48) implies that

$$\|\phi - \widehat{u}(T)\|_{0,2} \leq C\varepsilon^{1/2}e^{K_1 T} \tag{49}$$

From the other side since $\mathcal{A}^0$ is exponential then

$$\text{dist}_{L^2(\omega)}(\widehat{u}(T), \mathcal{A}^0) \leq C_1 e^{-\nu T} \tag{50}$$

Combining (49) and (50) we deduce that

$$\text{dist}_{L^2(\omega)}(\phi, \mathcal{A}^0) \leq C_1 e^{-\nu T} + C\varepsilon^{1/2}e^{K_1 T} \tag{51}$$

Taking the optimal value for T (solving the equation $C_1 e^{-\nu T} = C\varepsilon^{1/2}e^{K_1 T}$) in the estimate (51) we will have

$$\text{dist}_{L^2(\omega)}(\phi, \mathcal{A}^0) \leq C_2\varepsilon^\kappa, \quad \kappa = \frac{\nu}{2(K_1 + \nu)} \tag{52}$$

Since $\phi \in \mathcal{A}^\varepsilon$ is arbitrary then (52) proves Theorem 0.2.

Remark 2.1. The assumption (16) can be weakened in the following way:

$$\|G(T)\|_{0,2} \leq CT^{1-\beta}, \quad \beta > 0 \tag{53}$$

Then $\|G_\varepsilon(T)\|_{0,2} \leq C\varepsilon^\beta T^{1-\beta}$ and arguing as in the proof of Theorem 1.2 and 0.2 we derive the estimate (17) with $\kappa = \frac{\beta\nu}{2(K_1+\nu)}$. Note that (53) looks not restrictive because this estimate with $\beta = 0$: $\|G(T)\|_{0,2} \leq CT$ is evidently valid for every almost periodic function g.

The main results of the paper are obtained jointly with Prof. M.I. Vishik.

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