

GLOBAL ATTRACTOR AND STABILIZATION FOR A COUPLED PDE-ODE SYSTEM

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ABSTRACT. We study the asymptotic behavior of solutions of one coupled PDE-ODE system arising in mathematical biology as a model for the development of a forest ecosystem.

In the case where the ODE-component of the system is monotone, we establish the existence of a smooth global attractor of finite Hausdorff and fractal dimension.

The case of the non-monotone ODE-component is much more complicated. In particular, the set of equilibria becomes non-compact here and contains a huge number of essentially discontinuous solutions. Nevertheless, we prove the stabilization of any trajectory to a single equilibrium if the coupling constant is small enough.

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1. INTRODUCTION

We study the following coupled ODE-PDE system

$$(1.1) \quad \begin{cases} \partial_t^2 v + \varphi(v)\partial_t v + f(v) = \alpha w \\ \partial_t w - \Delta_x w + w = v, \quad \partial_n w|_{\partial\Omega} = 0 \end{cases}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$. Here $(v, w) = (v(t, x), w(t, x))$ are unknown functions, Δ_x is a Laplacian with respect to x , $\alpha > 0$ is a given parameter and f and φ are given nonlinearities which are assumed to satisfy some natural assumptions formulated in Section 2.

Our interest to that problem is motivated by the following system arising in the mathematical biology:

$$(1.2) \quad \begin{cases} \partial_t u = \beta\delta w - \gamma(v)u - fu, \\ \partial_t v = fu - hv, \\ \partial_t w - d\Delta_x w + \beta w = \alpha v, \quad \partial_n w|_{\partial\Omega} = 0 \end{cases}$$

where $\alpha, \beta, \delta, d, f, h$ are given positive parameters and $\gamma(v)$ is a given nonlinearity. This system has been introduced in [13] in order to describe the development of a forest ecosystem and has been recently studied analytically and numerically in [5, 6, 15]. Expressing $u = (\partial_t v + hv)/f$

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from the second equation of (1.2) and insert it into the first one, we end up with the system of the type (1.1) with respect to the variables (v, w) .

The main aim of the present paper is to study the long-time behavior of solutions of (1.1) using the ideas and methods of the attractors theory. From the mathematical point of view, the problem considered is a coupled system of a second order ODE with a linear PDE (heat-like equation). Heuristically, it is clear that the dynamics of such coupled dissipative systems should depend drastically on the monotonicity properties of the ODE component. In the case where this ODE is "monotone", i.e., it cannot produce the internal instability (and all of the instability is driven by the coupling with the PDE component), one expects the asymptotic compactness and the existence of a smooth finite-dimensional global attractor with "good" properties. In contrast to that, in the non-monotone case, the ODE-instability may produce the asymptotic discontinuities and even may completely destroy the initially smooth spatial profile. Thus, in that case, the smoothing effect from the PDE component is not strong enough in order to suppress the development of discontinuities provided by the internal instabilities of the ODE-component and, as a result, an extremely complicated (in a sense, pathological) dynamics may appear. We also mention that, although the existence and uniqueness of a solution of (1.2) has been rigorously proved in the above mentioned papers [5, 6], very few has been done concerning the asymptotic behavior of solutions as $t \rightarrow \infty$. To be more precise, different types of ω -limit sets of a single trajectory were considered there (associated with the different choice of the topology in the phase space) and their simplest properties were formulated, but the even the question whether or not they are empty remained open. As we will see below, some of them are indeed empty for the most part of the trajectories if the monotonicity assumption is essentially violated, see Remark 4.5

In the present paper, we justify the above heuristics in a mathematically rigorous way on the example of the ODE-PDE coupled system (1.1). In particular, we show (in Section 3) that the monotonicity arguments work perfectly if

$$(1.3) \quad f'(v) \geq \kappa_0 > 0, \quad v \in \mathbb{R}.$$

In this case, problem (1.1) possesses indeed a smooth global attractor \mathcal{A} of finite fractal dimension in the proper phase space Φ_∞ . Moreover, due to the presence of a global Lyapunov function, this attractor generically has very nice properties (it is the so-called regular attractor in the terminology of Babin and Vishik [2]). Namely, it is a finite collection of the finite-dimensional unstable manifolds associated with the equilibria:

$$\mathcal{A} = \cup_{u_0 \in \mathcal{R}} \mathcal{M}_{u_0}^+$$

(where \mathcal{R} denotes a (generically finite) set of equilibria of problem (1.1) and $\mathcal{M}_{u_0}^+$ is an unstable manifold associated with the equilibrium $u_0 \in \mathcal{R}$, see Section 3). Moreover, every trajectory of (1.1) converges *exponentially* to one of that equilibria. We also mention that the first equation of (1.1) is a *second* order ODE and, therefore, the monotonicity of f does not automatically imply the absence of the internal instability. For instance, the ODE

$$y'' + \varphi(y)y' + f(y) = h(t)$$

may produce the non-trivial dynamics even if f is monotone and φ is strictly positive, say for the case of a given time-periodic external force h . By this reason, our proof of the monotonicity of the ODE component is based on rather delicate arguments related with the existence of the global Lyapunov function and associated dissipative integrals, see Section 3.

The case where the monotonicity assumption is violated (which is considered in Section 4) occurs to be (as predicted by the heuristics) much more complicated. In contrast to the monotone case, there is a very few hope to develop a reasonable global attractor theory here (no matter in a strong or weak topology of the phase space), since, as a rule, even the equilibria set \mathcal{R} is already not compact in the strong topology of the phase space and not closed in the weak

topology. In addition, we see indeed a huge (uncountable) number of well-separated essentially discontinuous equilibria here, see Section 4.

Nevertheless, in the particular case of small coupling constant α , we succeed to give a complete description of the equilibria set $\mathcal{R} = \mathcal{R}_\alpha$ and verify that every trajectory of (1.1) converges as $t \rightarrow \infty$ to one of that equilibria. We mention that the standard Lojasiewicz technique for proving the stabilization seems to be non-applicable here even in the case of analytic non-linearities, since the equilibria set is not compact in any reasonable topology. By this reason, we develop a new method of proving the stabilization, based on the theory of non-autonomous perturbations of regular attractors, see Appendix. It is also worth to mention that, a posteriori, our stabilization result is somehow close to [12] where the pointwise stabilization for the non-smooth temperature driven phase separation model is proved (although our methods seem to be completely different).

The paper is organized as follows. Section 2 is devoted to the study of the analytical properties of problem (1.1) such as existence and uniqueness, dissipative estimates in different norm, etc. The case of monotone nonlinearity f is considered in Section 3, in particular, the existence of smooth regular attractor is proved here. In Section 4 we deal with the non-monotone case and, in particular, prove here the above mentioned stabilization result for the weakly coupled case. Finally, the Appendix is devoted to the derivation of the key estimate for our stabilization method which is, in turns, based on the perturbation theory of regular attractors.

To conclude, we mention that it would be interesting to consider the regularization of (1.1) in a spirit of a damped wave equation with displacement depending damping (see [8, 16, 17]):

$$(1.4) \quad \begin{cases} \partial_t^2 v + \varphi(v)\partial_t v + f(v) - \varepsilon \Delta_x v = \alpha w \\ \partial_t w - \Delta_x w + w = v \end{cases}$$

with $0 < \varepsilon \ll 1$. We return to that problem somewhere else.

2. A PRIORI ESTIMATES, EXISTENCE AND UNIQUENESS

We consider the following coupled system of a second order ODE with a heat equation:

$$(2.1) \quad \begin{cases} \partial_t^2 v + \varphi(v)\partial_t v + f(v) = \alpha w, & v(0) = v_0, \quad \partial_t v(0) = v'_0, \\ \partial_t w - \Delta_x w + w = v, & \partial_n w|_{\partial\Omega} = 0, \quad w|_{t=0} = w_0 \end{cases}$$

in a bounded 3D domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary. Here, $(v, w) = (v(t, x), w(t, x))$ are unknown functions, Δ_x is a Laplacian with respect to the variable x , $\alpha > 0$ is a given constant and φ and f are given nonlinearities, which satisfy the following assumptions:

$$(2.2) \quad \begin{cases} 1. \quad \varphi, f \in C^2(\mathbb{R}), \\ 2. \quad \varphi(v) \geq \beta_0 > 0, \\ 3. \quad f(v)v \geq -C + \gamma_0|v|^{2+\delta}, \\ 4. \quad f'(v) \geq -K \end{cases}$$

for some positive constants C, K, β_0, δ and γ_0 .

Finally, we assume that the initial data (v_0, v'_0, w_0) is taken from the $L^\infty(\Omega)$:

$$(2.3) \quad (v_0, v'_0, w_0) \in \Phi_\infty := [L^\infty(\Omega)]^2 \times [L^\infty(\Omega) \cap H^1(\Omega)].$$

The aim of that section is to establish a number of basic a priori estimates for that system which will allow us to verify the existence and uniqueness of a solution and to study its behavior as $t \rightarrow \infty$. We start with the following lemma which gives the global Lyapunov function for that problem.

Proposition 2.1. *Let the above assumptions hold and let $(v(t), w(t))$ be a sufficiently regular solution of problem (2.1). Introduce a functional*

$$(2.4) \quad \mathcal{L}(v, w) := \|\partial_t v\|_{L^2}^2 + 2(F(v), 1) - 2\alpha(v, w) + \alpha\|\nabla_x w\|^2 + \alpha\|w\|_{L^2}^2$$

where $F(v) := \int_0^v f(s) ds$ and (\cdot, \cdot) is used for the inner product in L^2 . Then, the following equality holds

$$(2.5) \quad \frac{d}{dt} \mathcal{L}(v(t), w(t)) = -2(\varphi(v(t)) \partial_t v(t), \partial_t v(t)) - \alpha \|\partial_t w(t)\|_{L^2}^2$$

Indeed, multiplying the first and the second equations of (2.1) by $\partial_t v$ and $\alpha \partial_t w$ respectively, taking a sum and integrating over Ω , we arrive at (2.5).

Corollary 2.2. *Let the above assumptions hold and let $(v(t), w(t))$ be a solution of (2.1). Then, the following estimate holds:*

$$(2.6) \quad \|\partial_t v(t)\|_{L^2} + \|v(t)\|_{L^2} + \|w(t)\|_{H^1} \leq Q(\|(v, w)\|_{\Phi_\infty})$$

for some monotone function Q independent of t and the solution

Indeed, according due to our dissipativity assumption (2.2).3,

$$(2.7) \quad F(u) \geq -C_1 + \gamma_1 |u|^{2+\delta}$$

for some new constants C_1 and γ_1 . Using this inequality, we easily check that

$$(2.8) \quad \gamma_1 (\|\partial_t v\|_{L^2}^2 + \|v\|_{L^2}^2 + \|w\|_{H^1}^2) - C_2 \leq \mathcal{L}(v, w) \leq Q(\|(v, w)\|_{\Phi_\infty})$$

for some constants $\gamma_1, C_2 > 0$ and some monotone function Q . Integrating now equation (2.5) by t and using that $\varphi(v) \geq 0$ and $\alpha > 0$, we arrive at (2.6).

The next corollary gives the L^2 -dissipation integral for that problem.

Corollary 2.3. *Let the above assumptions hold. Then,*

$$(2.9) \quad \int_0^\infty \|\partial_t v(t)\|_{L^2}^2 + \|\partial_t w(t)\|_{L^2}^2 dt \leq Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty})$$

for some monotone function Q .

Indeed, this estimate is an immediate corollary of (2.5), (2.8) and the assumption that $\varphi(v) \geq \gamma_0 > 0$.

We are now going to verify that the solution is globally bounded in Φ_∞ .

Proposition 2.4. *Let the above assumptions hold. Then, the following estimate is valid:*

$$(2.10) \quad \|v(t)\|_{L^\infty} + \|\partial_t v(t)\|_{L^\infty} + \|w(t)\|_{L^\infty \cap H^1} \leq Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty})$$

for some monotone function Q independent of t and the solution.

Proof. We first establish the L^∞ -bound for the w -component. Indeed, according to Corollary 2.2, the right-hand side v of the second equation of (2.1) is bounded in $L^\infty(\mathbb{R}_+, L^2(\Omega))$. Consequently, the standard regularity result for the heat equation gives

$$(2.11) \quad \|w(t)\|_{L^\infty} \leq C \|w(0)\|_{L^\infty} e^{-t} + C \|v\|_{L^\infty(\mathbb{R}_+, L^2)} \leq Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty}).$$

(here we have implicitly used the restriction on the space dimension).

Thus, we only need to establish the L^∞ -bounds for the v -component. To this end, we will use the L^∞ -bounds for the w -component obtained before and will consider the equation for the v -component as an ODE for every (almost every, being a pedant) fixed $x \in \Omega$. Indeed, let $y(t) := v(t, x_0)$. Then, this function solves

$$(2.12) \quad y''(t) + \varphi(y)y' + f(y) = h(t) = h_{w, x_0}(t) := \alpha w(t, x_0).$$

Multiplying this equation by $y' + \varepsilon y$, we have

$$(2.13) \quad [(y')^2 + 2F(y) + 2\varepsilon y y' + 2\varepsilon R(y)]' + 2(\varphi(y) - \varepsilon)(y')^2 + 2\varepsilon f(y)y = 2h(y' + \varepsilon y)$$

where $R(y) := \int_0^y \varphi(s)s ds$. Using that $\varphi(v)$ is strictly positive and f is dissipative, we deduce from this equation that, for sufficiently small $\varepsilon > 0$

$$(2.14) \quad \frac{d}{dt} S(y, y') + \gamma((y')^2 + y^2) \leq C|h(t)|^2$$

where

$$S(y, y') := (y')^2 + 2F(y) + 2\varepsilon yy' + 2\varepsilon R(y).$$

and γ is positive. Moreover,

$$\varepsilon_0(y^2 + (y')^2) - C \leq S(y, y') \leq Q(y^2 + (y')^2)$$

for some positive ε_0 and C and monotone Q . Applying the Gronwall lemma to inequality (2.14), we conclude that

$$(2.15) \quad y(t)^2 + (y'(t))^2 \leq Q(y(0)^2 + y'(0)^2) + \|h\|_{L^\infty(\mathbb{R}_+)}^2$$

for some monotone function Q which is independent of t and y , see eg, [3]. Taking now the supremum with respect to all $x_0 \in \Omega$ and using (2.11) for estimating h , we deduce estimate (2.10) and finish the proof of the proposition. \square

We are now ready to verify the existence and uniqueness of a solution for the problem (2.1).

Definition 2.5. A pair of functions $(v(t), w(t))$ is a solution of problem (2.1) if

$$(v(t), \partial_t v(t), w(t)) \in \Phi_\infty$$

for every $t \geq 0$ and (2.1) is satisfied in the sense of distributions.

Note that, from the first equation of (2.1), we see that $\partial_t v(t) \in L^\infty(\Omega)$. Therefore $v(t) \in W^{2,\infty}([0, T], L^\infty(\Omega))$ and the initial data for v is well-defined. Analogously, the w -component is continuous as a function with values, say, in $L^2(\Omega)$ and the initial data is again well-defined.

Theorem 2.6. *Let the above assumptions hold. Then, for every $(v_0, v'_0, w_0) \in \Phi_\infty$, problem (2.1) possesses a unique solution in the sense of Definition 2.5 and this solution satisfies estimate (2.10). Moreover, any two solutions $(v_1(t), w_1(t))$ and $(v_2(t), w_2(t))$ satisfy the following estimate:*

$$(2.16) \quad \begin{aligned} \|(v_1(t), \partial_t v_1(t), w_1(t)) - (v_2(t), \partial_t v_2(t), w_2(t))\|_{\Phi_\infty} &\leq \\ &\leq C e^{Kt} \|(v_1(0), \partial_t v_1(0), w_1(0)) - (v_2(0), \partial_t v_2(0), w_2(0))\|_{\Phi_\infty} \end{aligned}$$

where positive constants C and K depend only on the norms of the initial data.

Proof. Let us first verify the uniqueness and estimate (2.16). Indeed, let

$$(v(t), w(t)) := (v_1(t), w_1(t)) - (v_2(t), w_2(t)).$$

Then, these functions solve

$$(2.17) \quad \begin{cases} \partial_t^2 v + \varphi(v_1) \partial_t v + [\varphi(v_1) - \varphi(v_2)] \partial_t v_2 + [f(v_1) - f(v_2)] = \alpha w, \\ \partial_t w - \Delta_x w + w = v \end{cases}$$

Multiplying now the first equation of (2.17) by $\partial_t v + \varepsilon v$, $\varepsilon > 0$ is a small positive number, using the fact that $v_i, \partial_t v_i$ are globally bounded in L^∞ and applying the Gronwall inequality in a standard way (without integration by $x!$), we conclude that

$$(2.18) \quad \begin{aligned} \|\partial_t v(t)\|_{L^\infty(\Omega)}^2 + \|v(t)\|_{L^\infty}^2 &\leq \\ &\leq C e^{Kt} (\|\partial_t v(0)\|_{L^\infty(\Omega)}^2 + \|v(0)\|_{L^\infty(\Omega)}^2) + C \int_0^t e^{K(t-s)} \|w(s)\|_{L^\infty(\Omega)}^2 ds \end{aligned}$$

for some positive constants C and K depending only on the L^∞ -norms of v_i and $\partial_t v_i$. Furthermore, due to the maximum principle for the heat equation, we have the estimate

$$(2.19) \quad \|w(t)\|_{L^\infty} \leq e^{-t} \|w(0)\|_{L^\infty} + \int_0^t e^{-(t-s)} \|v(s)\|_{L^\infty} ds$$

Inserting this estimate into the right-hand side of (2.18), we arrive at

$$(2.20) \quad \|\partial_t v(t)\|_{L^\infty(\Omega)}^2 + \|v(t)\|_{L^\infty}^2 \leq \\ \leq C' e^{Kt} (\|\partial_t v(0)\|_{L^\infty(\Omega)}^2 + \|v(0)\|_{L^\infty(\Omega)}^2 + \|w(0)\|_{L^\infty}^2) + C' \int_0^t e^{K(t-s)} \|v(s)\|_{L^\infty(\Omega)}^2 ds$$

Applying again the Gronwall inequality to that relation, we conclude that

$$(2.21) \quad \|\partial_t v(t)\|_{L^\infty(\Omega)}^2 + \|v(t)\|_{L^\infty}^2 \leq C' e^{2Kt} (\|\partial_t v(0)\|_{L^\infty(\Omega)}^2 + \|v(0)\|_{L^\infty(\Omega)}^2 + \|w(0)\|_{L^\infty}^2)$$

This estimate, together with (2.19), give the desired L^∞ -estimate for the triple $(v, \partial_t v, w)$. In order to finish the proof of estimate (2.16), it remains to note that the desired estimate H^1 -norm of the w -component is immediate, since the L^∞ -control for the right-hand side of the heat equation for w is already obtained. Thus, the uniqueness and Lipschitz continuity (2.16) are proved.

So, we only need to prove the existence of a solution. It can be done in a standard way, based on a priori estimate (2.10), using the Banach fixed point theorem for proving the existence of a local solution and estimate (2.10) for extending this solution globally in time, see eg, [11] for the details. \square

Our next aim is to establish the basic dissipative estimate in the phase space Φ_∞ .

Theorem 2.7. *Let the above assumptions hold. Then, a solution $(v(t), w(t))$ of problem (2.1) satisfies the following dissipative estimate:*

$$(2.22) \quad \|(v(t), v'(t), w(t))\|_{\Phi_\infty} \leq Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty}) e^{-\beta t} + C_*$$

for some positive constants β and C_* and monotone function Q .

Proof. As we see from the proof of the previous proposition, the only problem is to obtain a *dissipative* estimate for the L^2 -norm of $v(t)$. Indeed, if this estimate is obtained, analyzing the equation for the w -component analogously to (2.11), we deduce the *dissipative* estimate for the L^∞ -norm of $w(t)$. This, in turns, gives the dissipative estimate for the right-hand side $h(t)$ of (2.12) and the Gronwall lemma applied to inequality (2.14) will finish the derivation of estimate (2.15).

So, we only need to obtain the dissipative estimate for the L^2 -norm. To this end, we multiply the first equation of (2.1) by $2(\partial_t v + \varepsilon v)$, $\varepsilon > 0$ is a small number, which will be fixed below, and *integrate* over $x \in \Omega$, after that we multiply the second equation of (2.1) by $2\alpha(\partial_t w + \varepsilon w)$, integrate over $x \in \Omega$ and take a sum of these two equations. Then, after the standard transformations, we end up with

$$(2.23) \quad \frac{d}{dt} Z(t) + 2\alpha \|\partial_t w\|^2 + 2((\varphi(v) - \varepsilon)\partial_t v, \partial_t v) + 2\varepsilon f(v).v = 2\alpha\varepsilon(v, w)$$

where

$$(2.24) \quad Z(t) := \|\partial_t v(t)\|_{L^2}^2 + 2(F(v(t)), 1) + 2\varepsilon(R(v(t)), 1) + 2\varepsilon(v(t), \partial_t v(t)) - \\ - 2\alpha(v(t), w(t)) - \alpha(\|\nabla_x w(t)\|_{L^2}^2 + (1 - \varepsilon)\|w(t)\|_{L^2}^2).$$

We now fix $\varepsilon > 0$ so small that

$$\varepsilon|(R(v(t)), 1)| \leq 1$$

(it is possible to do due to estimate (2.10), of course, ε will depend on the norm of the initial data). Then, due to (2.7), we have

$$(2.25) \quad \beta_2 [\|\partial_t v\|_{L^2}^2 + \|\nabla_x w\|_{L^2}^2 + \|w\|_{L^2}^2 + (|F(v)|, 1)] - C_2 \leq Z(t) \leq \\ \leq \beta_1 [\|\partial_t v\|_{L^2}^2 + \|\nabla_x w\|_{L^2}^2 + \|w\|_{L^2}^2 + (|F(v)|, 1)] + C_1$$

where the positive constants C_i and β_i are independent of $\varepsilon \rightarrow 0$ and (v, w) . Moreover, due to the fourth assumption of (2.2),

$$(2.26) \quad F(v) \leq f(v) \cdot v + Kv^2/2$$

Inserting estimates (2.25) and (2.26) into (2.20) and using again the third assumption of (2.2), we deduce the differential inequality:

$$(2.27) \quad \partial_t Z(t) + \beta\varepsilon Z(t) \leq C\varepsilon$$

where ε depends on the norm of the initial data, but the positive constants β and C are independent of v and w . Integrating this inequality, we arrive at

$$(2.28) \quad Z(t) \leq [Z(0) - \frac{C}{\beta}]e^{-\beta\varepsilon t} + \frac{C}{\beta}.$$

We see that, although the rate of convergence to the absorbing ball depends on the initial data (through the choice of $\varepsilon > 0$), the radius of the absorbing ball is *independent* of ε and, consequently, is independent of the norm of the initial data. This observation, together with estimate (2.25) implies that

$$(2.29) \quad \|\partial_t v(t)\|_{L^2} + \|v(t)\|_{L^2} + \|w(t)\|_{H^1} \leq Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty})e^{-\gamma t} + C_*$$

for some positive γ and C and a monotone function Q which are independent of t , v and w . Thus, the desired dissipative estimate in L^2 is obtained and Theorem 2.7 is proved. \square

We now formulate several auxiliary results on the smoothing property for the w -component and the existence of dissipative integrals in stronger norms which will be essentially used in the next sections.

Proposition 2.8. *Let the assumptions of Theorem (2.6) hold. Then, $w(t) \in W^{2,p}(\Omega)$ and $\partial_t w(t) \in W^{2,p}(\Omega)$ for any $t > 0$ and any $p < \infty$ and the following estimate is valid:*

$$(2.30) \quad \|w(t)\|_{W^{2,p}(\Omega)} + \|\partial_t w(t)\|_{W^{2,p}(\Omega)} \leq (1 + t^{-N})Q_p(\|(v_0, v'_0, w_0)\|_{\Phi_\infty})$$

for some positive exponent N and some monotone function Q (depending only on p).

Proof. Indeed, due to the smoothing property of the heat equation, the solution $\theta(t)$ of

$$(2.31) \quad \partial_t \theta - \Delta_x \theta + \theta = 0, \quad \theta|_{t=0} = w_0$$

satisfies the following estimate

$$(2.32) \quad \|\theta(t)\|_{W^{2,p}(\Omega)} + \|\partial_t \theta(t)\|_{W^{2,p}(\Omega)} \leq C_p t^{-N} \|w_0\|_{H^1}$$

for some exponent N and positive constant C_p depending only on p , see eg, [14]. The remainder $z(t) := w(t) - \theta(t)$ solve the heat equation with zero initial data

$$\partial_t z - \Delta_x z + z = v(t), \quad z|_{t=0} = 0.$$

Moreover, using estimate (2.10) together with the first equation of (2.1), we conclude that

$$(2.33) \quad \|v(t)\|_{L^\infty(\Omega)} + \|\partial_t v(t)\|_{L^\infty(\Omega)} + \|\partial_t^2 v(t)\|_{L^\infty(\Omega)} \leq Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty}).$$

Using that estimate together with the $W^{2,p}$ -regularity estimate for the heat equation, we arrive at

$$\|\theta(t)\|_{W^{2,p}(\Omega)} + \|\partial_t w(t)\|_{W^{2,p}(\Omega)} \leq Q_p(\|(v_0, v'_0, w_0)\|_{\Phi_\infty})$$

which, together with estimate (2.32), finishes the proof of the proposition. \square

Proposition 2.9. *Let the assumptions of Theorem 2.6 hold. Then, the following stronger version of dissipative integrals exist:*

$$(2.34) \quad \begin{cases} 1) & \int_1^\infty \|\partial_t w(t)\|_{C(\Omega)}^2 dt \leq Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty}) \\ 2) & \int_0^T |\partial_t^2 v(t, x_0)|^2 + |\partial_t v(t, x_0)|^2 dt \leq \varepsilon T + C_\varepsilon Q(\|(v_0, v'_0, w_0)\|_{\Phi_\infty}) \end{cases}$$

where $\varepsilon > 0$ is arbitrary, $x_0 \in \Omega$ is almost arbitrary, $C_\varepsilon > 0$ depends only on ε and Q is some monotone function.

Proof. We first note that, due to Lemma 2.8, we may assume without loss of generality that $\partial_t w(0) \in H^1$. Differentiating the equation for the w -component by t and denoting $z := \partial_t w$, we get

$$(2.35) \quad \partial_t z - \Delta_x z + z = \partial_t v, \quad z|_{t=0} = \partial_t v(0)$$

Applying the L^2 -regularity theorem for that heat equation, we will have

$$(2.36) \quad \int_0^T \|\Delta_x z(s)\|_{H^2}^2 ds \leq C \|z(0)\|_{H^1}^2 + C \int_0^T T \|\partial_t v(s)\|_{L^2}^2 ds$$

where the constant C is independent of T . Together with estimate (2.9) and embedding $H^2 \subset L^\infty$, it gives the desired first estimate of (2.34). Let us now prove the second estimate of (2.34). To this end, we multiply equation (2.12) by $2y'$ (without integration by $t!$). This gives

$$(2.37) \quad ((y')^2 + 2F(y) - 2\alpha v y)' + 2\varphi(y)(y')^2 = -2\alpha y \partial_t w(t, x_0)$$

Integrating this equality over $t \in [0, T]$, estimating

$$|2\alpha y \partial_t w(t, x_0)| \leq C \|v(t)\|_{L^\infty} \|\partial_t w(t)\|_{L^\infty} \leq \varepsilon + C_\varepsilon \|\partial_t w(t)\|_{L^\infty}^2,$$

and using the first estimate of (2.34) together with the strict positivity of φ , we deduce the second estimate of (2.34) and finish the proof of the proposition. \square

We conclude this section by showing that, if the initial data (v_0, v'_0, w_0) is smooth, the solution $(v(t), w(t))$ remains smooth for all t .

Proposition 2.10. *Let the assumptions of Theorem (2.6) hold. Assume, in addition, that*

$$(2.38) \quad (v_0, v'_0, w_0) \in W^{1,\infty}(\Omega).$$

Then, the solution $(v(t), \partial_t v(t), w(t)) \in W^{1,\infty}(\Omega)$ for any $t \geq 0$ and the following estimate holds:

$$(2.39) \quad \|v(t)\|_{W^{1,\infty}} + \|\partial_t v(t)\|_{W^{1,\infty}} + \|w(t)\|_{W^{1,\infty}} \leq C \|(v_0, v'_0, w_0)\|_{[W^{1,\infty}]^3} e^{Kt}$$

for some positive constants C and K (which depend on the L^∞ -norms of the initial data).

Proof. The desired estimate for the w -component is factually obtained in Proposition 2.8, thus, we only need to estimate the v component. Let x_1 and x_2 be two arbitrary points of Ω and let $z(t) := v(t, x_1) - v(t, x_2)$. Then, this function satisfies the following ODE

$$(2.40) \quad z''(t) + \varphi(v(t, x_1))z'(t) + [\varphi(v(t, x_1)) - \varphi(v(t, x_2))] \partial_t v(t, x_2) + \\ + [f(v(t, x_1)) - f(v(t, x_2))] = h_{x_1, x_2}(t) := \alpha(w(t, x_1) - w(t, x_2)).$$

Multiplying this equation by $\partial_t z(t)$ and arguing exactly as in (2.18), we arrive at

$$(2.41) \quad |z(t)|^2 + |z'(t)|^2 \leq C(|z(0)|^2 + |z'(0)|^2) e^{Kt} + C \int_0^t e^{K(t-s)} |h_{x_1, x_2}(s)|^2 ds$$

where the constants C and K depends on the L^∞ -norm of the solution. Furthermore, since the $W^{1,\infty}$ -estimate for the w -component is already obtained, we have

$$\sup_{x_1, x_2 \in \Omega} \frac{1}{|x_1 - x_2|^2} \int_0^t e^{K(t-s)} |h_{x_1, x_2}(s)|^2 ds \leq C_1 \int_0^t e^{K(t-s)} \|w(s)\|_{W^{1,\infty}}^2 ds \leq C e^{Kt}.$$

Dividing finally inequality (2.41) by $|x_1 - x_2|^2$ and taking the supremum over $x_1, x_2 \in \Omega$ from the both parts of the obtained inequality, we obtain the desired estimate for the $W^{1,\infty}$ -norms of v and $\partial_t v$ and finish the proof of the proposition. \square

Remark 2.11. Arguing analogously, one can show that if the initial data belong to C^k , the solution will be of class C^k for every $t \geq 0$. Thus, the blow up in finite time of the higher norms cannot occur. However, there is a principal difference between estimates (2.22) for the L^∞ -norm and estimate (2.39) for the $W^{1,\infty}$ -norm of the solution. Indeed, the first estimate is dissipative and shows that the L^∞ -norm of the solution cannot grow and even gives the absorbing ball in that norm. In contrast to that, the $W^{1,\infty}$ -norm, a priori, may grow exponentially and, in this sense, the solution may become "less and less regular" as $t \rightarrow \infty$ (i.e., it may tend to a discontinuous limit). As we will see in the next sections, the answer on the question whether or not it really happens depends in a crucial way on the monotonicity of the nonlinearity f .

3. THE MONOTONE CASE: ASYMPTOTIC COMPACTNESS AND REGULAR ATTRACTOR

According to the results of the previous section, equation (2.1) is uniquely solvable in the phase space Φ_∞ and the solution operators

$$(3.1) \quad S(t)(v_0, v'_0, w_0) := (v(t), \partial_t v(t), w(t))$$

generate a dissipative semigroup in Φ_∞ . The aim of this section is to study the long-time behavior of solutions as $t \rightarrow \infty$ in the particular case where the nonlinearity f is strictly monotone:

$$(3.2) \quad f'(v) \geq \kappa_0 > 0.$$

As we will see, in that case, the associated semigroup is asymptotically compact and possesses a smooth global attractor \mathcal{A} in Φ_∞ . Moreover, due to the Lyapunov functional, this attractor can be described as a finite union of finite-dimensional unstable manifolds. Our proof of the asymptotic compactness is based on the following lemma which can be considered as a refinement of estimate (2.39).

Lemma 3.1. *Let the assumptions of Theorem 2.1 hold and let, in addition, (3.2) be satisfied. Let us also introduce, for any $h > 0$ the following (semi)norm on the space $L^\infty(\Omega)$:*

$$(3.3) \quad \|v\|_{W_h^{1,\infty}} := \sup_{x_1, x_2 \in \Omega, |x_1 - x_2| \geq h} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|}$$

(being pedants, we would write esssup instead of sup). Then, every solution $(v(t), w(t))$ of problem (2.1) satisfies

$$(3.4) \quad \|v(t)\|_{W_h^{1,\infty}} + \|\partial_t v(t)\|_{W_h^{1,\infty}} \leq \frac{C_1}{h} e^{-\beta t} + C_2$$

where the positive constants β, C_i depend on the L^∞ -norms of the initial data, but are independent of t and $h \rightarrow 0$.

Proof. Analogously to the proof of Proposition 2.10, we introduce a function $z(t) := v(t, x_1) - v(t, x_2)$ which solves equation (2.40). But, using the monotonicity assumption (3.2) and the dissipation integrals (2.34), we are now able to suppress the exponential divergence in estimate (2.41). To this end, we multiply equation (2.40) by z' and transform the term containing the nonlinearity f as follows:

$$(3.5) \quad [f(v(t, x_1)) - f(v(t, x_2))]z' = 1/2[l(t)z^2]' - 1/2l'(t)z^2$$

where $l(t) := \int_0^1 f'(sv(t, x_1) + (1-s)v(t, x_2)) ds \geq \kappa_0 > 0$ and its derivative can be estimated as follows:

$$(3.6) \quad |l'(t)| \leq C(|\partial_t v(t, x_1)| + |\partial_t v(t, x_2)|)$$

where the constant C depends on the L^∞ -norm of the initial data, but is independent of t and x_i . Then, using the positivity of φ and the L^∞ -bounds for v , we get

$$1/2((z')^2 + lz^2)' + \gamma(z')^2 \leq C|h_{x_1, x_2}|^2 + C(|\partial_t v(t, x_1)| + |\partial_t v(t, x_2)|)z^2$$

for some positive constants γ and C . Multiplying now equation (2.40) by εz (where $\varepsilon > 0$ is a sufficiently small positive number) and taking a sum with the above inequality, we infer

$$(3.7) \quad 1/2((z')^2 + lz^2 + 2\varepsilon z z')' + (\gamma - \varepsilon)(z')^2 + \varepsilon l z^2 \leq C|h_{x_1, x_2}|^2 + C(|\partial_t v(t, x_1)| + |\partial_t v(t, x_2)|)(z^2 + (z')^2)$$

Let now

$$\mathcal{L}_z(t) := (z')^2 + lz^2 + 2\varepsilon z z'.$$

Then, since $l(t) \geq \kappa_0 > 0$, we may fix $\varepsilon > 0$ to be small enough that

$$(3.8) \quad \kappa(z^2 + (z')^2) \leq \mathcal{L}_z(t) \leq \kappa_1(z^2 + (z')^2)$$

for some positive κ and κ_1 . This, inequality, together with the evident estimate $|x| \leq \beta + \beta^{-1}x^2$ allows to transform (3.7) to

$$(3.9) \quad \frac{d}{dt}\mathcal{L}_z(t) + (\gamma - C(|\partial_t v(t, x_1)|^2 + |\partial_t v(t, x_2)|^2))\mathcal{L}_z(t) \leq C|h_{x_1, x_2}(t)|^2.$$

Applying the Gronwall inequality to this relation and using the dissipation integral (2.34) together with the bounds (3.8), we finally deduce the non-divergent analogue of estimate (2.41):

$$(3.10) \quad |z(t)|^2 + |z'(t)|^2 \leq C(|z(0)|^2 + |z'(0)|^2)e^{-\gamma t} + C \int_0^t e^{-\gamma(t-s)} |h_{x_1, x_2}(s)|^2 ds$$

for some positive C and γ depending only on the L^∞ -norm of the solution.

In order to deduce the desired estimate (3.3) from (3.10), we note that, due to Proposition 2.8, we may assume without loss of generality that $\|w(t)\|_{W^{1, \infty}} \leq C$ for all $t \geq 0$ and, consequently,

$$\sup_{|x_1 - x_2| \geq h} \frac{1}{|x_1 - x_2|^2} \int_0^t e^{-\gamma(t-s)} |h_{x_1, x_2}(s)|^2 ds \leq \alpha \int_0^t e^{-\gamma(t-s)} \|w(s)\|_{W^{1, \infty}}^2 ds \leq C_1$$

Moreover, obviously,

$$\|v\|_{W_h^{1, \infty}} \leq \frac{2\|v\|_{L^\infty}}{h}.$$

Dividing now inequality (3.10) by $|x_1 - x_2|^2$ and taking the supremum over all $x_i \in \Omega$, $|x_1 - x_2| \geq h$, we deduce the desired estimate (3.3). Lemma 3.1 is proved. \square

Our next step is to verify the existence of a global attractor \mathcal{A} for semigroup (3.1) associated with problem (2.1). We recall that, by definition, the global attractor \mathcal{A} should satisfy the following properties:

- 1) \mathcal{A} is compact in Φ_∞ ;
- 2) \mathcal{A} is strictly invariant: $S(t)\mathcal{A} = \mathcal{A}$;
- 3) \mathcal{A} attracts the images of all bounded sets as $t \rightarrow \infty$, i.e., for any bounded set $B \subset \Phi_\infty$ and any neighbourhood $\mathcal{O}(\mathcal{A})$ of \mathcal{A} in Φ_∞ , there exists time $T = T(B, \mathcal{A})$ such that

$$S(t)B \subset \mathcal{O}(\mathcal{A}), \quad \text{for } t \geq T.$$

We also recall that the attraction property can be also reformulated in terms of the non-symmetric Hausdorff distance between sets in Φ_∞ :

$$(3.11) \quad \lim_{t \rightarrow \infty} \text{dist}(S(t)B, \mathcal{A}) = 0,$$

see eg, [2] for the details.

Theorem 3.2. *Let the assumptions of Lemma 3.1 hold. Then, the semigroup $S(t)$ associated with problem (2.1) possesses a global attractor \mathcal{A} in the phase space Φ_∞ . This attractor is bounded in $[W^{1,\infty}(\Omega)]^3$ and has the following structure:*

$$(3.12) \quad \mathcal{A} = \mathcal{K}|_{t=0}$$

where $\mathcal{K} \subset C_b(\mathbb{R}, \Phi_\infty)$ is a set of all solutions of problem (2.1) which are defined for all $t \in \mathbb{R}$ and are globally bounded.

Proof. In order to deduce the existence of a global attractor from Lemma 3.1, we will use the so-called Kuratowski measure of non-compactness. Recall that, by definition, the Kuratowski measure of non-compactness $\alpha(B)$ of a set B is an infimum of all $r > 0$ for which it can be covered by a finite number of r -balls, see e.g., [10] for details. To be more precise, we need the following lemma.

Lemma 3.3. *Let*

$$B := \{v \in L^\infty(\Omega), \|v\|_{L^\infty} + \|v\|_{W_h^{1,\infty}} \leq R\}$$

for some R and $h > 0$. Then, its Kuratowski measure of non-compactness of the set B can be estimated as follows:

$$(3.13) \quad \alpha(B) \leq Rh$$

Proof. Let \mathcal{S}_h be the standard averaging operator

$$(\mathcal{S}_h v)(x) := \int_{\mathbb{R}^3} D_h(x, y) v(y) dy$$

where the smooth non-negative kernels $D_h(x, z)$ are such that

$$(3.14) \quad \begin{cases} 1. & \text{supp } D_h(x, \cdot) \subset \{z \in \Omega, |z - x| \leq h\} \\ 2. & \int_{\mathbb{R}^3} D_h(x, y) dy \equiv 1, \\ 3. & |D_h(x, y)| + |\nabla_x D_h(x, y)| \leq C_h, \quad x, y \in \mathbb{R}^3 \end{cases}$$

(since Ω is assumed to be smooth, such kernels exist).

Let also $B_h := \mathcal{S}_h(B)$. Then, on the one hand, the set B_h consists of smooth functions and, in particular, is bounded in $C^1(\Omega)$. By the Arzela-Ascoli theorem, it means that B_h is compact in $L^\infty(\Omega)$.

On the other hand,

$$|(\mathcal{S}_h v)(x) - v(x)| \leq \int_{\mathbb{R}^3} D_h(x, y) |v(y) - v(x)| dy \leq \|v\|_{W_h^{1,\infty}} h \int_{\mathbb{R}^3} D_h(x, y) dy \leq Rh.$$

Thus, $B \subset B_h + Rh$ and B_h is compact. This gives estimate (3.13) and finishes the proof of the lemma. \square

We are now ready to finish the proof of the theorem. Indeed, due to Proposition 2.8, we know that the w -component is bounded in $W^{2,p}(\Omega)$ for every $t > 0$ and, consequently, the w -component of $S(t)B$ is precompact in $L^\infty \cap H^1$ for any bounded set B . So, the Kuratowski measure of non-compactness for $S(t)B$ is determined by the v -component only. Moreover, Lemma 3.1 guarantees, that

$$v(t), \partial_t v(t) \subset \{u \in L^\infty(\Omega), \|u\|_{L^\infty} + \|u\|_{W_h^{1,\infty}} \leq R\}$$

if $t \geq T(h)$ is large enough (but R is independent of h). This, gives that

$$(3.15) \quad \lim_{t \rightarrow \infty} \alpha(S(t)B) = 0$$

for any bounded set B .

Since the semigroup $S(t)$ is Lipschitz continuous with respect to the initial data (see Theorem 2.6) and dissipative (see Theorem 2.7), the convergence of the Kuratowski measure (3.15) to

zero implies the asymptotic compactness of the semigroup $S(t)$ and the existence of a global attractor \mathcal{A} , see [10]. The structure (3.12) of the attractor is also a corollary of that abstract theorem and the fact that \mathcal{A} is bounded in $W^{1,\infty}$ follows from estimate (3.3) (together with the fact that the constant C_2 is independent of h). Thus, Theorem 3.2 is proved. \square

Our next task is to establish the regular structure of the attractor \mathcal{A} provided by the Lyapunov functional. To this end, we need to make some preparations. As a first step, we establish the differentiability of the semigroup $S(t)$ with respect to the initial data.

Proposition 3.4. *Let the assumptions of Theorem 2.6 hold. Then, the associated semigroup $S(t)$ is Frechet differentiable with respect to the initial data for every fixed t and it's Frechet derivative $D_\xi[S(t)\xi] \in \mathcal{L}(\Phi_\infty, \Phi_\infty)$ is Lipschitz continuous with respect to the initial data $\xi \in \Phi_\infty$ and the following estimate holds for every bounded set $B \subset \Phi_\infty$:*

$$(3.16) \quad \|S(t)\|_{C^{1,1}(B, \Phi_\infty)} \leq Ce^{2Kt}$$

where the constants C and K depend only on the radius of B .

The proof of this proposition is straightforward and standard, so, in order to avoid the technicalities, we rest it to the reader.

At the next step, we need to study the equilibria of problem (2.1).

Proposition 3.5. *Let the assumptions of Theorem 3.2 hold. Then, any equilibrium $(v_0, w_0) \in \mathcal{R}$ (the set of all equilibria) of problem (2.1) solves the following semilinear elliptic equation:*

$$(3.17) \quad -\Delta_x w + w = f^{-1}(\alpha w), \quad \partial_n|_{\partial\Omega} = 0, \quad v = (-\Delta_x + 1)w$$

(f^{-1} exists since f is now assumed to be monotone). Moreover, the equilibrium (v_0, w_0) is hyperbolic if and only if w_0 is hyperbolic as a solution of (3.17), i.e., if the equation

$$(3.18) \quad -\Delta_x \theta + \theta = [f^{-1}]'(\alpha w_0)\alpha\theta$$

has only trivial solution $\theta = 0$. In particular, for generic f , all of the equilibria $(v_0, w_0) \in \mathcal{R}$ are hyperbolic and \mathcal{R} is finite.

Proof. Indeed, the equations on equilibria for problem (2.1)

$$f(v) = \alpha w, \quad -\Delta_x w + w = v$$

are equivalent to (3.17). Let us verify the assertion on hyperbolicity. Indeed, the asymptotic compactness of the semigroup $S(t)$ implies in a standard way that the essential spectrum of the operator $D_\xi S(1)$ lies strictly inside of the unit circle. Thus, only eigenvalues of finite multiplicity are possible on the unit circle. Any such eigenvalue generates a *time-periodic* solution (z, θ) of the associated equation of variations

$$(3.19) \quad \begin{cases} \partial_t^2 z + \varphi(v_0)\partial_t z + f'(v_0)z = \alpha\theta \\ \partial_t \theta - \Delta_x \theta + \theta = z \end{cases}$$

However, analogously to the nonlinear problem (2.1), the linearized problem (3.19) possesses a global Lyapunov function (in order to find it, one needs to multiply the first and the second equations by $\partial_t z$ and $\alpha\partial_t \theta$ respectively, take a sum and integrate over Ω). Thus, every periodic solution of that linearized problem must be an equilibrium:

$$f'(v_0)z = \alpha\theta, \quad z = -\Delta_x \theta + \theta$$

and, consequently, z must solve (3.18). Vise versa, any nontrivial solution z of (3.18) generates a non-trivial equilibrium of (3.19) by setting $z = -\Delta_x \theta + \theta$.

Finally, the last assertion that generically \mathcal{R} is finite and all of the equilibria are hyperbolic is a standard corollary of the Sard theorem, see eg, [2]. Proposition 3.5 is proved. \square

Thus, we will assume from now on that all of the equilibria $(v_0, w_0) \in \mathcal{R}$ are hyperbolic (which automatically implies that \mathcal{R} is finite). Furthermore, for any $\xi_0 := (v_0, w_0) \in \mathcal{R}$, we define the associated unstable set $\mathcal{M}_{\xi_0}^+$ by the usual expression

$$(3.20) \quad \mathcal{M}_{\xi_0}^+ := \{(v_0, v'_0, w_0) \in \Phi_\infty, \exists(v(t), w(t)) \text{ which solves (2.1) for } t \leq 0 \text{ such that} \\ (v(0), \partial_t v(0), w(0)) = (v_0, v'_0, w_0) \text{ and } \lim_{t \rightarrow -\infty} (v(t), \partial_t v(t), w(t)) = (v_0, v'_0, w_0)\}$$

in other words, the unstable set $\mathcal{M}_{\xi_0}^+$ consists of all complete trajectories of (2.1) which stabilize to ξ_0 as $t \rightarrow -\infty$.

It is well known (see e.g., [2]) that, for hyperbolic equilibrium $\xi_0 \in \mathcal{R}$, the set $\mathcal{M}_{\xi_0}^+$ is *locally* (near ξ_0) a finite-dimensional submanifold of Φ_∞ and its dimension equals to the instability index of ξ_0 . But, in order to prove that the whole $\mathcal{M}_{\xi_0}^+$ is a submanifold of Φ_∞ , one needs the semigroup $S(t)$ to be injective, see again [2].

Proposition 3.6. *Let the assumptions of Theorem 2.6 hold. Then, the semigroup $S(t)$ associated with equation (2.1) is injective, i.e., the equality $S(T)\xi_1 = S(T)\xi_2$, for some $T > 0$, implies that $\xi_1 = \xi_2$.*

Proof. Indeed, let $(v_1(t), w_1(t))$ and $(v_2(t), w_2(t))$ be two solutions of problem (2.1) and let $(z(t), w(t))$ be their difference. Then, these functions solve (2.17). Let us rewrite this equation in the form

$$(3.21) \quad \partial_t \xi + \mathcal{B}\xi = \mathcal{P}(t)\xi$$

where $\xi(t) := (z(t), \partial_t z(t), \theta(t))$,

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\Delta_x + 1 \end{pmatrix}, \quad \mathcal{P}(t) := \begin{pmatrix} 1 & 1 & 0 \\ -[l_\varphi(t)\partial_t v_2(t) + l_f(t)] & 1 - \varphi(v_1(t)) & \alpha \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$l_\varphi(t) := \int_0^1 \varphi'(sv_1(t) + (1-s)v_2(t)) ds, \quad l_f(t) := \int_0^1 \varphi'(sv_1(t) + (1-s)v_2(t)) ds.$$

Let us consider equation (3.21) in a Hilbert space $H := [L^2(\Omega)]^3$. Then, obviously, \mathcal{B} is a positive self-adjoint (unbounded) operator in H and the operator $\mathcal{P}(t)$ is uniformly bounded for all $t \geq 0$. Then, the classical backward uniqueness theorem of Agmon and Nirenberg (see [1]) is applicable here and, consequently, $\xi(T) = 0$ implies that $\xi(0) = 0$. Proposition 3.6 is proved. \square

Corollary 3.7. *Let the assumptions of Theorem 3.2 hold. Then, for any hyperbolic equilibrium $\xi_0 := (v_0, w_0)$ of problem (2.1), the associated unstable manifold $\mathcal{M}_{\xi_0}^+$ is a finite-dimensional C^1 -submanifold of Φ_∞ diffeomorphic to \mathbb{R}^N , where N is the instability index of the equilibrium ξ_0 .*

This result is a standard corollary of the existence of a Lyapunov function, injectivity and smoothness of the semigroup $S(t)$, see [2].

We are now ready to formulate a theorem on the regular structure of the attractor \mathcal{A} which can be considered as the main result of the section.

Theorem 3.8. *Let the assumptions of Theorem 3.2 hold and let, in addition, all of the equilibria $\xi_0 \in \mathcal{R}$ be hyperbolic. Then,*

1) *any non-equilibrium complete trajectory $\xi(t)$, $t \in \mathbb{R}$ of the semigroup $S(t)$ belonging to the attractor is a heteroclinic orbit between two different equilibria ξ_- and ξ_+ :*

$$(3.22) \quad \lim_{t \rightarrow \pm\infty} \xi(t) = \xi_\pm,$$

where $\xi_- \neq \xi_+$;

2) The attractor \mathcal{A} is a finite union of finite-dimensional submanifolds of Φ_∞ :

$$(3.23) \quad \mathcal{A} = \cup_{\xi_0 \in \mathcal{R}} \mathcal{M}_{\xi_0}^+$$

where $\mathcal{M}_{\xi_0}^+$ is an unstable manifold of $\xi_0 \in \mathcal{R}$;

3) The rate of attraction to \mathcal{A} is exponential, i.e., for any bounded set $B \subset \Phi_\infty$,

$$(3.24) \quad \text{dist}(S(t)B, \mathcal{A}) \leq Q(\|B\|_{\Phi_\infty})e^{-\gamma t}$$

for some monotone function Q and positive constant γ .

Indeed, this theorem follows from the abstract result of Babin and Vishik on regular attractors, see [2] (see also [18]) and Propositions 2.1, 3.4 and 3.6.

Remark 3.9. Theorem 3.8 shows that the long-time behavior of solutions of problem (2.1) is "extremely good" if the monotonicity assumption (3.2) holds. As we will see in the next section, this behavior is "extremely bad" if the monotonicity assumption is essentially violated.

4. THE NON-MONOTONE f : STABILIZATION FOR THE CASE OF WEAK COUPLING

The aim of this section is to understand how the asymptotic behavior of (2.1) may look like when the monotonicity assumption (3.2) is violated. To this end, we will consider below the case of small coupling constant α , where the dynamics is, in a sense, determined by the limit ODE

$$(4.1) \quad y''(t) + \varphi(y(t))y'(t) + f(y(t)) = 0.$$

In particular, in that limit case, the value of $v(t, x)$ at $x = x_0$ is determined by the value of $(v_0(x), v'_0(x))$ at $x = x_0$ only and, consequently $v(t, x)$ and $v(t, y)$ evolve independently if $x \neq y$. Thus, if (4.1) has more than one equilibrium, the most part of trajectories will tend to a *discontinuous* equilibria, no matter whether or not the initial data (v_0, v'_0) is continuous. As we will see, the same property preserves for the case of small positive coupling constant α .

To be more precise, we assume that the limit equation (4.1) possesses a regular attractor in \mathbb{R}^2 , i.e., that

$$(4.2) \quad f'(u_0) \neq 0, \quad \text{for all } u_0 \text{ such that } f(u_0) = 0$$

(since the existence of a global Lyapunov function and dissipativity are immediate, only the hyperbolicity assumption on the equilibria should be postulated).

We start our exposition by verifying that the dissipative estimate (2.22) is uniform with respect to $\alpha \rightarrow 0$.

Proposition 4.1. *Let the assumptions of Theorem 2.7 hold. Then the positive constants β and C_* and monotone function Q in the dissipative estimate (2.22) are independent of $\alpha \rightarrow 0$.*

Proof. In order to verify this assertion, we need to check that the most part of estimates of Section 2 are uniform with respect to $\alpha \rightarrow 0$. We start with estimate (2.6). From the first point of view (based on the form of the global Lyapunov function (2.5)), one may guess that it is non-uniform and only $\alpha \|w(t)\|_{L^2}^2$ is uniformly bounded. However, the Lyapunov function gives the uniform bound for the L^2 -norm of the v -component. The standard L^2 -estimate for second equation

$$(4.3) \quad \partial_t w - \Delta_x w + w = v(t)$$

of (2.1) gives after that the uniform L^2 and L^∞ -bounds for the w -component.

Thus, the uniformity as $\alpha \rightarrow 0$ is verified for estimates (2.6) and (2.11). The uniformity of the L^∞ -bound (2.10) follows from (2.11) exactly as in Proposition 2.4.

So, it only remains to check the dissipative estimate (2.22) and, following the proof of Theorem 2.7, we see that only the uniformity of the L^2 -estimate (2.29) is necessary.

Analogously to (2.6) the function $Z(t)$ (defined by (2.24)) can be estimated as follows

$$(4.4) \quad \beta_2 [\|\partial_t v\|_{L^2}^2 + \alpha(\|\nabla_x w\|_{L^2}^2 + \|w\|_{L^2}^2) + (|F(v)|, 1)] - C_2 \leq Z(t) \leq \\ \leq \beta_1 [\|\partial_t v\|_{L^2}^2 + \alpha(\|\nabla_x w\|_{L^2}^2 + \|w\|_{L^2}^2) + (|F(v)|, 1)] + C_1$$

where β_i and C_i are now uniform with respect to $\alpha \rightarrow 0$. By this reason, estimates (2.27) and (2.28) do not give immediately the uniform analogue of (2.29), but only the uniform dissipative estimate for the L^2 -norm of $v(t)$. Combining after that this estimate with the usual L^2 -estimate for the heat equation (4.3), we verify that estimate (2.29) is indeed uniform as $\alpha \rightarrow 0$. Exactly as in Theorem 2.7, this gives the uniformity of estimate (2.22) and finishes the proof of the proposition. \square

Thus, due to the previous proposition, the radius of the absorbing ball in Φ_∞ for problem (2.1) is uniform with respect to $\alpha \rightarrow 0$. In particular, the Φ_∞ -norm of any equilibria of that problem is uniformly bounded. Denoting the set of equilibria for problem (2.1) by \mathcal{R}_α , we may conclude that

$$(4.5) \quad \|\mathcal{R}_\alpha\|_{\Phi_\infty} \leq C$$

where the constant C is independent of α .

This observation together with the hyperbolicity assumption (4.2) allow to give a complete description of the equilibria set \mathcal{R}_α if $\alpha > 0$ is small enough.

Proposition 4.2. *Let the assumptions of Theorem 2.6 hold and let, in addition, the limit hyperbolicity assumption (4.2) be valid. Denote these hyperbolic equilibria by $\{u_1, \dots, u_N\}$. Then, there exists $\alpha_0 > 0$ such that, for every $\alpha \leq \alpha_0$ and every partition*

$$(4.6) \quad \Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_N$$

on disjoint measurable sets: $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, there exists a unique equilibrium $(v_0, 0, w_0) \in \Phi_\infty$ of problem (2.1) such that

$$(4.7) \quad v_0 = \tilde{v}_0 + \theta, \quad \tilde{v}_0(x) := \sum_{i=1}^N u_i \chi_{\Omega_i}(x), \quad \|\theta\|_{L^\infty} \leq C\alpha$$

where $\chi_V(x)$ is a characteristic function of the set V and the constant C is independent of α . Moreover, every equilibrium $(v_0, 0, w_0) \in \Phi_\infty$ can be presented in such form.

Proof. Indeed, in order to find the equilibrium, we need to solve

$$f(v_0) = \alpha w_0, \quad \Delta_x w_0 - w_0 = v_0$$

which we rewrite in the form of a single equation on v_0 in $L^\infty(\Omega)$:

$$(4.8) \quad f(v_0) = \alpha(-\Delta_x + 1)^{-1}v_0.$$

We note that the function

$$F(v, \alpha) := f(v) - \alpha(-\Delta_x + 1)^{-1}v$$

belongs to $C^2(L^\infty(\Omega) \times \mathbb{R}, L^\infty(\Omega))$. Moreover, its derivative

$$D_v F(\tilde{v}_0, 0) := f'(\tilde{v}_0)$$

is invertible in L^∞ (due to the hyperbolicity assumption (4.2) and the norm of the inverse operator is uniformly bounded with respect to the choice of a partition. In addition, $F(\tilde{v}_0, 0) = 0$. Thus, the existence and uniqueness of the equilibrium v_0 in a small neighbourhood of \tilde{v}_0 if α is small follows from the implicit function theorem.

Let us now verify that any equilibrium (v_0, w_0) can be presented in that form. Indeed, let $(v_0, 0, w_0) \in \Phi_\infty$ be an arbitrary equilibrium. Then, according to (4.5), $\|w_0\|_{L^\infty} \leq C$ where C is independent of α . Therefore,

$$|f(v_0(x))| \leq C\alpha, \quad x \in \Omega$$

Since all of the roots $f(z) = 0$ are hyperbolic, for sufficiently small α , we conclude from here that

$$(4.9) \quad v_0(x) \in \mathcal{O}_{C\alpha}(u_{k(x)}), \quad x \in \Omega$$

for some root $u_{k(x)}$ of $f(z) = 0$. Fixing now

$$(4.10) \quad \Omega_i := \{x \in \Omega, k(x) = i\}$$

we see that the equilibrium (v_0, w_0) indeed has the form of (4.7) and the proposition is proved. \square

Remark 4.3. We see that, in contrast to the case of monotone f , we now have the *uncountable* number of different equilibria (all of them are hyperbolic in Φ_∞) most of which are *discontinuous* (we have only finite number of continuous equilibria associated with trivial partitions of Ω). Moreover, using the explicit description given in the previous proposition, it is not difficult to show that the set \mathcal{R}_α is not compact in the strong topology of the space Φ_∞ and is not closed in the weak-star topology of that space. By this reason, the possibility to apply the strong/weak global attractor theory to that problem seems very problematic. However, as the next theorem shows, any trajectory $(v(t), w(t))$ still converges to one of the equilibrium from \mathcal{R}_α as $t \rightarrow \infty$.

Theorem 4.4. *Let the assumptions of Proposition 4.2 hold. Then, there exists $\alpha_0 > 0$ such that, for every $\alpha \leq \alpha_0$ every trajectory $(v(t), \partial_t v(t), w(t))$ of problem (2.1) stabilizes as $t \rightarrow \infty$ to some equilibrium $(\bar{v}, 0, \bar{w}) \in \mathcal{R}_\alpha$ in the topology of $L^p(\Omega)$:*

$$(4.11) \quad \lim_{t \rightarrow \infty} \|(v(t), \partial_t v(t), w(t)) - (\bar{v}, 0, \bar{w})\|_{[L^p(\Omega)]^3} = 0$$

for any $1 \leq p < \infty$.

Proof. The proof of that convergence is strongly based on the perturbation theory of regular attractors and Proposition 5.1 (see Appendix). Indeed, due to Propositions 4.1 and 2.8, we may assume without loss of generality that $(v(0), \partial_t v(0), w(0))$ belongs to the absorbing ball B_R in Φ_∞ with the radius R independent of α and that

$$(4.12) \quad \|w\|_{C_b(\mathbb{R}_+ \times \Omega)} + \|\partial_t w\|_{C_b(\mathbb{R}_+ \times \Omega)} \leq C$$

where the constant C is also independent of α . Thus, the first equation of (2.1)

$$(4.13) \quad \partial_t^2 v(t, x) + \varphi(v(t, x))\partial_t v(t, x) + f(v(t, x)) = \alpha w(t, x)$$

can be treated as an ODE for every fixed $x \in \Omega$. Moreover, due to the hyperbolicity assumption (4.2) and uniform estimate (4.12) the right-hand side of (4.13) can be treated as small non-autonomous perturbation of the ODE

$$(4.14) \quad u'' + \varphi(u)u' + f(u) = 0.$$

Thus, the assumptions of Proposition 5.1 hold for problem (4.13) for every fixed $x \in \Omega$ if $\alpha \leq \alpha_0$ for sufficiently small positive α_0 . Due to this Proposition, we have the estimate:

$$(4.15) \quad \int_0^T |\partial_t^2 v(t, x)| + |\partial_t v(t, x)| dt \leq C_1 + C_2 \alpha \int_0^T |\partial_t w(t, x)| dt$$

where the positive constants C_1 and C_2 are independent of T , α and $x \in \Omega$. Integrating this inequality by $x \in \Omega$, we arrive at

$$(4.16) \quad \int_0^T \|\partial_t^2 v(t)\|_{L^1} + \|\partial_t v(t)\|_{L^1} dt \leq C_1 |\Omega| + C_2 \alpha \int_0^T \|\partial_t w(t)\|_{L^1} dt.$$

In order to estimate the integral into the right-hand side of (4.16), we differentiate the second equation of (2.1) by t , denote $\theta := \partial_t w$, multiply it by $\text{sgn } \theta(t)$ and integrate over Ω . Then, due to the Kato inequality, we arrive at

$$(4.17) \quad \partial_t \|\theta(t)\|_{L^1} + \|\theta\|_{L^1} \leq \|\partial_t v(t)\|_{L^1}$$

Integrating this inequality, we have

$$\|\partial_t w(t)\|_{L^1} \leq \|\partial_t w(0)\|_{L^1} e^{-t} + \int_0^t e^{-(t-s)} \|\partial_t v(s)\|_{L^1} ds.$$

Integrating the obtained inequality once more over $t \in [0, T]$ and using that $\partial_t w(0)$ is uniformly bounded, we arrive at

$$(4.18) \quad \int_0^T \|\partial_t w(t)\|_{L^1} dt \leq C + \int_0^T \|\partial_t v(t)\|_{L^1} dt$$

where C is again independent of α and T and the trajectory.

Inserting (4.18) into the right-hand side of (4.16) and assuming that α is small enough, we finally deduce the following L^1 -dissipation integral

$$(4.19) \quad \int_0^T \|\partial_t^2 v(t)\|_{L^1} + \|\partial_t v(t)\|_{L^1} + \|\partial_t w(t)\|_{L^1} dt \leq C$$

where the constant C is independent of T .

Thus, we have proved that $(v(t), \partial_t v(t), w(t))$ converges to some $\xi \in \Phi_\infty$ in the $L^1(\Omega)$ -norm. Moreover, since we have the control of the L^∞ -norm, the interpolation inequality gives the convergence in L^p for any $p < \infty$:

$$(4.20) \quad \lim_{t \rightarrow \infty} \|(v(t), \partial_t v(t), w(t)) - \xi\|_{[L^p(\Omega)]^3} = 0.$$

Thus, we only need to verify that $\xi \in \mathcal{R}_\alpha$ is an equilibrium. To this end, we will use the so-called trajectory approach (see [4] for the details) and consider positive semi-trajectories trajectories instead of points in the phase spaces. Indeed, arguing exactly as in the proof of estimate (2.16), but taking the L^p -norm instead of the L^∞ -norm, we see that

$$(4.21) \quad \begin{aligned} \|(v_1(t), \partial_t v_1(t), w_1(t)) - (v_2(t), \partial_t v_2(t), w_2(t))\|_{[L^p(\Omega)]^3} &\leq \\ &\leq C e^{kt} \|(v_1(0), \partial_t v_1(0), w_1(0)) - (v_2(0), \partial_t v_2(0), w_2(0))\|_{[L^p(\Omega)]^3} \end{aligned}$$

Define now the map $\mathbb{S} : \Phi_\infty \rightarrow L^\infty(\mathbb{R}_+, \Phi_\infty)$ via the expression

$$(4.22) \quad \mathbb{S} : (v_0, v_0', w_0) \rightarrow (v(\cdot), \partial_t v(\cdot), w(\cdot))$$

and let $\mathcal{K}_+ := \mathbb{S}(\Phi_\infty)$. Then, estimate (4.21) (together with the obvious fact that $\mathcal{K}_+ \subset L^\infty(\mathbb{R}_+, \Phi_\infty)$) shows that the map \mathbb{S} realizes a Lipschitz continuous homeomorphism between spaces Φ_∞ and \mathcal{K}_+ endowed by the topology of $[L^p(\Omega)]^3$ and $\Phi_{tr} := L^\infty(\mathbb{R}_+, [L^p(\Omega)]^3)$ respectively. The solution semigroup $S(t)$ is conjugated via that homeomorphism to the semigroup $T(t)$ of temporal shifts on Φ_{tr} :

$$(4.23) \quad S(t) = \mathbb{S}^{-1} \circ T(t) \circ \mathbb{S}, \quad (T(t)\xi)(s) := \xi(t+s), \quad \xi \in \mathcal{K}_+, \quad t, s \geq 0.$$

Thus, the convergence (4.20) implies that

$$T(t)\mathbb{S}\xi_0 := T(t)\mathbb{S}(v(0), \partial_t v(0), w(0)) \rightarrow \mathbb{S}\xi$$

in the space Φ_{tr} .

Let us first check that $\mathbb{S}\xi \in \mathcal{K}_+$, i.e., that the limit trajectory $\xi(t)$ solves equation (2.1). In other words, we need to show that \mathcal{K}_+ is closed in Φ_{tr} .

To this end, we need to show that it is possible to pass to the Φ_{tr} -limit in equations (2.1) in the sense of distributions for any sequence $\xi_n(t) := (v_n(t), \partial_t v_n(t), w_n(t))$ converging in Φ_{tr} to some $\xi(t) := (\bar{v}(t), \partial_t \bar{v}(t), \bar{w}(t))$ and bounded in $L^\infty(\mathbb{R}_+, \Phi_\infty)$. Indeed, the passage to the limit

in all linear terms are evident and only the passage to the limit in the nonlinear terms $\varphi(v)$ and $f(v)$ may a priori be problematic. But it is not the case, since convergence in Φ_{t_r} implies the convergence almost everywhere (up to extracting a subsequence) and this allows us to conclude in a standard way that $f(v_n) \rightarrow f(\bar{v})$ and $\varphi(v_n)\partial_t v_n \rightarrow \varphi(\bar{v})\partial_t \bar{v}$ (here we have implicitly used that v_n is uniformly bounded in L^∞). Thus, the limit function $\xi(t)$ solves indeed problem (2.1).

We are now ready to verify that $\xi(t)$ is an equilibrium which will finish the proof of the theorem. Indeed, due to the dissipation integral (2.9), we see that

$$\|\partial_t v(s + \cdot)\|_{L^2(\mathbb{R}_+ \times \Omega)} + \|\partial_t^2 v(s + \cdot)\|_{L^2(\mathbb{R}_+ \times \Omega)} + \|\partial_t w(s + \cdot)\|_{L^2(\mathbb{R}_+ \times \Omega)} \rightarrow 0$$

as $s \rightarrow \infty$. Thus, for the limit function $(\bar{v}(t), \partial_t \bar{v}(t), \bar{w}(t))$, we have $\partial_t \bar{v} \equiv \partial_t^2 \bar{v} \equiv \partial_t \bar{w} \equiv 0$ and ξ is indeed an equilibrium. Theorem 4.4 is proved. \square

Remark 4.5. Assume, in addition, that equation $f(z) = 0$ possesses at least two solutions v_1 and v_2 such that $f'(v_i) > 0$, $i = 0, 1$. Then, v_i will be exponentially stable equilibria of equation (4.1). Let now $v_0(x)$ be a smooth function such that

$$v_0(x) = v_1, \quad x \in \Omega_1, \quad v_0(x) = v_2, \quad x \in \Omega_2$$

for some non-empty $\Omega_i \subset \Omega$ of the nonzero measure.

Finally let us consider the initial data for problem (2.1) of the form

$$\xi_0 := (v_0(x), 0, 0).$$

Then, since $\alpha > 0$ is small and the equilibria v_i are exponentially stable, the solution $v(t, x)$ will remain close to v_i (for $x \in \Omega_i$) for all t . This shows that the smooth trajectory $S(t)\xi_0 = (v(t), \partial_t v(t), w(t))$ tends as $t \rightarrow \infty$ to the *discontinuous* equilibrium (in the L^p -topology, according to the last theorem).

This example shows that we cannot extend the assertion of the theorem to the case $p = \infty$ and obtain the convergence in the topology of the phase space Φ_∞ . Indeed, if the sequence of continuous functions converges in L^∞ to some limit function, this function is automatically continuous. Thus, the ω -limit set of the above constructed trajectory in the topology of the phase space is empty:

$$\omega_{\Phi_\infty}(\xi_0) = \emptyset.$$

Remark 4.6. It is clear from the proof of Proposition 4.2 that all of the equilibria \mathcal{R}_α are hyperbolic in the phase space Φ_∞ . Thus, we may construct the infinite-dimensional stable and unstable manifolds for any equilibrium belonging to \mathcal{R}_α if $\alpha > 0$ is small enough. However, it does not help much for the study of the limit dynamics since, as shown in the previous proposition, generically, we do not have the stabilization in the topology of Φ_∞ , but only in a weaker space $[L^p(\Omega)]^3$. And in this weaker space the solution semigroup $S(t)$ is *not differentiable*. By this reason, we are not able to extract the *exponential* convergence from the hyperbolicity of any equilibrium and do not know whether or not such exponential convergence takes place.

To conclude, we note that, arguing analogously to the proof of Proposition 4.2, one can extract some reasonable information about the equilibria \mathcal{R}_α even in the case where α is not small.

Proposition 4.7. *Let the assumptions of Theorem 2.6 hold. Assume, in addition, that $(\bar{v}, \bar{w}) \in \mathbb{R}^2$ is a spatially homogeneous hyperbolic (in Φ_∞) equilibrium of problem (2.1), i.e., that the equation*

$$(4.24) \quad f'(\bar{v})\theta - \alpha(-\Delta_x + 1)^{-1}\theta = h$$

is uniquely solvable for every $h \in L^\infty(\Omega)$. Assume, finally, that there exists another constant $\tilde{v} \neq \bar{v}$ such that

$$(4.25) \quad f(\bar{v}) = f(\tilde{v})$$

(this, of course, may happen only in the case of non-monotone f). Then, there exists $\delta_0 > 0$ such that, for any measurable partition $\Omega = \Omega_1 \cup \Omega_2$ on two disjoint sets where

$$(4.26) \quad |\Omega_2| \leq \delta_0,$$

there exists a hyperbolic equilibrium (v, w) such that v is close (in the L^∞ -metric) to

$$(4.27) \quad v_{12} := \bar{v}\chi_{\Omega_1}(x) + \tilde{v}\chi_{\Omega_2}(x).$$

Proof. We first check that the equation of variations

$$(4.28) \quad f'(v_{12})\theta - \alpha(-\Delta_x + 1)^{-1}\theta = h$$

is uniquely solvable if the measure of Ω_2 is small. To this end, we construct the approximative solution of this equation in the form $\tilde{\theta} := \theta_0 + \hat{\theta}$ where θ_0 solves equation (4.24) and

$$(4.29) \quad \hat{\theta}(x) := [f'(\bar{v}) - f'(\tilde{v})]\theta_0(x)\chi_{\Omega_2}(x).$$

Then, since \bar{v} is a hyperbolic equilibrium, we have

$$(4.30) \quad \|\theta_0\|_{L^\infty(\Omega)} \leq C\|h\|_{L^\infty(\Omega)}.$$

Moreover, the approximate solution $\tilde{\theta}$ thus constructed solves

$$(4.31) \quad f'(v_{12})\tilde{\theta} - \alpha(-\Delta_x + 1)^{-1}\tilde{\theta} = \tilde{h} := -\alpha(-\Delta_x + 1)^{-1}\hat{\theta}$$

Finally, since the measure of Ω_2 is small, we have

$$\|\hat{\theta}\|_{L^2} \leq C|\Omega_2|^{1/2}\|\theta_0\|_{L^\infty} \leq C_1\delta_0^{1/2}\|h\|_{L^\infty}$$

and, consequently,

$$(4.32) \quad \|\hat{h}\|_{L^\infty} \leq C\|\hat{h}\|_{H^2} \leq C'\|\hat{\theta}\|_{L^2} \leq C_2\delta_0^{1/2}\|h\|_{L^\infty}$$

where the constant C_2 is independent of h and of the concrete form of the partition $\Omega = \Omega_1 \cup \Omega_2$.

Thus, if $\delta_0 > 0$ is so small that $C_2\delta_0^{1/2} := \kappa < 1$, the norm of the reminder $\|\tilde{h}\|_{L^\infty}$ is estimated $\kappa\|h\|_{L^\infty}$ with $\kappa < 1$. Then, the standard iteration process gives the desired solution θ of equation (4.28) together with estimate

$$(4.33) \quad \|\theta\|_{L^\infty} \leq C\|h\|_{L^\infty}$$

with the constant C independent of $\delta_0 \rightarrow 0$. The uniqueness of a solution can be obtained in a standard way using the observation that the operator $f'(v_{12}) - \alpha(-\Delta_x + 1)^{-1}$ is self-adjoint in L^2 .

It is now not difficult to finish the proof of the proposition. Indeed, we seek for the desired equilibrium (v, w) in the form

$$v(x) = v_{12}(x) + \theta(x)$$

where θ is a small corrector which should satisfy the equation

$$(4.34) \quad f(v_{12} + \theta) - f(v_{12}) - \alpha(-\Delta_x + 1)^{-1}\theta = \hat{h} := \alpha(-\Delta_x + 1)[(\bar{v} - \tilde{v})\chi_{\Omega_2}]$$

and, arguing as before, we see that

$$\|\hat{h}\|_{L^\infty} \leq C|\Omega_2|^{1/2}.$$

Applying now the implicit function theorem to equation (4.34), we establish the existence of a unique solution θ ,

$$\|\theta\|_{L^\infty} \leq C_1|\Omega_2|^{1/2}$$

measure of Ω_2 is small enough. Proposition 4.7 is proved. \square

Remark 4.8. Although we are not able neither to give a complete description of the equilibria set \mathcal{R}_α nor to verify the stabilization if α is not small, we see that, under the assumptions of the last proposition (which are, in a sense, natural for the non-monotone case), the set of equilibria is not compact in Φ_∞ and is not closed in the weak-star topology of the phase space. These facts do not allow to extend the global attractor theory for the non-monotone case.

Note also that, although we formulate (for simplicity) Proposition 4.7 for the case of *spatially-homogeneous* hyperbolic equilibrium (\bar{v}, \bar{w}) it can be easily extended to the case of non-homogeneous equilibria. This shows that the conclusion of Proposition 4.7 is somehow "generic" for the non-monotone case.

5. APPENDIX. ESTIMATION OF TIME DERIVATIVES FOR NON-AUTONOMOUS PERTURBATIONS OF REGULAR ATTRACTORS.

The aim of that appendix is to verify the auxiliary estimate for non-autonomous perturbations of regular attractors. To be more precise, consider an ODE in \mathbb{R}^n :

$$(5.1) \quad u'(t) = F(u(t)), \quad u(0) = u_0$$

for some $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. We assume that, for every $u_0 \in \mathbb{R}^n$, this equation is globally (for $t \geq 0$) solvable and the associated semigroup $S(t)u_0 := u(t)$ is dissipative, i.e.,

$$(5.2) \quad \|S(t)u_0\| \leq Q(\|u_0\|)e^{-\beta t} + C_*$$

for some positive β and C_* and monotone Q . Therefore, equation (5.1) possesses a global attractor \mathcal{A} in \mathbb{R}^n . Our main assumption is that this attractor is *regular* in the sense of Theorem 3.8, i.e., all of the equilibria $u_0 \in \mathcal{R}$ are hyperbolic, every trajectory, belonging to the attractor \mathcal{A} is a heteroclinic orbit connecting two *different* equilibria and the attractor \mathcal{A} is a finite union of the unstable manifolds $\mathcal{M}_{u_0}^+$ associated with the equilibria $u_0 \in \mathcal{R}$:

$$(5.3) \quad \mathcal{A} = \cup_{u_0 \in \mathcal{R}} \mathcal{M}_{u_0}^+.$$

Finally, we assume that every unstable manifold $\mathcal{M}_{u_0}^+$ is a smooth submanifold of \mathbb{R}^n .

Consider now the following small non-autonomous perturbation of equation (5.1)

$$(5.4) \quad u' = F(u) + h(t), \quad u(0) = u_0, \quad t \geq 0$$

where the non-autonomous external force is uniformly small:

$$(5.5) \quad \|h\|_{W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)} \leq \varepsilon \ll 1.$$

The main result of this appendix is the following estimate.

Proposition 5.1. *Let the above assumptions hold and let the external force $h \in W^{1,\infty}(\mathbb{R}_+)$ satisfy estimate (5.5) for sufficiently small $\varepsilon > 0$. Then, any solution $u(t)$ of the perturbed problem (5.4) satisfies the following estimate:*

$$(5.6) \quad \int_0^T \|u'(t)\| dt \leq C_1 + C_2 \int_0^T \|h'(t)\| dt$$

where the positive constants C_1 and C_2 depend only on the norm of $u(0)$ and are independent of T and the concrete choice of $u(0)$ and $h(t)$.

Proof. Indeed, using the standard regular attractor perturbation arguments, one can check that for every bounded set B of \mathbb{R}^n and every $\delta > 0$, there exist $T = T(B, \delta)$ and $\varepsilon_0 = \varepsilon_0(B, \delta)$ such that, for every $\varepsilon \leq \varepsilon_0$ and every trajectory $u(t)$ starting from B , we can find a sequence u, \dots, u_N of *different* equilibria $u_i \in \mathcal{R}$ (of problem (5.1)!) and a sequence of times

$$0 = T_0^+ \leq T_1^- \leq T_1^+ < T_2^- \leq T_2^+ < \dots < T_N^- < T_N^+ = \infty$$

such that

$$(5.7) \quad u(t) \in \mathcal{O}_\delta(u_i), \quad t \in (T_i^-, T_i^+), \quad T_i^- - T_{i-1}^+ \leq T, \quad i = 1, \dots, N.$$

In other words, the sequence of equilibria u_i and the values of T_i^\pm depend on the concrete choice of $u(0)$ and h , but the number N of equilibria is bounded by the whole number $\#\mathcal{R} < \infty$ of possible equilibria (since the equilibria must be different) and the differences $T_i^+ - T_{i-1}^-$ are also uniformly bounded by T , see [2, 7, 18] for the details.

Thus, any trajectory starting from B spends at most time $T_{out} := \#\mathcal{R} \cdot T$ outside of the δ -neighbourhood $\mathcal{O}_\delta(\mathbb{R})$ of the equilibria set \mathcal{R} and this time depends only on B and δ . By this reason, the part of the trajectory, lying outside of $\mathcal{O}_\delta(\mathbb{R})$ gives only a *finite and uniformly bounded* impact to the integral (5.6) (which can be included into the constant C_1). So, we only need to estimate the left-hand side of (5.6) for the case where $u(t)$ belongs to a small neighbourhood of a single fixed equilibrium $u_0 \in \mathcal{R}$ only.

To this end, we will use the hyperbolicity assumption on u_0 . Indeed, the implicit function theorem implies the existence of $\varepsilon_0 > 0$ and $\delta > 0$ such that, for every $\varepsilon \leq \varepsilon_0$, there exists a unique solution $U_{u_0, h}(t)$ of (5.4) belonging to the δ -neighbourhood of u_0 for all t . Moreover, this solution, in a fact belongs to the $C\varepsilon$ -neighbourhood of u_0 and the following estimate holds:

$$(5.8) \quad |U'_{u_0, h}(t)| \leq C \int_{\mathbb{R}} e^{-\kappa|t-s|} |h'(s)| ds$$

where the constant C and the hyperbolicity exponent κ are independent of the concrete choice of $u_0 \in \mathcal{R}$ and the external force h satisfying (5.5), see [9, 18].

Furthermore, since u_0 is hyperbolic, the trajectory $U_{u_0, h}(t)$ will be also hyperbolic and we will have an exponential dichotomy in a small δ -neighbourhood of $U_{u_0, h}$. In particular, every trajectory $u(t)$ belonging to $\mathcal{O}_\delta(u_0)$ for $t \in [0, S]$, $S \gg 1$, will tend exponentially to $U_{u_0, h}(t)$ inside of the interval

$$(5.9) \quad |u(t) - U_{u_0, h}(t)| + |u'(t) - U'_{u_0, h}(t)| \leq C(e^{-\kappa|t|} + e^{-\kappa|S-t|})$$

and C and κ are independent of the concrete choice of u and h , see [9, 18] for the details. Therefore,

$$(5.10) \quad \int_0^t |u'(s)| ds \leq C + \int_0^t |U'_{u_0, h}(s)| ds$$

for $t \in [0, S]$ and $u(t) \in \mathcal{O}_\delta(u_0)$.

Thus, we have proved that

$$(5.11) \quad \int_0^T |u'(s)| ds \leq C_1 + C_2 \sum_{u_0 \in \mathcal{R}} \int_0^T |U'_{u_0, h}(s)| ds$$

where the constants C_i depend only on the radius of B .

In order to deduce (5.6) from (5.11), we will use estimate (5.8). Indeed, integrating it over $t \in [0, T]$ and using that h can be extended for $t \leq 0$ by zero, we have

$$(5.12) \quad \int_0^T |U'_{u_0, h}(t)| dt \leq C \int_0^T |h'(t)| dt + C \int_T^\infty e^{-\kappa|t-T|} |h'(s)| ds \leq C_1 + C \int_0^T |h'(t)| dt.$$

Inserting this estimate into the right-hand side of (5.11), we obtain (5.6) and finish the proof of the proposition. \square

Remark 5.2. As we can see from the proof, estimate (5.6) has a general nature which is not restricted by the class of ordinary differential equations. However, since we use it in the paper only for the ODE case, in order to avoid the technicalities related with the formulation of a "general" PDE, we restrict our consideration to the case of a ODE as well.

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