GLOBAL AND EXPONENTIAL ATTRACTORS FOR NONLINEAR REACTION-DIFFUSION SYSTEMS IN UNBOUNDED DOMAINS

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Abstract. We study the long time behavior of solutions of autonomous and nonautonomus reaction-diffusion equations in unbounded domains of $\mathbb{R}^3$. It is shown that, under appropriate assumptions on the nonlinear interaction function and on the external forces, these equations possess compact global (uniform) attractors in the corresponding phase space. Estimates for the Kolmogorov's ε-entropy of these attractors in terms of the Kolmogorov's entropy of the external forces are given. Moreover, (infinite dimensional) exponential attractors with the same entropy estimate as that of the corresponding global (uniform) attractor are also constructed.

Contents

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INTRODUCTION.

We consider the following system of reaction-diffusion equations:

$$
\begin{cases} \partial_t u - a\Delta_x u + f(u, \nabla_x u) + \lambda_0 u = g, \\ u\big|_{t=0} = u_0; \quad u\big|_{\partial\Omega} = 0, \end{cases} \tag{0.1}
$$

where Ω is an unbounded domain of $\mathbb{R}^3$ with a sufficiently regular boundary $\partial\Omega$ (see Section 1), $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to x, $f(u, \nabla_x u)$ and $g = g(x)$ are given interaction function and external forces respectively and $\lambda_0 > 0$ and $a > 0$ are given constants.

The long time behavior of the solutions of (0.1) is of a great current interest. It is well known that, under appropriate assumptions on the nonlinear term $f(u, \nabla_x u)$, on the external forces g and on the domain Ω, this behavior can be described in terms of the global or/and exponential attractors of the dynamical system generated by (0.1) (see e.g. [1-4], [11-13], [15-17], [20-21], [24-27], [30-31], [34-36] and the references therein). In particular, when Ω is bounded, the global attractor $\mathcal{A}$ of problem (0.1) has usually finite Hausdorff and fractal dimensions (see [3] and [27]). In that case, infinite dimensional attractors can naturally appear only in the case of *nonautonomous* equations (0.1) (e.g. with $g = g(t)$) and only when the time dependence of the external forces is described by an infinite number of parameters (e.g. in the case of almost-periodic external forces, see [5-6]). In contrast to this, in the case of unbounded domains, the attractor of problem (0.1) has usually infinite Hausdorff and fractal dimensions, even in the autonomous case and, consequently, in that case, the infinite dimensionality of the attractor has an internal nature and can appear even when the external forces vanish (see [9], [12], [25] and [34-36]; see also [10], [19] and [32] for analogous results for damped hyperbolic equations in unbounded domains).

Nevertheless, there exist several particular cases of equations (0.1) in *unbounded* domains Ω for which the associated dynamics is finite dimensional (and very similar to the case of bounded domains, see [1-2], [8], [11] and [13]).

In the present paper, we give a systematic study of one of the particular cases mentioned above, which is determined by the following assumption on the nonlinearity $f(u, \nabla_x u)$:

$$
f(v, w).v \geq 0, \quad \text{for all } v \in \mathbb{R}^k \text{ and } w \in \mathbb{R}^{3k} \tag{0.2}
$$

(here and below, $u.v$ denotes the standard inner product in $\mathbb{R}^k$) and assuming that the external forces $g(x)$ tend to zero in an appropriate sense as $|x| \to \infty$. For instance, the case $\Omega = \mathbb{R}^n$ and g belonging to a weighted Sobolev space $L^2_{\phi_\alpha}(\mathbb{R}^n)$, with $\phi_\alpha(x) := (1 + |x|^2)^\alpha$ and $\alpha > 0$, was studied in [1-2], [4], [8] and [11] and the case $\alpha = 0$ was considered in [13] (exponential attractors for (0.1) in weighted Sobolev spaces $W^{2,2}_{\phi_\alpha}(\mathbb{R}^n)$ were constructed in [4] and [11] under additional strong growth restrictions on the nonlinearity $f(u, \nabla_x u)$).

In this paper, we remove the growth restrictions on $f(u, \nabla_x u)$ (see Section 2 for the precise conditions on $f(u, \nabla_x u)$) and weaken the assumptions on the external forces g by taking

$$
g \in \dot{L}^2_b(\Omega) := \{g \in L^2_{loc}(\Omega), \lim_{|x| \to \infty} \|g\|_{L^2(\Omega \cap B^1_x)} = 0\}, \tag{0.3}
$$

where B_x^R denotes the open R-ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$.

We note that (0.3) requires no additional assumption on the rate of convergence of $g(x) \to 0$ as $|x| \to \infty$ and, consequently, the space $\dot{L}^2_b(\Omega)$ is larger than $L^2(\Omega)$. On the other hand, as elementary examples show (see [2] and [13]), the dimension of the attractor may be infinite (even under assumption (0.2)) if (0.3) is violated. Thus, assumption (0.3) looks like a sharp border between finite and infinite dimensional dynamics.

The following theorem gives the spatial asymptotics for the attractor $\mathcal{A}$ up to exponentially small terms as $|x| \to \infty$ and clarifies the nature of its finite dimensionality (see also [33] for an analogous result for nonlinear damped hyperbolic equations in an unbounded domain).

Theorem 1. *Let the above assumptions hold and let z_0 be an arbitrary equilibrium of* (0.1). *Then, there exists an effective radius $R_{eff} > 0$ which is determined by the rate of convergence in* (0.3) *such that*

$$|v(x) - z_0(x)| \le Ce^{-\beta \operatorname{dist}(x, B_0^{R_{eff}})}, \quad \text{for all } \ v \in \mathcal{A}, \tag{0.4}$$

where the constants $\beta > 0$ and $C > 0$ are independent of $x \in \Omega$, $v \in \mathcal{A}$ and R_{eff}.

Estimate (0.4) suggests that the fractal dimension of the attractor should be estimated in terms of the effective radius R_{eff}. The next theorem gives such estimates for the case of global $\mathcal{A}$ and exponential $\mathcal{M}$ attractors of problem (0.1).

Theorem 2. *Let the above assumptions hold. Then, there exists a finite dimensional exponential attractor $\mathcal{M}$ for problem* (0.1) *such that*

$$\dim_F(\mathcal{A}, C(\Omega)) \le \dim_F(\mathcal{M}, C(\Omega)) \le C \operatorname{vol}(\Omega \cap B_0^{R_{eff}}), \tag{0.5}$$

where the constant C is independent of R_{eff}.

Moreover, examples of equations (0.1) for which estimate (0.5) is sharp with respect to the radius R_{eff} are also given (see Section 6). We also note that the proof of Theorem 2 essentially uses the construction of exponential attractors in Banach spaces proposed in [15].

We then apply Theorem 2 to the following RDS with a small diffusion parameter $\nu \ll 1$:

$$\partial_t u = \nu \Delta_x u - f(u) - \lambda_0 u + g, \quad u\big|_{\partial\Omega} = 0, \tag{0.6}$$

where, in contrast to (0.1), the interaction function f does not depend explicitly on the gradient $\nabla_x u$. Indeed, after the rescaling $x \to x'\nu^{1/2}$ in (0.6), we obtain equation (0.6) with $\nu' = 1$ and with external forces $g_\nu(x') := g(x'\nu^{1/2})$. Applying then Theorem 2 to this new equation and noting that $R_{eff}(g_\nu) = \nu^{-1/2} R_{eff}(g)$, we obtain the following result.

Corollary 1. *Let the above assumptions hold an let, in addition, g belong to $\dot{L}^\infty(\Omega)$ (which is defined analogously to* (0.3)*). Then, equation* (0.6) *possesses a uniform family of exponential attractors $\mathcal{M}_\nu$ whose fractal dimensions can be estimated by*

$$\dim_F(\mathcal{A}_\nu, C(\Omega)) \le \dim_F(\mathcal{M}_\nu, C(\Omega)) \le C_1 \nu^{-3/2}, \tag{0.7}$$

where C_1 is independent of ν, and this estimate is sharp (see Section 6 for details).

We also consider the nonautonomous analogue of equation (0.1):

$$
\begin{cases} \partial_t u - a\Delta_x u + f(u, \nabla_x u) + \lambda_0 u = g(t), \\ u\big|_{t=0} = u_0; \quad u\big|_{\partial\Omega} = 0, \end{cases} \tag{0.8}
$$

where the external forces $g(t)$ depend explicitly on t. Following the general scheme of construction of the (uniform) attractor for nonautonomous equations via the skew-product technique (see [5-7] and [21]), we introduce the hull $g \in C(\mathbb{R}, \dot{L}^2_b(\Omega))$:

$$
\mathcal{H}(g) := \big[T_h g,\ h \in \mathbb{R}\big]_{C_{loc}(\mathbb{R}, \dot{L}^2_b(\Omega))}, \quad (T_h g)(t) := g(t+h), \tag{0.9}
$$

where $[\cdot]_V$ denotes the closure in the space V. A standard assumption on the nonautonomous external forces g (see [6-7] and [34]) is that the hull $\mathcal{H}(g)$ be compact in the appropriate topology, namely

$$
\mathcal{H}(g) \subset\subset C_{loc}(\mathbb{R}, \dot{L}^2_b(\Omega)). \tag{0.10}
$$

Some necessary and sufficient conditions to have such a compactness are given in Section 3.

As usual, in order to construct the (uniform) attractor for the nonautonomous equation (0.8), we consider a family of equations of type (0.8) for all external forces $\xi \in \mathcal{H}(g)$ and define the extended skew-product semigroup associated with problem (0.8) by

$$
\mathbb{S}_t : \Phi_b \times \mathcal{H}(g) \to \Phi_b \times \mathcal{H}(g), \quad \mathbb{S}_t(u_0, \xi) := (u_\xi(t), T_t g), \tag{0.11}
$$

where Φ_b is an appropriate phase space for problem (0.8) (see Section 2) and $u_\xi(t)$ denotes the solution of equation (0.8) with external forces $\xi \in \mathcal{H}(g)$ and with $u_\xi(0) = u_0$. Then, by definition, the projection $\mathcal{A} := \Pi_1 \mathbb{A}$ of the global attractor $\mathbb{A}$ of semigroup (0.11) onto the first component is called the uniform attractor of the nonautonomous problem (0.8), see [6-7], [21], [34] and Section 3 for details.

In contrast to the autonomous case, even in bounded domains Ω, the uniform attractor $\mathcal{A}$ of problem (0.8) has naturally infinite fractal and Hausdorff dimensions if the hull $\mathcal{H}(g)$ is in some sense infinite dimensional and, consequently, new quantitative characteristics are required in order to study such attractors. One possible approach to handle this problem, which was suggested in [7], is to study the Kolmogorov's ε-entropy of the uniform attractor.

We recall (see [22] for details) that, if K is a precompact set in a metric space M, then, for every $\varepsilon > 0$, it can be covered by a finite number of ε-balls in M. Let $N_\varepsilon(K, M)$ be the minimal number of such balls. Then, by definition, the Kolmogorov's ε-entropy of K in M is the following number:

$$
\mathbb{H}_\varepsilon(K, M) := \log_2 N_\varepsilon(K, M). \tag{0.12}
$$

It is worth to emphasize that, in contrast to the fractal dimension, quantity (0.12) is finite for every precompact set K in M and, in particular, it is finite for the uniform attractor $\mathcal{A}$ of problem (0.8).

The entropy of infinite dimensional uniform attractors of (0.8) in *bounded* domains Ω was studied in [6-7]. The case of autonomous RDEs in $\mathbb{R}^n$ was considered in [9] and [30]. The entropy of attractors of autonomous and nonautonomous RDEs in unbounded domains was investigated in [34-36]. Analogous results for damped hyperbolic equations were obtained in [32-33].

We now recall that an exponential attractor $\mathcal{M}$ (if it exists) always contains the global (uniform) attractor, see e.g. [4] or [15-18]. Consequently, a *finite dimensional* exponential attractor $\mathcal{M}$ cannot exist for problem (0.8) if the dimension of the associated uniform attractor is infinite. Therefore, it is natural, following [16], to introduce the notion of infinite dimensional exponential attractor and use the Kolmogorov's ε-entropy in order to control its 'size'. It is also natural to seek for an exponential attractor with the same type of asymptotics of the ε-entropy as that of the uniform attractor:

$$\mathbb{H}_\varepsilon(\mathcal{A},\Phi_b)\sim\mathbb{H}_\varepsilon(\mathcal{M},\Phi_b).$$

Infinite dimensional exponential attractors for nonautonomous reaction-diffusion equations in bounded domains Ω were constructed in [16]. In the present paper, we extend this result to the case of unbounded domains Ω.

Theorem 3. *Let the above assumptions hold. Then, there exists an infinite dimensional exponential attractor $\mathcal{M}$ for the nonautonomous problem* (0.8) *which possesses the following entropy estimate:*

$$\begin{aligned}(0.13)\quad \mathbb{H}_\varepsilon(\mathcal{A},C(\Omega))\le\mathbb{H}_\varepsilon(\mathcal{M},C(\Omega))\le C\,\mathrm{vol}(\Omega\cap B_0^{R_{eff}})\log_2\frac{1}{\varepsilon}+\\ +\mathbb{H}_{\varepsilon/K}\left(\mathcal{H}(g)\big|_{[0,L\log_2\frac{1}{\varepsilon}]\times\Omega},C([0,L\log_2\frac{1}{\varepsilon}],\dot{L}_b^2(\Omega))\right),\end{aligned}$$

where the constants C, K and L are independent of ε and the effective radius R_{eff} is determined in the same way as in Theorem 1.

Although the exponential attractor $\mathcal{M}$ constructed in Theorem 3 is a priori *infinite dimensional*, estimate (0.13) guarantees that it is *finite dimensional* if the hull $\mathcal{H}(g)$ is finite-dimensional; this is in particular the case for quasiperiodic external forces.

Corollary 2. *Let the above assumptions hold and let the external forces $g(t)$ be quasiperiodic with respect to t, with m rationally independent frequencies. Then,* (0.8) *possesses a finite dimensional exponential attractor $\mathcal{M}$ such that*

$$(0.14)\qquad \dim_F(\mathcal{A},C(\Omega))\le\dim_F(\mathcal{M},C(\Omega))\le C\,\mathrm{vol}(\Omega\cap B^{R_{eff}})+m,$$

where C is the same as in (0.5).

As in the autonomous case, applying Corollary 2 to the nonautonomous version of problem (0.6), we obtain the following result.

Corollary 3. *Let the above assumptions hold and let, in addition, the external forces $g(t)$ be quasiperiodic with respect to t, with m rationally independent frequencies, and belong to the space $C_b(\mathbb{R}\times\Omega)$. Then, equation* (0.8) *possesses a* *uniform family of nonautonomous exponential attractors $\mathcal{M}_\nu$ and the following estimate holds:*

$$\dim_F(\mathcal{A}_\nu, C(\Omega)) \le \dim_F(\mathcal{M}_\nu, C(\Omega)) \le C_1\nu^{-3/2} + m, \tag{0.15}$$

where the constant C_1 is independent of ν and m.

Finally, in Section 6, we also verify the sharpness of (0.15) and indicate examples of nonlinearities f for which we have

$$C_1\nu^{-3/2} + m \le \dim_F(\mathcal{A}_\nu, C(\Omega)) \le \dim_F(\mathcal{M}_\nu, C(\Omega)) \le C_2\nu^{-3/2} + m, \tag{0.16}$$

where C_1 and C_2 are independent of ν and m.

This paper is organized as follows. The definitions of the functional spaces which are necessary to study problem (0.8) in an unbounded domain and their basic properties are briefly indicated in Section 1. Analytic properties of the solutions of problem (0.8) in a finite time interval, such as existence and uniqueness, regularity and smoothing property, are established in Section 2. The global (uniform) attractor for problems (0.1) and (0.8) is constructed in Section 3. Estimate (0.13) for the ε-entropy of the *global* attractor $\mathcal{A}$ is obtained in Section 4. The construction of an *exponential* attractor $\mathcal{M}$, which satisfies estimate (0.13), is given in Section 5. Finally, in Section 6, we apply the above results to equation (0.6) and indicate analogous results for more general classes of reaction-diffusion equations with nonscalar diffusion matrices a.

The results of the present paper have been partially announced in [15].

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§1 Functional spaces.

In this section, we introduce several classes of Sobolev spaces in unbounded domains and briefly recall some of their properties which will be essential in the sequel. For a detailed study of these spaces, see [13] and [32-34].

Definition 1.1. *A function $\phi \in L^\infty_{loc}(\mathbb{R}^n)$ is called a weight function with growth rate $\mu \ge 0$ if the condition*

$$\phi(x+y) \le C_\phi e^{\mu|x|}\phi(y),\ \ \phi(x) > 0, \tag{1.1}$$

is satisfied for every $x, y \in \mathbb{R}^n$.

Remark 1.1. *It is not difficult to deduce from* (1.1) *that*

$$\phi(x+y) \ge C_\phi^{-1} e^{-\mu|x|}\phi(y), \tag{1.2}$$

is also satisfied for every $x, y \in \mathbb{R}^n$.

The following example of weight functions are of fundamental significance for our purposes:

$$\phi_{\{\beta\},x_0}(x) = e^{-\beta|x-x_0|},\ \ \beta \in \mathbb{R},\ \ x_0 \in \mathbb{R}^n.$$

(Obviously, this weight has the growth rate $|\beta|$.)

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be some (unbounded) domain in $\mathbb{R}^n$ and let ϕ be a weight function with growth rate μ. We set*

$$L^p_\phi(\Omega) = \left\{ u \in D'(\Omega) \ : \ \|u, \Omega\|^p_{\phi,0,p} \equiv \int_\Omega \phi(x)|u(x)|^p \, dx < \infty \right\}.$$

Analogously, the weighted Sobolev space $W^{l,p}_\phi(\Omega)$, $l \in \mathbb{N}$, is defined as the space of distributions whose derivatives up to order l belong to $L^p_\phi(\Omega)$.

In order to simplify the notations, we will write throughout the paper $W^{s,p}_{\{\beta\}}(\Omega)$ instead of $W^{s,p}_{e^{-\beta|x|}}(\Omega)$.

We also define another class of weighted Sobolev spaces as follows:

$$W^{l,p}_{b,\phi}(\Omega) = \left\{ u \in D'(\Omega) \ : \|u, \Omega\|^p_{b,\phi,l,p} = \sup_{x_0 \in \Omega} \{\phi(x_0)\|u, \Omega \cap B^1_{x_0}\|^p_{l,p}\} < \infty \right\}.$$

Here and below, we denote by $B^R_{x_0}$ the ball in $\mathbb{R}^n$ of radius R centered at x_0 and $\|u, V\|_{l,p}$ stands for $\|u\|_{W^{l,p}(V)}$.

We will write $W^{l,p}_b(\Omega)$ instead of $W^{l,p}_{b,1}(\Omega)$.

Proposition 1.1.

1. Let u belong to $L^p_\phi(\Omega)$, where ϕ is a weight function with growth rate μ. Then, for any $1 \le q \le \infty$, the following estimate holds:

$$\left(\int_\Omega \phi(x_0)^q \left(\int_\Omega e^{-\beta|x-x_0|} |u(x)|^p \, dx \right)^q dx_0 \right)^{1/q} \le C \int_\Omega \phi(x)|u(x)|^p \, dx, \tag{1.3}$$

for every $\beta > \mu$, where the constant C depends only on β, μ and the constant C_ϕ in (1.1) *(and is independent of Ω).*

2. Let u belong to $L^\infty_\phi(\Omega)$. Then, for every $\beta \ge \mu$, the following analogue of estimate (1.3) *is valid:*

$$\sup_{x_0 \in \Omega} \left\{ \phi(x_0) \sup_{x \in \Omega} \{e^{-\beta|x-x_0|}|u(x)|\} \right\} \le C \sup_{x \in \Omega} \{\phi(x)|u(x)|\}. \tag{1.4}$$

The proof of this Proposition can be found in [13] or [34].

In order to study reaction-diffusion equations (0.1) and (0.8), we need to impose some regularity assumptions on the unbounded domain $\Omega \subset \mathbb{R}^n$, which are assumed to be valid throughout the paper.

We assume that there exists a positive number $R_0 > 0$ such that, for every point $x_0 \in \Omega$, there exists a smooth domain $V_{x_0} \subset \Omega$ such that

$$B^{R_0}_{x_0} \cap \Omega \subset V_{x_0} \subset B^{R_0+1}_{x_0} \cap \Omega. \tag{1.5}$$

Moreover, we also assume that there exists a diffeomorphism $\theta_{x_0} : B^2_0 \to B^{R_0+2}_{x_0}$ such that $\theta_{x_0}(x) = x_0 + p_{x_0}(x)$, $\theta_{x_0}(B^1_0) = V_{x_0}$ and

$$\|p_{x_0}\|_{C^N} + \|p^{-1}_{x_0}\|_{C^N} \le K, \tag{1.6}$$

where the constant K is independent of $x_0 \in \Omega$ and N is large enough. For simplicity, we assume from now on that (1.5) and (1.6) hold for $R_0 = 2$.

We note that, in case Ω is bounded, conditions (1.5) and (1.6) are equivalent to the following: the boundary $\partial\Omega$ is a smooth manifold. Now, for unbounded domains, the sole smoothness of the boundary is not sufficient to obtain the regular structure of Ω as $|x| \to \infty$, since some uniform (with respect to $x_0 \in \Omega$) smoothness conditions are required. It is however more convenient to formulate these conditions in the form (1.5) and (1.6).

Proposition 1.2. *Let the domain* Ω *satisfy conditions* (1.5) *and* (1.6) *and the weight function* ϕ *satisfy condition* (1.1) *and let* R *be some positive number. Then, the following estimates hold:*

$$(1.7)\quad C_1\int_\Omega \phi(x)|u(x)|^p\,dx \le \int_\Omega \phi(x_0)\int_{\Omega\cap B^R_{x_0}}|u(x)|^p\,dx\,dx_0 \le \le C_2\int_\Omega \phi(x)|u(x)|^p\,dx.$$

Proof. We have

$$(1.8)\quad \int_\Omega \phi(x_0)\int_{\Omega\cap B^R_{x_0}}|u(x)|^p\,dx\,dx_0 = \int_\Omega |u(x)|^p\left(\int_\Omega \chi_{\Omega\cap B^R_x}(x_0)\phi(x_0)\,dx_0\right)dx.$$

Here, $\chi_{\Omega\cap B^R_x}$ denotes the characteristic function of the set $\Omega\cap B^R_x$.
It follows from inequalities (1.1) and (1.2) that

$$(1.9)\qquad C_1\phi(x)\le \inf_{x_0\in B^R_x}\phi(x_0)\le \sup_{x_0\in B^R_x}\phi(x_0)\le C_2\phi(x),$$

and assumptions (1.5) and (1.6) imply that

$$(1.10)\qquad 0<C_1\le \operatorname{mes}(\Omega\cap B^R_x)\le C_2,$$

uniformly with respect to $x\in\Omega$.
Estimate (1.7) is an immediate corollary of estimates (1.8)–(1.10) and Proposition 1.2 is proved.

Corollary 1.1. *Let* (1.5) *and* (1.6) *hold. Then, the following norm is equivalent to the usual norm in* $W^{l,p}_\phi(\Omega)$*:*

$$(1.11)\qquad \|u,\Omega\|_{\phi,l,p}=\left(\int_\Omega \phi(x_0)\|u,\Omega\cap B^R_{x_0}\|^p_{l,p}\,dx_0\right)^{1/p}.$$

In particular, all the norms (1.11) *are equivalent, for* $R>0$.

To study equation (0.1), we also need weighted Sobolev spaces of fractional order $s\in\mathbb{R}_+$ (and not for $s\in\mathbb{Z}$ only). We first recall (see [28] for details) that, if V is a bounded domain, a classical norm in the space $W^{s,p}(V)$, $s=[s]+l$, $0<l<1$, $[s]\in\mathbb{Z}_+$, can be defined by

$$(1.12)\qquad \|u,V\|^p_{s,p}=\|u,V\|^p_{[s],p}+\sum_{|\alpha|=[s]}\int_V\int_V \frac{|D^\alpha u(x)-D^\alpha u(y)|^p}{|x-y|^{n+lp}}\,dx\,dy.$$

It is not difficult to prove, arguing as in Proposition 1.2 and using this representation, that, for any bounded domain V with a sufficiently smooth boundary

$$(1.13)\qquad C_1\|u,V\|^p_{s,p}\le\int_V \|u,V\cap B^R_{x_0}\|^p_{s,p}\,dx_0\le C_2\|u,V\|^p_{s,p}.$$

This justifies the following definition.

Definition 1.3. *We define the space $W^{s,p}_{\phi}(\Omega)$, for $s \in \mathbb{R}_+$, as the space of distributions whose norm* (1.11) *is finite.*

It is not difficult to check that these norms are also equivalent for different $R > 0$. We now note that the weight functions

$$\phi_{\{\beta\},x_0} = e^{-\beta|x-x_0|}, \tag{1.14}$$

satisfy conditions (1.1) *uniformly* with respect to $x_0 \in \mathbb{R}^n$ and, consequently, all the estimates obtained above for arbitrary weights will be valid for family (1.14) uniformly with respect to $x_0 \in \mathbb{R}^n$. Since these estimates are of fundamental significance for what follows, we write them explicitly in several corollaries formulated below.

Corollary 1.2. *Let u belong to $L^p_{\{\delta\}}(\Omega)$, for $0 < \delta < \beta$. Then, the following estimate holds uniformly with respect to $y \in \mathbb{R}^n$:*

$$\left(\int_\Omega e^{-q\delta|x_0-y|}\left(\int_\Omega e^{-\beta|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \le \\ \le C_{\beta,\delta}\int_\Omega e^{-\delta|x-y|}|u(x)|^p\,dx. \tag{1.15}$$

Moreover, if $u \in L^\infty_{\{\delta\}}(\Omega)$, $\delta \le \beta$, then

$$\sup_{x_0\in\Omega}\left\{e^{-\delta|x_0-y|}\sup_{x\in\Omega}\{e^{-\beta|x-x_0|}|u(x)|\}\right\} \le C_{\beta,\delta}\sup_{x\in\Omega}\{e^{-\delta|x-y|}|u(x)|\}. \tag{1.16}$$

Corollary 1.3. *Let u belong to $W^{l,p}_{b,\phi}(\Omega)$ and ϕ be a weight function with growth rate $\mu < \beta$. Then*

$$C_1\|u,\Omega\|^p_{b,\phi,l,p} \le \\ \le \sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega} e^{-\beta|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx\right\} \le C_2\|u,\Omega\|^p_{b,\phi,l,p}. \tag{1.17}$$

For the proof of this corollary, see [34].

We will essentially use the subspaces of $W^{l,p}_b(\Omega)$ which consist of functions decaying when $|x| \to \infty$ below.

Definition 1.4. *We define the space $\dot W^{l,p}_b(\Omega)$ as follows:*

$$\dot W^{l,p}_b(\Omega) := \{u \in W^{l,p}_b(\Omega) : \lim_{|x_0|\to\infty}\|u,\Omega\cap B^1_{x_0}\|_{l,p} = 0\}. \tag{1.18}$$

The following proposition gives simple compactness criteria for sets in $\dot W^{l,p}_b(\Omega)$.

Proposition 1.3. *A set $B \in \dot W^{l,p}_b(\Omega)$ is compact if and only if:*

1. For every $x_0 \in \Omega$, the restriction $B\big|_{V_{x_0}}$ of the set B to V_{x_0} is compact in $W^{l,p}(V_{x_0})$.

2. The set B possesses a uniform 'tale' estimate, i.e. there exists a continuous function $R_B(z) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{z\to\infty} R_B(z) = 0$ and

$$\|u, \Omega \cap B^1_{x_0}\|_{l,p} \le R_B(|x_0|), \quad \forall u \in B. \tag{1.19}$$

This proposition can be easily proved by using the Hausdorff criterium.

The next proposition will be useful in order to verify that a function belongs to the space $\dot W_b^{l,p}(\Omega)$. In order to formulate it, we need the following definition.

Definition 1.5. For every $\beta > 0$ and every $u \in W_b^{l,p}(\Omega)$, we introduce the function $\mathcal{R}^\beta_{l,p}(u, z)$, $z \in \mathbb{R}_+$, as follows:

$$\mathcal{R}^\beta_{l,p}(u, z) := \sup_{x\in\Omega} \left\{ e^{-\beta \operatorname{dist}(x,\mathbb{R}^n\setminus B^z_0)} \|u, \Omega \cap B^1_x\|_{l,p} \right\}. \tag{1.20}$$

Proposition 1.4.

1) Let $u \in W_b^{l,p}(\Omega)$. Then, the following estimate holds for every $z \in \mathbb{R}_+$ and $\beta > 0$:

$$\mathcal{R}^\beta_{l,p}(u, z) \le \max\left\{ \|u, \Omega\|_{b,l,p} e^{-\beta z/2}, \sup_{|x|\ge z/2}\{\|u, \Omega \cap B^1_x\|_{l,p}\} \right\}. \tag{1.21}$$

In particular, $u \in \dot W_b^{l,p}(\Omega)$ if and only if

$$\mathcal{R}^\beta_{l,p}(u, z) \to 0, \quad \textit{as} \quad |z| \to \infty. \tag{1.22}$$

2) Let, in addition, a function $v \in W_b^{l_1,p_1}(\Omega)$ satisfy the estimate

$$\|v, \Omega \cap B^1_{x_0}\|_{l_1,p_1} \le C \sup_{x\in\Omega} \left\{ e^{-\beta|x-x_0|} \|u, \Omega \cap B^1_x\|_{l,p} \right\}, \tag{1.23}$$

for an appropriate $\beta > 0$. Then

$$\mathcal{R}^\beta_{l_1,p_1}(v, z) \le C\mathcal{R}^\beta_{l,p}(u, z), \quad z \in \mathbb{R}_+, \tag{1.24}$$

where the constant C is independent of l, p, l_1, p_1, $\beta > 0$, $z \in \mathbb{R}_+$, u and v. In particular, if $u \in \dot W_b^{l,p}(\Omega)$, then $v \in \dot W_b^{l_1,p_1}(\Omega)$.

Proof. The proof of estimate (1.21) is straightforward and we omit it here. In order to prove estimate (1.24), there remains to observe that the weight function

$$\varphi_{\beta,z}(x) := e^{-\beta \operatorname{dist}(x,\mathbb{R}^n\setminus B^z_0)} \tag{1.25}$$

has the growth rate β and satisfies inequality (1.1) with $C_{\varphi_{\beta,z}} \equiv 1$. Multiplying now inequality (1.23) by $\phi_{\beta,z}(x_0)$, applying the operator $\sup_{x_0\in\Omega}$ and using inequality (1.4), we obtain estimate (1.24) and Proposition 1.4 is proved.

§2 A priori estimates. Existence and uniqueness of solutions.

In this section, we consider the following parabolic boundary value problem:

$$
\begin{cases} \partial_t u - a\Delta_x u + f(u, \nabla_x u) + \lambda_0 u = g(t), \\ u\big|_{t=0} = u_0; \quad u\big|_{\partial\Omega} = 0, \end{cases} \tag{2.1}
$$

in an unbounded domain $\Omega \subset \mathbb{R}^3$ which satisfies conditions (1.5) and (1.6).

We recall that $u = (u^1, \cdots, u^k)$, $a, \lambda_0 > 0$, $f = (f^1, \cdots, f^k)$, $g = (g^1, \cdots, g^k)$ and that the nonlinear term f satisfies the following conditions:

$$
\begin{cases} 1.\ f \in C^1(\mathbb{R}^k \times \mathbb{R}^{3k}, \mathbb{R}^k), \\ 2.\ f(v,p).v \geq 0, \\ 3.\ |f(v,p)| \leq |v|Q(|v|)(1 + |p|^r),\ r < 2, \\ 4.\ |f'_v(v,p)| \leq |v|Q(|v|)(1 + |p|^r), \\ 5.\ |f'_p(v,p)| \leq |v|Q(|v|)(1 + |p|^{r-1}), \end{cases} \tag{2.2}
$$

for every $(v, p) \in \mathbb{R}^k \times \mathbb{R}^{3k}$ and for some monotonic function $Q : R_+ \to \mathbb{R}_+$.

We further assume that the right-hand side $g(t) = g(t, x)$ belongs to the space

$$
\Sigma := C_b(\mathbb{R}, \dot{L}^2_b(\Omega)) \tag{2.3}
$$

(we will assume below that g is translation-compact in Σ, endowed with the local topology, see Section 3). The phase space Φ_b for problem (2.1) is defined as

$$
\Phi_b := W_b^{2-\delta,2}(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}, \tag{2.4}
$$

i.e. $u_0 \in \Phi_b$, where the fixed exponent $\delta > 0$ is such that $\delta < \min\{\frac{1}{r} - \frac{1}{2}, \frac{1}{2}\}$. A solution of equation (2.1) is defined as a function u belonging to the space

$$
L^\infty(\mathbb{R}_+, \Phi_b) \cap C_b([0, \infty), L^2_b(\Omega)),
$$

which satisfies equation (2.1) in the sense of distributions.

The main aim of this section is to derive several useful a priori estimates for the solutions of (2.1) and to prove, based on these estimates, the unique solvability of the problem under consideration. We start by formulating the following L^∞-bounds on the solution u.

Theorem 2.1. *Let u be a solution of* (2.1)*. Then, the following estimate holds, uniformly with respect to $x_0 \in \Omega$:*

$$
\begin{aligned} |u(T, x_0)|^2 \leq C \sup\nolimits_{x\in\Omega}\{e^{-\beta|x-x_0|}|u(0,x)|^2\}e^{-\gamma T} + \\ + C \sup_{t\in[0,T]} \left\{ e^{-\gamma(T-t)} \int_\Omega e^{-\beta|x-x_0|}|g(t,x)|^2\, dx \right\}, \end{aligned} \tag{2.5}
$$

for some $\gamma > 0$ and sufficiently small $\beta > 0$.

Inequality (2.5) follows from a standard application of the maximum principle to the function $w(t, x) = u(t, x).u(t, x)$, see [13] or [23] for details.

The next theorem gives a dissipative estimate for the solution $u(t)$ in the phase space Φ_b.

Theorem 2.2. *Let u be a solution of problem* (2.1)*. Then, the following estimate is valid, uniformly with respect to $x_0 \in \Omega$:*

$$
(2.6)\quad \|u(T), \Omega \cap B^1_{x_0}\|_{2-\delta,2} \le
$$
$$
\le e^{-\gamma T} Q_1(\|u_0\|_{\Phi_b}) \sup_{x\in\Omega}\left\{ e^{-\beta|x-x_0|}\|u_0, \Omega\cap B^1_x\|_{2-\delta,2}\right\} +
$$
$$
+ Q_1(\|g\|_\Sigma) \sup_{t\in[0,T]}\left\{ e^{-\gamma(T-t)} \sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g(t), \Omega\cap B^1_x\|_{0,2}\right\}\right\},
$$

for some monotonic function Q_1 and small constants $\gamma > 0$ and $\beta > 0$.

Proof. We rewrite equation (2.1) as a linear equation:

$$
(2.7)\qquad \partial_t u - a\Delta_x u + \lambda_0 u = h(t) := g(t) - f(u(t), \nabla_x u(t)).
$$

The following analogue of parabolic regularity theorems for weighted Sobolev spaces is proved in [13]:

$$
(2.8)\quad \int_\Omega e^{-2\beta|x-y_0|}\|u(T), V_x\|^2_{2-\delta,2}\,dx \le
$$
$$
\le Ce^{-\gamma T}\int_\Omega e^{-2\beta|x-y_0|}\|u_0, V_x\|^2_{2-\delta,2}\,dx +
$$
$$
+ C\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\int_\Omega e^{-2\beta|x-y_0|}\|h(t), V_x\|^2_{0,2}\,dx\right\},
$$

for some positive constant C and small positive constants γ and β which are independent of $y_0 \in \Omega$ and V_x defined in (1.5) and (1.6). Applying the operator $\sup_{y_0\in\Omega} e^{-\beta|x_0-y_0|}$ to both sides of (2.8) and using (1.17), we obtain the following estimate:

$$
(2.9)\quad \sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u(T), V_x\|^2_{2-\delta,2}\right\} \le
$$
$$
\le C'e^{-\gamma T}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_0, V_x\|^2_{2-\delta,2}\right\} +
$$
$$
+ C'\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g(t), V_x\|^2_{0,2}\right\}\right\} +
$$
$$
+ C'\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|f(u(t), \nabla_x u(t)), V_x\|^2_{0,2}\right\}\right\},
$$

where C' and the small positive constants γ and β are also independent of x_0.

Thus, there only remains to estimate the last integral in the right-hand side of (2.9). According to (2.2), we have

$$
(2.10)\quad \|f(u(t), \nabla_x u(t)), V_x\|^2_{0,2} \le C\|u(t), V_x\|^2_{0,\infty} Q_2(\|u(t)\|_{L^\infty(\Omega)})(\|u, V_x\|^{2r}_{1,2r} + 1).
$$

Using a standard interpolation inequality (see [28]), we obtain

$$
(2.11)\qquad \|u, V_x\|_{1,2r} \le C\|u, V_x\|^{1-\theta}_{0,\infty}\|u, V_x\|^\theta_{2-\delta,2},
$$

where $\theta = \frac{1}{2-\delta} \in (0,1)$ (we also note that, due to assumptions (1.5) and (1.6), the constant C in (2.11) is independent of x). It follows from our assumptions on δ that $2r\theta < 2$ and, consequently, (2.10) and (2.11) imply

$$(2.12)\quad \|f(u(t),\nabla_x u(t)),V_x\|_{0,2}^2 \le \|u(t),V_x\|_{0,\infty}^2 Q_\mu(\|u(t)\|_{L^\infty(\Omega)}) + \mu\|u(t),V_x\|_{2-\delta,2}^2.$$

Here, $\mu > 0$ is an arbitrary positive number and Q_μ is an appropriate monotonic function, depending on μ. According to Theorem 2.1, we have

$$(2.13)\qquad Q_\mu\left(\|u(t)\|_{L^\infty(\Omega)}\right) \le e^{-\gamma t}\tilde{Q}_\mu\left(\|u_0\|_{L^\infty(\Omega)}\right) + \tilde{Q}_\mu\left(\|g\|_\Sigma\right),$$

for an appropriate new function $\tilde{Q}_\mu$. Consequently, due to (2.5), (2.9), (2.12) and (2.13), we obtain

$$(2.14)\quad Z_{x_0}(T) \le e^{-\gamma T} Q_\mu(\|u_0\|_{\Phi_b}) \sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_0,V_x\|_{2-\delta,2}^2\right\} + \\ + Q_\mu(\|g\|_\Sigma) \sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g(t),V_x\|_{0,2}^2\right\}\right\} + \\ + C\mu \sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}Z_{x_0}(t)\right\},$$

where $Z_{x_0}(t) := \sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u(t),V_x\|_{2-\delta,2}^2\right\}$ and where $\mu > 0$ can be chosen arbitrarily small. We now fix $\mu > 0$ such that $C\mu < 1/2$. Then, (2.14) implies that

$$(2.15)\quad Z_{x_0}(T) \le C_1 e^{-\gamma T} Q(\|u_0\|_{\Phi_b}) \sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_0,V_x\|_{2-\delta,2}^2\right\} + \\ + C_1 Q(\|g\|_\Sigma) \sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g(t),V_x\|_{0,2}^2\right\}\right\},$$

for an appropriate monotonic function Q (see [13]). Estimate (2.15), together with the obvious estimate

$$(2.16)\quad C^{-1}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|v,\Omega\cap B_x^1\|_{l,p}\right\} \le \\ \le \sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|v,V_x\|_{l,p}\right\} \le C\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|v,\Omega\cap B_x^1\|_{l,p}\right\},$$

imply (2.6) and Theorem 2.2 is proved.

Corollary 2.1. *Let the assumptions of Theorem 2.2 hold. Then, the following estimate is valid:*

$$(2.17)\qquad \|u(t)\|_{\Phi_b} \le Q_1(\|u_0\|_{\Phi_b})e^{-\gamma t} + Q_1(\|g\|_\Sigma),$$

for some monotonic function Q_1 and positive constant γ.

Indeed, applying $\sup_{x_0\in\Omega}$ to (2.6), we derive (2.17).

Corollary 2.2. *Let u be a solution of* (2.1) *and let the assumptions of Theorem 2.2 be satisfied. Then*

$$\begin{aligned}(2.18)\quad &\|f(u(T),\nabla_x u(T)),\Omega\cap B^1_{x_0}\|^2_{0,2}\le\\ &\le e^{-\gamma T}Q_2(\|u_0\|_{\Phi_b})\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{2-\delta,2}\right\}+\\ &+Q_2(\|g\|_\Sigma)\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g(t),\Omega\cap B^1_x\|^2_{0,2}\right\}\right\},\end{aligned}$$

for some monotonic function Q_2 and positive constants γ and β.

Indeed, (2.18) follows from estimates (2.5), (2.6) and (2.10).

Corollary 2.3. *Let u be a solution of* (2.1) *and let the assumptions of the previous theorem be valid. Then, $u\in C^{1-\delta/2}(\mathbb{R}_+,L^2_b(\Omega))$ and*

$$\begin{aligned}(2.19)\quad &\|u\|^2_{C^{1-\delta/2}([T,T+1],L^2(\Omega\cap B^1_{x_0}))}\le\\ &\le e^{-\gamma T}Q_3(\|u_0\|_{\Phi_b})\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{2-\delta,2}\right\}+\\ &+Q_3(\|g\|_\Sigma)\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g(t),\Omega\cap B^1_x\|^2_{0,2}\right\}\right\}.\end{aligned}$$

Indeed, rewriting equation (2.1) in the form (2.7) and applying the parabolic regularity theorem, together with estimate (2.18), we derive estimate (2.19) (see also [13] for details).

We now derive the smoothing property for equation (2.1).

Corollary 2.4. *Let u be a solution of* (2.1) *and let us assume that the assumptions of Theorem 2.2 hold. Then, for every fixed $0<\delta_1<\delta$*

$$u\in C([1,\infty),W^{2-\delta_1,2}_b(\Omega)),$$

and, for arbitrary $T\ge 1$, the following estimate is satisfied:

$$\begin{aligned}(2.20)\quad &\|u(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta_1,2}\le\\ &\le e^{-\gamma T}Q_4(\|u_0\|_{\Phi_b})\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{2-\delta,2}\right\}+\\ &+Q_4(\|g\|_\Sigma)\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g(t),\Omega\cap B^1_x\|^2_{0,2}\right\}\right\},\end{aligned}$$

where the monotonic function Q_4 depends on δ_1, but is independent of $x_0\in\Omega$.

Proof. We introduce the function $w(t)=(t-T+1)u(t)$. Then, this function satisfies

$$(2.21)\quad \begin{cases}\partial_t w-a\Delta_x w+\lambda_0 w=(t-T+1)g-(t-T+1)f(u,\nabla_x u)+u\equiv\tilde h(t),\\ w\big|_{t=T-1}=0;\ w\big|_{\partial\Omega}=0.\end{cases}$$

Applying the parabolic regularity theorem to the linear equation (2.21), we derive (2.20), analogously to (2.9), but replacing δ by δ_1, using estimates (2.9) and (2.18) in order to estimate the right-hand side of (2.21) and taking into account that $w(T-1)=0$.

The next corollary shows that the trajectory $u(t)$ is Hölder continuous with respect to $t\in[1,\infty)$ in the space Φ_b.

Corollary 2.5. *Let the assumptions of Theorem 2.2 hold and let u be a solution of* (2.1). *Then, the following estimate is valid, for $T\ge 1$:*

$$\|u\|_{C^s([T,T+1],\Phi_b)}\le Q_5(\|u_0\|_{\Phi_b})e^{-\gamma T}+Q_5(\|g\|_\Sigma), \tag{2.22}$$

where $s:=\frac12(\delta-\delta_1)\frac{2-\delta}{2-\delta_1}$, Q_5 is an appropriate monotonic function and $\gamma>0$ is a constant.

Indeed, according to (2.19) and (2.20), we have

$$\|u\|_{C^{1-\delta/2}([T,T+1],L^2_b(\Omega))}+\|u\|_{L^\infty([T,T+1],W^{2-\delta_1,2}_b(\Omega))}\le Q(\|u_0\|_{\Phi_b})e^{-\gamma T}+Q(\|g\|_\Sigma).$$

Interpolating between $C^{1-\delta/2}(L^2_b)$ and $L^\infty(W^{2-\delta_1,2}_b)$, we obtain estimate (2.22).

The next theorem shows that, under the assumptions of Theorem 2.1, equation (2.1) generates a Lipschitz continuous dynamical system in Φ_b.

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold. Then, for every $u_0\in\Phi_b$, equation* (2.1) *possesses a unique solution $u(t)\in\Phi_b$, $t\ge0$, i.e. this equation generates a dynamical process $\{U_g(t,\tau), t\ge\tau,\ \tau\in\mathbb{R}\}$ in the phase space Φ_b:*

$$U_g(t,\tau):\Phi_b\to\Phi_b,\quad U_g(t,\tau)u(\tau):=u(t),\quad t\ge\tau. \tag{2.23}$$

Moreover, if $u_1(t)$ and $u_2(t)$ are two solutions of (2.1) *with right-hand sides g_1 and g_2 respectively, then*

$$\begin{aligned}\|u_1(T)-u_2(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta,2}\le\\ \le Ce^{KT}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{2-\delta,2}\right\}+\\ +C\sup_{t\in[0,T]}\left\{e^{K(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g_1(t)-g_2(t),\Omega\cap B^1_x\|^2_{0,2}\right\}\right\},\end{aligned} \tag{2.24}$$

where the constants $C>0$ and $K>0$ depend on $\|u_i(0)\|_{\Phi_b}$ and $\|g_i\|_\Sigma$, but are independent of $x_0\in\Omega$.

Proof. The existence of a solution for problem (2.1) can be verified, based on estimate (2.17), in a standard way (e.g. by approximating the unbounded domain Ω by bounded ones Ω_n, solving (2.1) in Ω_n by the Leray-Schauder principle and finally passing to the limit $n\to\infty$, see [13] for details).

So, there only remains to verify estimate (2.24). Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.1) with right-hand sides $g_1(t)$ and $g_2(t)$ respectively. Then, the function $v(t):=u_1(t)-u_2(t)$ satisfies the equation

$$\partial_t v=a\Delta_x v-\lambda_0 v-\widehat L_1(t,x)v-\widehat L_2(t,x)\nabla_x v+h(t),\quad v\big|_{t=0}=u_1(0)-u_2(0), \tag{2.25}$$

where $h(t):=g_1(t)-g_2(t)$ and

$$\begin{cases}\widehat L_1(t,x):=\int_0^1 f'_u(su_1(t)+(1-s)u_2(t),s\nabla_x u_1(t)+(1-s)\nabla_x u_2(t))\,ds,\\ \widehat L_2(t,x):=\int_0^1 f'_{\nabla_x u}(su_1(t)+(1-s)u_2(t),s\nabla_x u_1(t)+(1-s)\nabla_x u_2(t))\,ds.\end{cases} \tag{2.26}$$

We note that, due to Hölder's inequality, assumptions (2.2) on the derivatives of f and the Sobolev's embeddings

$$W^{2-\delta,2} \subset W^{1,3r} \subset W^{1,6(r-1)} \text{ and } W^{2-\delta,2} \subset C, \tag{2.27}$$

(we recall that $n = 3$ and $\delta < \frac{1}{r} - \frac{1}{2}$), we have

$$\begin{aligned} &\|\widehat{L}_1(t), V_x\|_{0,3} + \|\widehat{L}_2(t), V_x\|_{0,6} \le \\ &\le (\|u_1(t), V_x\|_{2-\delta,2} + \|u_2(t), V_x\|_{2-\delta,2})\, Q\left(\|u_1(t), V_x\|_{2-\delta,2} + \|u_2(t), V_x\|_{2-\delta,2}\right), \end{aligned} \tag{2.28}$$

where the monotonic function Q is independent of $x \in \Omega$. Consequently, according to (2.17)

$$\|\widehat{L}_1(t), V_x\|_{0,3} + \|\widehat{L}_2(t), V_x\|_{0,6} \le M, \tag{2.29}$$

where the constant M depends on $\|u_i\|_{\Phi_b}$ and $\|g_i\|_\Sigma$, $i = 1, 2$, but is independent of x.

We derive estimate (2.24) in two steps. In a first step, we obtain an estimate of the L^2-norm of v, similar to (2.24).

Lemma 2.1. *Let the assumptions of Theorem 2.3 hold. Then, for every $x_0 \in \Omega$, the following estimate is satisfied:*

$$\begin{aligned} &\sup_{x_0\in\Omega} \left\{e^{-\beta|x-x_0|}\|v(T), V_x\|^2_{0,2}\right\} \le \\ &\quad \le C_M e^{K_M T} \sup_{x_0\in\Omega} \left\{e^{-\beta|x-x_0|}\|v(0), V_x\|^2_{0,2}\right\} + \\ &\quad\quad + C_M \sup_{t\in[0,T]} \left\{e^{K_M(T-t)} \sup_{x_0\in\Omega} \left\{e^{-\beta|x-x_0|}\|h(t), V_x\|^2_{0,2}\right\}\right\}, \end{aligned} \tag{2.30}$$

where $\beta > 0$ is a small constant and constants C_M and K_M depend on the constant M in (2.29)*, but are independent of x_0.*

Proof of Lemma 2.1. We multiply equation (2.25) by $ve^{-2\beta|x-y_0|}$, integrate over Ω and use the obvious inequality

$$|\nabla_x e^{-2\beta|x-y_0|}| \le 2\beta e^{-2\beta|x-y_0|}. \tag{2.31}$$

Then, we obtain, for a sufficiently small $\beta > 0$

$$\begin{aligned} &\partial_t \int_\Omega e^{-2\beta|x-y_0|}|v(t,x)|^2\,dx + \lambda_0 \int_\Omega e^{-2\beta|x-y_0|}|v(t,x)|^2\,dx + \\ &+ a\int_\Omega e^{-2\beta|x-y_0|}|\nabla_x v(t,x)|^2\,dx \le \int_\Omega e^{-2\beta|x-y_0|}|\widehat{L}_1(t,x)|\cdot|v(t,x)|\cdot|v(t,x)|\,dx + \\ &\quad + \int_\Omega e^{-2\beta|x-y_0|}|\widehat{L}_2(t,x)|\cdot|v(t,x)|\cdot|\nabla_x v(t,x)|\,dx + \\ &\quad\quad + \int_\Omega e^{-2\beta|x-y_0|}|h(t,x)|^2\,dx. \end{aligned} \tag{2.32}$$

We estimate the first two terms in the right-hand side of (2.32) by using representation (1.17), Hölder's inequality, classical Sobolev's embedding theorems and estimate (2.29). Indeed, the first term can be estimated as follows:

$$
\begin{aligned}
(2.33)\quad I_1(t) &\le C_1\int_\Omega e^{-\beta|x-x_0|}\|\,|\widehat{L}_1(t)|\cdot|v(t)|\cdot|v(t)|,V_x\|_{0,1}\,dx\le\\
&\le C_2\int_\Omega e^{-2\beta|x-y_0|}\|\widehat{L}_1(t),V_x\|_{0,3}\cdot\|v(t),V_x\|_{0,2}\cdot\|v(t),V_x\|_{1,2}\,dx\le\\
&\le C_3M^2\int_\Omega e^{-2\beta|x-y_0|}|v(t,x)|^2\,dx+a/4\int_\Omega e^{-2\beta|x-y_0|}|\nabla_x v(t,x)|^2\,dx.
\end{aligned}
$$

Analogously, using, in addition, the interpolation inequality

$$
\|v,V_x\|_{0,3}\le C\|v,V_x\|_{0,2}^{1/2}\cdot\|v,V_x\|_{1,2}^{1/2},
$$

we find

$$
\begin{aligned}
(2.34)\quad I_2(t) &\le C\int_\Omega e^{-2\beta|x-y_0|}\|\,|\widehat{L}_2(t)|\cdot|\nabla_x v(t)|\cdot|v(t)|,V_x\|_{0,1}\,dx\le\\
&\le C_1\int_\Omega e^{-2\beta|x-y_0|}\|\widehat{L}_2(t),V_x\|_{0,6}\cdot\|v(t),V_x\|_{0,3}\cdot\|v(t),V_x\|_{1,2}\,dx\le\\
&\le C_2\int_\Omega e^{-2\beta|x-y_0|}\|\widehat{L}_2(t),V_x\|_{0,6}\cdot\|v(t),V_x\|_{0,2}^{1/2}\cdot\|v(t),V_x\|_{1,2}^{3/2}\,dx\le\\
&\le C_3M^4\int_\Omega e^{-2\beta|x-y_0|}|v(t,x)|^2\,dx+a/4\int_\Omega e^{-2\beta|x-y_0|}|\nabla_x v(t,x)|^2\,dx.
\end{aligned}
$$

Inserting (2.33) and (2.34) into (2.32), we have

$$
\begin{aligned}
(2.35)\quad \partial_t\int_\Omega e^{-2\beta|x-y_0|}|v(t,x)|^2\,dx-K_M\int_\Omega e^{-2\beta|x-y_0|}|v(t,x)|^2\,dx+\\
+a/2\int_\Omega e^{-2\beta|x-y_0|}|\nabla_x v(t,x)|^2\,dx\le\int_\Omega e^{-2\beta|x-y_0|}|h(t,x)|^2\,dx,
\end{aligned}
$$

where $K_M:=-\lambda_0+C_3M^2(1+M^2)$. Applying Gronwall's lemma to relation (2.35), we obtain

$$
\begin{aligned}
(2.36)\quad \|v(T),V_{x_0}\|_{0,2}^2\le C_Me^{K_MT}\int_\Omega e^{-2\beta|x-y_0|}|v(0,x)|^2\,dx+\\
+\int_0^T e^{K_M(T-t)}\int_\Omega e^{-2\beta|x-y_0|}|h(t,x)|^2\,dx\,dt.
\end{aligned}
$$

Multiplying (2.36) by $e^{-\beta|x_0-y_0|}$, taking the supremum over $y_0\in\Omega$ of both sides of the obtained inequality and using inequalities (1.17), we finally find (2.30) and Lemma 2.1 is proved.

We are now in a position to complete the proof of the theorem. Indeed, applying the parabolic regularity theorem to equation (2.25) (see [13]), we obtain, analogously to (2.8) and (2.9)

$$(2.37)\quad \sup_{x_0\in\Omega}\left\{e^{-\beta|x-x_0|}\|v(T),V_x\|^2_{2-\delta,2}\right\}\le Ce^{-\gamma T}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|v(0),V_x\|^2_{2-\delta,2}\right\}$$
$$+C\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|h(t),V_x\|^2_{0,2}\right\}\right\}+$$
$$C\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}(\|\widehat{L}_1(t)v(t),V_x\|^2_{0,2}+\|\widehat{L}_2(t)\nabla_x v(t),V_x\|^2_{0,2})\right\}\right\}.$$

Thus, there remains to estimate the last supremum in the right-hand side of (2.37). To this end, we note that, due to Hölder's inequality, the Sobolev's embedding theorems and estimate (2.29)

$$(2.38)\quad \|\widehat{L}_1(t)v(t),V_x\|^2_{0,2}+\|\widehat{L}_2(t)\nabla_x v(t),V_x\|^2_{0,2}\le$$
$$\le C_\mu\|v(t),V_x\|^2_{0,2}+\mu\|v(t),V_x\|^2_{2-\delta,2},$$

which is valid for every $\mu>0$; see also (2.33) and (2.34). Inserting these estimates into (2.37) and using (2.24), we find

$$(2.39)\quad Z_{x_0}(T)\le C'_\mu e^{K'_M T}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|v(0),V_x\|^2_{2-\delta,2}\right\}+$$
$$+C'\sup_{t\in[0,T]}\left\{e^{K'_M(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|h(t),V_x\|^2_{0,2}\right\}\right\}+$$
$$+C\mu\sup_{t\in[0,T]}\left\{e^{-\gamma(T-t)}Z_{x_0}(t)\right\},$$

where $K'_M:=\max\{-\gamma,K_M\}$ and $Z_{x_0}(t):=\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|v(t),V_x\|^2_{2-\delta,2}\right\}$. Fixing $\mu<1/(2C)$ in (2.39), we obtain (2.24) as in (2.14) and Theorem 2.3 is proved.

Remark 2.1. We note that the constant $K=K'_M$ in (2.24) is defined by

$$(2.40)\qquad K'_M=\max\{-\gamma,-\lambda_0+CM^2(1+M^2)\},$$

where M is defined in (2.29) and γ and C are independent of M. Therefore, the exponent K'_M is *negative*, if M is small enough. This will be essential for the next sections.

The next corollary is an analogue of Corollary 2.4 for equation (2.25).

Corollary 2.6. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.1) with right-hand sides $g_1(t)$ and $g_2(t)$ respectively. Then, for every fixed $0<\delta_1<\delta$ and for an arbitrary $T\ge1$, the following estimate holds:*

$$(2.41)\quad \|u_1(T)-u_2(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta_1,2}\le$$
$$\le Ce^{KT}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_1(0)-u_2(0),V_x\|^2_{2-\delta,2}\right\}+$$
$$+C\sup_{t\in[0,T]}\left\{e^{K(T-t)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|g_1(t)-g_2(t),V_x\|^2_{0,2}\right\}\right\},$$

where the constant C depends on δ_1, but is independent of $x_0 \in \Omega$.

Indeed, we set $w(t) = (t - T + 1)v(t)$. Then

$$\partial_t w - a\Delta_x w + \lambda_0 w = v(t) + (t - T + 1)h(t)- \\ - (t - T + 1)\left(\widehat{L}_1(t)v(t) + \widehat{L}_2(t)\nabla_x v(t)\right) \equiv \tilde{h}(t), \quad w\big|_{t=T-1} = 0, \quad w\big|_{\partial\Omega} = 0. \tag{2.42}$$

Applying the parabolic regularity theorem to the linear equation (2.42), we obtain (2.41), analogously to (2.9), but replacing δ by δ_1, using estimates (2.24) and (2.38) in order to estimate the right-hand side of (2.42) and taking into account that $w(T-1) = 0$.

§3 The global (uniform) attractor.

In this section, we prove that equation (2.1) possesses the global (uniform) attractor in the phase space Φ_b. To this end, following [6-7], [21] and [34], we consider a family of equations of type (2.1), generated by all time translations of the initial equation and their limits in the corresponding topology. To be more precise, we consider the following family of problems:

$$\partial_t u = a\Delta_x u - \lambda_0 u - f(u, \nabla_x u) + \xi(t), \quad \xi \in \mathcal{H}(g), \tag{3.1}$$

where $\mathcal{H}(g)$ is the hull of the initial external forces g, defined via

$$\mathcal{H}(g) := \left[T_h g, h \in \mathbb{R}\right]_{\Sigma_{loc}}, \quad (T_h g)(t) := g(t+h), \tag{3.2}$$

$\Sigma_{loc} := C_{loc}(\mathbb{R}, \dot{L}^2_b(\Omega))$ and $[\cdot]_{\Sigma_{loc}}$ denotes the closure in the space Σ_{loc}.

In order to construct the attractor for family (3.2), we further assume that the hull (3.2) is compact in the space Σ_{loc}:

$$\mathcal{H}(g) \subset\subset \Sigma_{loc} \tag{3.3}$$

(following [6], such functions are called translation-compact in Σ_{loc}). We now introduce the function

$$\mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), z) := \sup_{t\in\mathbb{R}} \sup_{\xi\in\mathcal{H}(g)} \mathcal{R}^{\beta}_{0,2}(\xi(t), z), \tag{3.4}$$

where $\beta > 0$ is a sufficiently small parameter which will be specified below and the function $\mathcal{R}^{\beta}_{0,2}(\xi, z)$ is defined by (1.18). Then, due to (3.3) and due to Propositions 1.3 and 1.4, we have

$$\mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), z) \to 0 \text{ as } z \to \infty. \tag{3.5}$$

We now define the semigroup $\{\mathbb{S}_h,\ h \geq 0\}$ in the extended phase space $\Phi_b \times \mathcal{H}(g)$ via

$$\mathbb{S}_h : \Phi_b \times \mathcal{H}(g) \to \Phi_b \times \mathcal{H}(g), \quad \mathbb{S}_h(u_0, \xi) := (U_\xi(h, 0)u_0, T_h\xi), \tag{3.6}$$

where $U_\xi(t, \tau)$ is the solving operator of problem (2.1) with the external forces g replaced by $\xi \in \mathcal{H}(g)$.

We recall that a set $\mathbb{A} \subset \Phi_b \times \mathcal{H}(g)$ is the global attractor of semigroup (3.6) if

1. The set $\mathbb{A}$ is compact in $\Phi_b \times \mathcal{H}(g)$.
2. The set $\mathbb{A}$ is strictly invariant, i.e. $\mathbb{S}_h\mathbb{A} = \mathbb{A}$.
3. $\mathbb{A}$ is an attracting set for the semigroup $\mathbb{S}_h$, i.e. for every neighborhood $\mathcal{O}(\mathbb{A})$ of the attractor $\mathbb{A}$ in the space $\Phi_b \times \mathcal{H}(g)$ and every bounded subset $\mathbb{B} \subset \Phi_b \times \mathcal{H}(g)$, there exists $T = T(\mathcal{O}, \mathbb{B})$ such that

$$\mathbb{S}_h\mathbb{B} \subset \mathcal{O}(\mathbb{A}) \text{ for } h \geq T, \tag{3.7}$$

(see e.g. [6-7] for details). If the global attractor $\mathbb{A}$ exists, then the projection $\mathcal{A} := \Pi_1\mathbb{A}$ onto the first component of $\Phi_b \times \mathcal{H}(g)$ is called the uniform attractor of equation (2.1).

Theorem 3.1. *Let the assumptions of Theorem 2.2 hold and let in addition the external forces g be translation-compact in Σ_{loc} (see* (3.3)*). Then, the semigroup $\mathbb{S}_h$ possesses the global attractor $\mathbb{A}$ in the space $\Phi_b \times \mathcal{H}(g)$. Moreover, the associated uniform attractor $\mathcal{A}$ belongs to the space*

$$\dot{\Phi}_b := \dot{W}_b^{2-\delta,2}(\Omega), \tag{3.8}$$

and has the following structure:

$$\mathcal{A} = \cup_{\xi \in \mathcal{H}(g)} \mathcal{K}_\xi\big|_{t=0}, \tag{3.9}$$

where $\mathcal{K}_\xi$, $\xi \in \mathcal{H}(g)$, is the set of all solutions $u(t)$ of (3.1) *defined and bounded for all $t \in \mathbb{R}$ ($u \in L^\infty(\mathbb{R}, \Phi_b)$).*

Proof. We first prove that $\mathbb{S}_h$ possesses a *locally compact* attractor in the space $\Phi_b \times \mathcal{H}(g)$, i.e. an attractor that is a priori only bounded in $\Phi_b \times \mathcal{H}(g)$, but compact in the local topology of $\Phi_{loc} \times \mathcal{H}(g)$, where

$$\Phi_{loc} := W_{loc}^{2-\delta,2}(\overline{\Omega}), \tag{3.10}$$

and attracts bounded subsets of $\Phi_b \times \mathcal{H}(g)$ also in the local topology (i.e. the neighborhood $\mathcal{O}(\mathbb{A})$ in (3.7) must be understood in the space $\Phi_{loc} \times \mathcal{H}(g)$, see e.g. [34]). We recall that, due to the abstract attractor's existence theorem (see [3]), it suffices, in order to obtain such an attractor, to verify the following conditions:

1. The operators $\mathbb{S}_t$ are $\Phi_{loc} \times \mathcal{H}(g)$-continuous on every $\Phi_b \times \mathcal{H}(g)$-bounded subset B and every fixed $t \geq 0$.
2. The semigroup $\mathbb{S}_t$ possesses a bounded in $\Phi_b \times \mathcal{H}(g)$ and compact in $\Phi_{loc} \times \mathcal{H}(g)$ attracting set K.

Let us verify these conditions for our semigroup. The first condition is an immediate corollary of estimate (2.24). Applying the operator $\sup_{x_0 \in \Omega}$ to both sides of (2.20) and using (1.4), we have, for $T \geq 1$

$$\|u(T)\|_{W_b^{2-\delta_1,2}(\Omega)} \leq e^{-\gamma T} Q(\|u_0\|_{\Phi_b}) + Q(\|g\|_\Sigma), \tag{3.11}$$

for an appropriate monotonic function Q and fixed constants $0 < \delta_1 < \delta$ and $\gamma > 0$. Estimate (3.11), together with (3.3) and the obvious inequality

$$\|\xi\|_\Sigma \leq \|g\|_\Sigma, \quad \text{for every } \xi \in \mathcal{H}(g), \tag{3.12}$$

imply that the set

$$K := \left\{ u_0 \in W_b^{2-\delta_1,2}(\Omega) \,:\, \|u_0\|_{W_b^{2-\delta_1,2}} \le 2Q(\|g\|_\Sigma) \right\} \times \mathcal{H}(g) \tag{3.13}$$

is indeed an absorbing set for the semigroup $\mathbb{S}_h$. We note that the embedding $W_b^{2-\delta_1,2}(\Omega) \subset \Phi_{loc}$ is compact (since $\delta_1 < \delta$). Thus, the second condition is also satisfied and, consequently, the semigroup $\mathbb{S}_h$ possesses a *locally compact* attractor $\mathbb{A}$. Relation (3.9) then follows from the general global attractor's description theorem, see [3] and [6].

Let us now prove that the obtained attractor is in fact compact in the uniform topology of $\Phi_b \times \mathcal{H}(g)$. To this end, we note that, according to (2.6), every complete bounded solution $u(t) \in \mathcal{K}_\xi$ of equation (3.1) satisfies the estimate

$$\begin{aligned} &\|u(T), \Omega \cap B^1_{x_0}\|^2_{2-\delta,2} \le \\ &\quad \le Q(\|g\|_\Sigma) \sup_{t \in (-\infty, T]} \left\{ e^{-\gamma(T-t)} \sup_{x \in \Omega} \left\{ e^{-\beta|x-x_0|} \|\xi(t), \Omega \cap B^1_x\|^2_{0,2} \right\} \right\}. \end{aligned} \tag{3.14}$$

Proposition 1.4 now implies that

$$\|u(T), \Omega \cap B^1_{x_0}\|_{2-\delta_1,2} \le Q(\|g\|_\Sigma) \mathcal{R}^\beta_{0,2}(\mathcal{H}(g), x_0), \tag{3.15}$$

for some function Q and, consequently

$$\|\mathcal{A}, B^1_{x_0} \cap \Omega\|_{2-\delta_1,2} \le Q(\|g\|_\Sigma) \mathcal{R}^\beta_{0,2}(\mathcal{H}(g), x_0). \tag{3.16}$$

Estimate (3.16), together with condition (3.5), imply that $\mathcal{A} \subset \dot{\Phi}_b$ and possesses a uniform 'tale' estimate, i.e.

$$\lim_{|x_0| \to \infty} \|\mathcal{A}, \Omega \cap B^1_{x_0}\|_{2-\delta,2} = 0. \tag{3.17}$$

We also note that, by construction, $\mathcal{A}$ is compact in Φ_{loc}. Thus, due to Proposition 1.3, the set $\mathcal{A}$ is compact in $\dot{\Phi}_b$ and, therefore, $\mathbb{A}$ is compact in $\Phi_b \times \mathcal{H}(g)$.

So, there remains to verify that $\mathbb{A}$ is an attracting set for $\mathbb{S}_h$ in the uniform topology of $\Phi_b \times \mathcal{H}(g)$. Let us assume the converse. Then, there exist a sequence $\xi_n \in \mathcal{H}(g)$ of external forces, a sequence of solutions $u_n(t)$ of (3.1) with ξ replaced by ξ_n, $u_n(0)$ belonging to some bounded subset $B \subset \Phi_b$, and a sequence $t_n \to \infty$ such that

$$\operatorname{dist}_{\Phi_b}(u_n(t_n), \mathcal{A}) \ge \beta_0 > 0. \tag{3.18}$$

Since $\mathcal{A}$ is a *locally compact* attractor, then there exists $u_0 \in \mathcal{A}$ such that

$$\|u_n(t_n) - u_0, \Omega \cap B^R_0\|_{2-\delta,2} \to 0, \quad \text{for every } R > 0. \tag{3.19}$$

It follows from (3.6) and Proposition 1.4 that

$$\|u_n(t_n), \Omega \backslash B^R_0\|_{2-\delta,2} \le e^{-\gamma t_n} Q(\|B\|_{\Phi_b}) + Q(\|g\|_\Sigma) \mathcal{R}^\beta_{0,2}(\mathcal{H}(g), R), \tag{3.20}$$

and, consequently, due to (3.16) and (3.20)

$$\limsup_{n\to\infty} \|u_n(t_n) - u_0, \Omega\backslash B_0^R\|_{2-\delta,2} \le Q(\|g\|_\Sigma)\mathcal{R}^\beta_{0,2}(\mathcal{H}(g), R). \tag{3.21}$$

Convergences (3.5), (3.19) and (3.21) contradict (3.18). Therefore, $\mathbb{A}$ is an attracting set in the uniform topology of $\Phi_b \times \mathcal{H}(g)$ and Theorem 3.1 is proved.

Remark 3.1. The uniform attractor $\mathcal{A}$ of equation (2.1) can be defined without using the extended semigroup $\mathbb{S}_h$ and the skew-product technique. Namely, the set $\mathcal{A}$ is called the uniform attractor of equation (2.1) if

1. $\mathcal{A}$ is compact in Φ_b.
2. $\mathcal{A}$ is a uniformly attracting set for the process $\{U_g(t,\tau),\, t \ge \tau,\, \tau \in \mathbb{R}\}$, i.e. for every neighborhood $\mathcal{O}(\mathcal{A})$ in Φ_b and for every bounded subset $B \subset \Phi_b$, there exists $T = T(\mathcal{O}, B)$ such that, for every $\tau \in \mathbb{R}$

$$U_g(\tau + t, \tau)B \subset \mathcal{O}(\mathcal{A}) \quad \text{for } t \ge T. \tag{3.22}$$

3. $\mathcal{A}$ is minimal among the closed sets enjoying properties 1 and 2.

The equivalence of the two definitions is proved in [6].

Remark 3.2. We note that every periodic, quasiperiodic and almost-periodic with values in $\dot{L}^2_b(\Omega)$ external force is obviously translation-compact in Σ_{loc} and, consequently, equation (2.1) with right-hand sides from these classes possesses the uniform attractor $\mathcal{A}$. We also note that the class of translation-compact functions is larger than the class of almost-periodic functions. Indeed, it is not difficult to verify, using the Arzela-Ascoli theorem, that every

$$g \in C^\alpha_b(\mathbb{R}, \dot{W}^{\alpha,2}_b(\Omega)), \quad \alpha > 0, \tag{3.33}$$

is translation-compact in $L^\infty_{loc}(\mathbb{R}, \dot{L}^2_b(\Omega))$. So, the translation compactness can be interpreted as some regularity assumption on g. Moroever, it is possible to verify that g is translation-compact in Σ_{loc} if and only if

$$g \in \left[C^1_b(\mathbb{R}, C^1_{00}(\mathbb{R}^3)\big|_\Omega)\right]_{C_b(\mathbb{R}, L^2_b(\Omega))}, \tag{3.34}$$

where $C^1_{00}(\mathbb{R}^3)$ denotes the subspace of $C^1(\mathbb{R}^3)$ consisting of functions with finite support in $\mathbb{R}^3$ (see [34] for details).

Now, in order to study the Kolmogorov's entropy of the obtained attractor, we need some estimates on the difference of solutions belonging to a neighborhood of the attractor $\mathcal{A}$. First, we construct a special neighborhood of $\mathcal{A}$ in Φ_b. To this end, analogously to (3.4), we introduce, for every $u_0 \in \Phi_b$, the function

$$\mathcal{R}^\beta_{\Phi_b}(u_0, z) := \mathcal{R}^\beta_{2-\delta,2}(u_0, z), \tag{3.35}$$

where the constant $\beta > 0$ is the same as in (2.6). Then, due to (2.6), (1.23) and (1.24), we have the following 'tale' estimate for the solution $u(t)$ of equation (3.1):

$$\mathcal{R}^\beta_{\Phi_b}(u(t), z) \le Q(\|u(0)\|_{\Phi_b})e^{-\gamma t}\mathcal{R}^\beta_{\Phi_b}(u(0), z) + Q(\|g\|_\Sigma)\mathcal{R}^\beta_{0,2}(\mathcal{H}(g), z). \tag{3.36}$$

We finally set, for every $R \geq 0$

$$\mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), z, R) := 2Q(\|g\|_{\Sigma}) \begin{cases} \mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), z), & z < R, \\ \mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), R), & z \geq R, \end{cases} \tag{3.37}$$

and

$$\mathcal{V}_R := \left\{ u_0 \in \Phi_b : \mathcal{R}^{\beta}_{\Phi_b}(u_0, z) \leq \mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), z, R), \ \forall z \in \mathbb{R}_+ \right\}. \tag{3.38}$$

Then, on the one hand, it follows from (3.36) that set (3.38) is a uniformly absorbing set for the family of equations (3.1). On the other hand, (3.36) implies that, for every $u_0 \in \mathcal{V}_R$, the corresponding solution $u(t)$ of (3.1) satisfies

$$\mathcal{R}^{\beta}_{\Phi_b}(u(t), z) \leq \tilde{Q}(\|g\|_{\Sigma}) \mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), z, R), \tag{3.39}$$

where $\tilde{Q}$ is independent of R.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. Then, there exists $R = R_{eff} > 0$ such that, for every solutions $u_1(t)$ and $u_2(t)$ of equation* (3.1) *with right-hand sides $g_1 \in \mathcal{H}(g)$ and $g_2 \in \mathcal{H}(g)$ respectively and with initial data $u_1(0), u_2(0) \in \mathcal{V}_{R_{eff}}$*

$$u_1(t) - u_2(t) = \mathcal{L}_{u_1,u_2}(t) + \mathcal{N}_{u_1,u_2}(t), \tag{3.40}$$

where the first term in (3.40) *satisfies*

$$\|\mathcal{L}_{u_1,u_2}(t)\|_{\Phi_b} \leq Ce^{-\gamma t}\|u_1(0) - u_2(0)\|_{\Phi_b} + Ce^{Kt}\|g_1 - g_2\|_{C([0,t],L^2_b)}, \tag{3.41}$$

and the second one can be estimated as follows:

$$\|\mathcal{N}_{u_1,u_2}(t), \Omega \cap B^1_{x_0}\|_{2-\delta_1,2} \leq Ce^{Kt-\beta \operatorname{dist}(x_0, B_0^{R_{eff}})}\|u_1(0) - u_2(0)\|_{\Phi_b}, \tag{3.42}$$

where the constant R_{eff} can be found from the following equation:

$$\mathcal{R}^{\beta}_{0,2}(\mathcal{H}(g), R_{eff}) = \mu, \quad \mu := \mu(f, a, \lambda_0, \|g\|_{\Sigma}), \tag{3.43}$$

and the positive constants C, γ, β, μ, K and $\delta_1 < \delta$ depend only on $\|g\|_{\Sigma}$, but are independent of x_0, R_{eff} and $u_1(0), u_2(0) \in \mathcal{V}_{R_{eff}}$.

Proof. We set $v(t) := u_1(t) - u_2(t)$ and $h(t) := g_1(t) - g_2(t)$. Then, the function v obviously satisfies equation (2.25). Let us split the function $v(t)$ into a sum of three functions $v_i(t)$, $i = 1, 2, 3$:

$$v(t) = v_1(t) + v_2(t) + v_3(t), \tag{3.44}$$

where $v_1(t)$ is solution of

$$\partial_t v_1 = a\Delta_x v_1 - \lambda_0 v_1 - \widehat{L}_1(t)v_1 - \widehat{L}_2(t)\nabla_x v_1 + h(t), \quad v_1\big|_{t=0} = 0, \tag{3.45}$$

(the functions $\widehat{L}_1(t)$ and $\widehat{L}_2(t)$ are defined by (2.26)), the function $v_2(t)$ is solution of

$$(3.46)\quad \partial_t v_2 = a\Delta_x v_2 - \lambda_0 v_2 - (1-\chi_R)\widehat{L}_1(t)v_2 - (1-\chi_R)\widehat{L}_2(t)\nabla_x v_2, \ \ v_2\big|_{t=0} = v(0),$$

$\chi_R = \chi_R(x)$ being the characteristic function of the R-ball B_0^R in $\mathbb{R}^n$, and the function $v_3(t)$ is solution of

$$(3.47)\qquad \partial_t v_3 = a\Delta_x v_3 - \lambda_0 v_3 - \widehat{L}_1(t)v_3 - \widehat{L}_2(t)\nabla_x v_3 + h_R(t), \ \ v_3\big|_{t=0} = 0,$$

with $h_R(t) := -\chi_R(x)\left(\widehat{L}_1(t)v_2(t) + \widehat{L}_2(t)\nabla_x v_2(t)\right)$.

We note that equation (3.45) is very similar to (2.25) and, consequently, estimate (2.24) implies that

$$(3.48)\qquad \|v_1(t)\|_{\Phi_b} \le Ce^{Kt}\|h\|_{C([0,t],L_b^2(\Omega))},$$

for appropriate constants C and K, depending only on $\|g\|_\Sigma$. Let us then investigate equation (3.46). To this end, we observe that, due to (2.28) and (3.39), we have

$$(3.49)\quad \|(1-\chi_R)\widehat{L}_1(t),\Omega\cap B_{x_0}^1\|_{0,3} + \|(1-\chi_R)\widehat{L}_2(t),\Omega\cap B_{x_0}^1\|_{0,6} \le \\ \le \tilde{Q}(\|g\|_\Sigma)\mathcal{R}_{0,2}^\beta(\mathcal{H}(g),R) := M_R,$$

for some monotonic function $\tilde{Q}$. We note that equation (3.46) is also of the form (2.25) and, consequently, the function $v_2(t)$ satisfies the following analogue of (2.24):

$$(3.50)\qquad \|v_2(t)\|_{\Phi_b} \le Ce^{K_{M_R}t}\|v(0)\|_{\Phi_b},$$

where C depends only on $\|g\|_\Sigma$ and the coefficient K_M satisfies

$$(3.51)\qquad K_M := C_1 M_R^2(1+M_R^2) - C_2\lambda_0.$$

Since $\mathcal{R}_{0,2}^\beta(\mathcal{H}(g),R) \to 0$ as $R \to \infty$, then it is possible to fix $R = R_{eff}$ such that $K_M \le -\gamma := -C_2\lambda_0/2$. Moreover, the minimal number R which possesses this property obviously satisfies (3.43). In that case, (3.50) reads

$$(3.52)\qquad \|v_2(t)\|_{\Phi_b} \le Ce^{-\gamma t}\|v(0)\|_{\Phi_b}.$$

So, there only remains to study equation (3.47). As above, equation (3.47) is of the form (2.24). Applying the smoothing property (2.40) and using estimates (2.38) and (2.53), we deduce that, for some $0 < \delta_1 < \delta$ and for all $t \ge 1$

$$(3.53)\quad \|v_3(t),\Omega\cap B_{x_0}^1\|_{2-\delta_1,2} \le \\ \le C\sup_{s\in[0,t]}\left\{e^{K(t-s)}\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|h(s),\Omega\cap B_{x_0}^1\|_{0,2}\right\}\right\} \le \\ \le C_1 e^{Kt}\|v_2(t)\|_{\Phi_b}\sup_{|x|\le R}\left\{e^{-\beta|x-x_0|}\right\} \le C_2 e^{Kt-\beta\,\mathrm{dist}(x_0,B_0^R)}\|v(0)\|_{\Phi_b}.$$

We now set

$$(3.54)\qquad \mathcal{L}_{u_1,u_2}(t) := v_1(t) + v_2(t), \quad \mathcal{N}_{u_1,u_2}(t) := v_3(t).$$

Then, estimates (3.48), (3.52) and (3.53) give (3.41) and (3.42) and Theorem 3.2 is proved.

The next theorem shows that the 'size' of the attractor $\mathcal{A}\big|_{\Omega\cap B_{x_0}^1}$ decays exponentially as $|x_0| \to \infty$.

Theorem 3.3. *Let the assumptions of Theorem 3.2 hold and let $u_1(t)$ and $u_2(t)$, $t \in \mathbb{R}$, be two arbitrary solutions belonging to the set $\mathcal{K}_\xi$ (defined in (3.9)), for some $\xi \in \mathcal{H}(g)$. Then, the following estimate is valid:*

$$\|u_1(t) - u_2(t), \Omega \cap B^1_{x_0}\|_{2-\delta,2} \le Ce^{-\beta \operatorname{dist}(x_0, B_0^{R_{eff}})}, \quad t \in \mathbb{R}, \tag{3.55}$$

where the constant C is independent of $u_1, u_2 \in \mathcal{K}_\xi$, $t \in \mathbb{R}$ and $\xi \in \mathcal{H}(g)$ and the constants $\beta > 0$ and R_{eff} are the same as in Theorem 3.2.

Proof. We set $v(t) := u_1(t) - u_2(t)$. Then, this function satisfies the equation

$$\partial_t v = a\Delta_x v - \lambda_0 v - (1-\chi_{R_{eff}})\widehat{L}_1(t)v - (1-\chi_{R_{eff}})\widehat{L}_2(t)\nabla_x v = h_v(t), \ \ t \in \mathbb{R}, \tag{3.56}$$

where the functions $\widehat{L}_1$ and $\widehat{L}_2$ are defined in (2.26) and

$$h_v(t) := \chi_{R_{eff}}(x)\widehat{L}_1(t)v(t) + \ \chi_{R_{eff}}(x)\widehat{L}_2(t)\nabla_x v(t). \tag{3.57}$$

Since the attractor $\mathcal{A}$ is bounded in Φ_b, then

$$\|h_v(t), B^1_{x_0}\|_{L^2} \le C\chi_{R_{eff}+1}(x_0), \tag{3.58}$$

where the constant C is independent of t, u_1, u_2 and ξ. We now recall that we fix the effective radius R_{eff} such that (see (3.51)) equation (3.56) (or, equivalently, equation (3.46)) is exponentially stable and, consequently, according to (2.24), we have

$$\|v(t), \Omega \cap B^1_{x_0}\|^2_{2-\delta,2} \le \\ \le C \sup_{s\in(-\infty,t]} \left\{ e^{-\beta(t-s)} \left\{ \sup_{x\in\Omega} e^{-\beta|x-x_0|} \|h_v(s), \Omega \cap B^1_x\|^2_{0,2} \right\} \right\}, \tag{3.59}$$

where $\beta > 0$ and C are independent of t, u_1, u_2 and ξ. Inserting estimate (3.58) into the right-hand side of (3.59), we finally obtain inequality (3.55). This finishes the proof of Theorem 3.3.

Corollary 3.1. *Let the assumptions of Theorem 3.2 hold and let, in addition, equation (2.1) be autonomous:*

$$g(t,x) := g(x) \in \dot{L}^2_b(\Omega). \tag{3.60}$$

We further assume that $z_0 \in \dot{W}^{2,2}(\Omega)$ is an arbitrary equilibrium of equation (2.1):

$$a\Delta_x z_0 - \lambda_0 z_0 - f(z_0, \nabla_x z_0) = g, \quad z_0\big|_{\partial\Omega} = 0. \tag{3.61}$$

Then, the following estimate holds:

$$\|\mathcal{A} - z_0, \Omega \cap B^1_{x_0}\|_{2-\delta,2} \le Ce^{-\beta \operatorname{dist}(x_0, B_0^{R_{eff}})}, \tag{3.62}$$

where $\beta > 0$ and $C > 0$ are independent of z_0.

Indeed, taking $u_2(t) := z_0$ in (3.55), we find (3.62).

Remark 3.3. Estimate (3.62) describes the asymptotic behavior of the attractor $\mathcal{A}$ as $|x| \to \infty$ in the autonomous case. Moreover, this estimate shows that, outside the effective domain $\Omega_{eff} := \Omega \cap B_0^{R_{eff}}$, the attractor $\mathcal{A}$ coincides with the equilibrium z_0 up to exponentially small terms as $|x| \to \infty$. That is the reason why we can predict that the dynamics generated by (0.1) in an unbounded domain Ω can be more or less well approximated by equation (0.1) in the bounded domain Ω_{eff}. In the next section, we partially prove this conjecture by showing that $\mathcal{A}$ has indeed a finite fractal dimension which can be estimated from above in terms of R_{eff}:

$$\dim_F(\mathcal{A}, \Phi_b) \le C \operatorname{vol}(\Omega_{eff}), \tag{3.63}$$

where C is independent of R_{eff}. We also note that the right-hand side of (3.63) coincides with the typical asymptotics for the dimension of the attractor $\mathcal{A}$ of equation (0.1) in the *bounded* domain Ω_{eff} (see e.g. [3]).

§4 Kolmogorov's ε-entropy of the global (uniform) attractor.

This section is devoted to the study of the Kolmogorov's ε-entropy of the global (uniform) attractor $\mathcal{A}$ of equation (2.1). First, we briefly recall the definition of Kolmogorov's ε-entropy and give some classical examples (see [22] and [28] for a more detailed study).

Definition 4.1. Let K be a (pre)compact set in a metric space M. Then, due to Hausdorff's criteria, it can be covered by a finite number of ε-balls in M. Let $N_\varepsilon(K, M)$ be the minimal number of ε-balls that cover K. Then, by definition, the Kolmogorov's ε-entropy of K is defined as follows:

$$\mathbb{H}_\varepsilon(K, M) := \log_2 N_\varepsilon(K, M). \tag{4.1}$$

We now give several examples of typical asymptotics for the ε-entropy.

Example 4.1. We assume that $K = [0,1]^n$ and $M = \mathbb{R}^n$ (more generally, K is a n-dimensional compact Lipschitz manifold of a metric space M). Then

$$\mathbb{H}_\varepsilon(K, M) = (n + \overline{\overline{o}}(1)) \log_2 \frac{1}{\varepsilon}, \text{ as } \varepsilon \to 0. \tag{4.2}$$

This example justifies the definition of the fractal dimension.

Definition 4.2. The fractal dimension $\dim_F(K, M)$ is defined as

$$\dim_F(K, M) := \operatorname{limsup}_{\varepsilon \to 0} \frac{\mathbb{H}_\varepsilon(K, M)}{\log_2 1/\varepsilon}. \tag{4.3}$$

The following example shows that, for sets that are not manifolds, the fractal dimension may be noninteger.

Example 4.2. Let K be a standard ternary Cantor set in $M = [0, 1]$. Then

$$\dim_F(K, M) = \frac{\ln 2}{\ln 3} < 1.$$

The next example gives the typical behavior of the entropy in classes of functions with finite smoothness.

Example 4.3. Let V be a smooth bounded domain of $\mathbb{R}^n$ and let K be the unit ball in the Sobolev space $W^{l_1,p_1}(V)$ and M be another Sobolev space $W^{l_2,p_2}(V)$ such that the embedding $W^{l_1,p_1} \subset W^{l_2,p_2}$ is compact, i.e.

$$l_1 > l_2 \geq 0, \quad \frac{l_1}{n} - \frac{1}{p_1} > \frac{l_2}{n} - \frac{1}{p_2}.$$

Then, the entropy $\mathbb{H}_\varepsilon(K, M)$ has the following asymptotics (see [28]):

$$C_1 \left(\frac{1}{\varepsilon}\right)^{n/(l_1-l_2)} \leq \mathbb{H}_\varepsilon(K, M) \leq C_2 \left(\frac{1}{\varepsilon}\right)^{n/(l_1-l_2)}. \tag{4.4}$$

Finally, the last example shows the typical behavior of the entropy in classes of analytic functions.

Example 4.4. Let $V_1 \subset V_2$ be two bounded domains of $\mathbb{C}^n$. We assume that K is the set of all analytic functions ϕ in V_2 such that $\|\phi\|_{C(V_2)} \leq 1$ and that $M = C(V_1)$. Then

$$C_3 \left(\log_2 1/\varepsilon\right)^{n+1} \leq \mathbb{H}_\varepsilon(K\big|_{V_1}, M) \leq C_4 \left(\log_2 1/\varepsilon\right)^{n+1}, \tag{4.5}$$

(see [22]).

We now state the main result of this section.

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold. Then, the entropy of the attractor $\mathcal{A}$ satisfies (for $\varepsilon > 0$ small enough):*

$$\begin{aligned} \mathbb{H}_\varepsilon(\mathcal{A}, \Phi_b) \leq C \operatorname{vol}(\Omega \cap B_0^{R_{eff}+C_1}) \log_2 \frac{1}{\varepsilon} + \\ + \mathbb{H}_{\varepsilon/K}\left(\mathcal{H}(g)\big|_{[0, L\log_2 1/\varepsilon]\times\Omega}, C_b([0, L\log_2 1/\varepsilon], L^2_b(\Omega))\right), \end{aligned} \tag{4.6}$$

where R_{eff} is defined in Theorem 3.2 and constants C, C', L and K depend only on the left-hand side of equation (2.1) *and on $\|g\|_\Sigma$ and are independent of R_{eff}.*

Proof. We adapt the method suggested in [16] and [34] to our situation. Let $\mathcal{V}_R \subset \Phi_b$ be the same as in Theorem 3.2. Then, according to (3.41), (3.42) and (3.36), there exists a time $T = T(\|g\|_\Sigma)$ such that

$$U_\xi(\tau + T, T)\mathcal{V}_R \subset \mathcal{V}_R, \quad \tau \in \mathbb{R}, \quad \xi \in \mathcal{H}(g), \tag{4.7}$$

and, for every $u_1(0), u_2(0) \in \mathcal{V}_R$, decomposition (3.40) is valid:

$$\begin{cases} \|\mathcal{L}_{u_1,u_2}(T)\|_{\Phi_b} \leq 1/16\|u_1(0) - u_2(0)\|_{\Phi_b} + K_1\|g_1 - g_2\|_{C_b([0,T],L^2_b)}, \\ \|\mathcal{N}_{u_1,u_2}(T)\|_{\Phi_{eff}} \leq L_1\|u_1(0) - u_2(0)\|_{\Phi_b}, \end{cases} \tag{4.8}$$

where $\Phi_{eff} := W^{2-\delta_1,2}_{b,\phi_{eff}}(\Omega)$, $\phi_{eff}(x) := e^{+\beta \operatorname{dist}(x, B_0^{R_{eff}})}$, $\delta_1 < \delta$, and the constants K_1 and L_1 depend only on $\|g\|_\Sigma$. It is extremely important here that the embedding $\Phi_{eff} \subset \Phi_b$ be compact. Let now $B = B(R, u_0, \Phi_b)$, $u_0 \in \mathcal{V}_R$, be a Φ_b-ball of radius R centered at u_0 such that $\mathcal{A} \subset B$ (such a ball exists, since $\mathcal{A}$ is bounded in Φ_b). Our task is to construct, starting with the initial ball B, an $R/2^k$-covering of the attractor $\mathcal{A}$, for every fixed k. Estimating then the number of balls in these coverings, we will deduce estimate (4.6).

Let $k \in \mathbb{N}$ be fixed. We fix the minimal $\frac{R}{8K_1}\frac{1}{2^k}$-covering of the set $\mathcal{H}(g)\big|_{[0,(k+1)T]}$ for the metric of $C([0,(k+1)T], L^2_b(\Omega))$ (it is possible to do so, since g is translation-compact; the reason for taking the interval $[0,(k+1)T]$ instead of $[0,kT]$ will be clarified in the next section, devoted to the construction of an exponential attractor). Let $G := \{\xi_1, \cdots, \xi_N\} \subset \mathcal{H}(g)$ be the centers of this covering. Then, obviously

$$(4.8') \quad \log_2 N = \log_2 N(k) := \\ = \mathbb{H}_{R/(2^{k+3}K_1)}(\mathcal{H}(g)\big|_{[0,(k+1)T]}, C([0,(k+1)T], L^2_b(\Omega))).$$

Let us also fix the minimal covering of the unit ball $B(1,0,\Phi_{eff})$ for the Φ_{eff}-norm by a finite number P of $1/(8L_1)$-balls for the metric of Φ_b (such a covering exists, since $\Phi_{eff} \subset\subset \Phi_b$) and

$$(4.9) \qquad \log_2 P := \mathbb{H}_{1/(8L_1)}(B(1,0,\Phi_{eff}), \Phi_b).$$

It is important that, for every $r > 0$ and $v \in \Phi_b$, the ball $B(r, v, \Phi_{eff})$ of radius r in Φ_{eff} centered at v can be covered by *the same number* P of balls of radius $r/(8L_1)$ for the Φ_b-norm (this covering can be constructed from the initial covering of the unit ball $B(1,0,\Phi_{eff})$ by shifting and homotetion).

We now set $\mathcal{U}^j_0 := \{u_0\} \subset \mathcal{V}_R$, $j = 1, \cdots, N$, where u_0 is the center of our initial R-ball and we define the sets $\mathcal{U}^j_l \subset \mathcal{V}_R$ by induction.

Let us assume that the sets $\mathcal{U}^j_l \subset \mathcal{V}_R$, $j = 1, \cdots, N$, are already defined for some $l < k$. We now define the sets $\mathcal{W}^j_{l+1}$ as

$$(4.10) \qquad \mathcal{W}^j_{l+1} := U_{\xi_j}((l+1)T, lT)\mathcal{U}^j_l.$$

We consider, for every $j \in [1, N]$, the system of $L_1R/2^l$-balls for the Φ_{eff}-norm centered at the points of $\mathcal{W}^j_l$. We cover each of these balls by a finite number P of $R/(2^{l+3})$-balls for the metric of Φ_b and we denote by $\mathcal{U}'_{l+1}$ the set of all centers of these new balls. We finally construct the sets $\mathcal{U}^j_{l+1}$ from $\mathcal{U}'_{l+1}$ by the following rule: if $u_l \in \mathcal{U}'_{l+1}$ and $\mathcal{V}_R \cap B(R/2^{l+2}, u_l, \Phi_b) \neq \varnothing$, then we add one (arbitrary) point from this intersection to $\mathcal{U}^j_{l+1}$; if this intersection is empty, then we add nothing. Thus, obviously

$$\#\mathcal{U}^j_{l+1} \leq \#\mathcal{U}'_{l+1} \leq P\#\mathcal{U}^j_l, \quad l = 0, \cdots, k-1.$$

Thus, we have defined by induction the sets $\mathcal{U}^j_l$, for every $0 \leq l \leq k$ and $j \in [1, \cdots, N]$. We also note that the number of points in $\mathcal{U}^j_l$ is given by

$$(4.11) \qquad \#\mathcal{U}^j_l \leq P^l.$$

Lemma 4.1. *The $R/2^k$-balls in Φ_b centered at the points of the set $\mathcal{U}_k := \cup_{j=1}^N \mathcal{U}_k^j$ cover the attractor $\mathcal{A}$.*

Proof. Let w be an arbitrary point of $\mathcal{A}$. Then, according to Theorem 3.1, there exist $\xi \in \mathcal{H}(g)$ and a complete bounded solution $\widehat{u}(t)$ of (3.1) such that $\widehat{u}(kT) = w$.

According to our construction, there exist $j^* \in [1, \cdots, N]$ and $u_0^* := u_0 \in \mathcal{U}_0^{j^*}$ such that

$$\|\xi - \xi_{j^*}\|_{C([0,(k+1)T],L_b^2(\Omega))} \le \frac{R}{8K_1}\frac{1}{2^k}, \quad \|\widehat{u}(0) - u_0^*\|_{\Phi_b} \le R. \tag{4.12}$$

Let us assume that we have already proved that, for some $l < k$, there exists $u_l^* \in \mathcal{U}_l^{j^*}$ such that

$$\|\widehat{u}(lT) - u_l^*\|_{\Phi_b} \le \frac{R}{2^l}. \tag{4.13}$$

We now set $v_{l+1} := U_{\xi_{j^*}}((l+1)T, lT)u_l^* \in \mathcal{W}_{l+1}^{j^*}$. Then, due to (4.8)

$$\widehat{u}((l+1)T) \in \mathcal{O}_{R/2^{l+3}}(B(L_1 R/2^l, v_{l+1}, \Phi_{eff}), \Phi_b), \tag{4.14}$$

where $\mathcal{O}_\mu(V, \Phi_b)$ is a μ-neighborhood of the set V in the space Φ_b. Thus, due to our construction, there exists $u'_{l+1} \in \mathcal{U}'_{l+1}$ such that

$$\widehat{u}((l+1)T) \in \mathcal{O}_{R/2^{l+3}}(B(R/2^{l+3}, u'_{l+1}, \Phi_b)) = B(R/2^{l+2}, u'_{l+1}, \Phi_b).$$

We note that $\mathcal{V}_R \cap B(R/2^{l+2}, u_{l+1}, \Phi_b) \ne \varnothing$, since it contains $\widehat{u}(lT)$. Consequently, by definition of $\mathcal{U}_{l+1}^{j^*}$, there exists $u_{l+1}^* \in \mathcal{U}_{l+1}^{j^*}$ such that

$$\|\widehat{u}((l+1)T) - u_{l+1}^*\|_{\Phi_b} \le \frac{R}{2^{l+1}}. \tag{4.15}$$

Thus, by induction, we conclude that there exists u_k^* such that

$$\|\widehat{u}(kT) - u_k^*\|_{\Phi_b} \le \frac{R}{2^k}.$$

Since $w = u(kT) \in \mathcal{A}$ is arbitrary, Lemma 4.1 is proved.

Therefore, we have constructed the $R/2^k$-covering of the attractor $\mathcal{A}$ (centered at the points of $\mathcal{U}_k := \cup_{j=1}^N \mathcal{U}_k^j$). Consequently,

$$N_{R/2^k}(\mathcal{A}, \Phi_b) \le NP^k. \tag{4.16}$$

Taking the logarithm of both sides of (4.16) and writing the explicit expression of $N = N(k)$, we obtain, for every $k \in \mathbb{N}$

$$\begin{aligned} \mathbb{H}_{R/2^k}(\mathcal{A}, \Phi_b) \le k \log_2 P + \\ + \mathbb{H}_{R/(2^{k+2}L_1)}(\mathcal{H}(g)\big|_{[0,(k+1)T]\times\Omega}, C([0,(k+1)T], L_b^2(\Omega))). \end{aligned} \tag{4.17}$$

Estimate (4.6) is a simple corollary of (4.17). Indeed, let $R>\varepsilon>0$ be fixed and let k be such that

$$\frac{R}{2^{k-1}}>\varepsilon>\frac{R}{2^k}. \tag{4.18}$$

Then, noting that the ε-entropy is a nonincreasing function of ε, it follows from (4.17) that

$$\begin{aligned}\mathbb{H}_\varepsilon(\mathcal{A},\Phi_b)\le(1+\log_2\frac{R}{\varepsilon})\log_2 P+\\ +\mathbb{H}_{\varepsilon/(8L_1)}\left(\mathcal{H}(g)\big|_{[0,T(1+\log_2\frac{R}{\varepsilon})]\times\Omega},C([0,T(1+\log_2\frac{R}{\varepsilon})],L^2_b(\Omega))\right).\end{aligned} \tag{4.19}$$

In order to complete the proof of the theorem, there remains to verify that

$$\log_2 P\le C\,\mathrm{vol}(\Omega\cap B_0^{R_{eff}+C_1}). \tag{4.20}$$

Indeed, according to our choice of space Φ_{eff} and, due to (4.9), there exists a constant $C_1=C_1(L_1,\beta)$ such that

$$\log_2 P\le\mathbb{H}_{1/(16L_1)}\left(B(1,0,W_b^{2-\delta_1,2}(\Omega\cap B_0^{R_{eff}+C_1}),\Phi_b(\Omega\cap B_0^{R_{eff}+C_1})\right). \tag{4.21}$$

It is well known (see e.g. [32] or [34]) that the right-hand side of (4.21) can be estimated as follows:

$$\begin{aligned}\mathbb{H}_{1/(16L_1)}\left(B(1,0,W_b^{2-\delta_1,2}(\Omega\cap B_0^{R_{eff}+C_1}),\Phi_b(\Omega\cap B_0^{R_{eff}+C_1})\right)\le\\ \le C_2\,\mathrm{vol}(\Omega\cap B_0^{R_{eff}+C_1}),\end{aligned} \tag{4.22}$$

where the constant C_2 is independent of R_{eff}. Theorem 4.2 is proved.

Remark 4.1. We note that, in the autonomous case $g=g(x)$, (4.6) implies the following estimate for the fractal dimension of $\mathcal{A}$:

$$\dim_F(\mathcal{A},\Phi_b)\le C\,\mathrm{vol}(\Omega\cap B_0^{R_{eff}+C_1}). \tag{4.23}$$

We also recall that, for bounded domains Ω, we have the estimate

$$\dim_F(\mathcal{A},\Phi_b)\le C\,\mathrm{vol}(\Omega). \tag{4.24}$$

§5 Exponential attractors.

The main aim of this section is to construct an (infinite dimensional) exponential attractor for the nonautonomous problem (0.1). For the reader's convenience, we start our considerations by giving the standard definition of exponential attractors for semigroups (see [17]).

Definition 5.1. Let $\mathbb{S}_h:\Phi_b\to\Phi_b$, $h\ge0$, be a semigroup. Then, a set $\mathbb{M}$ is an exponential attractor for $\mathbb{S}_h$ if

1. $\mathbb{M}$ is compact in Φ_b.
2. $\mathbb{M}$ is *semi-invariant* with respect to $\mathbb{S}_h$, i.e. $\mathbb{S}_h\mathbb{M}\subset\mathbb{M}$, $\forall h\ge0$.

3. $\mathbb{M}$ attracts the bounded subsets of Φ_b *exponentially*, i.e. there exists $\gamma > 0$ such that, for every bounded subset $\mathbb{B} \subset \Phi_b$, there exists a constant $C = C(\mathbb{B})$ such that

$$\operatorname{dist}_{\Phi_b}(\mathbb{S}_h\mathbb{B}, \mathbb{M}) \le Ce^{-\nu h}, \quad \nu > 0, \quad h \ge 0. \tag{5.1}$$

4. The set $\mathbb{M}$ has finite fractal dimension, i.e.

$$\dim_F(\mathbb{M}, \Phi_b) < \infty. \tag{5.2}$$

For nonautonomous equations, a definition of exponential attractor can be obtained from Definition 5.1 by using the skew product technique and by projecting the exponential attractor for the extended semigroup onto the first component (see [16] and [18]). We give this definition in the case of equation (3.1).

Definition 5.2. Let $U_\xi(t,\tau) : \Phi_b \to \Phi_b$ be the solving operators for (3.1). Then, a set $\mathcal{M}$ is an exponential attractor for this equation if
1. $\mathcal{M}$ is compact in Φ_b.
2. $\mathcal{M}$ attracts exponentially the trajectories of the family of equations (3.1), i.e. for every bounded subset $B \subset \Phi_b$, there exists $C = C(B)$ such that

$$\sup_{\xi\in\mathcal{H}(g)} \operatorname{dist}_{\Phi_b}(U_\xi(t,0)B, \mathcal{A}) \le Ce^{-\nu t}, \quad \nu > 0. \tag{5.3}$$

3. For every $u_0 \in \mathcal{M}$, there exists $\xi \in \mathcal{H}(g)$ such that $U_\xi(t,0)u_0 \in \mathcal{M}$, for every $t \ge 0$.
4. The set $\mathcal{M}$ has finite fractal dimension: $\dim_F(\mathcal{M}, \Phi_b) < \infty$.

As in the autonomous case, it follows immediately from the definition that the uniform attractor is a subset of any exponential attractor:

$$\mathcal{A} \subset \mathcal{M}. \tag{5.4}$$

We note that, although this definition of nonautonomous exponential attractors is well adapted to the study of nonautonomous equations in bounded domains Ω with periodic or quasiperiodic external forces (see [11] or [16]), it is not convenient for more general translation-compact time dependences or/and for unbounded domains. Indeed, as already mentioned, there is no reason to expect the uniform attractor to have finite dimension in general and point 4 of Definition 5.2, together with the embedding (5.4), imply that $\dim_F \mathcal{A} < \infty$. Consequently, there is also no reason to expect the existence of an exponential attractor in the sense of Definition 5.2 in general.

Thus, condition 4 of Definition 5.2 should be modified. We note, however, that this condition cannot be dropped completely. Indeed, in that case, every compact (semi-invariant) absorbing set of (2.1) would be an exponential attractor, which does not make sense.

We note that, although an exponential attractor must have infinite dimension (if the same is true for the uniform attractor), it is reasonable to construct it as 'small' as possible. Using Kolmogorov's entropy in order to compare the 'size' of infinite dimensional sets, we come to the following problem: construct an exponential attractor $\mathcal{M}$ whose entropy $\mathbb{H}_\varepsilon(\mathcal{M})$ has in some sense the same type of asymptotics, as $\varepsilon \to 0$, as the entropy $\mathbb{H}_\varepsilon(\mathcal{A})$ of the uniform attractor:

$$\mathbb{H}_\varepsilon(\mathcal{M}, V_0) \sim \mathbb{H}_\varepsilon(\mathcal{A}, V_0). \tag{5.5}$$

Having estimate (4.6) for the entropy of the right-hand side of (5.5), it looks reasonable to give the following definition of (infinite dimensional) exponential attractors for equation (2.1).

Definition 5.3. A set $\mathcal{M}$ is an (infinite dimensional) exponential attractor for equation (2.1) if conditions 1–3 of Definition 5.2 are satisfied and if, in addition

$$(5.6)\quad \mathbb{H}_\varepsilon(\mathcal{M},\Phi_b)\le C_1\,\mathrm{vol}(\Omega\cap B_0^{R_{eff}+C'})\log_2\frac{1}{\varepsilon}+ \\ +\mathbb{H}_{\varepsilon/L_1}\left(\mathcal{H}(g)\big|_{[0,K_1\log_2 1/\varepsilon]\times\Omega}, C([0,K_1\log_2 1/\varepsilon],L^2_b(\Omega))\right),$$

for appropriate constants C_1, C', L_1 and K_1 depending only on $\|g\|_\Sigma$, a, λ_0 and f.

Remark 5.1. We note that, if the external force g is in some proper sense finite dimensional (e.g. if it is quasiperiodic; see also the examples in Section 6 below), then the second term in the right-hand side of (5.6) has the asymptotics $L''\log_2 1/\varepsilon$ and, consequently, (5.6) implies that $\mathcal{M}$ is finite dimensional (that is the reason why we put into parentheses the words 'infinite dimensional' in Definition 5.3). In this situation, Definition 5.3 for exponential attractors coincides with the standard definition (i.e. Definition 5.2).

The main result of this section is the following theorem.

Theorem 5.1. *Let the assumptions of Theorem 3.1 hold. Then, equation* (2.1) *possesses an (infinite dimensional) exponential attractor $\mathcal{M}$ in the sense of Definition 5.3.*

Proof. We first note that it is sufficient to verify the exponential attraction for an absorbing set for equation (2.1) (indeed, the image of every bounded subset of Φ_b enters the absorbing set after a finite time).

Let us construct an absorbing set $\mathbb{B}\subset\Phi_b\times\mathcal{H}(g)$ for the extended semigroup $\mathbb{S}_t:\Phi_b\times\mathcal{H}(g)\to\Phi_b\times\mathcal{H}(g)$ via the expression

$$(5.7)\qquad \mathbb{B}:=\mathcal{V}_R\times\mathcal{H}(g),$$

where $V_R:=V_{R_{eff}}$ is the same as in Theorem 3.2. We now fix the time $T=T(\|g\|_\Sigma)$ exactly as in the proof of Theorem 4.1 (in order to satisfy (4.7) and (4.8)). Then, obviously

$$(5.8)\qquad \mathbb{S}_T\mathbb{B}\subset\mathbb{B}.$$

As usual, we first construct an exponential attractor for the discrete semigroup $\mathbb{S}(n):=\mathbb{S}_{nT}$, $n\in\mathbb{N}$. Then, we extend this result to the continuous case.

We will use the notations of Theorem 4.1. Namely, for every $k\in\mathbb{N}$, we consider the $R/(K_1 2^{k+3})$-net $G(k):=\{\xi_1^k,\cdots,\xi_N^k\}$ in the hull $\mathcal{H}(g)\big|_{[0,(k+1)T]}$ of the right-hand side, restricted to the time interval $[0,(k+1)T]$. Then, the number $N=N_k$ satisfies estimate (4.8′). For every $j\in[1,\cdots,N(k)]$ and $l\le k$, we fix the sets $\mathcal{U}_l^j(k)\subset\mathcal{V}_R$ constructed in the proof of Theorem 4.1 (except that, in contrast to Section 4, we now indicate explicitly the dependence of sets $G:=G(k)$ and $\mathcal{U}_l^j:=\mathcal{U}_l^j(k)$ on k). We also recall that

$$(5.9)\qquad \#\mathcal{U}_l^j(k)\le P^l,\quad l\le k,\quad j\in[1,\cdots,N(k)],$$

where the constant P is defined in (4.9).

Then, it can be proved (repeating word by word the proof of Lemma 4.1) that the system of $R/(2^k)$-balls for the Φ_b-metric centered at the points of $\mathcal{U}_k(k) := \cup_{j=1}^{N(k)}\mathcal{U}_k^j(k)$ covers the set $\Pi_1\mathbb{S}_k\mathbb{B}$. We reformulate this property in the following equivalent way:

$$\operatorname{dist}_{\Phi_b}(\Pi_1(\mathbb{S}_k\mathbb{B})\,,\,\mathcal{U}_k(k)) \le \frac{R}{2^k}. \tag{5.10}$$

Roughly speaking, the main idea of our construction of exponential attractor is to take this attractor as the union $\left[\cup_{k\in\mathbb{N}}\mathcal{U}_k(k)\right]_{\Phi_b}$. Then, (5.10) gives the exponential attraction, whereas (5.9) gives the proper entropy estimate. However, in order to satisfy the other conditions of Definition 5.3, we have to be a little more accurate.

Let us first define the lifting of the sets $\mathcal{U}_k^j(k)$ (which will be denoted below by $\mathcal{U}^j(k)$ in order to simplify the notations) to the extended phase space $\Phi_b\times\mathcal{H}(g)$. To this end, for every $v\in\mathcal{U}^j(k)$, we take a point $(u_v,\xi_v)\in\mathbb{B}$ such that

$$\|v - U_{\xi_v}(k,0)u_v\|_{V_0} \le R/2^k, \quad \|\xi_j - \xi_v\|_{C([0,(k+1)T],L^2_b)} \le R/(2^{k+3}K_1). \tag{5.11}$$

(It is clear from the proof of Lemma 4.1 that the points $v\in\mathcal{U}^j(k)$ for which such a point does not exist can be dropped out of $\mathcal{U}_k^j(k)$, thus preserving property (5.10)).

We set $\mathbb{U}^j(k) := \{\mathbb{S}_k(u_v,\xi_v)\in\Phi_b\times\mathcal{H}(g)\,:\, v\in\mathcal{U}^j(k)\}$ and $\mathbb{U}(k) := \cup_{j=1}^N\mathbb{U}^j(k)$. Then (due to (5.9)–(5.11))

$$\operatorname{dist}_{\Phi_b}(\Pi_1(\mathbb{S}_k\mathbb{B})\,,\,\Pi_1\mathbb{U}(k)) \le R/2^{k-1},\ \mathbb{U}(k)\subset\mathbb{S}_k\mathbb{B},\ \#\mathbb{U}(k)\le N(k)P^k. \tag{5.12}$$

We are now in a position to construct the discrete exponential attractors $\mathbb{M}^d$ and $\mathcal{M}^d := \Pi_1\mathbb{M}^d$. To this end, we first define a sequence of sets $\mathbb{E}(k)$ by induction:

$$\mathbb{E}(0) := \mathbb{U}(0), \quad \mathbb{E}(k+1) := \mathbb{S}_1\mathbb{E}(k)\cup\mathbb{U}(k+1), \tag{5.13}$$

and we then define the exponential attractor $\mathbb{M}^d$ as follows:

$$\mathbb{M}^d := \left[\cup_{k\in\mathbb{N}}\,\mathbb{E}(k)\right]_{\Phi_b\times\mathcal{H}(g)}. \tag{5.14}$$

Indeed, the semi-invariance ($\mathbb{S}_1\mathbb{M}^d\subset\mathbb{M}^d$) is an immediate consequence of (5.13) and (5.14). Furthermore, the exponential attraction follows from the first formula of (5.12). Thus, there only remains to verify the entropy estimate. We recall that the entropy of a set coincides with that of its closure. Consequently, we will estimate the entropy of $\mathcal{M}_1^d := \cup_{k\in\mathbb{N}}\Pi_1\mathbb{E}(k)$.

Let us fix an arbitrary ε, $R>\varepsilon>0$, and compute $k=k(\varepsilon)$ from the inequality

$$\frac{R}{2^k} < \varepsilon < \frac{R}{2^{k-1}}. \tag{5.15}$$

We then split $\mathcal{M}_1^d$ as follows:

$$\mathcal{M}_1^d = \left(\cup_{l\le k}\Pi_1\mathbb{E}(l)\right)\cup\left(\cup_{l>k}\Pi_1\mathbb{E}(k)\right). \tag{5.16}$$

We note that, according to our construction, $\mathbb{E}(k)\subset\mathbb{S}_k\mathbb{B}$. Consequently (since $\mathbb{B}$ is semi-invariant), the second set in decomposition (5.16) is a subset of $\mathbb{S}_{k+1}\mathbb{B}$. We recall that (due to (5.12)) the system of $R/2^k(<\varepsilon)$-balls centered at the points of $\Pi_1\mathbb{U}(k+1)$ covers $\Pi_1(\mathbb{S}_{k+1}\mathbb{B})$ and, consequently, covers the second set in (5.16). Thus, the minimal number $N_\varepsilon(\mathcal{M}^d,\Phi_b)$ of ε-balls which cover the attractor satisfies

$$N_\varepsilon(\mathcal{M}^d,\Phi_b)\le\sum_{l\le k}\#\mathbb{E}(k)+\#\mathbb{U}(k+1)\le\sum_{l\le k+1}\#\mathbb{E}(k). \tag{5.17}$$

It follows immediately from the inductive definition of the sets $\mathbb{E}(l)$ that

$$\#\mathbb{E}(l)\le\sum_{m\le l}\#\mathbb{U}(m)\le l\#\mathbb{U}(l)\le(k+1)\#\mathbb{U}(k+1)\le(k+1)N(k+1)P^k. \tag{5.18}$$

Inserting this estimate into (5.17) and applying the $\log_2$, it follows that

$$\mathbb{H}_\varepsilon(\mathcal{M}^d,\Phi_b)\le 2\log_2(k+1)+\log_2 N(k+1)+(k+1)\log_2 P. \tag{5.19}$$

Expressing $k=k(\varepsilon)$ from (5.15) and inserting this expression into (5.19), we obtain (as in the end of the proof of Theorem 4.1) estimate (5.6). Thus, the set $\mathcal{M}^d$ is indeed a discrete exponential attractor. We also note that the upper bound (4.6) for the entropy of the global attractor differs from the estimate for the exponential attractor $\mathcal{M}^d$ by the term $\log_2(k(\varepsilon)+1)$, which is proportional to the logarithm of the logarithm of ε.

To complete the proof of the theorem, there remains to extend $\mathcal{M}^d$ to a continuous exponential attractor.

Lemma 5.1. *We set*

$$\mathbb{M}=\mathbb{M}^c:=\cup_{t\in[0,T]}\mathbb{S}_t\mathbb{M}^d. \tag{5.20}$$

Then, $\mathcal{M}:=\Pi_1\mathbb{M}$ is an (infinite dimensional) exponential attractor for equation (2.1).

Proof. Let us verify the conditions of Definition 5.3. The semi-invariance follows immediately from the discrete semi-invariance of $\mathbb{M}^d$ and from (5.20).

Let us now verify the exponential attraction. To this end, we fix an arbitrary $(v,\xi)\in\mathbb{B}$ with corresponding trajectory $u(t):=U_\xi(t,0)v$ and we consider the time $t=kT+s$, where $k\in\mathbb{N}$ and $0\le t<T$. Then, according to the construction of the sets $\mathbb{U}(k)$, there exists $(v^*,\xi^*)\in\mathbb{B}$ such that $\mathbb{S}_k(v^*,\xi^*)\in\mathbb{U}(k)$ and

$$\|u(k)-U_\xi(k,0)v^*\|_{\Phi_b}\le R/2^{k-1},\ \ \|\xi-\xi^*\|_{C([0,(k+1)T],L^2_b(\Omega))}\le R/(2^{k+2}K_1). \tag{5.21}$$

The second inequality follows from the second inequality in (5.11) and from the fact that $G(k)=\{\xi_j,\ j=1,\cdots,N\}$ is an $R/2^{k+2}K_1$-net. In particular, we note that this inequality implies that

$$\|T_k\xi-T_k\xi^*\|_{C([0,T],L^2_b(\Omega))}\le R/(2^{k+2}K_1). \tag{5.22}$$

(That was the reason why we have considered the time interval $[0,(k+1)T]$ instead of $[0,kT]$ from the very beginning.)

We note that, by definition of $\mathbb{M}$, the point $u^*(t) := U_{T_k\xi^*}(kT+s, kT)U_\xi(kT, 0)v^*$ belongs to $\mathcal{M}$. Moreover, estimate (2.24) implies that

$$(5.23)\quad \|u(t)-u^*(t)\|_{\Phi_b} \le L''\left(\|u(kT)-u^*(kT)\|_{\Phi_b} + \|T_k\xi - T_k\xi^*\|_{C([0,T],L^2_b(\Omega))}\right).$$

(We note that the constant $L'' \ge 1$ is chosen such that (5.23) holds uniformly with respect to $u(k), u^*(k) \in V_R$ and $0 < t - kT < T$. It is possible to do so thanks to Theorem 2.3.)

Inserting estimates (5.21) and (5.22) into the right-hand side of (5.23), we have

$$(5.24)\qquad \|u(t)-u^*(t)\|_{\Phi_b} \le L_1 2^{-k} \le L_1 2^{1-t/T}.$$

Since $u^*(t) \in \mathcal{M}$, we obtain the attraction property.

So, there only remains to verify the entropy estimate. To this end, it is convenient to introduce the operator

$$(5.25)\qquad \mathbb{S} : \Phi_b \times \mathcal{H}(g) \to C([0,T],\Phi_b),\quad \mathbb{S}(v,\xi) := U_\xi(t,0)v,$$

and to consider the set $\widehat{\mathcal{M}} := \mathbb{S}\mathbb{M} \subset C([0,T],\Phi_b)$. (Roughly speaking, we replace every point of $\mathcal{M}$ by the piece of the corresponding trajectory of length T.) We also need the following proposition.

Proposition 5.1. *The $L''R/2^{k-2}$-balls for the topology of $C([0,T],\Phi_b)$ centered at the points of $\mathbb{S}(\mathbb{U}(k))$ cover $\mathbb{S}(\mathbb{S}_k\mathbb{B})$.*

Proof. The proof of this proposition is similar to our proof of the attraction property. We take an arbitrary point $(v,\xi) \in \mathbb{B}$ and we set $u(t) := U_\xi(t,0)v$. Then, there exists a point $(v^*,\xi^*) \in \mathbb{B}$, $u^*(t) = U_{\xi^*}(t,0)v^*$, such that $(u^*(k), T_k\xi^*) \in \mathbb{U}(k)$ and

$$(5.26)\qquad \|u(kT)-u^*(kT)\|_{\Phi_b} \le R/2^{k-1},\quad \|\xi-\xi^*\|_{C([0,(k+1)T],L^2_b)} \le R/2^{k+1}.$$

Thus, according to estimate (5.23)

$$(5.27)\quad \|u(s+kT)-u^*(s+kT)\|_{\Phi_b} \le \\ \le L''(R/2^{k-1}+R/2^{k+1}) \le L''R/2^{k-2},\ 0<s<T,$$

and Proposition 5.1 is proved.

Let us now fix ε, $R > \varepsilon > 0$, and let us construct the ε-covering of $\widehat{\mathcal{M}}$. As in the discrete case, we compute $k = k(\varepsilon)$ from the inequality

$$(5.28)\qquad L''R/2^k < \varepsilon < L''R/2^{k-1}.$$

We then decompose the set $\widehat{\mathcal{M}}$ as follows:

$$(5.29)\qquad \widehat{\mathcal{M}} = \cup_{l\le k}\mathbb{S}(\mathbb{E}(l)) \cup \mathbb{S}\left(\cup_{l\ge k}\mathbb{E}(l)\right) := \widehat{\mathcal{M}}_1 \cup \widehat{\mathcal{M}}_2.$$

We note that, as in the discrete case, $\widehat{\mathcal{M}}_2 \subset \mathbb{S}(\mathbb{S}_{k+1}\mathbb{B})$ and, due to Proposition 5.1, it can be covered by the ε-balls centered at the points of $\mathbb{S}(\mathbb{U}(k+1)) \subset \mathbb{S}(\mathbb{E}(k+1))$.

Thus, we have proved that the system of ε-balls centered at the points of $\mathbb{S}(\mathbb{E}(l))$, $l = 1, \cdots, k+1$, covers $\widehat{\mathcal{M}}$. Consequently (compare with (5.19)), the entropy $\mathbb{H}_\varepsilon$ of this covering satisfies

$$(5.30)\quad \mathbb{H}_\varepsilon(\widehat{\mathcal{M}}, C([0,T],\Phi_b)) \le \mathbb{H}_\varepsilon \le (k+1)\log_2 P + \log_2 N(k) + 2\log_2(k+1).$$

Computing $k = k(\varepsilon)$ from (5.28) and inserting this value into (5.30), we obtain an entropy estimate of the form (5.6) for the set $\widehat{\mathcal{M}}$.

We are now in a position to estimate the entropy of $\mathcal{M}$ and to complete the proof of the lemma. To this end, we introduce the projector

$$\mathrm{PR} : C([0,T],\Phi_b) \to 2^{\Phi_b}, \quad \mathrm{Pr}(v) := \{v(t) : t \in [0,T]\}.$$

Obviously, $\mathcal{M} = \mathrm{PR}(\widehat{\mathcal{M}})$ and, consequently

$$(5.31)\qquad \mathrm{dist}_{\Phi_b}(\mathcal{M}, \cup_{l\le k+1}\mathrm{PR}(\mathbb{S}(\mathbb{E}(l)))) \le \varepsilon,$$

where $k = k(\varepsilon)$ is defined by (5.25). Thus, there remains to construct the covering of the sets $\mathcal{O}_\varepsilon(\mathrm{PR}(\mathbb{S}(\mathbb{E}(l))))$, where $\mathcal{O}_\varepsilon$ denotes the ε-neighborhood in Φ_b. We recall that the sets $\mathbb{E}(l)$ contain a finite number of points. Therefore, it is sufficient to know how to cover the sets

$$(5.32)\quad \mathcal{O}_\varepsilon(\mathbb{S}(v,\xi)) = \{w \in \Phi_b : \exists t \in [0,T],\ \|v(t) - w\|_{\Phi_b} \le \varepsilon,\ v(t) := U_\xi(t,0)v\},$$

for every $(v,\xi) \in \mathbb{E}(l)$. In order to construct such a covering, we note that, according to our construction, $\mathbb{M} \subset \mathbb{S}_1\mathbb{B}$ (since we take a union in (5.14) not from $k = 0$, but from $k = 1$). Then, according to Corollary 2.4, $\mathcal{M}$ is bounded in $W_b^{2-\delta',2}(\Omega)$ and, consequently (due to Corollary 2.5), every trajectory $u(t) := U_\xi(t,0)v$ starting from an arbitrary point $(v,\xi) \in \mathbb{M}$ is uniformly Hölder continuous in Φ_b, i.e.

$$(5.33)\qquad \|u(t+s) - u(t)\|_{V_0} \le Cs^\gamma, \quad t \ge 0,\ 0 < s < T.$$

In particular, (5.33) holds uniformly with respect to $(v,\xi) \in \mathbb{E}(l)$, $l \in \mathbb{N}$.

Let us fix $s_0 = s_0(\varepsilon) = (\varepsilon/C)^{1/\gamma}$ and let us consider the following discrete subset of (5.32):

$$(5.34)\qquad L_{(v,\xi)} := \{v(ns_0) : n = 0, 1, \cdots, [T/s_0]\}.$$

Then, obviously, the 2ε-balls centered at the points of $L_{(v,\xi)}$ cover (5.32). Moreover

$$(5.35)\qquad \#L_{(v,\xi)} = 1 + [T/s_0] \le (C/\varepsilon)^{1/\gamma},$$

and, consequently, the system of 2ε-balls centered at the points of $L_{(v,\xi)}$, $(v,\xi) \in \mathbb{E}(l)$, $l \le k+1$, covers $\mathcal{M}$. The entropy of this covering can be estimated by $\mathbb{H}_\varepsilon + 1/\gamma \log_2 C/\varepsilon$. Thus

$$(5.36)\quad \mathbb{H}_{2\varepsilon}(\mathcal{M},\Phi_b) \le (k+1)\log_2 P + \log_2 N(k) + 2\log_2(k+1) + \frac{1}{\gamma}\log_2 C/\varepsilon.$$

Computing $k(\varepsilon)$ from inequality (5.28) and inserting this expression into (5.36), we obtain estimate (5.6). Lemma 5.1, and thus Theorem 5.1, are proved.

§6 Examples and concluding remarks.

In this section, we derive several corollaries of the results obtained above which show in particular that estimates (4.6) and (5.6) are in a sense sharp. We restrict ourselves to the case where the attractors $\mathcal{A}$ and $\mathcal{M}$ are finite dimensional only (examples for the infinite dimensional case are given in [16]). We start our considerations with the autonomous case

$$g = g(x) \in L^2_b(\Omega). \tag{6.1}$$

Then, estimates (4.6) and (5.6) imply that

$$\dim_F(\mathcal{A}, \Phi_b) \le \dim_F(\mathcal{M}, \Phi_b) \le C \operatorname{vol}(\Omega \cap B_0^{R_{eff}+C'}), \tag{6.2}$$

where R_{eff} is defined in Theorem 3.2 and C and C' depend only on $\|g\|_{L^2_b}$. So, in that case, we have a finite dimensional exponential attractor in complete agreement with the standard definition. Moreover, in the particular case $\Omega = \mathbb{R}^3$, (6.2) reads

$$\dim_F(\mathcal{A}, \Phi_b) \le \dim_F(\mathcal{M}, \Phi_b) \le C_1(R_{eff}+1)^3, \tag{6.3}$$

where C_1 depends only on $\|g\|_{L^2_b}$ and is independent of R_{eff}.

Let us now verify the sharpness of estimate (6.3) with respect to the parameter $R_{eff} \to \infty$. To this end, we consider the following particular scalar ($k = 1$) case of equation (2.1)

$$\partial_t u = \Delta_x u - f(u) - \lambda_0 u + g_R(x), \tag{6.4}$$

where $f(u)$ is an interaction function satisfying the conditions

$$1.\ f \in C^2(\mathbb{R}, \mathbb{R}), \quad 2.\ f(u).u \ge 0, \quad 3.\ \exists \mu_0 \in \mathbb{R},\ f'(\mu_0) \le -\lambda_0 - 1. \tag{6.5}$$

For a given $R > 1$, we construct an equilibrium point $u_R(x)$ of equation (6.4) via

$$u_R(x) := \mu_0 \chi\left(|x|/(2R)\right), \tag{6.6}$$

where $\chi(z) \in C^\infty(\mathbb{R})$ is such that $0 \le \chi(z) \le 1$, $\chi(z) = 1$ for $|z| \le 1$ and $\chi(z) = 0$ for $|z| \ge 2$. Having the functions $u_R(x)$, we finally define a family of external forces $g_R(x)$ such that $u_R(x)$ is the equilibrium point of equation (6.4), namely

$$g_R(x) := f(u_R(x)) + \lambda_0 u_R(x) - \Delta_x u_R(x). \tag{6.7}$$

Then, $\|g_R\|_{L^\infty} \le C$, uniformly with respect to $R \ge 1$. Moreover, $\operatorname{supp} g_R \subset B_0^R$ and, consequently, we have the following estimate for the effective radius $R_{eff} = R_{eff}(g_R)$:

$$C_1 R \le R_{eff}(g_R) \le C_2 R. \tag{6.8}$$

In order to obtain a lower bound on the dimension of the global attractor $\mathcal{A}$, we use the well-known fact that this dimension is greater than the unstable index of any equilibrium point of equation (6.4):

$$\dim_F(\mathcal{A}, \Phi_b) \ge \operatorname{Ind}^+_{u_R}, \tag{6.9}$$

(see e.g. [3] or [27]). On the other hand, it is not difficult to verify, using the min-max principle, that

$$\operatorname{Ind}^+_{u_R} \ge \operatorname{Ind}^+(\Delta_x + \operatorname{Id}, B_0^{R/2}) \ge C_3 R^3, \tag{6.10}$$

(see [3] for details). Thus, we have proved the following result.

Proposition 6.1. *Let the above assumptions hold. Then, the following estimates are satisfied for the global and exponential attractors of* (6.4)*:*

$$C'R^3 \le \dim_F(\mathcal{A}, \Phi_b) \le \dim_F(\mathcal{M}, \Phi_b) \le C''R^3, \tag{6.11}$$

where C' *and* C'' *are independent of* $R \ge 1$.

Thus, (6.11) shows that (4.6) and (5.6) are sharp as $R_{eff} \to \infty$.

For our next applications, it will be useful to have an exponential attractor not only in the phase space Φ_b, but also in the space $L^\infty(\Omega)$.

Proposition 6.2. *Let the assumptions of Theorem 3.1 hold and let, in addition,* (6.1) *be satisfied. Then, there exists an exponential attractor* $\mathcal{M} \in \Phi_b$ *of equation* (2.1) *such that:*

1. There exists a positive constant $\gamma > 0$ *and a monotonic function* Q*, which are independent of* R_{eff}*, such that, for every bounded set* $B \subset L^\infty(\Omega)$

$$\operatorname{dist}_{L^\infty(\Omega)}(S_tB, \mathcal{M}) \le Q(\|B\|_{L^\infty(\Omega)})e^{-\gamma t}. \tag{6.12}$$

2. The fractal dimension of $\mathcal{M}$ *satisfies the estimate:*

$$\dim_F(\mathcal{A}, L^\infty(\Omega)) \le C \operatorname{vol}(\Omega \cap B_0^{R_{eff}+C'}), \tag{6.13}$$

where C *and* C' *are also independent of* R_{eff}.

Proof. The assertion of Proposition 6.2 for the phase space Φ_b is proved in Theorem 5.1. In order to extend this result to $L^\infty(\Omega)$, there remains to note that, as proved in [13], equation (2.1) possesses a $L^\infty(\Omega) \to \Phi_b$ smoothing property of the following form:

$$\|u(1)\|_{\Phi_b} \le Q(\|u(0)\|_{L^\infty}) + Q(\|g\|_{L^2_b}), \tag{6.14}$$

where the function Q is independent of R_{eff}, which finishes the proof of Proposition 6.2.

In our next application, we consider the following reaction-diffusion system of type (2.1), with a small diffusion parameter $\nu \ll 1$, in the domain Ω:

$$\partial_t u = \nu \Delta_x u - \lambda_0 u - f(u) + g(x), \tag{6.15}$$

where the nonlinearity is independent of $\nabla_x u$.

Proposition 6.3. *Let the function* f *satisfy*

$$1.\ f \in C^2(\mathbb{R}^k, \mathbb{R}^k), \quad 2.\ f(u).u \ge 0, \tag{6.16}$$

and let, in addition, the external forces satisfy

$$g \in \dot{L}^\infty(\Omega). \tag{6.17}$$

Then, equation (6.15) *possesses an exponential attractor* $\mathcal{M}_\nu$*, for every* $\nu > 0$*, in* $L^\infty(\Omega)$*. Moreover, these attractors satisfy* (6.12) *uniformly with respect to* $\nu > 0$ *and their fractal dimension can be estimated as follows:*

$$\dim_F(\mathcal{A}_\nu, L^\infty(\Omega)) \le \dim_F(\mathcal{M}_\nu, L^\infty(\Omega)) \le C\nu^{-3/2}, \tag{6.18}$$

where the constant C is independent of $\nu > 0$.

Proof. Indeed, the form of equation (6.15) is preserved under the rescaling $x \to x'\nu^{1/2}$ (which obviously also preserves the L^∞-norm). We should then only replace the domain Ω by $\Omega' = \nu^{-1/2}\Omega$, the diffusion coefficient ν by 1 and the external forces $g(x)$ by $g_\nu(x') := g(x'\nu^{1/2})$. We also note that the L^2_b-norms of g_ν are uniformly bounded as $\nu \to 0$ (to this end, we need the assumption $g \in L^\infty(\Omega)$). Thus, constructing an exponential attractor for the rescaled equation, returning then to the initial problem and using the invariance of the L^∞-norm, we construct exponential attractors $\mathcal{M}_\nu$ enjoying property (6.12) uniformly with respect to ν and satisfying the following estimates:

$$(6.19)\quad \dim_F(\mathcal{M}_\nu, L^\infty(\Omega)) \le C\,\mathrm{vol}(\nu^{-1/2}\Omega \cap B^{R_{eff}(g_\nu)+C'}) \le C_1(R_{eff}(g_\nu)+1)^3,$$

where the constant C_1 is independent of ν. So, there remains to estimate the quantity $R_{eff}(g_\nu)$ with respect to ν. To this end, we note that the assumption $g \in \dot{L}^\infty(\Omega)$, the obvious formula

$$\mathcal{R}^\beta_{0,2}(\mathcal{H}(g), z) \le C\mathcal{R}^\beta_{0,\infty}(\mathcal{H}(g), z) := C \sup_{\xi\in H(g)} \sup_{x\in\Omega} \left\{ e^{-\beta\,\mathrm{dist}(x,\mathbb{R}^3\setminus B^z_0)}|\xi(0,x)| \right\},$$

where C is independent of z and g, the equality

$$\mathcal{R}^\beta_{0,\infty}(\mathcal{H}(g_\nu), z) = \mathcal{R}^\beta_{0,\infty}(\mathcal{H}(g), z\nu^{1/2})$$

and the definition of R_{eff} (see Theorem 2.3) immediately imply the estimate

$$(6.19')\qquad R_{eff}(g_\nu) \le C_g\nu^{-1/2}.$$

Inserting (6.19′) into (6.19), we finally obtain (6.18) and Proposition 6.3 is proved.

Remark 6.1. It is well known (see e.g. [3], [20] and [27]), that estimate (6.18) is sharp with respect to the parameter ν.

Remark 6.2. We note that we have rigorously proved estimate (6.18) *only* for scalar diffusion matrices a. Nevertheless, we have used this assumption only in order to derive the dissipative estimate (2.5) for the L^∞-norm (applying the maximum principle). Consequently, if this estimate is a priori known, then we can extend the result of Proposition 6.3 to larger classes of reaction-diffusion systems. In particular, instead of (6.15), we can consider the system

$$(6.20)\qquad \partial_t u = a\nu\Delta_x u - \lambda_0 u - f(u) + g(x),$$

with an arbitrary diffusion matrix a satisfying $a + a^* > 0$ and a nonlinearity f satisfying the assumptions

$$(6.21)\quad 1.\ f \in C^2(\mathbb{R}^k, \mathbb{R}^k),\ \ 2.\ f'(u) \ge -K,\ \ 3.\ f(u).u \ge 0,\ \ 4.\ |f(u)| \le C(1+|u|^p),$$

where p is arbitrary. The dissipative estimate for the L^∞-norm, of the form (2.5) (for $\nu = 1$ after rescaling), has been obtained in [35]. Thus, equations (6.20) possess exponential attractors $\mathcal{M}_\nu$ which satisfy (6.12) uniformly with respect to ν and their fractal dimension satisfies the sharp estimate (6.18).

Let us now consider the case of quasiperiodic external forces $g(t)$. More precisely, we assume that the function g has the following structure:

$$g(t) := G(T_t^\omega \phi_0),\ G \in C^1(\mathbb{T}^m, \dot{L}_b^2(\Omega)), \quad \phi_0 \in \mathbb{T}^m, \tag{6.22}$$

where $\mathbb{T}^m$ is the m-dimensional torus, $\omega = (\omega^1, \cdots, \omega^m)$ is a rationally independent frequency vector and $T_t^\omega \phi_0 := (\phi_0 + \omega t) \,(\mathrm{mod}\, (2\pi)^m)$ is a standard linear flow on the torus. In that case, the hull $\mathcal{H}(g)$ possesses the following description:

$$\mathcal{H}(g) = \big\{G(T_t^\omega \phi), \quad \phi \in \mathbb{T}^m\big\}, \tag{6.23}$$

and, consequently, using the obvious fact that the standard linear flow preserves the metric on the torus and the smoothness of G ($G \in C^1$), we have

$$\begin{aligned}\mathbb{H}_{\varepsilon/L_1}\big(\mathcal{H}(g), L^\infty([0, K_1 \ln 1/\varepsilon], L_b^2(\Omega)\big) \le \mathbb{H}_{\varepsilon/L_1}\big(\mathcal{H}(g), C(\mathbb{R}, L_b^2(\Omega))\big) \le \\ \le \mathbb{H}_{\varepsilon/(C_G L_1)}(\mathbb{T}^m, \mathbb{R}^m) \le \big(m + \overline{\overline{o}}(1)\big) \ln \frac{1}{\varepsilon},\end{aligned} \tag{6.24}$$

where C_G only depends on $\|G\|_{C^1(\mathbb{R}, L_b^2(\Omega))}$. Thus, Theorem 5.1 and (6.24) imply the following result.

Proposition 6.4. *Let the assumptions of Theorem 3.1 hold and let, in addition, the external forces g have structure* (6.22) *(i.e. let g be quasiperiodic with m independent frequencies). Then, there exists finite dimensional uniform $\mathcal{A}$ and exponential $\mathcal{M}$ attractors of equation* (3.1)*, whose fractal dimensions satisfy*

$$\dim_F(\mathcal{A}, \Phi_b) \le \dim_F(\mathcal{M}, \Phi_b) \le C\, \mathrm{vol}(\Omega \cap B_0^{R_{eff}+C'}) + m, \tag{6.25}$$

where the first term in the right-hand side of (6.25) *is the same as in the autonomous case, see* (6.2)*.*

As in the autonomous case, it is not difficult to verify the sharpness of estimate (6.25).

Proposition 6.5. *Under the assumptions of Proposition 6.4, there exists a family of equations of the form* (3.1) *in $\Omega = \mathbb{R}^3$ such that*

$$C_1 R_{eff}^3 + m \le \dim_F(\mathcal{A}, \Phi_b) \le \dim_F(\mathcal{M}, \Phi_b) \le C_2 R_{eff}^3 + m, \tag{6.26}$$

where C_i, $i = 1, 2$, are independent of R_{eff} and m.

Proof. Let us consider equation (6.4) in $\Omega = \mathbb{R}^3$ under assumptions (6.5) on the nonlinearity and with the external forces $g_R(x)$ defined in (6.7). Then, without loss of generality, we may assume that the equilibrium $u_R(x)$ is hyperbolic (it is well-known, see e.g. [3] and [33], that the hyperbolicity of an equilibrium is a generic property, so, if it is violated for the initial g_R, one can find a new $\tilde{g}_R$ arbitrarily close to g_R for which this assumption is satisfied). Thus, we have

$$\Phi_b = \Phi_b^- + \Phi_b^+, \tag{6.27}$$

where the subspaces $\Phi_b^+ := \Pi_+\Phi_b$ and $\Phi_b^- := \Pi_-\Phi_b$ correspond to the stable and unstable parts of the linear operator $\Delta_x - \lambda_0 - f'(u_R)$. Moreover, it is well-known that the unstable subspace is finite dimensional:

$$\dim \Phi_b^+ < \infty, \tag{6.28}$$

and there exists the finite dimensional local unstable manifold $\mathcal{M}^+(u_R)$, defined by

$$\begin{aligned} \mathcal{M}^+(u_R) := \{u_0 \in \Phi_b \,:\, \|u_0 - u_R\|_{\Phi_b} \le \varepsilon,\ \exists u \in C_b(\mathbb{R}_-, \Phi_b) \\ \text{such that } u(t) \text{ solves (6.4)}, u(0) = u_0 \text{ and } \lim_{t\to-\infty} u(t) = u_R\}, \end{aligned} \tag{6.29}$$

where $\varepsilon > 0$ is small enough (see e.g. [2] or [33] for details). This manifold is C^1-diffeomorphic to Φ_b^+, i.e. there exists

$$\begin{aligned} M \in C^1(B(\varepsilon, 0, \Phi_b^+), \Phi_b^-), \quad M(0) = M'(0) = 0, \quad \text{such that} \\ \mathcal{M}^+(u_R) = \{v \in \Phi_B \,:\, \exists u_0^+ \in B(\varepsilon, 0, \Phi_b^+),\ v = u_R + u_0^+ + M(u_0^+)\}. \end{aligned} \tag{6.30}$$

Let us now construct the small nonautonomous perturbation of equation (6.4). To this end, we fix an arbitrary quasiperiodic function $w(t, \phi_0)$ satisfying

$$w(t, \phi_0) := W(T_t^\omega \phi_0), \quad W \in C^2(\mathbb{T}^m, \dot{C}^2(\mathbb{R}^3)), \tag{6.31}$$

define the nonautonomous perturbation of the external forces $g_R(x)$, for every small $\delta \ge 0$, by

$$g_{R,m,\phi_0}(t) := g_R + \delta g_1(t, \delta, \phi_0), \tag{6.32}$$

where

$$g_1(t, \delta, \phi_0) := \partial_t w(t, \phi_0) - a\Delta_x w(t, \phi_0) + \frac{1}{\delta}[f(u_R + \delta w(t, \phi_0)) - f(u_R)], \tag{6.33}$$

and consider the following equation:

$$\partial_t u = \Delta_x u - f(u) + g_{R,m,\phi_0}(t). \tag{6.34}$$

We note that the external forces in (6.34) are defined such that the function $\tilde{u}(t) := u_R + \delta w(t, \phi_0)$ is a solution of (6.34).

Then, according to the standard theory of nonautonomous perturbations of unstable manifolds (see [14] and [29]), for every $\phi_0 \in \mathbb{T}^m$, the set

$$\begin{aligned} \mathcal{M}_\delta^+(u_R, \phi_0) := \{u_0 \in \Phi_b \,:\, \exists u \in C_b(\mathbb{R}_-, \Phi_b) \\ \text{such that } u(t) \text{ solves (6.34)}, u(0) = u_0 \text{ and } \|u(t) - u_R\|_{\Phi_b} \le \varepsilon, \forall t \le 0\ \}, \end{aligned} \tag{6.35}$$

is a C^1-submanifold of Φ_b that is diffeomorphic to Φ_b^+ if $\delta > 0$ is small enough. Moreover, these manifolds tend to (6.29) as $\delta \to 0$; more precisely, there exists

$$\begin{aligned} N \in C^1([0, \delta_0] \times B(\varepsilon, 0, \Phi_b^+) \times \mathbb{T}^m, \Phi_b) \ \text{ such that} \\ \mathcal{M}_\delta^+(u_R, \phi_0) = \{v \in \Phi_b,\ \ \exists u_0^+ \in B(\varepsilon, 0, \Phi_b^+), \\ v = V(\delta, u_0^+, \phi_0) := u_R + u_0^+ + M(u_0^+) + \delta N(\delta, u_0^+, \phi_0)\}. \end{aligned} \tag{6.36}$$

We now recall that, according to the construction of the uniform attractor $\mathcal{A}$ of problem (6.34)

$$\cup_{\phi_0\in\mathbb{T}^m}\mathcal{M}^+_\delta(u_R,\phi_0)\subset\mathcal{A}. \tag{6.37}$$

Consequently, there only remains to find an $m+\dim\Phi_b^+$-dimensional submanifold of Φ_b which is contained in the left-hand side of (6.37). Thanks to the implicit function theorem, such a submanifold exists if

$$\begin{aligned}\operatorname{rank}\{D_{u_0^+}V(\delta,0,\phi_0),D_{\phi_0}V(\delta,0,\phi_0)\} &=\\ =\operatorname{rank}&\begin{pmatrix}E_\kappa & \delta D_{u_0^+}N(\delta,0,\phi_0)\\ \delta\Pi_+D_{\phi_0}N(\delta,0,\phi_0) & \delta\Pi_-D_{\phi_0}N(\delta,0,\phi_0)\end{pmatrix}=\kappa+m,\end{aligned} \tag{6.38}$$

at least for one point $\phi_0\in\mathbb{T}^m$, where $\kappa:=\operatorname{Ind}^+_{u_R}$ and E_κ is the identity matrix.

It is not difficult to verify that condition (6.38) is satisfied for sufficiently small $\delta>0$ if

$$\operatorname{rank}\{\Pi_-D_{\phi_0}N(0,0,\phi_0)\}=m. \tag{6.39}$$

There remains to note that the derivative in (6.39) can be found as a unique bounded solution of the variation equation associated with (6.34). More precisely, let $\gamma\in\mathbb{R}^m$ and let $v_-^\gamma(t)$ be the unique bounded solution of

$$\partial_t v_-^\gamma=\Pi_-(\Delta_x v_-^\gamma-f'(u_R)v_-^\gamma)+\Pi_-D_{\phi_0}g_1(t,0,\phi_0)\gamma,\quad t\le 0. \tag{6.40}$$

Then, $D_{\phi_0}N(0,0,\phi_0)\gamma=v_-^\gamma(0)$. On the other hand, (6.33) implies that

$$D_{\phi_0}g_1(t,0,\phi_0)=\partial_t D_{\phi_0}w(t,\phi_0)-a\Delta_x D_{\phi_0}w(t,\phi_0)+f'(u_R)D_{\phi_0}w(t,\phi_0). \tag{6.41}$$

Comparing (6.40) and (6.41), we conclude that

$$\Pi_-D_{\phi_0}N(0,0,\phi_0)=\Pi_-D_{\phi_0}W(\phi_0). \tag{6.42}$$

Therefore, condition (6.39) is obviously satisfied if the nonautonomous perturbation satisfies the additional generic assumption

$$\operatorname{rank}\{\Pi_-D_{\phi_0}W(\phi_0)\}=m,\quad\text{for some } \phi_0\in\mathbb{T}^m. \tag{6.43}$$

Thus, under assumption (6.43), the uniform attractor $\mathcal{A}$ of equation (6.34) contains a submanifold that is diffeomorphic to $R^{\operatorname{Ind}^+_{u_R}+m}$, if $\delta>0$ is small enough, and, consequently

$$\dim_F(\mathcal{A},\Phi_b)\ge\operatorname{Ind}^+_{u_R}+m. \tag{6.44}$$

Combining now (6.10), (6.44) and Proposition 6.4, we finish the proof of Proposition 6.5. Thus, (6.25) is indeed sharp with respect to R_{eff} and m.

Let us formulate, to conclude, the nonautonomous analogue of Proposition 6.3.

Proposition 6.6. *Let the function $f = f(u)$ satisfy* (6.16) *and let, in addition, the external forces g be quasiperiodic with m independent frequencies:*

$$g(t) := G(T_t^\omega \phi_0), \quad G \in C^1(\mathbb{T}^m, \dot{L}^\infty(\Omega)). \tag{6.45}$$

Then, there exists a family $\mathcal{M}_{\nu,m}$ of exponential attractors for equations (6.15) *(with g replaced by $g(t)$) such that*

1. There exists a positive constant $\gamma > 0$ and a monotonic function Q which are independent of ν and m (but which obviously depend on $\|G\|_{C^1}$) such that, for every bounded set $B \subset L^\infty(\Omega)$

$$\operatorname{dist}_{L^\infty(\Omega)} (U_g(\tau + t, \tau)B, \mathcal{M}_{\nu,m}) \le Q(\|B\|_{L^\infty(\Omega)})e^{-\gamma t}. \tag{6.46}$$

2. The fractal dimension of $\mathcal{M}_{\nu,m}$ satisfies the inequalities

$$\dim_F(\mathcal{A}_{\nu,m}, L^\infty(\Omega)) \le \dim_F(\mathcal{M}_{\nu,m}, L^\infty(\Omega)) \le C_1\nu^{-3/2} + m, \tag{6.47}$$

where C is independent of ν and m.

The proof of Proposition 6.6 is based on a rescaling argument and is very similar to that of Proposition 6.3 and we omit it here.

Remark 6.3. Applying the rescaling arguments to Proposition 6.5, we obtain examples of nonautonomous equations (6.15) for which

$$\begin{aligned} C_1\nu^{-3/2} + m \le \dim_F(\mathcal{A}_{\nu,m}, L^\infty(\Omega)) \le \\ \le \dim_F(\mathcal{M}_{\nu,m}, L^\infty(\Omega)) \le C_2\nu^{-3/2} + m, \end{aligned} \tag{6.48}$$

where C_1 and C_2 are independent of ν and m. We also note that estimates of the form (6.48) for *uniform* attractors in *bounded* domains Ω are obtained in [6].

Remark 6.4. To conclude, we note that, although we have considered in this paper only the case of three dimensional domains $\Omega \subset \mathbb{R}^3$, the main results of the paper remain true (after minor changes) for an arbitrary space dimension n. Obviously, in that case, instead of assumption (2.3), we should require that

$$g \in C_b(\mathbb{R}, L_b^q(\Omega)), \quad \text{with } q > \frac{n}{2}, \tag{6.49}$$

and is translation-compact in the local topology of this space, and consider problem (0.1) in the phase space

$$\Phi_b^q := W_b^{2-\delta,q}(\Omega)$$

(see also [12] and [34]).

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