A GRONWALL-TYPE LEMMA WITH PARAMETER AND DISSIPATIVE ESTIMATES FOR PDES

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Abstract. We discuss the problem of establishing dissipative estimates for certain differential equations for which the usual methods apparently do not work. The main tool is a new Gronwall-type lemma with parameter. As an application, we consider a semilinear equation of viscoelasticity with low dissipation.

1. Introduction

Dissipative partial differential equations play a central role in modern mathematical physics, providing an accurate description for a large variety of phenomena occurring in natural sciences. This class of equations includes, among others, Navier-Stokes systems, reaction-diffusion equations, Cahn-Hilliard and Kuramoto-Sivashinsky equations, damped wave equations. Such PDEs are usually reformulated as Cauchy problems in a suitable Banach space $(X, \|\cdot\|)$ (the *phase-space*) of the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\xi(t) = A(\xi(t),t), \quad t>0, \\ \xi(0)=x\in X, \end{cases}$$

where, for every $t \geq 0$, $A(\cdot,t)$ is some operator densely defined on X. The global well-posedness for all initial data $x \in X$ defines the *solution operator* $S(t)$, namely, a one-parameter family of operators $S(t) : X \to X$ such that $S(t)x = \xi(t)$ is the unique solution at time $t \geq 0$ to the Cauchy problem with initial datum x. Further continuity properties of the solutions reflect into analogous continuity properties of $S(t)$. In the autonomous case (i.e., when A does not depend explicitly on t) the maps $S(t)$ form a semigroup of operators.

For such problems, it is well known that the *dissipative* estimate

$$\|S(t)x\| \leq \mathcal{Q}(\|x\|)\mathrm{e}^{-\nu t} + C \tag{1}$$

is crucial in order to establish analytical and dynamical properties of the solutions (see [1, 10, 11, 14] and references therein). Here, $\nu > 0$, $C \geq 0$ and the nonnegative increasing function $\mathcal{Q}$ are independent of the initial datum $x \in X$. Indeed, on the one hand, this estimate often provides the necessary a priori bounds for the solutions, needed to prove the global solvability of the problem. On the other hand, it allows to verify the existence of a so-called *absorbing set* in the phase-space, which is essential for the investigation of the asymptotic properties of the solutions, in terms of global or exponential attractors. Recall that an absorbing set is a bounded set $\mathcal{B} \subset X$ such that, for every $\|x\| \leq R$,

$$S(t)x \in \mathcal{B}, \qquad \forall t \geq t_R,$$

where the *entering time* $t_R \geq 0$ depends (increasingly) only on R.

In many relatively simple situations, (1) can be verified through a differential inequality of the type

$$\psi'(t) + \nu\psi(t) \leq C, \tag{2}$$

where $\psi(t) = \Psi(S(t)x)$ and $\Psi : X \to [0, \infty)$ is an appropriate coercive (energy) functional. Then, the desired dissipative estimate follows from an application of the standard Gronwall lemma. In order to treat more complicated equations, several improvements of the basic differential inequality (2), along with related Gronwall-type estimates, can be found in the literature. As an example, we mention here one possible nonlinear generalization, namely,

$$\varphi'(t) + \nu\psi(t) \leq C,$$

where $\varphi(t) = \Phi(S(t)x)$ and the functional $\Phi : X \to \mathbb{R}$ satisfies

$$\Phi(x) - C_0 \leq \Psi(x) \leq \mathcal{Q}(\Phi(x)),$$

for some $C_0 \geq 0$ and some nonnegative increasing function $\mathcal{Q}$, both independent of the initial datum $x \in X$. As shown in [2], these conditions are sufficient to obtain the dissipative estimate (1).

The aim of this note is to present a new Gronwall-type inequality, somehow already implicitly exploited in some recent papers [13, 15, 16], which allows to establish dissipativity when the standard techniques do not apply (for instance, for the infinite-energy solutions to Navier-Stokes equations in cylindrical domains). The main feature of this method, compared to the usual ones, is that it is not focused on a *single* differential inequality like (2), but rather on the whole family of such inequalities, depending on a parameter $\varepsilon > 0$. In fact, we use different values of the parameter ε in different regions of the phase-space and, even in order to verify estimate (1) for a single trajectory, we need to consider many differential inequalities corresponding to different values of the parameter ε. To be more precise, we study a family of differential inequalities of the form[1]

$$\psi'(t) + \varepsilon\psi(t) \leq C\varepsilon^\alpha[\psi(t)]^\beta + C, \tag{3}$$

depending on a small parameter $\varepsilon > 0$. Here $C > 0$ and $\alpha > \beta \geq 1$. Obviously, the solutions of the ordinary differential equation associated with (3), for any *fixed* $\varepsilon > 0$, may blow up in finite time (if $\beta > 1$) and, for this reason, any single inequality is not strong enough to give the dissipative estimate for ψ. However, as we will see, if the function ψ satisfies *simultaneously* (3) for all $\varepsilon \in (0, \varepsilon_0]$, for some $\varepsilon_0 > 0$, then the conclusion

$$\psi(t) \leq \mathcal{Q}(\psi(0))\mathrm{e}^{-\nu t} + C_\star \tag{4}$$

is still true, for some $\nu > 0$, $C_\star \geq 0$ and some nonnegative increasing function $\mathcal{Q}$ (which can be explicitly written in terms of the parameters α, β and C), so yielding the desired dissipativity.

[1]Actually, (3) is only a particular instance of the more general family of differential inequalities considered in the next Lemma 1.

We will illustrate this method on the following example of an integrodifferential equation arising in the theory of isothermal viscoelasticity

$$\partial_{tt}u - 2\Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\mathrm{d}s + g(u) = f, \tag{5}$$

where g is a dissipative nonlinearity, f is a given time-independent external force and the memory kernel μ satisfies some natural decaying assumptions.

It is worth noting that the dissipative estimate (1) for this problem has been already obtained in [3, 8]. However, in those papers, the proof is strongly based on the existence of a global Lyapunov function, and it does not provide any clue of how to compute nor estimate the monotone function $\mathcal{Q}$ appearing in (1). Thus, although the existence of an absorbing set $\mathcal{B}$ is established, the entering time of a given trajectory into $\mathcal{B}$ can only be predicted theoretically, but not actually computed. For practical purposes, this fact has some quite unpleasant consequences; for instance, if one wants to prove the existence of an *exponential attractor*. This is a compact set $\mathcal{E} \subset X$, of finite fractal dimension, which is positively invariant for $S(t)$ and attracts (with respect to the Hausdorff semidistance $\boldsymbol{\delta}$ in X) bounded sets at an exponential rate; that is, for every $\|x\| \leq R$,

$$\boldsymbol{\delta}\big(S(t)x, \mathcal{E}\big) \leq \mathcal{Q}_\star(R)\mathrm{e}^{-\omega t},$$

for some $\omega > 0$ and some nonnegative increasing function $\mathcal{Q}_\star$ (see [6, 7]). Of course, the notion becomes much more effective, especially in view of numerical computations, if ω and $\mathcal{Q}_\star$ are known. In the situation above, the exponential rate ω can be explicitly calculated, but the function $\mathcal{Q}_\star$ remains unknown. Another essential drawback is that the method cannot be extended to the case of nonautonomous external forces $f = f(t)$, since nonautonomous systems do not possess any Lyapunov functional.

In contrast to that, the application of a Gronwall-type lemma with parameter allows to derive the dissipative estimate (1), with explicit expressions for ν, C and $\mathcal{Q}$, even in the nonautonomous case.

2. A Gronwall-Type Lemma

We now proceed to state our result.

Lemma 1. *Let $\alpha > \beta \geq 1$ and $\gamma \geq 0$ be such that*

$$\frac{\beta - 1}{\alpha - 1} < \frac{1}{1+\gamma}.$$

Let ψ be a nonnegative absolutely continuous function on $[0,\infty)$ which fulfills, for some $K \geq 0$, $Q \geq 0$, $\varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$, the differential inequality

$$\psi'(t) + \varepsilon\psi(t) \leq K\varepsilon^\alpha[\psi(t)]^\beta + \varepsilon^{-\gamma}q(t),$$

where q is any nonnegative function satisfying

$$\sup_{t\geq 0}\int_t^{t+1} q(y)\mathrm{d}y \leq Q.$$

Then, there exists $R_0 > 0$ with the following property: for every $R \geq 0$, there is $t_R \geq 0$ such that

$$\psi(t) \leq R_0, \qquad \forall t \geq t_R,$$

whenever $\psi(0) \le R$. Both R_0 and t_R can be explicitly computed.

Proof. The hypothesis on q implies that, for any $t \ge 0$,

$$\int_t^{t+\tau} q(y)\mathrm{d}y \le Q(1+\tau), \qquad \forall \tau > 0.$$

Due to the assumptions on α, β, γ, we can select $\vartheta \in (0,1)$ satisfying the inequality

$$1 - \vartheta > \max\{\beta - \alpha\vartheta, \gamma\vartheta\}.$$

Calling $\omega = 1 - \gamma\vartheta > \vartheta$, we consider the function

$$J(r) = -\omega r^{\omega - \vartheta} + \omega K r^{\beta - \alpha\vartheta - \gamma\vartheta}.$$

As $\lim_{r\to\infty} J(r) = -\infty$, we can choose $\varrho \ge \omega Q$ such that $\varrho^{-\vartheta/\omega} \le \varepsilon_0$ and

$$J(r) \le -1 - 2\omega Q, \qquad \forall r \ge \varrho^{1/\omega}.$$

Then, we introduce the auxiliary function

$$\varphi(t) = [\psi(t)]^\omega.$$

We preliminarily note that, for (almost) every t such that $\varphi(t) \ge \varrho$, we have

$$\varphi'(t) \le -1 - 2\omega Q + \omega q(t). \tag{6}$$

Indeed, for (almost) any fixed t, setting $\varepsilon = [\varphi(t)]^{-\vartheta/\omega}$ (note that $\varepsilon \le \varepsilon_0$ when $\varphi(t) \ge \varrho$), the differential inequality reads

$$\varphi'(t) \le J\big([\varphi(t)]^{1/\omega}\big) + \omega q(t).$$

• If $\varphi(t) \le \varrho$ for some $t \ge 0$, then $\varphi(t+\tau) \le 2\varrho$, for every $\tau \ge 0$. If not, let $\tau_1 > 0$ be such that $\varphi(t+\tau_1) > 2\varrho$, and set $\tau_0 = \sup\{\tau \in [0,\tau_1] : \varphi(t+\tau) \le \varrho\}$. Integrating (6) on $[t+\tau_0, t+\tau_1]$, we obtain the contradiction

$$2\varrho < \varphi(t+\tau_1) \le \varrho - (\tau_1 - \tau_0) - 2\omega Q(\tau_1 - \tau_0) + \omega Q(1 + \tau_1 - \tau_0) < 2\varrho.$$

• If $\varphi(0) > \varrho$, then $\varphi(t_\star) \le \varrho$, for some $t_\star \le \varphi(0)(1+\omega Q)^{-1}$. Indeed, let $t > 0$ be such that $\varphi(\tau) > \varrho$ for all $\tau \in [0,t]$. Integrating (6) on $[0,t]$, we are led to

$$\varrho < \varphi(t) \le \varphi(0) - t - 2\omega Q t + \omega Q(1+t) \le \varphi(0) - (1+\omega Q)t + \varrho.$$

Therefore, it must be $t < \varphi(0)(1+\omega Q)^{-1}$.

In order to come back to the original $\psi(t)$, just define

$$R_0 = (2\varrho)^{1/\omega} \qquad \text{and} \qquad t_R = R^{1/\omega}(1+\omega Q)^{-1}.$$

By applying the two points discussed above, the proof of the lemma follows. □

As a straightforward consequence, we have

Corollary 2. *Within the assumptions of Lemma 1, ψ fulfills the dissipative estimate* (4)*. Moreover, the function $\mathcal{Q}$ and the constants ν and $C_\star$ can be explicitly computed.*

3. An Application: Viscoelasticity with Low Dissipation

3.1. **The model equation.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For $t \in \mathbb{R}^+ = (0, \infty)$, we consider the evolution system arising in the theory of isothermal viscoelasticity [3, 8]

$$\begin{cases} \partial_{tt}u - \Delta u - \displaystyle\int_0^\infty \mu(s)\Delta\eta(s)\mathrm{d}s + g(u) = f, \\ \partial_t \eta = -\partial_s \eta + \partial_t u, \end{cases} \tag{7}$$

where $u = u(\boldsymbol{x}, t) : \Omega \times [0, \infty) \to \mathbb{R}$, $\eta = \eta^t(\boldsymbol{x}, s) : \Omega \times \mathbb{R}^+ \times [0, \infty) \to \mathbb{R}$, supplemented with the boundary and initial conditions

$$\begin{cases} u(t)|_{\partial\Omega} = \eta^t|_{\partial\Omega} = \eta^t(0) = 0, \\ u(0) = u_0, \quad \partial_t u(0) = v_0, \quad \eta^0(s) = \eta_0(s), \end{cases} \tag{8}$$

$u_0, v_0, \eta_0(s)$ being given data. Here, $\mu : \mathbb{R}^+ \to [0, \infty)$ is a summable absolutely continuous function, with $\mu'(s) < 0$ almost everywhere. Besides, the inequality

$$\mu(s + \sigma) \leq \Theta \mathrm{e}^{-\delta\sigma} \mu(s) \tag{9}$$

holds for some $\Theta \geq 1$ and $\delta > 0$, every $\sigma \geq 0$ and almost every $s \in \mathbb{R}^+$. Without loss of generality, we also assume that

$$\int_0^\infty \mu(s)\mathrm{d}s = 1.$$

The Cauchy problem (7)-(8) is cast in the so-called *memory setting* [4, 5], and is equivalent (see [9]) to the integrodifferential equation (5), with boundary and initial conditions

$$\begin{cases} u(t)|_{\partial\Omega} = 0, \\ u(0) = u_0, \quad u(t)|_{t<0} = u_0 - \eta_0(-t), \quad \partial_t u(0) = v_0. \end{cases}$$

Notation. We set $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ with the usual inner products, and we interpret $-\Delta$ as the positive selfadjoint operator on H of domain $H^2(\Omega) \cap H_0^1(\Omega)$. We also consider the Hilbert space $M = L^2_\mu(\mathbb{R}^+; V)$ of square summable functions on $\mathbb{R}^+$ with values in V, with respect to the measure $\mu(s)\mathrm{d}s$. To account for the boundary conditions on η, we view $-\partial_s$ as the linear operator on M of domain

$$\mathrm{dom}(T) = \{\eta = \eta(s) \in M \,:\, \partial_s \eta \in M, \, \eta(0) = 0\}.$$

Then, $-\partial_s$ is the infinitesimal generator of the right-translation semigroup on M. Finally, we introduce the product Hilbert space $X = V \times H \times M$.

3.2. **Earlier results.** Taking $f \in H$ independent of time and $g \in C^2(\mathbb{R})$, with $g(0) = 0$, such that the growth condition

$$|g''(u)| \leq C\big(1 + |u|\big) \tag{10}$$

and the dissipation condition

$$\liminf_{|u|\to\infty} \frac{g(u)}{u} > -\lambda \tag{11}$$

are satisfied ($\lambda > 0$ being the first eigenvalue of $-\Delta$), problem (7)-(8) generates a strongly continuous semigroup $S(t)$ on X which possesses the global attractor $\mathcal{A}$, see [8]. As a byproduct, this entails the existence of an absorbing set $\mathcal{B}$, but no information is available about the actual entering time in $\mathcal{B}$, starting from a bounded subset of X. On the other hand, due to the very low dissipation, it seems out of reach to prove the existence of $\mathcal{B}$ by means of standard estimates, without appealing to the gradient-system structure (except in the simpler case when g is sublinear). However, with a further (albeit quite general) assumption on the nonlinearity, Lemma 1 allows us to find the absorbing set in a direct way.

3.3. **The absorbing set.** We take $g \in C^1(\mathbb{R})$, with $g(0) = 0$. Instead of (11), we assume the slightly less general condition

$$\liminf_{|u|\to\infty} g'(u) > -\lambda. \tag{12}$$

Moreover, we replace (10) with

$$|g(u)|^{6/5} \leq C\mathcal{G}(u) + C, \tag{13}$$

where

$$\mathcal{G}(u) = \int_0^u g(y)\mathrm{d}y.$$

Remark 3. Conditions (12)-(13) are satisfied, for instance, by any function of the form

$$g(u) = u|u|^p + g_0(u), \qquad p \in (0, 4],$$

with $g_0(0) = 0$ and

$$|g_0'(u)| \leq C\big(1 + |u|^q\big), \qquad q < p.$$

This includes the physically significant case of the derivative of the double-well potential $g(u) = u^3 - u$.

Remark 4. It is easy to check that (13) yields the bound

$$|g(u)| \leq C\big(1 + |u|^5\big). \tag{14}$$

In fact, the sole (12) and (14) are enough to ensure the existence of (possibly nonunique) solutions for all initial data $x \in X$, using a standard Galerkin approximation scheme. We agree to call *Galerkin solutions* those solutions obtained as limits in the approximation scheme, for which formal estimates apply. It is also worth mentioning that it seems impossible to obtain dissipative estimates for Galerkin solutions in presence of a nonlinearity of supercritical growth rate (that is, with reference to the remark above, when $p > 2$) using the Lyapunov function approach, since the asymptotic compactness of such solutions is not known.

For $u = u(\boldsymbol{x}) \in V$, we define

$$G(u) = \int_\Omega \mathcal{G}(u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}.$$

The following lemma is a straightforward consequence of (12).

Lemma 5. *There exist $\kappa \in (0,1)$ and $C \geq 0$ such that*

$$\langle g(u), u\rangle_H \geq G(u) - \frac{\kappa}{2}\|u\|_V^2 - C, \tag{15}$$

$$G(u) \geq -\frac{\kappa}{2}\|u\|_V^2 - C, \tag{16}$$

for every $u \in V$.

Given a Galerkin solution $\xi(t) = (u(t), \partial_t u(t), \eta^t)$, with $\xi(0) = (u_0, v_0, \eta_0)$, we define the corresponding energy by

$$E(t) = \frac{1}{2}\|\xi(t)\|^2.$$

Then, we have

Theorem 6. *Assuming* (12) *and* (13)*, there exists $R_0 > 0$ such that, for every $R \geq 0$ and every $x = (u_0, v_0, \eta_0) \in X$ with $\|x\| \leq R$, the energy $E(t)$ of a related Galerkin solution fulfills the relation*

$$E(t) \leq R_0, \qquad \forall t \geq t_R,$$

for some $t_R \geq 0$ depending only on R. Both R_0 and t_R can be explicitly computed.

Proof. To simplify the calculations, we assume that (9) holds with $\Theta = 1$. In that case, (9) is equivalent to

$$\mu'(s) + \delta\mu(s) \leq 0. \tag{17}$$

Working within the approximation scheme, the (Lyapunov) functional

$$L(t) = E(t) + G(u(t)) - \langle f, u(t)\rangle_H$$

fulfills the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}L = -2I, \tag{18}$$

having set

$$I(t) = -\int_0^\infty \mu'(s)\|\eta^t(s)\|_V^2 \mathrm{d}s \geq \delta\|\eta^t\|_M^2,$$

where the latter inequality follows from (17). Following [12], choosing now $\nu > 0$ small and $s_\nu > 0$ such that $\int_0^{s_\nu} \mu(s)\mathrm{d}s \leq \nu/2$, we put

$$\mu_\nu(s) = \mu(s_\nu)\chi_{(0,s_\nu]}(s) + \mu(s)\chi_{(s_\nu,\infty)}(s),$$

and we introduce the further functionals

$$\Phi_1(t) = -\int_0^\infty \mu_\nu(s)\langle \partial_t u(t), \eta^t(s)\rangle_H \mathrm{d}s,$$
$$\Phi_2(t) = \langle \partial_t u(t), u(t)\rangle_H.$$

Then, exploiting (15) and (17), we have the inequalities (cf. [8, 12])

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 \leq \varepsilon_\nu\|u\|_V^2 - (1-\varepsilon_\nu)\|\partial_t u\|_H^2 + c_\nu I + c_\nu + \int_0^\infty \mu(s)|\langle g(u), \eta(s)\rangle_H|\mathrm{d}s,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2 \leq -(1-\kappa-\varepsilon_\nu)\|u\|_V^2 + \|\partial_t u\|_H^2 - \frac{\kappa}{2}\|u\|_V^2 - G(u) + c_\nu I + c_\nu,$$

for some $c_\nu \geq 0$ and some $\varepsilon_\nu > 0$ such that $\varepsilon_\nu \to 0$ as $\nu \to 0$ (both c_ν and ε_ν can be explicitly computed). It is then clear that, upon fixing ν small enough, the functional

$$\Phi(t) = 2\Phi_1(t) + \Phi_2(t)$$

fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + 2\omega E + \frac{\kappa}{2}\|u\|_V^2 + G(u) \leq cI + 2\int_0^\infty \mu(s)|\langle g(u), \eta(s)\rangle_H|\mathrm{d}s + c, \tag{19}$$

for some $\omega > 0$. Here and in the sequel, $c \geq 0$ stands for a *generic* constant, independent of the initial data. At this point, for (any) $\varepsilon \in (0, \varepsilon_0]$ and $k \geq 0$, we set

$$\Lambda(t) = L(t) + \varepsilon\Phi(t) + k.$$

In light of (16), it is apparent that, provided that $\varepsilon_0 > 0$ is small enough and k is large enough,

$$\frac{1-\kappa}{2}E \leq \Lambda \leq 2E + G(u) + c. \tag{20}$$

In particular, appealing again to (16),

$$2\omega E + \frac{\kappa}{2}\|u\|_V^2 + G(u) \geq \omega\Lambda - c.$$

Thus, collecting (18)-(19) and using (17), up to possibly further reducing ε_0, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \omega\varepsilon\Lambda \leq -\delta\|\eta\|_M^2 + 2\varepsilon\int_0^\infty \mu(s)|\langle g(u), \eta(s)\rangle_H|\mathrm{d}s + c.$$

We now observe that, in view of (16) and (20),

$$|G(u)| \leq G(u) + \kappa E + c \leq \Lambda + c.$$

Therefore, exploiting (13), we find the estimate

$$\begin{aligned}
2\varepsilon\int_0^\infty \mu(s)|\langle g(u), \eta(s)\rangle_H|\mathrm{d}s &\leq c\varepsilon\int_0^\infty \mu(s)\|g(u)\|_{L^{6/5}}\|\eta(s)\|_V\mathrm{d}s \\
&\leq c\varepsilon\|\eta\|_M\|g(u)\|_{L^{6/5}} \\
&\leq c\varepsilon\|\eta\|_M + c\varepsilon\|\eta\|_M|G(u)|^{5/6} \\
&\leq \delta\|\eta\|_M^2 + c\varepsilon^2\Lambda^{5/3} + c.
\end{aligned}$$

In conclusion, for every $\varepsilon \in (0, \omega\varepsilon_0]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \varepsilon\Lambda \leq c\varepsilon^2\Lambda^{5/3} + c.$$

Since by (14) and (20)

$$E(t) \leq \frac{2}{1-\kappa}\Lambda(t), \qquad \Lambda(0) \leq c\big(1 + R^6\big),$$

we obtain the desired result by an application of Lemma 1. □

Remark 7. Theorem 6 holds, with minor changes in the proof, if we consider a time-dependent external force f which is L^2-translation bounded in H.

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